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The Recovery Potential for underfunded Pension Plans

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The Recovery Potential for underfunded Pension Plans*

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Abstract

We investigate whether risk-taking for resurrection type of risk preference (non-constant risk aversion) can increase the probability of achieving inflation-indexed pension benefits at retirement, especially when the starting position is underfunded. By maximizing the expected utility of the ratio of final wealth to a close approximation of this inflation-indexed target fund, we find that this non-constant risk aversion type of utility gives a high degree of certainty about achieving a certain percentage of this desired target fund. The CRRA utility is too risk-averse to overcome under-funding.

Keywords: Inflation-indexed pension benefits, life-cycle investment, state-dependent utility, stochastic optimal control, underfunded pension plans
JEL Classification: C61, D15, D53, G4, G11

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1 Introduction

Insufficient fund to provide adequate benefits at the time of retirement is a major concern for corporations or governments that currently sponsor defined benefit (DB) pension plan for their employees. DB plan sponsor uses funding ratio to measure the proportion of liability that available assets cover. When funding ratio is below 100%, the plan is underfunded. In the period from 2008 to 2021, the American state retirement systems have been funded blow 80%, estimated by Pew Charitable Trusts (2021); Aubry & Wandrei (2021) showed that the aggregate actuarial funding ratio for American state and local pension plans was below 75% in the same period. The 2022 edition of the Milliman Corporate Pension Funding Study (PFS) by Waida, Perry & Clark (2022) studied the largest DB pension plans sponsored by the 100 U.S. public companies, and showed that the aggregate funding ratio was below 90% in the period from 2008 to 2021. These funding ratios improved significantly in 2021, however, pension funding remains volatile due to energy crisis, high inflation and slower economic growth. The issue of insufficient fund is also common in countries such as Iceland and Mexico; the funding ratio aggregated at the national level in countries such as Indonesia and the United Kingdom deteriorated over the last decade and its values were below 100% in 2019 and 2020, reported by OECD (2021). To ease the burden, a transition from DB pension plan to defined contribution (DC) pension plan has been taken place worldwide. OECD (2021) estimated that the fastest shift away from DB plans happened in Nigeria (from 45% of assets in 2010 to 13% in 2020) and Israel (from 77% in 2010 to 51% in 2020). The Netherlands and the United Kingdom are also transitioning from a DB to a DC system. Instead of employers, in a DC pension plan, employees bear most of the risk of planned contributions being insufficient to fund the benefits. The size of the fund at retirement strongly depends on the amount and frequency of contributions and the returns the fund generates, hence participants in a DC pension plan are not confident about the level of retirement income.

In periods with high inflation, participants in a DC pension plan also have deep concerns about the stability of their purchasing power. Namely, the purchasing power of a certain amount of money reduces dramatically when the annual inflation rate is high. For instance, with an annual inflation rate of three percent, the purchasing power of a certain amount of money loses around twenty-five percent of its initial value after ten years. Merton (2014) suggested that an inflation-indexed annuity that pays the same amount (adjusted for inflation) each year after retirement guarantees a specific level of retirement income. It also pointed out that investment value and asset volatility are not the right measures if our goal is to obtain a particular future income. Our paper studies to which extend the inflation-indexed pension benefits at retirement could be obtained in a DC pension plan when we start from an underfunded position. In our study, the position is underfunded when the current value of the fund is lower than the expected value of the future payments under risk neutral measure.

In classical life cycle problems, the optimal criteria which determine the investment strategy are often stated in terms of nominal wealth at a certain future time. Risk preferences are then expressed in terms of a utility function of an individual agent, and investment policy is chosen which maximizes the expected utility of the terminal wealth. A well-studied utility function is power utility function. Its relative risk aversion is constant, i.e., CRRA risk preference. The usual stochastic optimization problem, which considers the expected utility of terminal wealth under CRRA preferences, was first studied in Merton (1969) and Merton (1971). The agent has limited choice of investing the wealth in only two different assets: a risky asset (for example a stock) and a risk-free asset (for example a bank account). Such problems can be solved by dynamic programming techniques when a Markovian assumption on the state process is satisfied. Optimal strategies can then be derived after solving a non-linear partial differential equation, the Hamilton-Jacobi-Bellman equation, which characterizes the best possible expected utility of terminal wealth. With CRRA utility function, these dynamic programming techniques lead to closed form solution. The optimal allocation strategy is to hold a constant fraction of the wealth in the risky asset, also known as the Merton ratio. Later, Cox & Huang (1989) showed that in complete markets this dynamic optimization problem can be reduced to a static problem. This leads to explicit solutions based on a dual approach and even allows the Markov assumption on state variables to be dropped.

An extension in which the agent optimizes expected utility but must, under all circumstances, outperform a certain (stochastic) benchmark was treated as a constraint on the terminal wealth, for instance in Grossman & Zhou (1996) and Tepla (2001). Tepla (2001) shows that the optimal investment policy under such a constraint can be interpreted as an investment in the benchmark plus an investment of the remaining wealth in a contingent claim which has a positive value. If the value of agent's initial wealth and the benchmark are equal, the solution to the maximizing problem is simply to invest in the benchmark for the entire investment period. Therefore, this approach doesnot cope with under-funding.

Including a reference level in the utility function may be more suitable for an agent with the desire to achieve some target, since such functions evaluate outcomes relative to this reference level, see Warren (2019). A reference dependent utility function could be defined over the difference between final wealth and target wealth or over the ratio. Earlier research for DC pension plans, Cairns et al. (2006), linked a stream of contribution premiums to salary level and maximized the expect CRRA utility of the ratio of the terminal wealth to the salary level at retirement. This approach has the advantages of taking into account the participant's attitude to risk and the correlation between the salary level and asset returns. Instead, Han & Huang (2012) maximized the expected CRRA utility of the terminal real wealth in excess of a stochastic guarantee. The optimal asset allocations with stock, nominal bond, indexed bond and cash were intensively studied in their study. Both papers solved the optimal asset allocation by dynamic programming techniques. Though, neither of them considered underfunded starting position.

In the framework of prospect theory (PT), Blake, Wright & Zhang (2013) and Donnelly, Khemka & Lim (2022) assumed agent to be loss-averse instead of risk-averse with respect to the present value of the target retirement pension fund. The utility function in this framework is "S"-shaped: convex below the reference point and concave above it, implying that participant with this utility is risk seeking in the domain of losses and risk-averse in the domain of gains. With this strategy, Blake, Wright & Zhang (2013) showed that participant increases proportional wealth in risky assets if the accumulating fund is below the present value of the target fund and decreases the proportion if the fund is above it, unless the fund is very much above target. Compared to CRRA preference, this strategy is more focused on achieving the specified target fund level. Donnelly, Khemka & Lim (2022) extended the framework by incorporating a stochastic non-tradable labour income process and imposed time-dependent upper and lower bounds to ensure the participant's fund value is between particular bounds at retirement. It showed that the participant's retirement outcomes are robustly centered around the target fund; and that imposing terminal wealth constraints does not improve the certainty of achieving the desired target, while it increases the chance of obtaining a lower income. One drawback of this framework is that it is not easy to implement because it requires the solution of a nonlinear dynamic programming problem whenever there is new information about the key state variables, stated in Blake, Wright & Zhang (2013).

To investigate the effect of risk preference on the probability of achieving the target fund, we look beyond the classical choice of CRRA risk preferences. In this paper we make use of the class Symmetric Asymptotically Hyperbolic Absolute Risk Aversion (SAHARA) utility functions. This class of utility functions is first introduced by Chen, Pelsser & Vellekoop (2011). In contrast to what holds in the CRRA case, SAHARA risk preferences show decreasing risk aversion as a function of wealth when the level of wealth is below a pre-specified wealth level: the threshold. This implies that investment strategies will prescribe more instead of less risk-taking once funding ratios fall below this threshold. However, the agent stays risk-averse all the time. This "risk-taking for resurrection" prevents that excessive de-risking leads to an almost risk-less investment strategy, which makes it impossible to leave the situation of under-funding. SAHARA utility functions also enable the agent to have a high level of risk aversion around the threshold, which could be interpreted as locking-in the desired outcomes. With SAHARA utility functions a variety of risk preferences could be specified by setting a threshold and modeling risk preference around it.

In our paper, we adapt the financial market that considered in Brennan & Xia (2002), which accommodates time-varying interest rates and inflation rates. Our target fund at retirement is the expected present value of inflation-indexed future pension income at retirement under risk neutral measure. This income is generated by a series of indexed zero bonds. We maximize the expected utility of the ratio of final wealth to a close approximation of the target fund. This stochastic approximation is contingent to the interest rate and the price level at retirement. It is the benchmark in our model. The ratio of final wealth to this benchmark is called the replacement ratio at retirement. With the assumption of complete market and no-arbitrage, the optimal asset allocation policies for a CRRA or SAHARA agent is derived in closed-form.

We find that the probability of overcoming under-funding could be improved when the agent sets a threshold, and becomes less risk-averse when the replacement ratio falls below this threshold. When the agent tends to lock-in the desired outcomes, modeled by high level of risk aversion around the threshold, the probability density distribution of the replacement ratios are centered around the threshold. Compared to CRRA risk preference, the optimal investment strategies derived for these types of risk preference ensure that the probability density distribution of the replacement ratio is more towards the benchmark.

The rest of the paper is organized as follows. We present the financial model and the benchmark in Section 2. Section 3 introduces briefly SAHARA risk preference. Section 4 derives the solutions to the optimal portfolio problem for agent with power and SAHARA risk preferences. Section 5 shows numeric examples. In this section we make a comparison of the performance with and without a benchmark for power utility; then explore the probability density distribution when the agent has different types of risk preference, in particular, below and around the threshold. We also study the corresponding investment strategies. Section 6 summarizes the results and draws the conclusion.

2 Financial market and Benchmark

We adapt the financial market considered by Brennan & Xia (2002). The financial market consists of three state variables: stochastic real interest rate r_t, stochastic instantaneous expected inflation rate π_t and stochastic stock price S_t. The loading on the innovations are constant. The market is complete, so that all the state variables can be spanned by the asset returns. We add a desired benchmark at retirement to the model. The agent can invest in a nominal instantaneous risk free asset, stock, nominal/indexed bond.

2.1 Financial market

The (commodity) price level Π follows a diffusion process:

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \xi_S dW_{S,t} + \xi_r dW_{r,t} + \xi_\pi dW_{\pi,t}, \tag{1}$$

where π_t is the instantaneous expected rate of inflation and follows an Ornstein-Uhlenbeck process:

$$\begin{aligned} d\pi_t =& \kappa_\pi(\bar{\pi} - \pi_t)dt + \sigma_\pi dW_{\pi,t}, \\ \pi_s =& \bar{\pi} + e^{-\kappa_\pi(s-t)}(\pi_t - \bar{\pi}) + \sigma_\pi \int_t^s e^{-\kappa_\pi(s-u)} dW_{\pi,u}. \end{aligned} \tag{2}$$

The Ornstein-Uhlenbeck process is mean-reverting, which means that in the long run, the process tends to drift towards its long-term mean $\bar{\pi}$. The intensity of this mean-reverting tendency is scaled by the parameter κ_π. ξ_S, ξ_r and ξ_π represent the constant loadings on the stochastic innovations. All the Ws are standard Brownian motions under a probability measure $\mathbb{P}$. $dW_t = [dW_{S,t}, dW_{r,t}, dW_{\pi,t}]'$. The Brownian motions dW_S, dW_r, dW_π are assumed to be correlated with correlation coefficient ρ as follows:

$$\rho = \begin{pmatrix} 1 & \rho_{Sr} & \rho_{S\pi} \\ \rho_{Sr} & 1 & \rho_{r\pi} \\ \rho_{S\pi} & \rho_{r\pi} & 1 \end{pmatrix}$$

Consequently,

$$\frac{\Pi_s}{\Pi_t} = \exp\left\{ \int_t^s \left(\pi_u - \frac{1}{2}\xi' \rho \xi\right) du + \int_t^s \xi' dW_u \right\}, \tag{3}$$

where $\xi = [\xi_S, \xi_r, \xi_\pi]'$.

Similarly, the instantaneous real riskless interest rate, r_t, follows the Ornstein-Uhlenbeck process:

$$\begin{aligned} dr_t =& \kappa_r(\bar{r} - r_t)dt + \sigma_r dW_{r,t}, \\ r_s =& \bar{r} + e^{-\kappa_r(s-t)}(r_t - \bar{r}) + \sigma_r \int_t^s e^{-\kappa_r(s-u)} dW_{r,u}, \end{aligned} \tag{4}$$

where $\bar{r}$ is the long term mean level of the real interest rates. The nominal stock price S_t follows a Geometric Brownian motion,

$$\frac{dS_t}{S_t} = (R_t + \sigma_S \lambda_S)dt + \sigma_S dW_{S,t}, \tag{5}$$

where λ_S is the constant unit risk premium associated with the innovation $dW_{S,t}$, and R_t is the nominal interest rate at time t. $R_t = r_t + \pi_t - \xi_S\lambda_S - \xi_r\lambda_r - \xi_\pi\lambda_\pi$, where λ_r and λ_π are the constant unit risk premium associated with the innovation $dW_{r,t}$ and $dW_{\pi,t}$, respectively. Moreover, $\lambda = \rho\xi - \rho\omega$ with ω being $\omega = [\omega_S, \omega_r, \omega_\pi]'$.

The real pricing kernel of the economy which determines the expected returns on all securities, M_t is given by:

$$\begin{aligned} \frac{dM_t}{M_t} =& - r_t dt + \omega_S dW_{S,t} + \omega_r dW_{r,t} + \omega_\pi dW_{\pi,t} \\ =& - r_t dt + \omega' dW_t, \end{aligned} \tag{6}$$

where ωs represent the constant loadings on the stochastic innovations in the economy. The pricing kernel relative can be written as follows:

$$\frac{M_s}{M_t} = \exp\left\{ \int_t^s \big(- r_\tau - \frac{1}{2}\omega'\rho\omega\big)d\tau + \int_t^s \omega' dW_\tau \right\}. \tag{7}$$

If there are at least four securities whose instantaneous variance-covariance matrix has rank three, the state variables can be spanned. Moreover, the variance-covariance matrix of real returns on cash, stock and two finite maturity bonds with different maturities has rank three.

The nominal price at time t of a bond which matures at time T with a nominal payoff of 1, satisfies:

$$P(t,T) = \exp\ \{A(t,T) - B(t,T)r_t - C(t,T)\pi_t\}, \tag{8}$$

where $A(t,T)$, $B(t,T)$, and $C(t,T)$ are time-dependent constants, in particular $B(t,T) = \kappa_r^{-1}(1-e^{\kappa_r(t-T)})$, $C(t,T) = \kappa_\pi^{-1}(1 - e^{\kappa_\pi(t-T)})$ and the expression for $A(t,T)$ is in Appendix 2. As a result,

$$\frac{dP(t,T)}{P(t,T)} = \bigg(R_t - B(t,T)\sigma_r\lambda_r - C(t,T)\sigma_\pi\lambda_\pi\bigg)dt - B(t,T)\sigma_r dW_{r,t} - C(t,T)\sigma_\pi dW_{\pi,t}, \tag{9}$$

The component $-B(t,T)\sigma_r\lambda_r - C(t,T)\sigma_\pi\lambda_\pi$ represents its nominal risk premium. The derivations of this equation can be found in Brennan & Xia (2002).

In addition, the real price at time t of a indexed bond which matures at time T with a real payoff of 1, is given as follow:

$$p^*(t,T) = \exp\{A^*(t,T) - B(t,T)r_t\}, \tag{10}$$

where $A^*(t,T)$ is a time-dependent constant and its expression is in Appendix 2. Similarly,

$$\frac{dp^*(t,T)}{p^*(t,T)} = \big(r_t - B(t,T)\sigma_r(-\omega_s\rho_{Sr} - \omega_r - \omega_\pi\rho_{r\pi})\big)dt - B(t,T)\sigma_r dW_{r,t}. \tag{11}$$

Its nominal return is $P^*(t,T) = \Pi_t\ p^*(t,T)$:

$$\frac{dP(t,T)}{P^*(t,T)} = (r_t + \pi_t - B(t,T)\sigma_r\lambda_r)\,dt + \xi_S dW_{S,t} + (\xi_r - B(t,T)\ \sigma_r)dW_{r,t} + \xi_\pi dW_{\pi,t}. \tag{12}$$

2.2 Benchmark

To model the pension assets at retirement T, we assume the pensioner can live τ more years after reaching the retirement age. In particular, we consider an individual who starts working at age 25, retires at age 65 and passes away at age 85. The current time is at age 25. Then the investment horizon is 40 years, i.e. T = 40, and $\tau = 20$. To take purchasing power into account, the pensioner is assumed to receive an annuity that pays an amount of 1 in real terms or Π_{T+i} ($i \in [0, \tau]$) in nominal terms at the end of each year while the pensioner is alive. Early withdrawal before the retirement age is not allowed. We use τ indexed bonds to generate these payments. To buy these indexed bonds at retirement T, the required amount of asset in real terms at time T is:

$$\begin{aligned} a_T &= p^*(T, T+1) + p^*(T, T+2) + \cdots + p^*(T, T+\tau) \\ &= e^{A^*(T,T+1) - B(T,T+1) r_T} + e^{A^*(T,T+2) - B(T,T+2) r_T} + \cdots + e^{A^*(T,T+\tau) - B(T,\mathrm{T}+\tau) r_T}. \end{aligned} \tag{13}$$

At time 0, the following equations hold:

$$\begin{aligned} \mathrm{E}[a_T] &= \sum_{i=1}^{\tau} e^{A(T,T+i) - B(T,T+i)\mathrm{E}_t[r_T] + \frac{1}{2} B(T,T+i)^2 \mathrm{Var}[r_T]}, \\ \mathrm{E}[a_T^2] &= \sum_{i=1}^{\tau} \sum_{j=1}^{\tau} e^{\left(A(T+i) + A(T,T+j)\right) - \left(B(T,T+i) + B(T,T+j)\right)\mathrm{E}[r_T] + \frac{1}{2}\left(B(T,T+i) + B(T,T+j)\right)^2 \mathrm{Var}[r_T]}, \\ \mathrm{Var}[a_T] &= \mathrm{E}[a_T^2] - \left(\mathrm{E}[a_T]\right)^2, \end{aligned} \tag{14}$$

where $\mathrm{E}[r_T] = \bar{r} + e^{-\kappa_r T}(r_0 - \bar{r})$, $\mathrm{Var}[r_T] = \frac{\sigma_r^2}{2\kappa_r}(1 - e^{-2\kappa_r T})$, and r_T is normally distributed with $\mathrm{E}[r_T]$ and $\mathrm{Var}[r_T]$. This functional form of a_T doesnot permit integration in closed-form, hence, we use the principle of Fenton-Wilkinson (FW) Approximation method to generate a log-normal variable L_T such that

$$\begin{aligned} L_T &= e^{m - n r_T} \ (m, \, n \text{ constant}), \\ \mathrm{E}[a_T] = \mathrm{E}[L_T] &= e^{m - n\mathrm{E}[r_T] + \frac{1}{2} n^2 \mathrm{Var}[r_T]}, \\ \mathrm{Var}[a_T] = \mathrm{Var}[L_T] &= \left(\mathrm{E}[L_T]\right)^2 (e^{n^2 \mathrm{Var}[r_T]} - 1). \end{aligned} \tag{15}$$

Solving for m and n gives

$$\begin{aligned} n &= \sqrt{\frac{\ln\left(\mathrm{Var}[a_T]/(\mathrm{E}[a_T])^2 + 1\right)}{\mathrm{Var}[r_T]}}, \\ m &= \ln\left(\mathrm{E}[a_T]\right) + n\mathrm{E}[r_T] - \frac{1}{2} n^2 \mathrm{Var}[r_T]. \end{aligned} \tag{16}$$

L_T is our benchmark. Its value approximates that of the required amount of asset at time T, i.e. a_T. Important to note that Fenton-Wilkinson (FW) Approximation method is a lognormal approximation for sums of log-normal variables based on the first and the second moment-matching. It is introduced in Fenton (1960). This approximation method is satisfactory in our case. We will illustrate it with numeric examples.

2.3 Numeric Examples for a_T and L_T

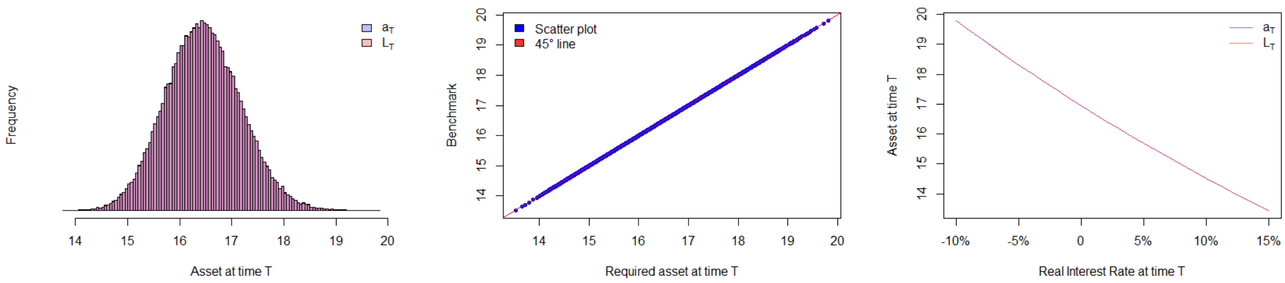
In this subsection we give numeric examples to show how well the value of a_T can be approximated by that of L_T. Table 1 shows model parameters for the simulation. In particular, the starting value of the real interest rate r_0 is 1%, and its long-term mean $\bar{r}$ is 2%. The intensity of its mean-reverting tendency take the value from $\{0.1, 0.6\}$. The Brownian motion increments underlying the real interest rate level is negatively correlated with that underlying the stock price level and the instantaneous expected rate of inflation.

Table 1: Overview of model parameters.

$r_0 = 1\%$	$\bar{r} = 2\%$	$\kappa_r = \{0.1, 0.6\}$
$\omega_S = -0.20$	$\omega_r = 0.17$	$\omega_\pi = 0.12$
$\rho_{Sr} = -0.12$	$\rho_{r\pi} = -0.06$	$\sigma_r = 0.026$

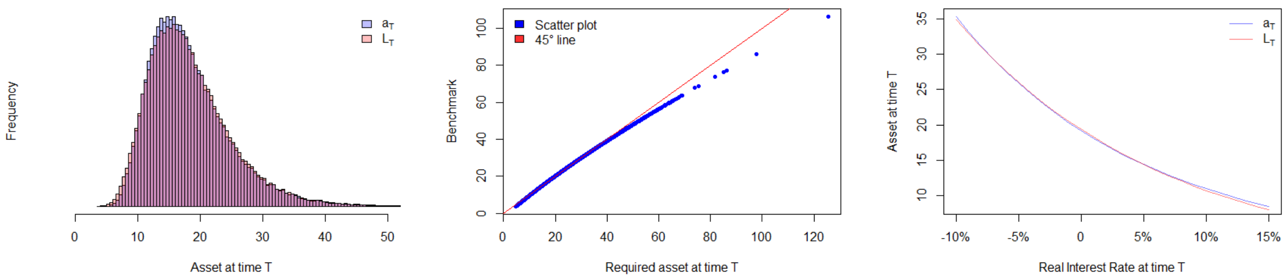
We run 100,000 simulations and draw the histogram of a_T and L_T to study their distributions, also make plots and tables to study the gap between the value of a_T and L_T.

Figure 1: Asset a_T & L_T with $\kappa_r = 0.6$.



(a) Histogram of a_T & L_T. (b) Scatter plot of a_T & L_T. (c) Plot of a_T & L_T.

Figure 2: Asset a_T & L_T with $\kappa_r = 0.1$.



(a) Histogram of a_T & L_T. (b) Scatter plot of a_T & L_T. (c) Plot of a_T & L_T.

Panel a of Figure 1 and Figure 2 show that the distribution of the benchmark L_T closely matches that of the required asset at time T, i.e. a_T. Panel b and panel c of Figure 1 reveal that the value of L_T and a_T are close to each other when the intensity of the real interest rate's mean-reverting tendency is high (for instance 0.6). Table 2 displays that there is almost no gap between the values of a_T and L_T for all potential value of the real interest rate at time T. When this intensity is low (for instance 0.1), the gap between the value of a_T and L_T increases when the value of the real interest rate at time T is very low or very high, indicated on panel b and panel c of Figure 2. Table 3 shows that the gap is below 0.5 when the real interest rate at time T is in the range of -10% and 15%, which is around 95% of the cases. When the real interest rate is very high or very low, the benchmark L_T serves as a lower bound of the required asset at time T.

Table 2: Gap between a_T and L_T with $\kappa_r = 0.6$.

	r_T					
	−10%	−5%	0%	5%	10%	15%
a_T	19.80	18.32	16.95	15.69	14.52	13.45
L_T	19.79	18.32	16.95	15.69	14.52	13.44

Table 3: Gap between a_T and L_T with $\kappa_r = 0.1$.

	r_T					
	−10%	−5%	0%	5%	10%	15%
a_T	35.29	25.86	19.19	14.43	11.00	8.52
L_T	34.87	25.98	19.36	14.43	10.75	8.01

3 Modeling Preferences

In this section, we briefly introduce the SAHARA utility functions. Different from the CRRA class of utility functions, these utility functions are defined for all wealth levels (also negative wealth levels) and enable a threshold wealth where the absolute risk aversion reaches a finite maximal value, in other words, the agent is the most risk-averse at this threshold wealth. Its absolute risk aversion function $A(x)$ satisfies

$$A(x) := -\frac{U''(x)}{U'(x)} = \frac{\alpha}{\sqrt{\beta^2 + (x - w_0)^2}} > 0 \tag{17}$$

for a certain scale parameter $\beta > 0$, risk aversion parameter $\alpha > 0$ and threshold wealth $w_0 \in R$. Clearly, $A(x)$ is strictly positive for all values of $x \in R$ which shows that U presents preferences which are risk-averse for all wealth levels.

The utility functions U can be written in an explicit form[1] by solving the differential equation (17), which is as follows:

$$U(x) = \begin{cases} -\frac{1}{\alpha^2-1}\big((x - w_0) + \sqrt{\beta^2 + (x - w_0)^2}\big)^{-\alpha}\big((x - w_0) + \alpha\sqrt{\beta^2 + (x - w_0)^2}\big) & \alpha \neq 1 \\ \frac{1}{2}\ln\big((x - w_0) + \sqrt{\beta^2 + (x - w_0)^2}\big) + \frac{1}{2}\beta^{-2}(x - w_0)\big(\sqrt{\beta^2 + (x - w_0)^2} - (x - w_0)\big) & \alpha = 1 \end{cases}, \tag{18}$$

where $x \in R$. In addition,

$$U'(x) = \big((x - w_0) + \sqrt{\beta^2 + (x - w_0)^2}\big)^{-\alpha} = \beta^{-\alpha} e^{-\alpha \operatorname{arcsinh}\left((x-w_0)/\beta\right)}, \tag{19}$$

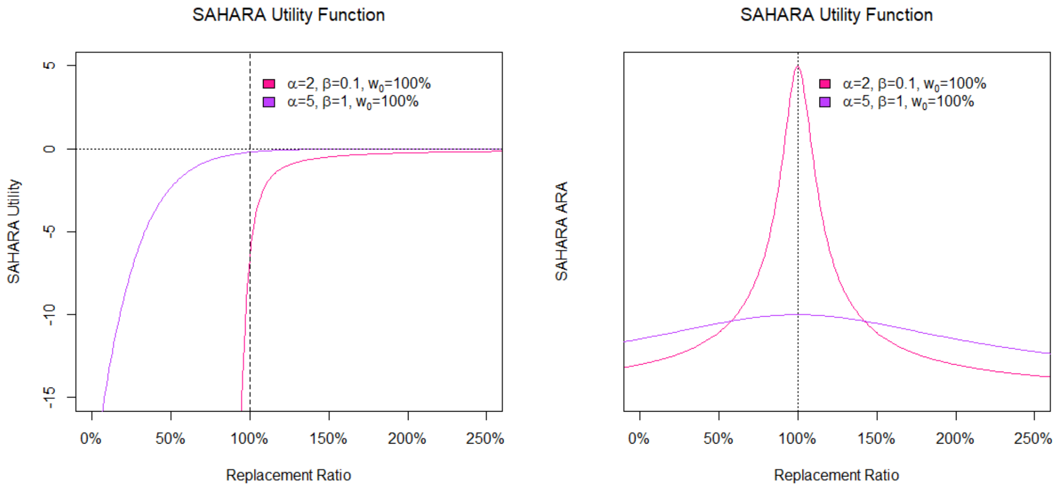
and

$$I(y) = \big(U'\big)^{-1}(y) = \beta \sinh\Big(-\frac{1}{\alpha}\ln y - \ln\beta\Big) + w_0 = \frac{1}{2}\big(y^{-1/\alpha} - \beta^2 y^{1/\alpha}\big) + w_0 \tag{20}$$

with domain $y \in R^+$.

[1]See Chen, Pelsser & Vellekoop (2011) for these closed-form expressions and a more extensive discussion of the properties of SAHARA utility functions.

Figure 3: Plot of SAHARA utility function and absolute risk aversion (ARA) with $w_0 = 100\%$.



(a) SAHARA utility function. (b) ARA SAHARA utility function.

Figure 3 shows two examples of the resulting SAHARA utility functions U and the corresponding absolute risk aversion functions. The SAHARA utility functions are defined on the whole real line and are concave everywhere, which indicates that it has a higher dis-utility for the same loss at low wealth than at high wealth. Define the replacement ratio at time T as $C_T := \frac{X_T}{\Pi_T L_T}$. To give some more intuition for the risk aversion parameters α and β in the SAHARA utility function, we can compare the absolute risk-aversion γ/C_T for the CRRA utility function, with the risk aversion $\alpha/\sqrt{\beta^2 + (C_T - 1)^2}$ for the SAHARA utility function. We see two important quantitative differences between the two utility functions:

- A CRRA utility function will always ensure that $C_T > 0$, as the risk aversion γ/C_T approaches infinity for $C_T \downarrow 0$. However, for SAHARA utility the risk aversion stays finite, even for $C_T < 0$. This means that negative wealth is possible for SAHARA utility, and we may have to control for this.
- The CRRA utility function attaches no specific significance to the replacement ratio $C_T = 1$. On the other hand, the SAHARA risk aversion peaks at $C_T = 1$, and decreases to zero for $C_T > 1$ or $C_T < 1$. Given α, the height of the peak in risk aversion is controlled by the parameter β: smaller β, higher peak. The peak in risk aversion at $C_T = 1$ implies that the optimal investment strategy will de-risk near $C_T = 1$ and will be more risk-taking for $C_T > 1$ or $C_T < 1$. For replacement ratios $C_T \ll 1$, we can explain this behavior as "risk-taking for resurrection"; for replacement ratios $C_T \ll 1$, we can explain the behavior as pure return-seeking behavior. This feature of positive but diminishing risk aversion when we reach low levels of wealth (or, in our case replacement ratios) helps to overcome an initial position of under-funding.

We remark that the SAHARA class of preferences contains the exponential and power utility functions as limiting cases. When $\beta \downarrow 0$, $w_0 = 0$, we find for $x > 0$ that $A(x) = \frac{\alpha}{x}$, which corresponds to the absolute risk aversion function of the power utility function. On the other hand, if $\alpha \uparrow \infty$ and $\beta \uparrow \infty$, while $\frac{\alpha}{\beta}$ stays constant, we find that $A(x) \to \frac{\alpha}{\beta}$ which corresponds to that of the exponential utility function.

When we maximize the expected utility of the replacement ratio $\frac{x}{L_t}$, the absolute risk aversion function for SAHARA utility functions becomes $A(x) = \frac{\alpha/L_t}{\sqrt{\beta^2 + (x/L_t - d)^2}}$. The ratio $\frac{\alpha}{L_t}$ can interpreted as the adjusted risk aversion parameter. For CRRA utility functions with risk aversion parameter γ, its $A(x)$ is $\frac{\gamma}{x}$, no matter the agent maximize with respect to wealth x or the replacement ratio $\frac{x}{L_t}$. Rewrite it as $\frac{\gamma/L_t}{x/L_t}$, then the absolute risk aversion function of CRRA preference and SAHARA preference can be compared with each other.

4 Optimizing the Replacement Ratio at Retirement

In our model, an agent tries to maximize the expected utility of the replacement ratio at time T, $C_T := \frac{X_T}{\Pi_T L_T}$. Furthermore, taking short positions is allowed. The objective function can be specified as follows:

$$\max_{X \in \mathscr{A}(X_0)} E\left[U\left(\frac{X_T}{\Pi_T L_T}\right)\right], \tag{21}$$

where $\mathscr{A}(X_0)$ denotes the class of all possible wealth processes that can be generated by self-financing strategies θ in this market with an initial capital X_0. Particularly, $X_0 = \phi E[M_T L_T]$ with ϕ between 0 and 1, indicating an underfunded starting position. Following Cox & Huang (1989) this dynamic stochastic optimal control problem (21) can be formulated as a static optimization problem. This implies that the objective function can be specified as follows:

$$\begin{aligned} &\max_{X_T} \; E\left[U\left(\frac{X_T}{\Pi_T L_T}\right)\right], \\ &\text{s.t. } E\left[M_T \cdot \frac{X_T}{\Pi_T}\right] = X_0 \end{aligned} \tag{22}$$

Closed-form solutions can be derived for CRRA risk preference and SAHARA risk preference in complete markets.

4.1 CRRA Preferences

Consider the problem of an agent with an CRRA utility function. The objective function becomes:

$$\begin{aligned} &\max_{X_T} \; E\left[\frac{\left\{X_T/(\Pi_T L_T)\right\}^{1-\gamma}}{1-\gamma}\right], \\ &\text{s.t. } E\left[M_T \cdot \frac{X_T}{\Pi_T}\right] = X_0. \end{aligned} \tag{23}$$

The optimal terminal wealth X_T^* is given by:

$$X_T^* = X_0 L_T^{1-1/\gamma} \, \frac{\Pi_T M_T^{-1/\gamma}}{F(0,T,1-1/\gamma)}, \tag{24}$$

where

$$F(t,T,\gamma) = \mathrm{E}_t\left[\left(\frac{M_T}{M_t} L_T\right)^{\gamma}\right] = e^{\gamma c_1(t,T) + \gamma^2 c_2(t,T)}, \tag{25}$$

$$\begin{aligned} c_1(t,T) &= -\frac{1}{2} V_M (T-t) + m - \left(n - n e^{\kappa_r (t-T)} + (T-t) - B(t,T)\right) \bar{r} - \left(B(t,T) + n e^{\kappa_r (t-T)}\right) r_t, \\ c_2(t,T) &= \frac{1}{2}\sigma_r^2 I_2(t,T) + \frac{1}{2} V_M (T-t) + \frac{1}{2}(n\sigma_r)^2 I_4(t,T) \\ &\quad - \sigma_r (\omega_S \rho_{Sr} + \omega_r + \omega_\pi)\left(I_1(t,T) + n I_3(t,T)\right) + n\sigma_r^2 I_5(t,T). \end{aligned} \tag{26}$$

The expression for I_i ($i \in [1,5]$) are in Appendix 3.

Assume at time t the agent invests an optimal proportion of wealth in stock, a nominal bond with maturity T_1, and a nominal bond with maturity T_2. The remaining wealth, $1 - i'\theta^*$, is invested in cash.

The vector of optimal proportional wealth allocation to the stock and two nominal bonds, $\theta_t^* = (\theta_S^*, \theta_1^*, \theta_2^*)'$, is given by

$$\theta_t^* = \frac{1}{\gamma}\Omega^{-1}\Lambda + (1-\frac{1}{\gamma})\Omega^{-1}\sigma\rho(\xi_S, \xi_r - \big(B(t,T) + ne^{\kappa_r(t-T)}\big)\sigma_r, \xi_\pi)', \tag{27}$$

where $\Omega = \sigma\rho\sigma'$, $\Lambda = \sigma\lambda$, with

$$\sigma = \begin{pmatrix} \sigma_S & 0 & 0 \\ 0 & -B(t,T_1)\sigma_r & -C(t,T_1)\sigma_\pi \\ 0 & -B(t,T_2)\sigma_r & -C(t,T_2)\sigma_\pi \end{pmatrix}.$$

The optimal proportional wealth allocation to the assets depend on the risk aversion γ, the maturity of the bond T_1 and T_2, the investment horizon T, also the life expectancy after retirement τ. The equation (27) expresses the optimal portfolio as the sum of two portfolios. The first portfolio $\Omega^{-1}\Lambda$ is the nominal mean-variance tangency portfolio. The amount is inversely related to the agent's relative risk aversion γ. The second portfolio with weight $1-\frac{1}{\gamma}$ has the largest correlation with the present value of the benchmark in nominal terms. In addition, the proportion $-\omega_S/(\sigma_S\gamma)$ invested in stock is constant.

In case the agent invests in stock, a nominal bond with maturity T_1, and an indexed bond with maturity T_2,

$$\sigma = \begin{pmatrix} \sigma_S & 0 & 0 \\ 0 & -B(t,T_1)\sigma_r & -C(t,T_1)\sigma_\pi \\ \xi_S & \xi_r - B(t,T_2)\sigma_r & \xi_\pi \end{pmatrix}.$$

The proportion invested in stock is constant only if ξ_S is zero.

The optimal final wealth without benchmark is a special case of that with benchmark, which can be achieved by equating m and n to 0. The optimal proportional wealth allocation is shown in Brennan & Xia (2002) as follows:

$$\theta_t^* = \frac{1}{\gamma}\Omega^{-1}\Lambda + (1-\frac{1}{\gamma})\Omega^{-1}\sigma\rho(\xi_S, \xi_r - B(t,T)\big)\sigma_r, \xi_\pi)'. \tag{28}$$

The difference between equation (27) and equation (28) is the term $ne^{\kappa_r(t-T)}$, which is originated from the element $c_1(t,T)$ of the benchmark L_T. The term $ne^{\kappa_r(t-T)}$ cancels partially the effect of $B(t,T)$ off when t gets close to T, i.e when the investment horizon get short.

4.2 SAHARA Preferences

For SAHARA utility functions, the optimal final wealth is:

$$X_T^* = I(vM_TL_T)\Pi_TL_T = \frac{1}{2}\left((vM_TL_T)^{-\frac{1}{\alpha}} - \beta^2(vM_TL_T)^{\frac{1}{\alpha}}\right)\Pi_TL_T + w_0\Pi_TL_T, \tag{29}$$

For some t $\in [0,T]$,

$$\begin{aligned}\frac{X_t^*}{\Pi_t} =& E_t\big[\frac{M_T}{M_t}\frac{X_T^*}{\Pi_T}\big] \\ =& \frac{1}{2}\mathrm{E}_t\left[(vM_t)^{-1/\alpha}\left(\frac{M_T}{M_t}L_T\right)^{1-1/\alpha}\right] \\ &- \frac{1}{2}\beta^2\mathrm{E}_t\left[(vM_t)^{1/\alpha}\left(\frac{M_T}{M_t}L_T\right)^{1+1/\alpha}\right] + w_0\mathrm{E}_t\left[\frac{M_T}{M_t}L_T\right] \\ =& e^{c_1(t,T)+c_2(t,T)}\left(e^{\frac{1}{\alpha^2}c_2(t,T)}\beta\sinh\left(-\frac{1}{\alpha}\ln\left(vM_te^{c_1(t,T)+2c_2(t,T)}\right) - \ln\beta\right) + w_0\right).\end{aligned} \tag{30}$$

Note that $L_t := F(t,T,1) = e^{c_1(t,T)+c_2(t,T)}$, then

$$C_t^* = e^{\frac{1}{\alpha^2}c_2(t,T)}\beta \sinh\left(-\frac{1}{\alpha}\ln\left(vM_t e^{c_1(t,T)+2c_2(t,T)}\right) - \ln\beta\right) + w_0. \tag{31}$$

v can be solved by setting $X_0^* = X_0 = \phi E[M_T L_T]$, i.e.,

$$\begin{aligned} \frac{X_0}{F(0,T,1)} - w_0 =& e^{\frac{1}{\alpha^2}c_2(0,T)}\beta \sinh\left(-\frac{1}{\alpha}\ln\left(ve^{c_1(0,T)+2c_2(0,T)}\right) - \ln\beta\right), \\ v =& \frac{\beta^{-\alpha}\, e^{-\alpha \operatorname{arcsinh}\frac{\frac{X_0}{F(0,T,1)}-w_0}{\beta\, e^{\frac{1}{\alpha^2}c_2(0,T)}}}}{e^{c_1(0,T)+2c_2(0,T)}}. \end{aligned} \tag{32}$$

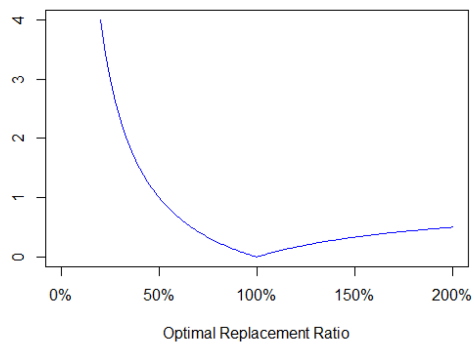
The vector of optimal proportional wealth allocation θ_t^* can be derived as follows:

$$\theta_t^* = \frac{1}{\alpha_t}\Omega^{-1}\Lambda + (1-\frac{1}{\alpha_t})\Omega^{-1}\sigma\rho(\xi_S, \xi_r - \left(B(t,T) + ne^{\kappa_r(t-T)}\right)\sigma_r, \xi_\pi)', \tag{33}$$

where $\alpha_t = \frac{C_t^*}{\sqrt{(\beta\, e^{\frac{1}{\alpha^2}c_2(t,T)})^2 + (C_t^* - w_0)^2}}\alpha$ $(C_t^* \neq 0)$.

Compare the optimal proportional wealth allocation equation (33) to equation (27), the optimal proportional wealth allocation for SAHARA utility is time-dependent and replacement-ratio-dependent. In particular, the function $c_2(t,T)$ decreases when time t approaches T, which implies that $1/\alpha_t$ decreases when the investment horizon gets short. $1/\alpha_t$ also changes with the replacement ratio. As an example, let w_0 be 100% and α be 1, ignore the term $(\beta\, e^{\frac{1}{\alpha^2}c_2(t,T)})^2$, then the following figure shows how $1/\alpha_t$ changes with the optimal replacement ratio.

Figure 4: $1/\alpha_t$ (with $w_0 = 100\%$, $\alpha = 1$, $\beta = 0$) w.r.t. optimal replacement ratio C_t^*.



The value of $1/\alpha_t$ reaches a minimum at the threshold w_0. Its value increases strongly when the replacement ratio falls below the threshold and slightly when the replacement ratio rises above the threshold. In case α is blow 1, the minimum decreases clearly with time t. At the retirement, $\alpha_T = C_T^* A(C_T^*)$ and $C_T^* A(C_T^*)$ is the relative risk aversion of the corresponding SAHARA utility function with respect to the optimal replacement ratio. Similarly, optimizing without benchmark is a special case of optimizing with benchmark, and the results can be generated by equating m and n to zero.

5 Numeric Examples

In this section, we explore in complete market the effects the benchmark imposes on the probability density distribution of the optimal replacement ratio, and the recovery potential by changing agent's risk preferences around a threshold.

Table 4: Overview model parameters continued.

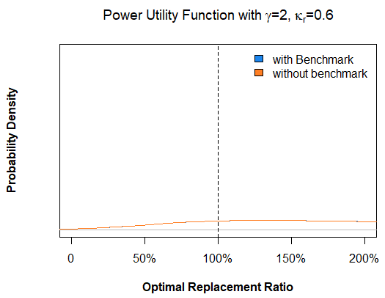
$\sigma_S = 0.16$	$\sigma_\pi = 0.014$	$\sigma_\Pi = 0.013$
$\rho_{S\pi} = -0.024$	$\phi = \{60\%, 80\%\}$	$w_0 = 100\%$
$\xi_S = 0$	$\xi_r = 0$	$\xi_\pi = 0.015$

Table 4 gives additional parameters used in the model. The wealth at current time, X_0, satisfies the equation: $X_0 = \phi F(0, T, 1)$. The function $F(0, T, 1)$ calculates the value of the benchmark L_T at current time and ϕ indicates the initial under-funding level. For the underfunded starting position, we consider two cases, with ϕ being 80% for case 1 and 60% for case 2. For power utility function, we take $\gamma = 5$. For SAHARA utility functions, we let the threshold wealth $w_0 = 100\%$. This implies that the agent is the most risk-averse at the replacement ratio C_T of 100% and willing to accept more risk if C_T moves away from 100%. Hence, the agent tends to lock-in the reference level. The parameter α takes value from $\{2, 5\}$ and β from $\{0.1, 1\}$. The maturity of the two nominal bonds are 5 year and 30 year respectively. In case a indexed bond is included, 30-year nominal bond is replaced by a 5-year indexed bond. By setting $\xi_S = 0$ and $\xi_r = 0$, $d\Pi_t/\Pi_t$ is perfectly correlated with $d\pi_t/\pi_t$ and the market is complete.

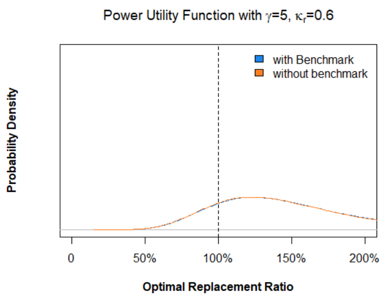
5.1 CRRA with vs without benchmark

In this subsection we explore the effect the benchmark imposes on the probability density distribution of the optimal replacement ratio with power utility in different scenarios. Risk aversion parameter γ chooses value from $\{2, 5, 10\}$, and the starting under-funding level ϕ from $\{60\%, 80\%\}$. First we consider the case where κ_r is 0.6. Figure 5 and Figure 6 display the probability density distribution of the optimal replacement ratio when the starting under-funding level is 80% and 60% respectively.

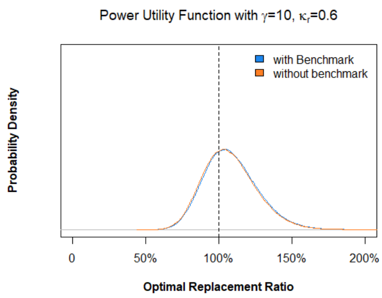
Figure 5: Probability density plot of optimal replacement ratio when $\kappa_r = 0.6$ and $\phi = 80\%$.



(a) Power with $\gamma = 2$.

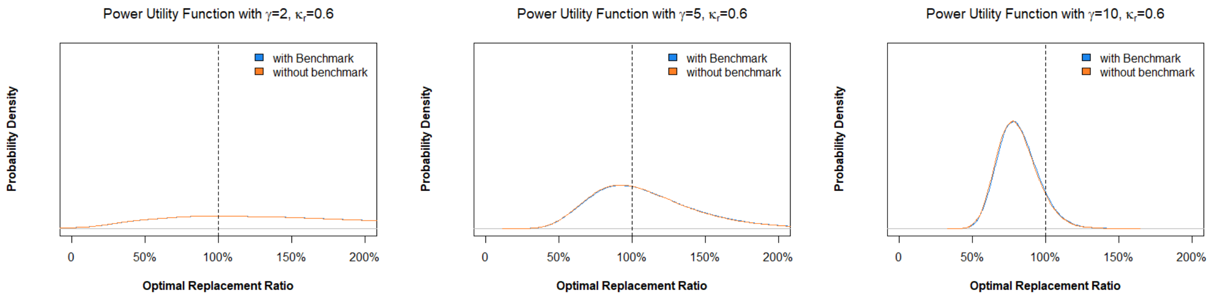


(b) Power with $\gamma = 5$.



(c) Power with $\gamma = 10$.

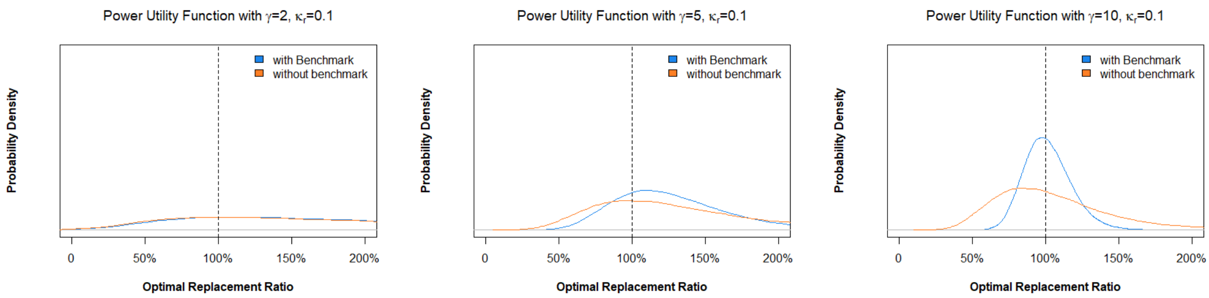
Figure 6: Probability density plot of optimal replacement ratio when $\kappa_r = 0.6$ and $\phi = 60\%$.



(a) Power with $\gamma = 2$. (b) Power with $\gamma = 5$. (c) Power with $\gamma = 10$.

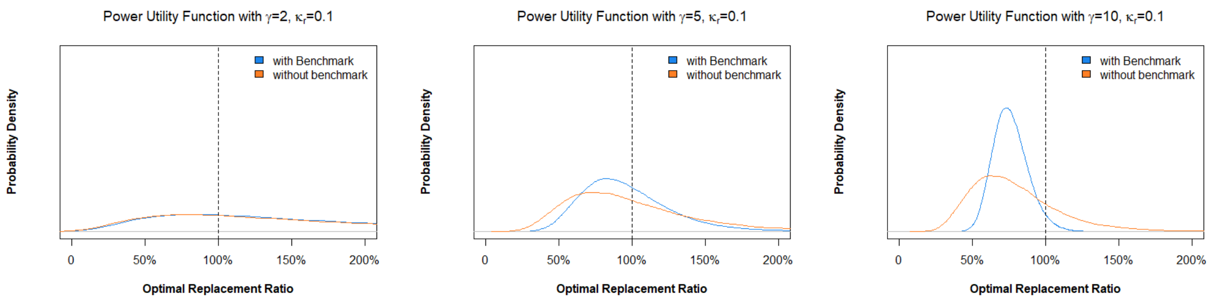
Figure 5 and Figure 6 show that when κ_r is 0.6, by varying the value of γ and the value of the underfunding level, the benchmark merely affects the probability density distribution of the optimal replacement ratio. Next we change κ_r to 0.1.

Figure 7: Probability density plot of optimal replacement ratio when $\kappa_r = 0.1$ and $\phi = 80\%$.



(a) Power with $\gamma = 2$. (b) Power with $\gamma = 5$. (c) Power with $\gamma = 10$.

Figure 8: Probability density plot of optimal replacement ratio when $\kappa_r = 0.1$ and $\phi = 60\%$.



(a) Power with $\gamma = 2$. (b) Power with $\gamma = 5$. (c) Power with $\gamma = 10$.

Figure 7 and Figure 8 show that the probability density distribution of the optimal replacement ratio is sensitive to κ_r. When κ_r decreases from 0.6 to 0.1, the mode of the distribution shits to the left. Furthermore, when γ takes high value, such as 5 or 10, the benchmark has clear effect on the probability density distribution that the left tail shifts to the right and the right tail to the left, such that the distribution is more centered around the reference level. Table 5 presents an overview of the optimal replacement ratio statistics for the case where κ_r is 0.1 and the starting under-funding level is 60%. Particularly, γ takes the value of 5 or 10.

Table 5: Overview C_T^* statistics with $\kappa_r = 0.1$, $\phi = 60\%$.

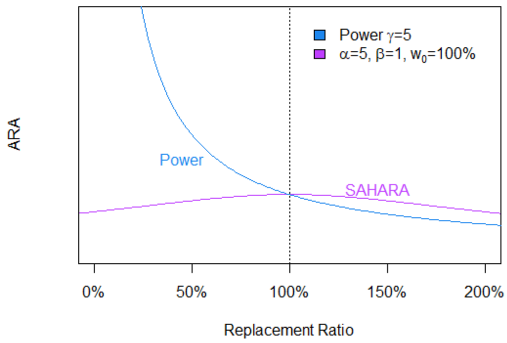
	Benchmark		Without Benchmark	
	$\gamma = 5$	$\gamma = 10$	$\gamma = 5$	$\gamma = 10$
$P(C_T^* \geq 100\%)$	39.01%	3.27%	40.72%	20.05%
$P(C_T^* \geq 90\%)$	52.37%	12.04%	50.10%	28.94%
$P(C_T^* \geq 80\%)$	67.04%	34.07%	60.67%	40.85%
$P(C_T^* \geq 60\%)$	91.55%	92.64%	82.06%	70.69%

When γ is 5, the probability of the optimal replacement ratio of being not below 90% or 80% or 60% improves with benchmark by 2.27%, 6.37% and 9.49% respectively. When γ is 10, the probability of the optimal replacement ratio of being not below 100% or 90% or 80% decreases, however, the probability of being 60% improves significantly by around 22%. The results imply that power utility function defined over the ratio of final wealth to a benchmark shifts probability density distribution of the optimal replacement ratio in such a way that it is more centered around the benchmark.

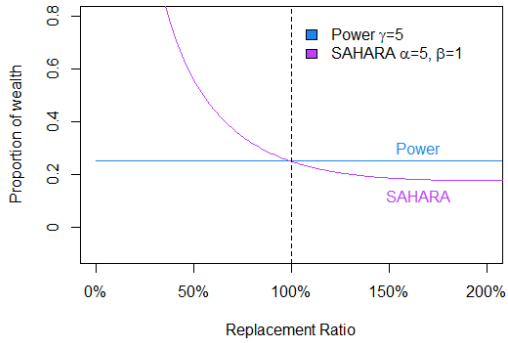
5.2 Less risk-averse below the threshold and more risk-averse above the threshold

In this subsection we choose parameters $\alpha = 5$ and $\beta = 1$ for SAHARA utility function to model this particular type of risk preferences: when the replacement ratio falls below the threshold, the agent becomes less risk-averse instead of more risk-averse (modeled by power utility function); while the replacement ratio goes above the threshold, the agent tends to be more risk-averse to lock-in the the desired outcome: the replacement ratio being 100%. The idea of agent being less risk-averse below the threshold is that excessive de-risking in case of power utility leads to an almost riskless investment strategy, makes it impossible to leave the situation of under-funding. The plot of the absolute risk aversion for these two utility functions are shown below on panel a of Figure 9, the proportional wealth allocation to stock with respect to the replacement ratio at time T on panel b of Figure 9. The probability density distribution of the optimal replacement ratio with start under-funding level of being 80% and 60% is on panel c and panel d of Figure 9 respectively.

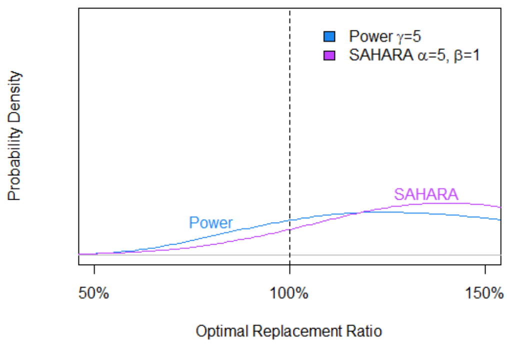
Figure 9: Power and SAHARA utility functions with benchmark case 1.



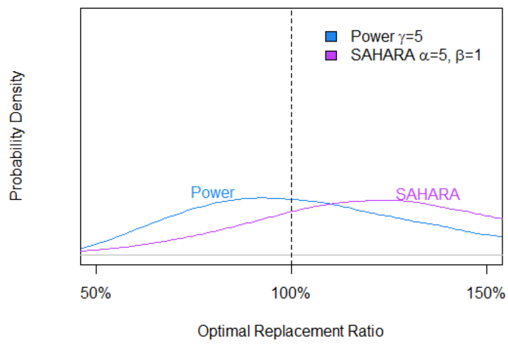
(a) Absolute Risk Aversion.



(b) Proportional wealth allocation to stock at time T.



(c) Probability Density plots C_T^* with initial under-funding level 80%.



(d) Probability Density plots C_T^* with initial under-funding level 60%.

Agent with CRRA risk preference invests a constant proportion of the wealth to stock, shown on panel a of Figure 9 by the blue line. Below the threshold 100%, the agent with SAHARA risk preference invests a larger proportion of the wealth in stock and above the threshold a smaller proportion in stock. Panel c and d of Figure 9 show that this SAHARA type of risk preference shits the mode of the probability distribution to the right slightly when the start under-funding level is 80%, significantly when the start under-funding level is 60%.

Table 6: Overview C_T^* statistics with $\kappa_r = 0.6$.

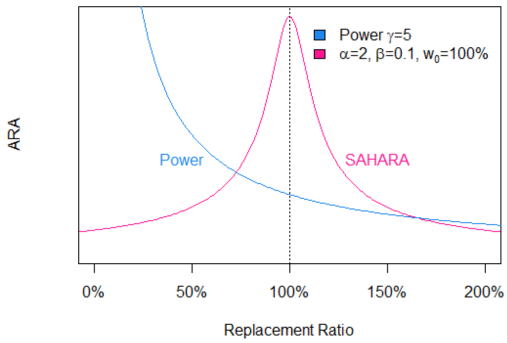
Statistics	$\phi = 80\%$		$\phi = 60\%$	
	Power $\gamma = 5$	SAHARA $\alpha = 5$ $\beta = 1$	Power $\gamma = 5$	SAHARA $\alpha = 5$ $\beta = 1$
$P(C_T^* \geq 100\%)$	83.43%	89.54%	54.63%	76.53%
$P(C_T^* \geq 90\%)$	90.00%	93.98%	66.67%	84.55%
$P(C_T^* \geq 80\%)$	94.90%	96.82%	78.21%	90.48%
$P(C_T^* < 0\%)$	0%	0%	0%	0.04%

Table 6 presents the statistics of the optimal replacement ratio at time T: C_T^* for both preferences when the starting under-funding level is 80% and 60%. Table 6 shows that with SAHARA risk preference, the probability of the optimal replacement ratio being not below 100% and not below 90% increase. When the start under-funding level is 60%, this increase is substantial and its value is around 20%. In short, the probability of overcoming under-funding could be improved by being less risk-averse instead of more risk-averse when the replacement ratio falls below the threshold. The lower the start under-funding level, the more obvious this improvement. Note that when the start under-funding level is very low, with SAHARA preference a small proportion of the optimal replacement ratios will end up being negative, because SAHARA utility functions are defined for all wealth levels. To avoid these negative optimal replacement ratios a lower bound could be added to the optimizing objective function. The consequence is that it will shift the distribution of the optimal replacement ratios to the left and a proportion of the improvement will be sacrificed. In this example, the proportion of negative optimal replacement ratios can be neglected.

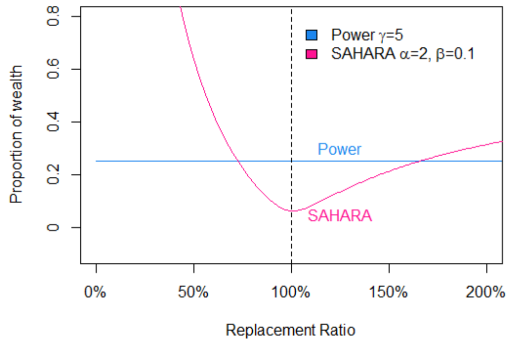
5.3 More risk-averse around the threshold, and less risk-averse below/above the threshold

In this subsection we choose parameters $\alpha = 2$ and $\beta = 0.1$ for SAHARA utility function to model the following risk preference: the agent has very high level of risk aversion around the threshold (100%), i.e. the agent's target; when the replacement ratio moves away from the threshold, the agent's risk aversion level reduces rapidly, such that the agent becomes much less risk-averse below the threshold while slightly less risk-averse above the threshold. This high level of risk aversion level at the threshold can be interpreted as the agent's great desire to lock-in the target. The plot of the absolute risk aversion for these two utility functions are shown below on panel a of Figure 10, the proportional wealth allocation to stock with respect to the replacement ratio at time T on panel b, and the probability density distribution of the optimal replacement ratio on panel c and d for the starting under-funding level of being 80% and 60% respectively.

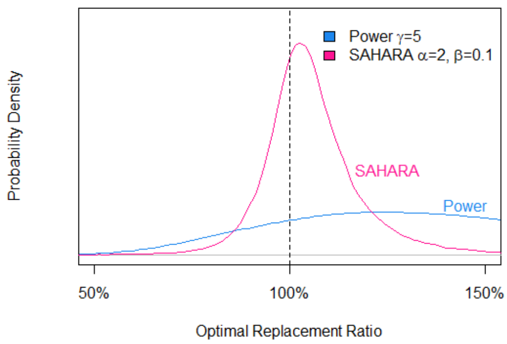
Figure 10: Power and SAHARA utility functions with benchmark case 2.



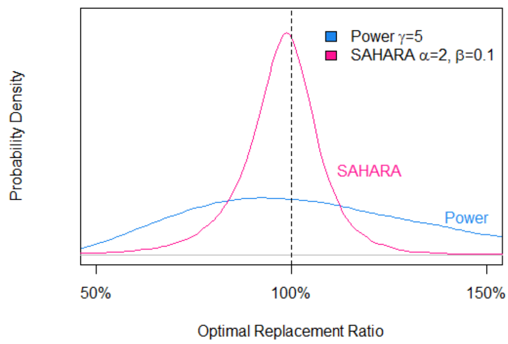
(a) Absolute Risk Aversion.



(b) Proportional wealth allocation to stock at time T.



(c) Probability Density plots of C_T^* with initial under-funding level 80%.



(d) Probability Density plots of C_T^* with initial under-funding level 60%.

Similarly, with SAHARA risk preference the proportion of wealth invested in stock varies with the replacement ratio, shown on panel b by the pink line. Around the threshold (100%), the agent with SAHARA risk preference has high level of risk aversion, consequently the agent invests a much smaller proportion of the wealth in stock, shown on panel b by the pink line. When the replacement ratio moves away from the threshold, a larger proportion is invested in stock. Panel c and d show that the probability distribution generated by this SAHARA type of risk preference is more centered around the threshold.

Table 7: Overview C_T^* statistics with $\kappa_r = 0.6$.

Statistics	$\phi = 80\%$		$\phi = 60\%$	
	Power $\gamma = 5$	SAHARA $\alpha = 2$ $\beta = 0.1$	Power $\gamma = 5$	SAHARA $\alpha = 2$ $\beta = 0.1$
$P(C_T^* \geq 100\%)$	83.43%	68.59%	54.63%	40.30%
$P(C_T^* \geq 90\%)$	90.00%	93.68%	66.67%	78.95%
$P(C_T^* \geq 80\%)$	94.90%	98.67%	78.21%	92.91%
$P(C_T^* < 0\%)$	0%	0%	0%	0.04%

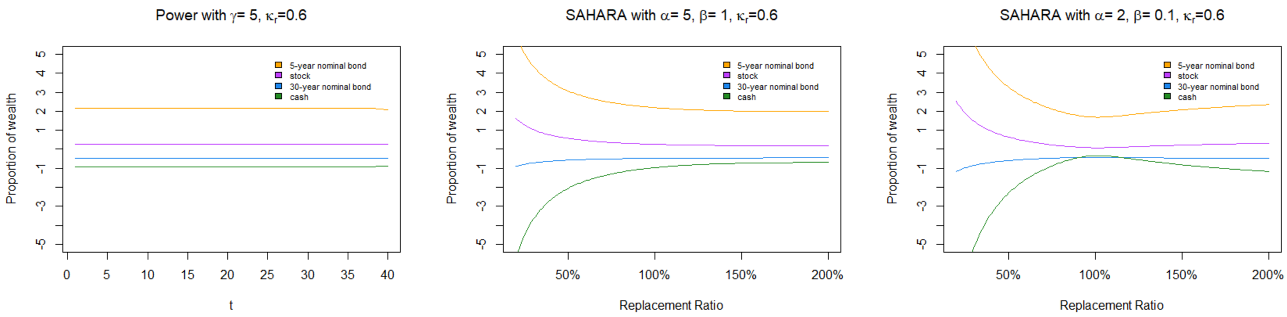
Table 7 presents the statistics of the optimal replacement ratio at time T: C_T^* for both preferences when the starting under-funding level is 80% and 60% respectively. With this type of SAHARA preference the probability of the optimal replacement ratio being not below 100% decreases, nevertheless, the probability of the optimal replacement ratio being not below 90% or 80% increases slightly when the starting under-funding level is 80%, and significantly when the starting under-funding level is 60%. The increase is around 10% and 15% respectively.

Results from this and previous subsection imply that SAHARA type of risk preference is more target-focused and shits the mode of the probability distribution of the optimal replacement ratios to the right even when the starting under-funding level is low. By agent being high risk-averse around a reference level and less risk-averse below the reference level, the optimal replacement ratios generated are more towards this level.

5.4 Proportional wealth allocation

Figure 11 displays the plot of the proportional wealth allocation to stock, 5-year nominal bond, 30-year nominal bond and cash for power utility function with $\gamma = 5$ for the whole period and SAHARA utility functions with $\alpha = 5$, $\beta = 1$ and $\alpha = 2$, $\beta = 0.1$ at time $T/2$. For SAHARA utility, there is an time-related element to the optimal proportional wealth allocations, though the effect is not significant when the value of α is big enough. The mean-reverting tendency is 0.6 and the start under-funding level is 60%. Table 11 presents the proportional wealth allocations for power utility and SAHARA utility at time $T/2$. The allocations for SAHARA utility are given for the following replacement ratios: 50%, 100% and 150%.

Figure 11: Proportional wealth allocation with $\kappa_r = 0.6$ and $\phi = 60\%$.



(a) Power with $\gamma = 5$.

(b) SAHARA with $\alpha = 5$, $\beta = 1$ at time $T/2$.

(c) SAHARA with $\alpha = 2$, $\beta = 0.1$ at time $T/2$.

Table 8: Proportional wealth allocations with $\kappa_r = 0.6$, $\phi = 60\%$.

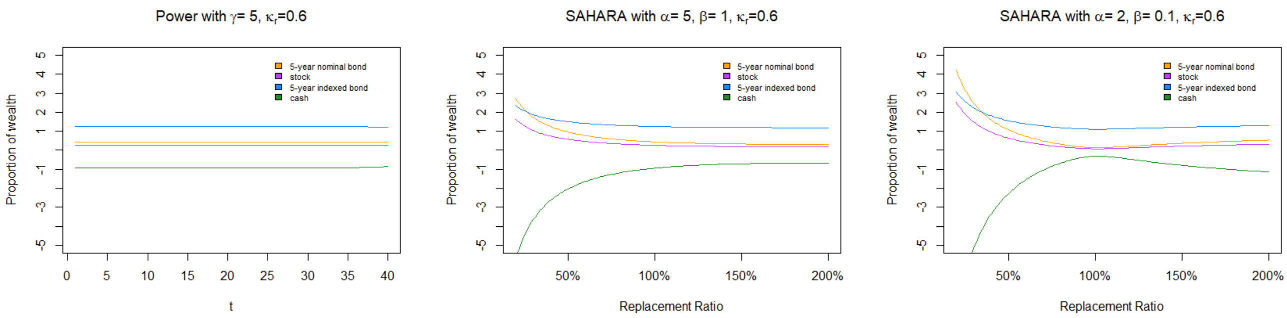
Prop. Allocation	Time $T/2$						
	Power	SAHARA $\alpha = 5$, $\beta = 1$			SAHARA $\alpha = 2$, $\beta = 0.1$		
	$\gamma = 5$	$C_T = 50\%$	$C_T = 100\%$	$C_T = 150\%$	$C_T = 50\%$	$C_T = 100\%$	$C_T = 150\%$
Stock	0.25	0.57	0.26	0.19	0.64	0.08	0.21
5-year nominal bond	2.17	3.07	2.19	2.00	3.26	1.68	2.07
30-year nominal bond	-0.48	-0.58	-0.48	-0.46	-0.60	-0.42	-0.47
Cash	-0.94	-2.07	-0.97	-0.74	-2.31	-0.34	-0.82

For power utility, the proportional allocation to assets doesnot vary with the level of the replacement ratio. Additionally, the proportion invested in stock also doesnot vary with the investment horizon. The allocation pattern indicates that the agent takes long position in 5-year nominal bond and stock, while short position in 30-year nominal bond and cash. About the short position in cash, Ma (2011) pointed out that the cash account is not a safe asset in real terms and the returns on cash are lower than the returns on bonds and stocks. An agent with high risk aversion would keep short position in cash to hedge against the inflation risk. The 5-year nominal bond turns out to be the most dominant asset in the pension portfolio. When the investment horizon gets short, the agent decreases weight in 5-year nominal bond and short less in 30-year nominal bond and cash, shown on panel a of Figure 11.

Panel b and c of Figure 11 show that for SAHARA utility the proportional wealth allocation varies with the level of replacement ratio. With $\alpha = 5$ and $\beta = 1$, this difference is obvious when replacement ratio is below 100%. Compared to the proportional wealth allocations at the threshold (100%), Table 8 shows that when the replacement ratio is 50% (low), the proportional wealth allocated to cash decreases significantly from -0.97 to -2.07 and that to 30-year nominal bond slightly from -0.48 to -0.58, while the proportional wealth invested in stock and 5-year nominal bond increases slightly from 0.26 to 0.57 and greatly from 2.19 to 3.07, respectively. The lower the replacement ratio, the more the extend. In other words, to overcome underfunded situation an agent with low risk aversion borrows more cash and invests heavily in 5-year nominal bond and stock. When the replacement ratio is 150% (high), the proportional wealth allocations for SAHARA utility with $\alpha = 5$ and $\beta = 1$ are comparable to that for power utility with $\gamma = 5$. This is in line with the feature of the absolute risk aversion level SAHARA utility prescribes, namely the agent becomes less risk-averse when the replacement ratio is low, and similar when the replacement ratio is high. With $\alpha = 2$, $\beta = 0.1$, the proportion the agent invests in stock at the threshold (100%) is at its lowest level (0.08 proportion of the wealth), since the agent is the most risk-averse at the threshold. When the replacement ratio continues to fall below the threshold, the agent borrows more to invest in 5-year nominal bond and stock to overcome under-funding. On the other hand, when the replacement ratio continues to rise above the threshold, the agent increases gradually the proportion invested in 5-year nominal bond and stock. The extent is less compared to the case when the replacement ratio falls below the threshold.

We also show the results when the agent invest in stock, 5-year nominal bond, 5-year indexed bond and cash in Figure 12 and Table 9.

Figure 12: Proportional wealth allocation with $\kappa_r = 0.6$ and $\phi = 60\%$.



(a) Power with $\gamma = 5$.

(b) SAHARA with $\alpha = 5$, $\beta = 1$ at time $T/2$.

(c) SAHARA with $\alpha = 2$, $\beta = 0.1$ at time $T/2$.

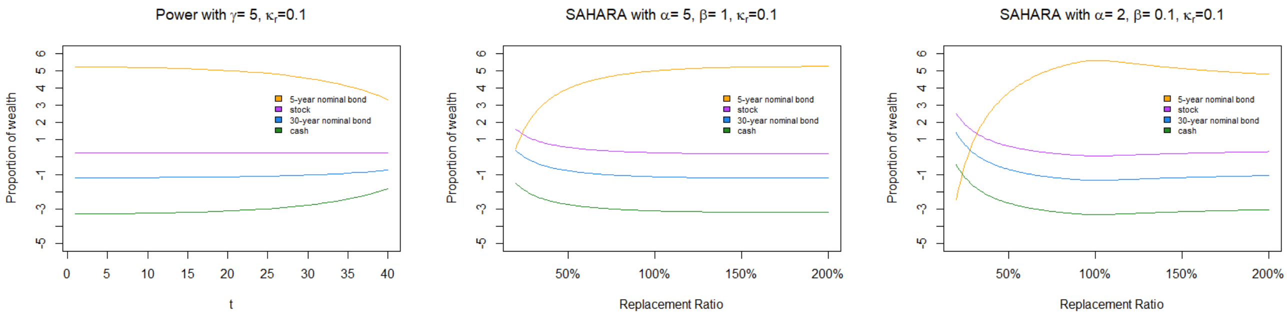
Table 9: Proportional wealth allocations with $\kappa_r = 0.6$, $\phi = 60\%$.

Prop. Allocation	Time $T/2$						
	Power	SAHARA $\alpha = 5$, $\beta = 1$			SAHARA $\alpha = 2$, $\beta = 0.1$		
	$\gamma = 5$	$C_T = 50\%$	$C_T = 100\%$	$C_T = 150\%$	$C_T = 50\%$	$C_T = 100\%$	$C_T = 150\%$
Stock	0.25	0.57	0.26	0.19	0.64	0.08	0.21
5-year nominal bond	0.43	0.96	0.44	0.33	1.08	0.14	0.37
5-year indexed bond	1.24	1.50	1.25	1.20	1.56	1.10	1.21
Cash	-0.92	-2.04	-0.94	-0.71	-2.28	-0.31	-0.79

Instead of the 5-year nominal bond, the 5-year indexed bond turns out to be the most dominant asset in the pension portfolio. The agent takes long position in stock and bonds while short position only in cash. For SAHARA utility, when the replacement ratio falls below the threshold, the agent borrows more cash to invest in stock and bonds to overcome under-funding. With $\alpha = 2$, $\beta = 0.1$, the agent also borrows more cash (compared to the proportion invested at the threshold) to invest in stock and bonds when the replacement ratio rises above the threshold, though with less extent. Notably, the proportion allocated to stock and cash are similar to that from earlier case.

As a comparison, we change the mean-reverting tendency from 0.6 to 0.1. The results are presented in Figure 13 and Table 10.

Figure 13: Proportional wealth allocation with $\kappa_r = 0.1$ and $\phi = 60\%$.



(a) Power with $\gamma = 5$.

(b) SAHARA with $\alpha = 5$, $\beta = 1$ at time $T/2$.

(c) SAHARA with $\alpha = 2$, $\beta = 0.1$ at time $T/2$.

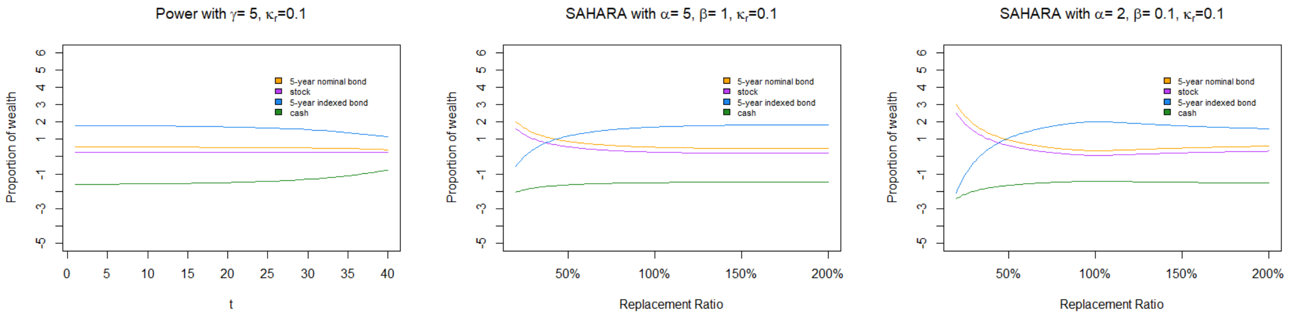
Table 10: Proportional wealth allocations with $\kappa_r = 0.1$, $\phi = 60\%$.

Prop. Allocation	Time $T/2$						
	Power	SAHARA $\alpha = 5$, $\beta = 1$			SAHARA $\alpha = 2$, $\beta = 0.1$		
	$\gamma = 5$	$C_T = 50\%$	$C_T = 100\%$	$C_T = 150\%$	$C_T = 50\%$	$C_T = 100\%$	$C_T = 150\%$
Stock	0.25	0.57	0.26	0.19	0.64	0.07	0.21
5-year nominal bond	5.02	3.97	5.00	5.22	3.73	5.61	5.14
30-year nominal bond	-1.15	-0.79	-1.14	-1.21	-0.71	-1.34	-1.19
Cash	-3.12	-2.75	-3.12	-3.19	-2.31	-3.33	-3.17

Panel a of Figure 13 shows that for power utility function the time effect on the proportion invested in bonds and cash becomes obvious: the absolute value of the proportion allocated to these individual assets decreases clearly with time. For both power and SAHARA utilities, the proportion invested in stock has the similar value as in the case when $\kappa_r = 0.6$. However, the proportional wealth allocated to bonds and cash changes with the mean-reverting tendency. Compared to the case with $\kappa_r = 0.6$, the agent with power utility shorts more in 30-year nominal bond and cash to invest in 5-year nominal bond to hedge real interest risk. For SAHARA utility with $\alpha = 5$, $\beta = 1$, the agent decreases the proportion invested in 5-year nominal bond when the replacement ratio decreases. With $\alpha = 2$, $\beta = 0.1$, the proportion allocated to 5-year nominal bond is the most at the threshold, and the agent tends to short 5-year nominal bond when the replacement ratio is low.

Then we show the results when the agent invest in stock, 5-year nominal bond, 5-year indexed bond and cash in Figure 14 and Table 11.

Figure 14: Proportional wealth allocation with $\kappa_r = 0.1$ and $\phi = 60\%$.



(a) Power with $\gamma = 5$. (b) SAHARA with $\alpha = 5$, $\beta = 1$ at time $T/2$. (c) SAHARA with $\alpha = 2$, $\beta = 0.1$ at time $T/2$.

Table 11: Proportional wealth allocations with $\kappa_r = 0.6$, $\phi = 60\%$.

Prop. Allocation	Time $T/2$						
	Power	SAHARA $\alpha = 5$, $\beta = 1$			SAHARA $\alpha = 2$, $\beta = 0.1$		
	$\gamma = 5$	$C_T = 50\%$	$C_T = 100\%$	$C_T = 150\%$	$C_T = 50\%$	$C_T = 100\%$	$C_T = 150\%$
Stock	0.25	0.57	0.26	0.26	0.64	0.07	0.07
5-year nominal bond	0.53	0.54	0.54	0.33	0.96	0.34	0.34
5-year indexed bond	1.71	1.71	1.71	1.20	1.06	2.02	2.02
Cash	-1.50	-1.63	-1.50	-1.48	-1.66	-1.43	-1.49

Similarly, the 5-year indexed bond turns out to be the most dominant asset in the pension portfolio. For power utility, the agent takes long position in stock, bonds while short position only in cash. For SAHARA utility, the agent tends to short the 5-year indexed bond when the replacement ratio is low, to earn the inflation premium. For both utilities, the agent short less in cash compared to the case where the 5-year nominal bond and 30-year nominal bond are included in the portfolio.

6 Conclusion

In this paper, we have investigated the recovery potential for underfunded pension plans by maximizing the expected utility of the ratio of its terminal real wealth to a stochastic benchmark in real terms. This benchmark closely approximates the value of a inflation-indexed annuity at retirement which protects participant's purchasing power after retirement. The real interest rate and inflation rate are stochastic. The agent can invest in cash, stock, nominal bonds and indexed bonds. With the assumptions of complete market and no arbitrage, we are able to model the underfunded starting position and derive closed-form solutions under CRRA and SAHARA utility functions. SAHARA utility function allows the agent to decrease risk aversion level below a threshold to overcome under-funding and increase risk aversion level around the threshold to secure desired outcomes. The optimal portfolio weights are represented by a weighted average of two respective portfolios. The first portfolio is the nominal mean-variance tangency portfolio. The second portfolio has the largest correlation with the present value of the benchmark in nominal terms. With SAHARA utility, the weight changes with the replacement ratio and through time.

In the numeric application, we have shown that when the starting position is underfunded, the combination of a non-constant risk aversion type of risk preference, in particular a risk-taking for resurrection type of risk preference, and a reference-dependent utility defined over the ratio of final wealth to a benchmark, lead to a higher chance of achieving a certain percentage of the desired benchmark. The CRRA utility turns out to be too risk-averse to overcome under-funding. All in all, with a underfunded starting position, optimizing based on SAHARA preferences could be a useful tool to recover undeundfunded position.

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Appendix

1. Power preferences

As a recap, this subsection shows a brief summary of Power utility functions. Power utility functions are defined as follows:

$$U(x) = \begin{cases} \frac{x^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \ \& \ \gamma \in R_+; \\ \ln(x) & \text{if } \gamma = 1, \end{cases} \tag{34}$$

where γ denotes the agent's level of risk aversion, and $x \in R_+$. Furthermore,

$$U'(x) = x^{-\gamma}, \ \ I(y) = y^{-\frac{1}{\gamma}} \text{ (for all } y \in R_+), \ \ A(x) = -\frac{U''}{U'} = \frac{\gamma}{x}. \tag{35}$$

where $B(t,T) = \kappa_r^{-1}(1 - e^{\kappa_r(t-T)})$, $C(t,T) = \kappa_\pi^{-1}(1 - e^{\kappa_\pi(t-T)})$.

2. $A(t,T)$ and $A^*(t,T)$

$$\begin{aligned} A(t,T) =& \left[B(t,T) - (T-t)\right]\bar{r}^* + \left[C(t,T) - (T-t)\bar{\pi}^*\right] \\ &- \frac{\sigma_\pi^2}{4\kappa_\pi^3}\left[2\kappa_\pi\big(C(t,T) - (T-t)\big) + \kappa_\pi^2 C^2(t,T)\right] \\ &- \frac{\sigma_\pi^2}{4\kappa_\pi^3}\left[2\kappa_\pi\big(C(t,T) - (T-t)\big) + \kappa_\pi^2 C^2(t,T)\right] \\ &+ \frac{\sigma_r\sigma_\pi\rho_{r\pi}}{\kappa_r\kappa_\pi}\left[(T-t) - C(t,T) - B(t,T) + \frac{1 - e^{(\kappa_r+\kappa_\pi)(t-T)}}{\kappa_r + \kappa_\pi}\right] \\ &+ (\xi'\rho\xi - \omega'\rho\xi)(T-t), \end{aligned} \tag{36}$$

where $\bar{r}^* = \bar{r} - \lambda_r(\sigma_r/\kappa_r)$, and $\bar{\pi}^* = \bar{\pi} - \lambda_\pi(\sigma_\pi/\kappa_\pi)$.

$$\begin{aligned} A^*(t,T) =& \Big(-\bar{r} - \frac{\sigma_r}{\kappa}(\omega_s\rho_{Sr} + \omega_r + \omega_\pi\rho_{r\pi})\Big)\big(T - t - B(t,T)\big) \\ &- \frac{\sigma_r^2}{4\kappa_r^3}\left[2\kappa_r\big(B(t,T) - (T-t)\big) + \kappa_r^2 B^2(t,T)\right]. \end{aligned} \tag{37}$$

3. Derivation of $F(t,T,\gamma)$

$$\begin{aligned} I_1(t,T) &= \int_t^T B(u,T)du = \frac{1}{\kappa_r}(T-t) - \frac{1}{\kappa_r}B(t,T), \\ I_2(t,T) &= \int_t^T B^2(u,T)du = \frac{1}{2\kappa_r^3}\big\{2\kappa_r\left[(T-t) - B(t,T)\right] - \kappa_r^2 B^2(t,T)\big\}, \\ I_3(t,T) &= \int_t^T e^{\kappa_r(u-T)}du = B(t,T), \\ I_4(t) &= \int_t^T e^{2\kappa_r(u-T)}du = B(t,T) - \frac{\kappa_r}{2}B^2(t,T), \\ I_5(t,T) &= \int_t^T B(u,T)e^{\kappa_r(u-T)}du = \frac{1}{2}B^2(t,T). \end{aligned} \tag{38}$$

$$
\begin{aligned}
&r_T = \bar{r} + (r_t - \bar{r})e^{\kappa_r(t-T)} + \sigma_r \int_t^T e^{\kappa_r(u-T)} dW_{r,u}, \\
&\int_t^T r_s ds = \bar{r}(T-t) + (r_t - \bar{r})B(t,T) + \sigma_r \int_t^T B(u,T) dW_{r,u}, \\
&\mathrm{E}_t[r_T] = \bar{r} + (r_t - \bar{r})e^{\kappa_r(t-T)}, \\
&\mathrm{Var}_t(r_T) = \sigma_r^2 \; I_4(t) = \sigma_r^2 \left(B(t,T) - \frac{\kappa_r}{2} B^2(t,T) \right), \\
&\mathrm{E}_t \left[\int_t^T r_s ds \right] = \bar{r}(T-t) + (r_t - \bar{r})B(t,T), \\
&\mathrm{Var}_t \left[\int_t^T r_s ds \right] = \sigma_r^2 \; I_2(t) = \frac{\sigma_r^2}{2\kappa_r^3} \left\{ 2\kappa_r \left[(T-t) - B(t,T) \right] - \kappa_r^2 B^2(t,T) \right\}.
\end{aligned}
\tag{39}
$$

Let $\omega' \rho \omega = V_M$,

$$
\begin{aligned}
&\ln\left(\frac{M_T}{M_t}\right) = \int_t^T (-r_u - \frac{1}{2}V_M) du + \int_t^T \omega' dW_u, \\
&\mathrm{E}_t \left[\ln(\frac{M_T}{M_t}) \right] = -\bar{r}(T-t) + (\bar{r} - r_t)B(t,T) - \frac{1}{2}V_M(T-t), \\
&\mathrm{Var}_t \left[\ln(\frac{M_T}{M_t}) \right] = \sigma_r^2 I_2(t,T) + V_M(T-t) - 2\sigma_r(\omega_S \rho_{Sr} + \omega_r + \omega_\pi \rho_{r\pi}) I_1(t,T).
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
F(t,T,\gamma) &= \mathrm{E}_t \left[\left(\frac{M_T}{M_t} L_T \right)^\gamma \right] \\
&= \mathrm{E}_t \left[e^{\gamma \int_t^T (-r_u - \frac{1}{2}V_M) du + \gamma \int_t^T \omega' dW_u} \; e^{\gamma m - \gamma n (\bar{r} + (r_t - \bar{r}) e^{\kappa_r(t-T)}) - \gamma n \sigma_r \int_t^T e^{\kappa_r(u-T)} dW_{r,u}} \right] \\
&= \mathrm{E}_t \left[e^{-\frac{\gamma}{2} V_M (T-t) + \gamma m - \gamma n \left(\bar{r} + (r_t - \bar{r}) e^{\kappa_r(t-T)} \right)} \; e^{\gamma \int_t^T -r_u du + \gamma \int_t^T \omega' dW_u - \gamma n \sigma_r \int_t^T e^{\kappa_r(u-T)} dW_{r,u}} \right] \\
&= e^{-\frac{\gamma}{2} V_M (T-t) + \gamma m - \gamma n \left(\bar{r} + (r_t - \bar{r}) e^{\kappa_r(t-T)} \right) - \gamma (\bar{r}(T-t) + (r_t - \bar{r}) B(t,T))} \\
&\quad \times e^{\frac{1}{2}\gamma^2 \sigma_r^2 I_2(t,T) + \frac{1}{2}\gamma^2 V_M (T-t) + \frac{1}{2}(\gamma n \sigma_r)^2 I_4(t,T) - \gamma^2 \sigma_r (\omega_S \rho_{Sr} + \omega_r + \omega_\pi)(I_1(t,T) + n I_3(t,T)) + \gamma^2 n \sigma_r^2 I_5(t,T)} \\
&= e^{\gamma \left(-\frac{1}{2} V_M (T-t) + m - n (\bar{r} + (r_t - \bar{r}) e^{\kappa_r(t-T)}) - (\bar{r}(T-t) + (r_t - \bar{r}) B(t,T)) \right)} \\
&\quad \times e^{\gamma^2 \left(\frac{1}{2}\sigma_r^2 I_2(t,T) + \frac{1}{2} V_M (T-t) + \frac{1}{2}(n\sigma_r)^2 I_4(t,T) - \sigma_r (\omega_S \rho_{Sr} + \omega_r + \omega_\pi \rho_{r\pi})(I_1(t,T) + n I_3(t,T)) + n \sigma_r^2 I_5(t,T) \right)} \\
&= e^{\gamma c_1(t,T) + \gamma^2 c_2(t,T)},
\end{aligned}
\tag{41}
$$

where

$$
\begin{aligned}
c_1(t,T) &= -\frac{1}{2} V_M (T-t) + m - n(\bar{r} + (r_t - \bar{r}) e^{\kappa_r(t-T)}) - (\bar{r}(T-t) + (r_t - \bar{r}) B(t,T)) \\
&= -\frac{1}{2} V_M (T-t) + m - \left(n - n e^{\kappa_r(t-T)} + (T-t) - B(t,T) \right) \bar{r} - \left(B(t,T) + n e^{\kappa_r(t-T)} \right) r_t, \\
c_2(t,T) &= \frac{1}{2}\sigma_r^2 I_2(t,T) + \frac{1}{2} V_M (T-t) + \frac{1}{2}(n\sigma_r)^2 I_4(t,T) \\
&\quad - \sigma_r (\omega_S \rho_{Sr} + \omega_r + \omega_\pi \rho_{r\pi}) \left(I_1(t,T) + n I_3(t,T) \right) + n \sigma_r^2 I_5(t,T).
\end{aligned}
\tag{42}
$$

4. Derivation of hedging strategy $\theta^* = (\theta_S^*, \theta_1^*, \theta_2^*)$

Power utility functions:

Let $t \in [0, T]$, then

$$\begin{aligned} \frac{X_t^*}{\Pi_t} = \mathrm{E}_t\left[\frac{M_T}{M_t}\frac{X_T^*}{\Pi_T}\right] =& \mathrm{E}_t\left[\frac{M_T}{M_t}\frac{X_0 L_T^{1-1/\gamma} M_T^{-1/\gamma}}{F(t,T,1-1/\gamma)}\right] \\ &= X_0 M_t^{-1/\gamma}\frac{F(t,T,1-1/\gamma)}{F(0,T,1-1/\gamma)}, \end{aligned} \tag{43}$$

The instantaneous real return on optimally invested wealth is

$$\begin{aligned} d\,\frac{X_t^*}{\Pi_t} =& f_1(r_t, T-t)dt - \frac{X_t^*}{\Pi_t}\frac{\omega_S}{\gamma}dW_{S,t} - \frac{X_t^*}{\Pi_t}\frac{\omega_\pi}{\gamma}dW_{\pi,t} \\ &- \frac{X_t^*}{\Pi_t}\left\{\frac{\omega_r}{\gamma} + (1-\frac{1}{\gamma})\left(B(t,T) + ne^{\kappa_r(t-T)}\right)\sigma_r\right\}dW_{r,t}. \end{aligned} \tag{44}$$

The real return on portfolio θ^*, which is a vector of optimal proportion of wealth invested in stock, a bond with maturity T_1, and a bond with maturity T_2. The remaining wealth, $1 - i'\theta^*$, is invested in cash (a nominal instantaneous risk-free asset). The nominal wealth process is given by

$$\frac{dX_t}{X_t} = (R_t + \theta'\Omega)dt + \theta'\sigma dW_t. \tag{45}$$

The real wealth process, $\dfrac{X_t}{\Pi_t}$, is given by

$$\begin{aligned} d\,\frac{X_t}{\Pi_t} =& f_2(r_t, T-t)dt + \frac{X_t}{\Pi_t}\left(x_S\sigma_S - \xi_S\right)dW_{S,t} - \frac{X_t}{\Pi_t}\left[(x_1B_1 + x_2B_2)\sigma_r + \xi_r\right]dW_{r,t} \\ &- \frac{X_t}{\Pi_t}\left[(x_1C_1 + x_2C_2)\sigma_\pi + \xi_\pi\right]dW_{\pi,t}. \end{aligned} \tag{46}$$

Then,

$$\theta_t^* = \frac{1}{\gamma}\Omega^{-1}\Lambda + (1-\frac{1}{\gamma})\Omega^{-1}\sigma\rho(\xi_S, \xi_r - \left(B(t,T) + ne^{\kappa_r(t-T)}\right)\sigma_r, \xi_\pi)'. \tag{47}$$

5. Derivation of v for SAHARA utility functions in complete market

For some t $\in [0, T]$,

$$\begin{aligned} \frac{X_t^*}{\Pi_t} =& \mathrm{E}_t\left[\frac{M_T}{M_t}\frac{X_T^*}{\Pi_T}\right] \\ =& \frac{1}{2}\mathrm{E}_t\left[(vM_t)^{-1/\alpha}\left(\frac{M_T}{M_t}L_T\right)^{1-1/\alpha}\right] \\ &- \frac{1}{2}\beta^2\mathrm{E}_t\left[(vM_t)^{1/\alpha}\left(\frac{M_T}{M_t}L_T\right)^{1+1/\alpha}\right] + w_0\mathrm{E}_t\left[\frac{M_T}{M_t}L_T\right] \\ =& e^{c_1(t,T)+c_2(t,T)+\frac{1}{\alpha^2}c_2(t,T)}\frac{1}{2}\left[\left(vM_te^{c_1(t,T)+2c_2(t,T)}\right)^{-1/\alpha} - \beta^2\left(vM_te^{c_1(t,T)+2c_2(t,T)}\right)^{1/\alpha}\right] + e^{c_1(t,T)+c_2(t,T)}w_0 \\ =& e^{c_1(t,T)+c_2(t,T)}\left(e^{\frac{1}{\alpha^2}c_2(t,T)}\beta\sinh\left(-\frac{1}{\alpha}\ln\left(vM_te^{c_1(t,T)+2c_2(t,T)}\right) - \ln\beta\right) + w_0\right). \end{aligned} \tag{48}$$

$$\begin{aligned} \frac{X_0}{F(0,T,1)} - w_0 =& e^{\frac{1}{\alpha^2}c_2(0,T)}\beta\sinh\left(-\frac{1}{\alpha}\ln\left(ve^{c_1(0,T)+2c_2(0,T)}\right) - \ln\beta\right), \\ v =& \frac{\beta^{-\alpha}\ e^{-\alpha\,\mathrm{arcsinh}\frac{\frac{X_0}{F(0,T,1)}-w_0}{\beta\ e^{\frac{1}{\alpha^2}c_2(0,T)}}}}{e^{c_1(0,T)+2c_2(0,T)}}. \end{aligned} \tag{49}$$

If take the derivative of $\frac{X_t^*}{\Pi_t}$, the vector of optimal proportional wealth allocation can be derived. The instantaneous real return on optimally invested wealth is

$$\begin{aligned} d\,\frac{X_t^*}{\Pi_t} =& f_1^{SA}(r_t, T-t)dt - \frac{\omega_S}{\alpha_t}dW_{S,t} - \frac{\omega_\pi}{\alpha_t}dW_{\pi,t} \\ & - \big\{\frac{\omega_r}{\alpha_t} + \big(1-\frac{1}{\alpha_t}\big)\big(B(t,T) + ne^{\kappa_r(t-T)}\big)\sigma_r\big\}dW_{r,t}. \end{aligned} \tag{50}$$

where $g_1(t) = -\frac{1}{\alpha}ln\left(vM_t e^{2c_2(t,T)}\right) - \ln\beta$, $g_2(t) = e^{\frac{1}{\alpha^2}c_2(t,T)}\beta\sinh\big(g_1(t)\big) + w_0 = C_t^*$, $\alpha_t = \frac{C_t^*}{e^{\frac{1}{\alpha^2}c_2(t,T)}\beta\, cosh(g_1)}\alpha = \frac{C_t^*}{\sqrt{(e^{\frac{1}{\alpha^2}c_2(t,T)}\beta)^2 + (C_t^* - w_0)^2}}\alpha$.

Then,

$$\theta_t^* = \frac{1}{\alpha_t}\Omega^{-1}\Lambda + (1-\frac{1}{\alpha_t})\Omega^{-1}\sigma\rho(\xi_S, \xi_r - \big(B(t,T) + ne^{\kappa_r(t-T)}\big)\sigma_r, \xi_\pi)'. \tag{51}$$

Similarly, when without benchmark,

$$\begin{aligned} \frac{X_t^*}{\Pi_t} = \mathrm{E}_t\big[\frac{M_T}{M_t}\frac{X_T^*}{\Pi_T}\big] =& \frac{1}{2}\mathrm{E}_t\big[(vM_t)^{-1/\alpha}(\frac{M_T}{M_t})^{1-1/\alpha}\big] \\ & - \frac{1}{2}\beta^2\mathrm{E}_t[(vM_t)^{1/\alpha}(\frac{M_T}{M_t})^{1+1/\alpha}] + w_0\mathrm{E}_t[\frac{M_T}{M_t}] \\ =& e^{\mathrm{E}_t\left[\ln(\frac{M_T}{M_t})\right] + \frac{1}{2}(1+\frac{1}{\alpha^2})\mathrm{Var}\left[\ln(\frac{M_T}{M_t})\right]}\frac{1}{2}\Big[\Big(vM_t e^{\mathrm{E}_t\left[\ln(\frac{M_T}{M_t})\right] + \mathrm{Var}\left[\ln(\frac{M_T}{M_t})\right]}\Big)^{-1/\alpha} \\ & - \beta^2\Big(vM_t e^{\mathrm{E}_t\left[\ln(\frac{M_T}{M_t})\right] + \mathrm{Var}\left[\ln(\frac{M_T}{M_t})\right]}\Big)^{1/\alpha}\Big] + e^{\mathrm{E}_t\left[\ln(\frac{M_T}{M_t})\right] + \frac{1}{2}\mathrm{Var}\left[\ln(\frac{M_T}{M_t})\right]}w_0 \\ =& e^{\mathrm{E}_t\left[\ln(\frac{M_T}{M_t})\right] + \frac{1}{2}\mathrm{Var}\left[\ln(\frac{M_T}{M_t})\right]} \\ & \times\left(e^{\frac{1}{2\alpha^2}\mathrm{Var}\left[\ln(\frac{M_T}{M_t})\right]}\beta\,\sinh\left(-\frac{1}{\alpha}\ln(vM_t e^{\mathrm{E}_t\left[\ln(\frac{M_T}{M_t})\right] + \mathrm{Var}\left[\ln(\frac{M_T}{M_t})\right]}) - \ln\beta\right) + w_0\right) \\ v =& \frac{\beta^{-\alpha}\, e^{-\alpha\,\mathrm{arcsinh}\frac{X_0/e^{\mathrm{E}[\ln(M_T)] + \frac{1}{2}\mathrm{Var}[\ln(M_T)]} - w_0}{\beta\, e^{\frac{1}{2\alpha^2}\mathrm{Var}[\ln(M_T)]}}}}{e^{\mathrm{E}[\ln(M_T)] + \mathrm{Var}[\ln(M_T)]}}. \end{aligned} \tag{52}$$