Netspar

Network for Studies on Pensions, Aging and Retirement

NETSPAR ACADEMIC SERIES

Robust Hedging of Terminal Wealth under Interest Rate Risk and Inflation Risk

Anne Balter, Lieske Coumans, Frank de Jong

DP 11/2021-023

Robust Hedging of Terminal Wealth under Interest Rate Risk and Inflation Risk*

Anne G. Balter
Tilburg University
and Netspar

Lieske Coumans†
Tilburg University
and Netspar

Frank de Jong
Tilburg University
and Netspar

This Version: November 9, 2021

Abstract

Investors often hedge their liabilities against nominal interest rate risk. However, inflation risk also plays an important role for real wealth outcomes, especially in the long-run. If both risks follow a bivariate mean-reverting process, optimal allocations in nominal bond strategies typically turn out to be extreme, in particular when the bond maturities lay close to each other. We show that this makes the investment strategy sensitive to small changes in the mean-reversion parameters and the feedback parameter that takes into account the impact of the inflation rate level on the nominal interest rate drift. We perform a numerical analysis to demonstrate that small estimation errors of these parameters might have a large impact on terminal real wealth. A range of values of the feedback parameter is applied to compare the resulting investment strategies to Brennan and Xia (2002), Van Bilsen et al. (2020), and Munk et al. (2004). We find that the optimal two bond strategies involve one medium term bond and one very long-term bond, but these strategies are very sensitivity to parameter uncertainty. One bond strategies are more robust, but cannot completely hedge inflation risk, which results in a large loss in the Certainty Equivalent Wealth of a risk averse investor.

Keywords: Interest Rate Risk Management, Inflation Risk Management, Life-Cycle Investment, Hedging Demand, Parameter Uncertainty, Robust Portfolio Choice.

*We are grateful to the valuable feedback of the discussants at the Netspar Pension Days and the Netspar International Pension Workshop, Matteo Bonetti (Maastricht University), Peter Schotman (Maastricht University), and Sijmen Duinveld. We are also grateful to Bas Werker, Theo Nijman, Antoon Pelsser, Bart Dees, Lieke Beekers and Jorgo Goossens, and other conference and seminar participants for their helpful comments and suggestions. The research in this paper has been funded by Netspar.

†Corresponding author: Department of Econometrics and Operations Research, Tilburg University. E-mail address: l.coumans@tilburguniversity.edu.

1 Introduction

Long-term investors typically hedge against nominal interest rate risk by positive positions in nominal bonds. However, inflation decreases the real value of wealth over time, especially when the investment horizon is long. Therefore, it is highly relevant to hedge both the real interest rate risk and the inflation risk on the long-term. In theory, this can be done by investing in an Index Linked Bond (ILB) that protects the real value of the principal (or coupon) in case of inflation. However, in practice ILBs are quite illiquid in both the US and the Eurozone (Ciocyte and Westerhout, 2017). Even if they are traded, not many maturities are present which can leave a significant share of real interest rate risk if the maturities do not match the investment horizon (Beetsma et al., 2020).

The academic literature has developed several strategies to invest in either one or two nominal bonds, aiming to hedge both the interest rate risk and inflation risk. An example is the model of Sangvinatsos and Wachter (2005) (SW), where both risk factors follow a mean-reverting process, with correlated shocks, and there is feedback from the level of the expected inflation rate to the change in the interest rate. Earlier studies show that the optimal investment strategy are substantially affected by the market prices of risk, e.g. Flor and Larsen (2011), Garlappi and Uppal (2007), and Feldhütter et al. (2012). We find that the feedback parameter, the mean-reversion parameters, and choice of bond maturities have a substantial impact as well. For example, specific values of the feedback parameter lead to considerable different models investigated by Brennan and Xia (2002), Munk et al. (2004) and Bilsen et al. (2020) (BX, MSV, and BBB respectively). The impact of the parameters is especially large if the investor invests in two bonds, resulting in extreme long-short positions.[1] When applying a one bond strategy, the position is not so extreme and less sensitive to the parameters, but it fails to sufficiently hedge inflation risk.

This paper makes several contributions to the existing literature. First, we show that the extreme positions in the two bond strategies can lead to a large sensitivity of the investor's utility to parameter uncertainty.[2] Second, we show that the extreme positions have an advantage though: for a risk averse investor, they lead to considerably higher utilities than a one bond strategy. Third, we show that one bond strategies lead to

[1] This sensitivity is noted in the study of Brennan and Xia (2002), and Martellini et al. (2015). Moreover, it is analysed in a related paper by Benzoni et al. (2007) where the optimal asset allocation is very sensitive to the speed of cointegration between dividends and wages.

[2] As specific values of the feedback parameter lead to special cases of the model such as the BX-model, parameter uncertainty about the feedback parameter can also be seen as a specific type of model uncertainty.

the opposite results: they are more robust to parameter uncertainty, but fail to hedge inflation risk well. Fourth, we show that it is important to determine the maturity of the bond(s). A robust choice turns out to be different from the average liability duration. This choice does not only affect the magnitude of the bond allocations, but can also enlarge the impact of parameter uncertainty.

We study investment strategies when only nominal bonds are available, but the investor wants to hedge both the real interest rate risk and inflation risk. In particular, this paper focuses on the impact of changes in bond maturities and model parameters on the (sub)optimal hedge demands for nominal bonds and on wealth. To determine the optimal demand in the maximising expected utility setting, we use the SW-model. It results in a simultaneous long-short composition of two nominal bonds that fully hedges inflation risk. We use a framework with no stocks, no borrowing constraints, no transaction costs, nor unhedgeable inflation.[3] In this way, we focus on the main advantage and disadvantage of the optimal allocations in a two bond strategy: they allow to fully hedge inflation risk, but they are extreme in absolute magnitude.[4]

The results in this paper are based on a numerical analysis using an initial parameter set, originated from Brennan and Xia (2002), and a variety of sensitivity parameter sets that impact the optimal hedge demand. We show that the bond allocations in the (two bond) SW-strategy are extreme, especially when the bond maturities or mean-reversion parameters lay close to each other, or the feedback parameter is close to zero. For example, if the feedback parameter equals -0.001, the optimal bond allocations for a speculative investor equal 7,632% and -4,818% of her wealth. But even when the feedback parameter is sufficiently far from zero, the allocations are large in absolute value and result in a large sensitivity of the investor's utility to parameter uncertainty: for small estimation errors of relevant parameters, the investor will face a considerable loss in her Certainty Equivalent (CE) of wealth compared to investing according to the correct parameter values. In an extreme case where the investor incorrectly believes that the mean-reversion parameters of the risk factors double, she looses even $90 - 100\%$ of her CE, depending on her risk aversion and the feedback parameter. Uncertainty about the feedback parameter appears to be relevant as well: if she has a relative risk aversion of 2, she can loose 15% of her

[3]In the literature, inflation risk is split into unhedgeable and hedgeable inflation. Two examples of unhedgeable risks and corresponding numerical solutions for optimal consumption and asset allocations are studied by Beetsma et al. (2020). However, we do not include unhedgeable inflation in our model, as we focus on hedging strategies. We verified analytically and numerically that this does not change the relative results, i.e. the percentage of loss in the Certainty Equivalent of wealth.

[4]Both this advantage and disadvantage are analysed by Martellini et al. (2015) for the nested BX-strategy, where leverage constraints are proposed to tackle the issue of extreme weights.

CE if she incorrectly believes that the feedback parameter is twice as large or small as it actually is.

As one bond strategies result in less extreme bond demands, these turn out to be more robust to parameter uncertainty than two bond strategies. As a consequence, if the investor has incorrect parameter beliefs, applying a one bond strategy might even lead to a higher CE than applying the two bond strategy. However, the disadvantage of one bond strategies is that these do not fully hedge inflation risk, so that the risk averse investor faces large losses in her CE compared to the two optimal bond strategies.[5] For example, we consider an investor with a relative risk aversion of 2 who invests according to the one bond SW-strategy. She looses about 25% of her CE compared to the two bond SW-strategy, slightly depending on the feedback parameter. She looses even more if the feedback parameter or the risk aversion increases. A final observations is that all results regarding parameter uncertainty depend on the chosen bond maturities.

Related work from Feldhütter et al. (2012) analyses these (dis)advantages of robustness against parameter uncertainty and the ability to hedge inflation of the one and three bond SW-strategy.[6] However, the analysis does not take into account inflation as a risk factor nor the impact of the bond maturities, which are important aspects of hedging long-term real wealth. Moreover, the loss in utility is measured by a Bayesian approach over the posterior distribution of the data generating process parameters.

We now briefly discuss some remaining related literature. Baker et al. (2020) show that based on swap markets in the United States it is likely that long-short bond constructions reflect the demand of pension funds, that represent long-term investors. In both practice and life-cycle investment literature, the problem of extreme demands is usually solved by applying lower and upper allocation constraints. Constraints of 0-100% prevent leverage 'funded' by non-tangible human capital. Bilsen et al. (2020) argue that reasonable negative positions can yet be possible in practice through applying swap contracts. A short position can then be replicated by a payer interest swap contract, receiving larger floating legs when the interest rate increases. Blommestein (2007) confirms that swaps are often preferred over bonds by investor as less capital is required. However, with constraints the maximised utility problem cannot be solved analytically anymore. Another way to decrease the positions in absolute magnitude is to invest in a long-term bond.[7]

[5] Mkaouar et al. (2017) note the large loss in utility due to not fully hedging inflation with one bond in the nested BX-model.

[6] In case of the one bond SW-strategy, the hedging strategy hedges the short rate. In case of the three bond strategy, it hedges the short rate, slope, and curvature of the yield curve.

[7] As long-term swaps are more liquid than long-time bonds, the positions in bonds in the BX-model

Quaedvlieg and Schotman (2020) empirically show that a naive hedging strategy that consists of a long position in a bond with the maturity equal to the Last Liquid Point (20-years) not only leads to less sensitive demands, but also outperforms multi-factor interest models in terms of fit and turnover. Yet, about 50% of the interest rate risk remains unhedged based on a horizon investment of 50 years (Quaedvlieg and Schotman, 2020)[p.22].

The structure of the paper is as follows. In Section 2, we explain the SW-model and the corresponding extreme long-short positions in the optimal nominal bond demands. We also show the corresponding optimal demands if the investor invests in only one bond, or if the model simplifies to the BX-, MSV-, or BBB-model. In Section 3, we derive an approximated robust bond maturity combination that minimises the sum of these positions. In Section 5, we explore the effect of parameter uncertainty by computing the impact of deviations in the feedback parameter on the bond allocations and CE. In Section 4, we explore this impact due to deviations in the mean-reversion parameters. Section 6 concludes.

2 Model description

We introduce the long-term optimisation problem of the investor in Section 2.1, the financial market in Section 2.2 and the optimal one and two bond investment strategies in Section 2.3. Section 2.4 explains three suboptimal strategies. Section 2.5 shows how we will measure the impact of parameters in (sub)optimal strategies on utility of terminal wealth.

2.1 Optimisation problem

As the investor cares about the real value of her wealth, the objective is to maximise expected utility from her terminal price-deflated wealth

$$\max_{W_T} \mathbb{E}_0 \left[u \left(\frac{W_T}{\Pi_T} \right) \right] \tag{1}$$

can be translated to long-term swaps in practice. Long-term swaps are sometimes debated because they can lock-in a fund, and they can be costly and complex. However, this is not the focus of our research. Furthermore, investing in these long-term swaps is commonly used in practice and complies with e.g. Dutch pension fund regulations (Blommestein, 2007)[p.181].

where $u()$ is the utility function of the investor, W_t is the nominal wealth at t, Π_t is the realised inflation at t, and T is some large number to represent a long-term investment. She optimises taking into account her budget constraint [8] with nominal pricing kernel ζ_t

$$\mathbb{E}_0[\zeta_T W_T] = W_0 \tag{2}$$

$$\frac{d\zeta_t}{\zeta_t} = -R_t^f \ dt - \tilde{\lambda}_R dB_t^R - \tilde{\lambda}_\pi dB_t^\pi \tag{3}$$

where $\tilde{\lambda} = (\tilde{\lambda}_R, \tilde{\lambda}_\pi)$ are the factor loadings. Moreover, we assume that the investor has a constant relative risk aversion (CRRA) utility function with coefficient of relative risk aversion γ

$$u(x) = \begin{cases} \frac{x^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(x) & \text{if } \gamma = 1 \end{cases} \tag{4}$$

The investor aims to hedge two risks: the nominal interest rate R_t and the realised inflation Π_t. A low R_t is disadvantageous for the investor as it will lead to a low return on wealth. A high Π_t is a risk for the investor as it will erode the real value of wealth.

We assume that the increase in Π_t equals the compounded stochastic expected inflation: $\frac{d\Pi_t}{\Pi_t} = \pi_t dt$. Hence, hereinafter inflation risk refers to the risk in π_t, unless explicitly referred to Π_t.[9] Furthermore, we assume that both risk factors follow an Ornstein-Uhlenbeck (OU) process

$$\begin{aligned} d\begin{bmatrix} R_t \\ \pi_t \end{bmatrix} &= -\begin{bmatrix} \kappa_R & \kappa_{R\pi} \\ 0 & \kappa_\pi \end{bmatrix} \begin{bmatrix} R_t - \bar{R} \\ \pi_t - \bar{\pi} \end{bmatrix} dt + \begin{bmatrix} \sigma_R & 0 \\ 0 & \sigma_\pi \end{bmatrix} d\begin{bmatrix} B_t^R \\ B_t^\pi \end{bmatrix} \\ &\equiv -K \begin{bmatrix} R_t \\ \pi_t \end{bmatrix} dt + \sigma_X d\begin{bmatrix} B_t^R \\ B_t^\pi \end{bmatrix} \end{aligned} \tag{5}$$

For the remainder of this paper, we define κ_R and κ_π as the mean-reversion parameters and $\kappa_{R\pi}$ as the feedback parameter that corresponds to the relation between the level of π_t and the drift adjustment of R_t. We assume that the mean-reversion parameters and volatilities of the factors are positive: $\kappa_R, \kappa_\pi, \sigma_R, \sigma_\pi > 0$; and $\kappa_R \neq \kappa_\pi$. Furthermore, we assume that the feedback parameter is negative: $\kappa_{R\pi} < 0$. This means that if π_t is

[8] Note that this budget constraint is expressed in the nominal wealth. It can be redefined in terms of real wealth and a real pricing kernel, but that will lead to the same optimal investment strategy.

[9] This distinction between the risk in Π_t and the risk in π_t is mentioned for comparison to the realised inflation risk in Π_t and expected inflation risk in π_t mentioned in the literature, where Π_t is not just the compounded value of π_t. (e.g. Brennan and Xia (2002)).

below its long-term average $\bar{\pi}$, it will decrease the drift adjustment of R_t.[10] The economic intuition is that if the inflation rate is low, the investor expects a lower nominal interest rate at the long-term since the nominal interest rate can be seen as the sum of the real inflation and the inflation. Finally, the Brownian Motion processes of the risk factors, B_t^R and B_t^π, are correlated by the correlation matrix ρ^R expressed in $\rho_{R\pi} \in (-1, 1)$

$$\rho^R = \begin{bmatrix} 1 & \rho_{R\pi} \\ \rho_{R\pi} & 1 \end{bmatrix} \tag{6}$$

2.2 Financial market

To focus on bond allocations, we consider a market without stocks. As we assume that no Index Linked Bonds are available, the investor can only invest in cash and nominal bonds with different maturities T_i for bond i. We denote the time to maturity at time t as $\tau_i = T_i - t$. The corresponding bond price $P_t(\tau_i)$ is:

$$P_t(\tau_i) = e^{-a(\tau_i) - b_R(\tau_i)R_t + \frac{\kappa_{R\pi}}{\kappa_\pi}\left(b_R(\tau_i) + \frac{1}{\kappa_R - \kappa_\pi}(e^{-\tau_i \kappa_R} - e^{-\tau_i \kappa_\pi})\right)\pi_t} \tag{7}$$

The derivations of $P_t(\tau_i)$ and $a(\tau_i)$ can be found in Appendix A of Sangvinatsos and Wachter (2005).[11] For later comparisons with nested models we define a function $c_\pi(\tau_i)$ as well. The functions $b_R(\tau_i)$ and $c_\pi(\tau_i)$ are given by

$$b_R(\tau_i) = \frac{1}{\kappa_R}(1 - e^{-\kappa_R(\tau_i)}) \tag{8}$$

$$c_\pi(\tau_i) = \frac{1}{\kappa_\pi}(1 - e^{-\kappa_\pi(\tau_i)}) \tag{9}$$

The remaining wealth that is not invested in nominal bonds is allocated to a bank account with the nominal interest rate R_t as return. As can be seen in equation (7), as a result of the two-factor model the bond price is affected by the risk factors R_t and π_t by different sensitivities. Under our assumptions about the negative feedback parameter and the positive mean-reversion parameters, and the additional assumption that $\kappa_R > \kappa_\pi$, it can be derived that both sensitivities (the factors in front of R_t and π_t) are negative.[12] Therefore, an infinitely risk averse investor can fully hedge π_t by investing in two nominal

[10] i.e. if $R_t < \bar{R}$, the nominal interest rate will increase with a slower rate to $\bar{R}$, or R_t will even decrease. If $R_t > \bar{R}$, it will decrease with a faster rate to $\bar{R}$.

[11] Sangvinatsos and Wachter (2005) show that the bond price equals $e^{A(\tau_i)X(t) + A_1(\tau_i)}$, where $A_1(\tau_i)$ is a scalar, and we derive the vector $A(\tau_i)$ in Appendix A.

[12] The additional assumption that $\kappa_R > \kappa_\pi$ is argued to be plausible in Section 3.

bonds. To see why, consider an investor that only cares about nominal wealth and hence fully hedges R_t, where we consider an infinitely risk averse investor to focus on the hedge demand. She will invest in only one nominal bond with time to maturity τ_1 to hedge against a low interest rate, because the bond price $P_t(\tau_1)$ increases if R_t decreases. Now suppose that the investor wants to optimise her real wealth. Only going long in one nominal bond exposes her to inflation risk, as a high π_t both decreases $P_t(\tau_1)$ and her real wealth. Therefore, simultaneously she goes short in a second nominal bond with time to maturity τ_2 to benefit from an increasing π_t due to the decreased $P_t(\tau_2)$. Based on her risk aversion, she can balance these long and short bond demands to result in optimal exposures to the risk factors.[13]

We now formalise some definitions in matrix notation. We let σ correspond to the factor exposures, expressed in the matrix A and factor volatility matrix σ_X, in line with Sangvinatsos and Wachter (2005). If the investor invests in two bonds, A is a 2x2 matrix and σ is defined as

$$\sigma = A\sigma_X = \begin{bmatrix} -b_R(\tau_1)\sigma_R & \left(\frac{\kappa_{R\pi}}{\kappa_R-\kappa_\pi} \cdot c_\pi(\tau_1) + \frac{\kappa_{R\pi}}{\kappa_\pi} b_R(\tau_1) - \frac{\kappa_{R\pi}}{\kappa_\pi(\kappa_R-\kappa_\pi)}\left(1-e^{-\tau_1\kappa_R}\right)\right)\sigma_\pi \\ -b_R(\tau_2)\sigma_R & \left(\frac{\kappa_{R\pi}}{\kappa_R-\kappa_\pi} \cdot c_\pi(\tau_2) + \frac{\kappa_{R\pi}}{\kappa_\pi} b_R(\tau_2) - \frac{\kappa_{R\pi}}{\kappa_\pi(\kappa_R-\kappa_\pi)}\left(1-e^{-\tau_2\kappa_R}\right)\right)\sigma_\pi \end{bmatrix} \tag{10}$$

where the expression of A can be found in Appendix A. If the investor only invests in one bond, A and therefore σ are 1x2 vectors only containing the upper row. The definitions and sizes of the remaining relevant variables $x_t^*, \lambda, \Omega, \Lambda$, and $\tilde{\lambda}$ can be found in Table 1.

2.3 Optimal bond allocations

Sangvinatsos and Wachter (2005) derive the optimal portfolio weights x_t^* and express these in a speculative and a hedge demand

$$x_t^* = \frac{1}{\gamma}\Omega^{-1}\Lambda - (1-\frac{1}{\gamma})\Omega^{-1}\sigma'\rho^R\sigma_X B(\tau)' \tag{11}$$

where $\tau = T - t$ and $B(\tau)$ is the vector with exposures to the shocks in R_t and π_t. The definition of B can be found in Appendix A. This section will analyse the resulting optimal bond allocations for two cases: if the investor uses either one or two nominal bonds as hedging assets.

When applying a two bond strategy, the matrix σ is invertible. In that case, using

[13] Section 2.3 gives an example for an infinitely risk averse investor.

the definitions in equations (5) and (10), and Table 1, we can simplify x_t^* to

$$x_t^* = \frac{1}{\gamma}(\sigma')^{-1}(\rho^R)^{-1}\lambda - \left(1 - \frac{1}{\gamma}\right)(A')^{-1}B(\tau)' \tag{12}$$

It shows that the hedge demand is independent of ρ^R and σ_X, and dependent on the mean-reversion parameters and time to maturities τ_1 and τ_2 (through $(A')^{-1}$ and $B(\tau)$). Hence, the bond maturity choice is not straightforward anymore: in case of hedging nominal wealth, she could just invests in a bond with maturity T. However, when hedging her real wealth the investor chooses τ_1 and τ_2, where $\tau_1 \neq \tau_2$. She then increases (decreases) the bond maturity to increase (decrease) the sensitivity of the bond price to the risk factors. Moreover, the factor $(A')^{-1}$ implies the short-long composition of bonds as explained in Section 2.2 and leads to extreme positions if $\kappa_R \approx \kappa_\pi$, or $\tau_1 \approx \tau_2$. The intuition is as follows: consider an infinitely risk averse investor that wants to fully hedge π_t. She aims to match the optimal exposures to the shocks in R_t and π_t respectively. When she increases her long position to increase her exposure to shocks in R_t, she needs to hold a larger short position to decrease her exposure to shocks in π_t, and vice versa. She needs to increase or decrease the positions more if the sensitivity of the bond is small. If the mean-reversion parameters or bond maturities lay close to each other, the difference between the sensitivities of the bonds is small. Therefore, in that case she needs large demands to exactly match the different risk exposures to the two risk factors.

For completeness, we analytically explain why the inverse of A' can result in such extreme positions. Recall that the formula of A is given in equation (10). The determinant equals

$$\text{Det}\,(A) = \frac{\kappa_{R\pi}}{\kappa_R - \kappa_\pi}(b_R(\tau_2)c_\pi(\tau_1) - b_R(\tau_1)c_\pi(\tau_2)) \tag{13}$$

This shows that the determinant converges to zero if one of the following conditions holds: $\kappa_{R\pi} \approx 0$, $\tau_1 \approx \tau_2$, $\kappa_R \approx \kappa_\pi$, or $\tau_i \approx 0$ (which results in $b_R(\tau_2)c_\pi(\tau_1) \approx b_R(\tau_1)c_\pi(\tau_2)$). As a result, the long and short position will become extremely large in absolute values.

Now, we shortly analyse the resulting optimal bond allocations if the investor uses only one nominal bond as hedging asset to partly hedge the inflation risk. In that case, σ is not invertible and the simplification to equation (12) does not hold anymore. Therefore, in contrast to the case with two nominal bonds, ρ^R and σ_X become relevant to compute the hedging demand. In line with the two bond strategy, the mean-reversion parameters and time to maturities remain relevant as well. The advantage of the optimal one bond strategy is that the position in the bond will not be so extreme as in the two bonds strategy. This can be seen by considering the determinant of A for the one bond case

$$\begin{aligned}\text{Det}\,(A) &= b_R^2(\tau_1)\sigma_R^2 + \frac{\kappa_{R\pi}}{\kappa_R - \kappa_\pi}[\kappa_{R\pi}\sigma_\pi(c_\pi^2(\tau_1) + b_R^2(\tau_1)) \\ &- 2b_R(\tau_1)\left(c_\pi(\tau_1)\rho^R\sigma_R(\kappa_R - \kappa_\pi) + c_\pi(\tau_1)\sigma_\pi\kappa_{R\pi} + b_R(\tau_1)\kappa_\pi\rho^R\sigma_R - b_R(\tau_1)\kappa_R\rho^R\sigma_R\right)] \quad (14)\end{aligned}$$

This shows that even if $\kappa_{R\pi} \approx 0$ or $\tau_1 \approx \tau_2$, the determinant does not converge to zero as in the two bond strategy. As a result, the long and short position will not become so extremely large in absolute values.

2.4 Suboptimal bond allocations

This section explains three suboptimal bond allocations. These are interesting to study, because those are optimal for specific settings within our general SW-model. First, we consider the two bond strategy derived by Brennan and Xia (2002) (BX) that is widely accepted in the literature. It is optimal in the BX-setting which is obtained if $\kappa_{R\pi} = \kappa_\pi - \kappa_r$ and $\kappa_R = \kappa_r$ in our model, where $r_t = R_t - \pi_t$ is the real interest rate that follows an OU-process. This results in a two-factor model where the factors r_t and π_t are only linked through their possibly correlated shocks, and not by a feedback parameter term in one of the drift terms. Converging ρ^R to ρ^r and σ_R to σ_r as explained in Appendix B, the optimal strategy in the BX-setting equals[14]

$$x_t^{BX} = \frac{1}{\gamma}\Omega^{-1}\Lambda - (1 - \frac{1}{\gamma})\left(\Omega^{-1}\sigma\rho^r\right)\begin{bmatrix}\sigma_r b_R(T-t) \\ 0\end{bmatrix} \quad (15)$$

In line with the optimal two bond strategy, the hedge demand depends on the mean-reversion parameters and the bond maturities, but is independent of ρ and σ_X. This is less evident from the solution of Brennan and Xia (2002).

The second suboptimal strategy is derived by Bilsen et al. (2020) (BBB) that is optimal for the same parameters as the BX-setting, but now the investor only invests in one nominal bond instead of two. The intuition is similar as in the optimal one bond strategy versus the two bond strategy: the BX-strategy fully hedges inflation risk, but can result in extreme positions. The BBB-strategy can only partly hedge inflation risk, but results in less extreme positions. Converging $\rho_{R\pi}$ to $\rho_{r\pi}$ as explained in Appendix B,

[14] See equation (34) of their paper. Note that their vector ξ only contains zeros in our market setting where realised inflation equals the compounded value of expected inflation.

the optimal strategy in the BBB-setting equals (Bilsen et al., 2020)[Appendix A.5]

$$x_t^{BBB} = \frac{1}{b_r^2(\tau_1)\sigma_r^2 + c_\pi^2(\tau_1)\sigma_\pi^2 + 2\rho_{r\pi} b_r(\tau_1) c_\pi(\tau_1)\sigma_r\sigma_\pi} \cdot \tag{16}$$
$$\left[\frac{1}{\gamma}\left(-b_r(\tau_1)\sigma_r\tilde{\lambda}_r - c_\pi(\tau_1)\sigma_\pi\tilde{\lambda}_\pi\right) + \left(1 - \frac{1}{\gamma}\right)\left(b_r^2(\tau_1)\sigma_r^2 + \rho_{r\pi} b_r(\tau_1) c_\pi(\tau_1)\sigma_r\sigma_\pi\right)\frac{b_r(\tau)}{b_r(\tau_1)}\right]$$

As the formula shows, the demand exists of a speculative demand and a hedging demand (the terms with $\frac{1}{\gamma}$ and $(1 - \frac{1}{\gamma})$ respectively). We recognize the simple strategy when no inflation risk is present: the hedging term then equals $(1 - \frac{1}{\gamma}) b_r(\tau)/b_r(\tau_1)$. When the investor is infinitely risk averse and chooses a bond with maturity $T_1 - t = T - t$, she invests 100% in that bond. However, when inflation risk enters, the investor cannot fully hedge that risk. She lowers the total bond demand, which decreases the risk of lower bond prices in case of high inflation. Note that in this way she cannot fully inflation risk anymore.

The third suboptimal strategy is derived by Munk et al. (2004) (MSV) and is optimal for an extreme case where $\kappa_{R\pi} = 0$. In that case, the nominal interest rate is only affected by one factor, and inflation cannot be fully hedged anymore by a long-short position in bonds. Therefore, the optimal strategy becomes a one bond strategy that contains a less extreme position than the positions in two bond strategies. Munk et al. (2004) show that the optimal strategy equals

$$x_t^{MSV} = -\frac{1}{\gamma}\frac{\lambda_R}{b_R(\tau_1)\sigma_R} + \left(1 - \frac{1}{\gamma}\right)\frac{b_R(T-t)}{b_R(\tau_1)} - \left(1 - \frac{1}{\gamma}\right)\frac{c(T-t)\rho_{R\pi}\sigma_\pi}{b_R(\tau_1)\sigma_R} \tag{17}$$

The optimal investment strategy consists of three terms: a speculative term, a nominal interest rate hedge term, and an inflation hedge term. As a consequence of the one factor model, the inflation hedge term can only partially hedge inflation risk. The investor makes a trade-off between increasing the bond demand to hedge against a low R_t (by the positive nominal interest rate hedge term) and decreasing the bond demand to decrease the exposure to risks in π_t (by the negative inflation hedge term if $\rho_{R\pi} > 0$).

2.5 Certainty Equivalent Wealth

This section explains how we measure the impact of (input parameters for) investment strategies on utility. Consider a market with the data generating processes based on the parameters in set $\Theta = \{\kappa_{R\pi}, \kappa_R, \sigma_R, \lambda_R, \kappa_\pi, \sigma_\pi, \lambda_\pi, \rho_{R\pi}\}$. The investor however thinks the correct parameters are in an alternative set k denoted by $\hat{\Theta}^k$ and applies her perceived optimal strategy $x^{\delta,k}$

$$x_t^{\delta,k} = \arg\max_{x_t} \ \mathbb{E}_0 \left[\frac{\left(\frac{W_T|x_t}{\Pi_T}\right)^{1-\gamma} - 1}{1-\gamma} \middle| \hat{\Theta}^k \right] \tag{18}$$

where δ indicates if a one ($\delta = 1$) or two ($\delta = 2$) bond strategy is applied. This is only the actual optimal one or two bond strategy if $\Theta = \hat{\Theta}^k$. We express the impact of this strategy on expected utility from terminal real wealth by the Certainty Equivalent (CE) of wealth, i.e. the amount of wealth at $t = 0$ that makes the investor indifferent between receiving the CE for certain or investing according to $x_t^{\delta,k}$

$$CE^{\delta,k} = u^{-1}\left(\mathbb{E}_0 \left[u\left(\frac{W_T|x_t^{\delta,k}}{\Pi_T} \right) \middle| \Theta \right] \right) \tag{19}$$

Recall that the utility function $u()$ is given in equation (4). To compute the nominal wealth based on $x_t^{\delta,k}$, we plug the bond allocations into the Stochastic Differential Equation (SDE) of the nominal wealth process

$$\frac{dW_t}{W_t} = \left[R_t + \left(x_t^{\delta,k}\right)' \Lambda \right] \ dt + \left(x_t^{\delta,k}\right)' \sigma \left[dB_t^R \quad dB_t^\pi \right]' \tag{20}$$

where Λ and σ are based on the parameters of the data generating process in Θ. The wealth process with respect to a suboptimal investment strategy cannot be exactly discretised.[15] Therefore, we apply the Euler approximation to compute it in discrete form

$$W_{t+1} = W_t \cdot [1 + (R_t + (x_t^{\delta,k})'\Lambda)\Delta + x_t^{\delta,k}\sigma\sqrt{\Delta}\epsilon_{t+1}] \tag{21}$$

where the definition of σ is given in equation (10), the step size is given by Δ, and $\epsilon_{t+1} \sim N(0, \Omega) = N(0, \sigma\rho\sigma')$, where ρ is based on Θ.

3 Impact bond maturities on optimal portfolio weights

We study the impact of the bond maturities on the demands in both the one- and two bond optimal strategies. In Section 3.1, we discuss the initial parameter set. In Section

[15] In case of an optimal strategy, writing out the expression of x_t^* shows that the model can be exactly discretised. However, this is not possible in case of suboptimal strategies. We choose the time step to be small to prevent large discretisation errors.

3.2, we show the impact of bond maturities on the optimal bond allocations.

3.1 Initial parameters

All results will be investigated for three relevant risk aversion coefficients $\gamma \in \{1, 2, 25\}$, where $\gamma = 1$ represents the 'log utility' or 'speculative' investor that only has a speculative demand, and $\gamma = 25$ is the 'hedge' investor corresponding to the almost infinitely risk averse investor who only has a hedge demand. The investor with $\gamma = 25$ is not only useful to investigate the hedge demand, but can also be related to practice in the case of a hedge fund. In reality, an investor will often be interested in both demands and therefore we include the moderately risk averse investor with $\gamma = 2$ as well which implies a sort of average bond demand of the speculative and hedge investor.

We assume that the investment horizon equals $T = 30$ years to represent a long-term investment. For simplification of notation we split the parameter set Θ into $\kappa_{R\pi}$ and $\Theta_R \equiv \{\kappa_R, \sigma_R, \lambda_R, \kappa_\pi, \sigma_\pi, \lambda_\pi, \rho_{R\pi}\}$. In this section we initially set $\kappa_{R\pi} = -0.078$ (leading to the BX-model), where we mention the impact of other values of $\kappa_{R\pi}$ as well. Moreover, we set the initial parameter set Θ_R equal to the values given in the first column of Table 2. These values are based on the empirical results of Brennan and Xia (2002).[16] Note that the estimates for the mean-reverting interest rate are based on the real interest rate $r_t = R_t - \pi_t$. Appendix B explains how we convert these values to estimates for the mean-reverting nominal interest rate R_t. Although the parameter values from the initial parameter set 0 will not be examined in detail, we performed a plausibility check on them. The magnitude of $\kappa_R > \kappa_\pi$ is supported by empirical evidence in the literature, as the mean-reversion of expected inflation tends to be low (e.g. Brennan and Xia (2002) and Van den End (2011)). Furthermore, it is economically intuitive that λ_R and λ_π are negative: Table 1 and equation (10) show that negative market prices of risk lead to positive risk premiums.

3.2 Results

We now show the most relevant results about the impact of the bond maturities on x_t^*. It appears that plausible bond maturities in practice lead to extreme bond allocations. For

[16] We use the second set of Table 1 of Brennan and Xia (2002) that are based on yearly observations from 1890-1985, as they argue that the first set based on monthly observations from 1970-1995 led to unrealistic much oscillations in the estimated real interest rate. Furthermore, we set $\rho_{r\pi}$=0, because that estimate tends to be very low and we did not see considerable differences between using either $\rho_{r\pi} = 0$ or the estimated (very small) value.

example; for $\tau_1 = 3, \tau_2 = 10$, and $\gamma = 2$ the investor should invest 1,850% of her wealth in bond 1 and -566% in bond 2. Therefore, it is relevant to analyse the impact of τ_1 and τ_2 by studying the 'extremeness' of bond allocations for different bond combinations. In the analysis, we assume that the possible bond maturities are in the range $\tau_i \in \{1, 2, ..., 50\}$ for $i \in \{1, 2\}$. This maximum maturity of 50 is in line with the maximum available swap maturity data.

To approach the 'least extreme' bond allocations by changing τ_1 and τ_2, we define a 'robust' combination of bond maturities $(\tau_1^{min}, \tau_2^{min})$. We let it minimise the sum of the absolute values of x_0^* of both bonds[17]

$$(\tau_1^{min}, \tau_2^{min}) = \arg\min_{\tau_1, \tau_2} \left(|x_{0,1}^*| + |x_{0,2}^*| \right) \tag{22}$$

where $x_{0,1}^*$ and $x_{0,2}^*$ denote the optimal bond allocations to bond 1 and 2 respectively. This leads to four main results that hold for all $\gamma \in \{1, 2, 25\}$. Furthermore, the example allocations mentioned in this section are based on $\kappa_{R\pi} = -0.078$, but the conclusions remain valid for some sensitivity values $\kappa_{R\pi} \in \{-0.117, -0.039, -0.001\}$.

First, the robust combination according to the definition in equation (22) equals $(\tau_1^{min}, \tau_2^{min}) = (11, 50)$. So if the investor can choose both τ_1 and τ_2, she will choose this combination to minimise the sum of absolute values of x_0^*. The choice of $\tau_1 = 11$ is roughly in line with the maturity of 10 years, shown in Figure 4 of the paper of Brennan and Xia (2002). Moreover, $\tau_2 = 50$ is in line with the common practice of long-term investors that invest in very long-term bonds.[18] Note that these maturities do not solve the issue of large allocations, for example at $t = 0$ these still are equal to $|349\%| + |-107\%| = 457\%$ of wealth for $\gamma = 2$.

Second, we compute the 'robust' choice of τ_2 that minimises the sum of absolute values of x_0^* of both bonds when τ_1 is fixed to a certain value $\tau_1 \in \{5, 10, ..., 50\}$. The results are shown in Table 3. It shows that for all fixed values of τ_1, the robust choice includes one long-term bond maturity of at least 30 years. Intuitively, this can be explained by the fact that a long-term bond is more sensitive to changes in R_t and π_t. Therefore, a smaller allocation to this bond is necessary to result in a certain risk exposure compared

[17]For $t > 0$, the problem can be considered as a new problem at $t = 0$, but with a decreased investment horizon T. We verified that for smaller $T \in \{1, 5, 10, 20\}$, the robust choice did not change. Therefore, it is sufficient to consider the bond allocations at $t = 0$ instead of for all $t \in [0, T]$.

[18]Note that long-term bonds are not very liquid in practice. However, as explained in Section 1, the long (short) position in a long-term bond can be replicated by an interest rate swap contract. These swaps are available for high maturities and therefore commonly used by long-term investors.

to bonds with smaller bond maturities.

Third, if the investor deviates from this robust choice substantially, this heavily affects the optimal bond allocations. For example, Figure 1 shows the optimal allocations for $\tau_2 = 50$, a τ_1 equal to the value of the x-axis, and several values of $\kappa_{R\pi}$. As we know from the 'robust' choice, the sum $|x^*_{0,1}| + |x^*_{0,2}|$ is the smallest for $\tau_1 = 11$. When the investor chooses a slightly different τ_1 than $\tau_1^{min} = 11$, this does not lead to large changes in the sum of absolute allocations. However, as Figure 1 shows, when τ_1 is considerably lower or higher than τ_1^{min}, the increase in the allocations can be large. This is in line with approaching the condition $\tau_1 \approx 0$ or $\tau_1 \approx \tau_2$ respectively. Recall that Section 2.3 explains that these conditions lead to extreme bond demands below (13). For example, for $\tau_1 = 5 << \tau_1^{min} = 11$, $\gamma = 2$ and $\kappa_{R\pi} = -0.078$ the sum of absolute bond allocations already increases to about 570%. And even more striking, when τ_1 is increased to more than 30 years, the sum becomes larger than 960% for $\gamma = 2$.

Fourth, all three results above not only hold for $T = 30$, but for the horizon investments $T \in \{1, 5, 10, 20\}$ as well. This is because under the initial parameters the extreme allocations are a result of the large (dominating) term $(A')^{-1}$ in the optimal demand given in equation (12). This term only depends on the bond maturities and mean-reversion parameters.

Hence, the results show that the bond maturities can have a large impact on the bond demand. We define a robust choice of bond maturities. Under the initial parameters, this choice always includes a long-term bond. If both maturities can be chosen, the robust choice equals $(\tau_1, \tau_2) = (11, 50)$. However, as it is just an approximation of a robust strategy, to prevent false accuracy we choose to use $(\tau_1, \tau_2) = (10, 50)$ instead for the remainder of this paper. As explained in the third result, this does not lead to substantial different results.

4 Impact feedback parameter

Because the optimal bond allocations of the SW-strategy are large in absolute numbers, they are expected to be sensitive to changes in parameters. This may lead to large utility losses due to small estimation errors, emphasizing the relevance of parameter uncertainty. This section shows the impact of the feedback parameter on the optimal bond allocations and the resulting Certainty Equivalent (CE) of wealth. Note that parameter uncertainty can be seen as model uncertainty, as Section 2.4 shows that specific values of the feedback parameter lead to special cases of the SW-model.

4.1 Impact on optimal portfolio weights

We use the initial parameters of set Θ_R as given in the first column of Table 2, and the robust bond maturity combination $(\tau_1, \tau_2) = (10, 50)$ as supported in Section 3.2.[19] We compute the optimal bond allocations for different values of the feedback parameter, namely $\kappa_{R\pi} \in \{-0.117, -0.078, -0.039, -0.001, 0\}$. The value of $\kappa_{R\pi} = 0$ is relevant because in this case the SW-strategy equals the MSV-strategy. Similarly, applying the SW-strategy based on $\kappa_{R\pi} = -0.078$ results in the BX-strategy (two bonds) or BBB-strategy (one bond).[20] The first and third values of $\kappa_{R\pi} = -0.117$ and $\kappa_{R\pi} = -0.039$ are corresponding to 150%- and 50%-sensitivities of the value of $\kappa_{R\pi} = -0.078$. The value of $\kappa_{R\pi} = -0.001$ is added to study the impact when the feedback parameter converges to 0.

Table 4 shows the optimal bond allocations in the two bond and one bond SW-strategy for the different values of $\kappa_{R\pi}$. For $\kappa_{R\pi} = 0$, the investor cannot apply the two bond strategy.[21] The table shows considerable differences between the extreme short-long positions in the two bond strategies and the less extreme (long) positions in the one bond strategies. This difference increases when $\kappa_{R\pi}$ converges to 0. The intuition is that in this case the investor needs these extreme positions in the two bond strategy to hedge the risks, because of the weaker relationship between the inflation rate level and the nominal interest rate drift. The impact of the feedback parameter on the one bond SW-strategy is less significant, because the investor cannot fully hedge the inflation rate anyway.

4.2 Impact on Certainty Equivalent Wealth

The large sensitivity of bond allocations with respect to the feedback parameter may result in a large impact on the CE. Recall that Section 2.5 explains how the CE is computed.[22] The data generating process is based on the actual parameters $\Theta = \{\kappa_{R\pi}, \Theta_R\}$, while the investment strategy is based on the investor's beliefs $\hat{\Theta} = \{\hat{\kappa}_{R\pi}, \hat{\Theta}_R\}$. To isolate the effect of the feedback parameter we let the investor have correct beliefs of $\hat{\Theta}_R = \Theta_R$, but she may have incorrect beliefs about the feedback parameter when $\hat{\kappa}_{R\pi} \neq \kappa_{R\pi}$.

[19] In case of a one bond strategy, we choose $\tau_1 = 10$ to keep the one and two bond strategies comparable.

[20] The value of -0.078 is based on $\kappa_\pi - \kappa_R$.

[21] In this case, the 2x2 matrix σ given in equation (10) is not invertible anymore, while the optimal allocations in equation (11) are based on this inverse.

[22] We assume a constant time to maturity of the bond τ_i. Furthermore, we assume that $\Delta = 1/12$ and that the investor rebalances her bond allocation to the perceived optimal strategy every time step. If $\gamma = 25$, we cannot directly apply equation (19) because of numerical inaccuracy. Therefore, for this risk aversion coefficient we compute the CE by assuming that W_T is log-normally distributed over all simulations at time T, where the mean and volatility of W_T are determined numerically.

Table 5 shows the impact on the CE if the investor applies the one or two bond SW-strategy based on $\hat{\kappa}_{R\pi}$ given in the first column, while the actual $\kappa_{R\pi}$ in the GDP equals the value given in the top row. We start to analyse the results for $\kappa_{R\pi} = -0.078$. In this case, the correct belief of $\hat{\kappa}_{R\pi} = -0.078$ results in the optimal investment strategy. This corresponds to the BX- or BBB-strategy in case of a two or one bond investment strategy respectively. However, if she has incorrect beliefs about the feedback parameter, she applies a suboptimal strategy not equal to the BX- or BBB-strategy. Note that this suboptimality can therefore also be interpreted as model uncertainty.

We observe two main results. First, having an incorrect belief about the feedback parameter leads to a considerable loss in CE for the two bond SW-strategy, especially when $\hat{\kappa}_{R\pi}$ is lower than $\kappa_{R\pi}$. For example, if $\hat{\kappa}_{R\pi} = -0.039$, this leads to a decrease in the CE from $2,517$ to 0 for a hedge investor ($\gamma = 25$). Second, this sensitivity to $\hat{\kappa}_{R\pi}$ considerably decreases if this investor would apply the one bond SW-strategy. For the same example with $\hat{\kappa}_{R\pi} = -0.039$ and $\gamma = 25$, the loss in CE would be smaller, i.e. decreasing from 251 to 200. Moreover, if $\hat{\kappa}_{R\pi} = -0.039$ and $\gamma = 2$ the one bond SW-strategy even leads to a higher CE of $2,316$ than the two bond SW-strategy with a CE of $1,592$, when these strategies are based on the incorrect belief about $\kappa_{R\pi}$. Hence, the one bond SW-strategy performs bad in hedging inflation, but is robust against parameter uncertainty about the feedback parameter.

We make two additional observations for other actual feedback parameter values of the GDP. First, for $\kappa_{R\pi} \in \{-0.039, -0.001, 0\}$, the one bond SW-strategy performs better than the two bond SW-strategy in case of incorrect beliefs about the feedback parameter: for all three risk aversion coefficients and $\hat{\kappa}_{R\pi} \neq \kappa_{R\pi}$, the CE is higher if the investor applies the one bond SW-strategy than if she applies the two bond SW-strategy. Second, the CE of the two bond SW-strategy is very sensitive to $\hat{\kappa}_{R\pi}$ if the actual $\kappa_{R\pi} = -0.001$. For example, having incorrect beliefs $\hat{\kappa}_{R\pi} \neq \kappa_{R\pi}$ always leads to a loss in the CE of 100%, even for $\gamma \in \{1, 2\}$ which lead to less sensitive CE for other $\kappa_{R\pi}$. Moreover, if the investor has the exact correct belief of $\hat{\kappa}_{R\pi} = -0.001$, the investor can still apply the two bond SW-strategy. However, if she makes a very small error in the estimation of $\kappa_{R\pi}$ by having the belief $\hat{\kappa}_{R\pi} = 0$, the two bond SW-strategy is not defined anymore, as mentioned in Section 4.1. The investor does not face this issue if she invests in one bond only, which is another advantage of the one bond SW-strategy.

For completeness, the analysis has been repeated with alternative combinations of bond maturities (τ_1, τ_2).[23] For both the one and two bond strategies, the bond positions

[23] The results are available upon request.

became more extreme, as the combinations differ from the derived robust combination used in the initial analysis. For the two bond strategies, this does not affect the CE for all values of $\kappa_{R\pi}$. However, it does impact the CE under the one bond strategies in two ways. First, an increasing bond maturity leads to a larger loss in the CE under a one bond strategy compared to a two bond strategy, although this is less visible if $\kappa_{R\pi}$ increases. Second, it makes the one bond SW-strategy more robust to parameter uncertainty about the feedback parameter: the optimal CE lays closer to the CE with a strategy where $\hat{\kappa}_{R\pi} \neq \kappa_{R\pi}$.

Hence, because the feedback parameter has a considerable impact on the optimal investment strategy, an estimation error in this parameter can result in a large loss in CE. As the underlying model changes with the feedback parameter as well, this can be seen as either parameter uncertainty or model uncertainty. Due to the less extreme bond positions, the one bond strategies are more robust for this uncertainty, although the results of the one bond strategies are dependent of the chosen bond maturity.

5 Robustness with respect to remaining parameters

In Section 4 we investigate the sensitivity of (sub)optimal investment strategies to the feedback parameter, keeping the remaining parameters constant. However, based on our earlier analytical analysis we expect the strategies to also be sensitive to changes in the mean-reversion parameters and to changes in the correlation between the shocks in the risk factors. Therefore, this section will show the impact of changes in these parameters on the optimal nominal bond demands and Certainty Equivalent (CE) of wealth.

5.1 Impact on optimal portfolio weights

This section shows the impact on the optimal bond allocations of changes in the parameters in Θ_R. To analyse this, we use the (robust) bond maturity combination $(\tau_1, \tau_2) = (10, 50)$ in line with Section 4.1 and we set $\kappa_{R\pi} = -0.078$.[24] We apply 8 alternative sensitivity sets $\hat{\Theta}_R^k (k \in \{1, 2, ..., 8\})$, shown in Table 2. As Section 2.3 shows that the mean-reversion parameters affect the hedge demand, sets 1 till 6 correspond to changes in κ_R and κ_π with respect to the initial parameters in set Θ. The corresponding volatilities

[24]This value of $\kappa_{R\pi}$ is relevant for the analysis in Section 5.2. Results for the other values of the feedback parameter can be delivered upon request.

σ_R and σ_π are changed simultaneously to keep the unconditional variance constant.[25] Sets 1 till 4 decrease or increase one mean-reversion parameter, sets 5 and 6 decrease or increase the variable for both risk factors instead of one. Moreover, sets 7 and 8 apply changes to the correlation between the shocks in the risk factors, $\rho_{R\pi}$. Recall that this factor is relevant, because Section 2.3 shows that this factor impacts the one bond strategies.[26]

We first analyse the impact on the two bond strategies, shown in Table 6. The two bond SW-strategy contains more extreme positions in particular for the sets where κ_R doubles. In these cases, the investor believes that R_t is less persistent than it actually is. Therefore, she holds more extreme positions, especially regarding the short position. Similarly, she holds less extreme (short) positions in case she thinks κ_R is smaller than it actually is. Moreover, in line with the mean-reversion parameter for R_t, for an increasing κ_π she holds more extreme (short) positions, and less extreme (short) positions when she thinks κ_π is smaller than it actually is. When $\rho_{R\pi}$ changes, based on the analytical analysis in Section 2.3 we expect that only the speculative demand changes. The small changes in the hedge demand for $\gamma = 25$ are just a result of the difference between $\gamma = 25$ and $\gamma = \infty$. The BX-strategy appears to change in the opposite direction of the two bond SW-strategy when the mean-reversion parameters change, i.e. it contains less extreme (short) positions for increases in κ_R and decreases in κ_π. The effect of an adjusted $\rho_{R\pi}$ is exact the same as for the two bond SW-strategy, as expected.

Consecutively, we analyse the optimal demands in the one bond strategies. Table 6 show that the one bond strategies are considerably more robust to parameter changes than the two bond strategies. This is especially the case if the investor has incorrect beliefs about κ_R or a higher $\rho_{R\pi}$ ($k = \{1, 3, 5, 6, 8\}$). The intuition is as follows. Suppose an investor with $\gamma > 1$ who invests in only one bond and believes that κ_R is less persistent than it actually is. Then she decreases her bond demand due to her smaller (positive) R_t-hedge demand. Because her demand is less extreme in absolute terms than when she would invest in two bonds, this decrease in her bond demand is also less extreme than when she would use a two bond strategy. A final observation is that the changes compared to set 0 are non-monotone in γ. For example, if κ_R doubles, the speculative demand increases compared to set 0 but the hedge demand decreases. This is because the decreasing (positive) R_t-hedge demand is only relevant if the investor is at least

[25] For example, set 1 contains a doubling of κ_R and an increase of $\sqrt{200\%}$ in σ_R simultaneously. Hence, the unconditional variance of $R_t(\mathrm{Var}(R_t) = \sigma_R^2/2\kappa_R)$ is in line with set 0.

[26] The factor $\rho_{R\pi}$ also affects the speculative demand of two bond strategies, but we focus on the hedge demand.

moderately risk averse ($\gamma > 1$).

5.2 Impact on Certainty Equivalent Wealth

Similar as in Section 4.2, we compute the impact on the CE. Recall that the CE-computations are explained in Section 2.5 and that the investor believes that the data generating process is based on $\hat{\Theta} = \{\hat{\Theta}_R, \hat{\kappa}_{R\pi}\}$, while it is actually based on $\Theta = \{\Theta_R, \kappa_{R\pi}\}$. Since we are now interested in the impact of the parameters in Θ_R, we set $\hat{\kappa}_{R\pi} = \kappa_{R\pi} = -0.078$.[27] Hence, the investor knows the correct value of the feedback parameter, but can have incorrect beliefs about the parameters in Θ_R. Table 7 shows the impact on the CE for every belief set $\hat{\Theta}^k$. It is shown for the five different strategies: the two bond SW-strategy, the BX-strategy, the one bond SW-strategy, the BBB-strategy, and the MSV-strategy. The corresponding CE-values are denoted by CE^*, CE^{BX}, $CE^{\tilde{*}}$, CE^{BBB}, and CE^{MSV} respectively.

We first analyse the results for the two bond strategies. The CE under the SW-strategy appears to be most sensitive to incorrect beliefs about κ_R, compared to changes in the other parameters in Θ_R. The most extreme loss in CE is present when $\hat{\kappa}_R > \kappa_R$ for $k = \{1, 5\}$. In these cases, Section 5.1 shows that the investor holds too extreme positions and therefore is exposed to more risk than under the actual optimal bond positions.[28] Similar as for κ_R, the SW-strategy appears to be more sensitive to beliefs of a higher κ_π ($\hat{\kappa}_\pi > \kappa_\pi$) than those of a lower κ_π ($\hat{\kappa}_\pi < \kappa_\pi$). The loss in CE due to incorrect beliefs about κ_π are particularly present for an at least moderately risk averse investor ($\gamma > 1$). Interestingly, the BX-strategy is more (less) robust to parameter uncertainty in κ_R (κ_π). For example, for the hedge investor ($\gamma = 25$) and $\hat{\kappa}_R > \kappa_R (k = 1)$ the SW-strategy leads to a loss in CE of 100%, while the BX-strategy leads to a smaller loss of 27%. Finally, both the SW- and BX-strategy are almost insensitive to beliefs of $\hat{\rho}_{R\pi} > \rho_{R\pi}$, but beliefs of $\hat{\rho}_{R\pi} < \rho_{R\pi}$ lead to a significant loss in CE for an investor with risk aversions $\gamma = 1, 2$. This is in line with the large changes in the speculative demands shown in Section 5.1.

Considering the results of Table 7 regarding the one bond strategies, we see that the CE appears to be less sensitive to incorrect parameter beliefs than the two bond strategies, in line with the less sensitive bond demands explained in Section 5.1. [29] This is especially the case if the investor has incorrect beliefs about κ_R or $\hat{\rho}_{R\pi} > \rho_{R\pi}$. However,

[27] The results for the other values of $\kappa_{R\pi}$ are in line with $\kappa_{R\pi} = -0.078$ and can be delivered upon request.

[28] This observation is even more relevant when $\kappa_{R\pi} = -0.001$ or $\kappa_{R\pi} = 0$.

[29] This effect is even more visible when $\kappa_{R\pi} = -0.001$ or $\kappa_{R\pi} = 0$.

even while the one bond strategies appear to be more robust to parameter uncertainty than the two bond strategies, the CE remains considerably lower than the CE of $2,157$ under the optimal two bond strategy without parameter uncertainty. This is because the one bond strategy fails to fully hedge the inflation risk. For the investor with risk aversions $\gamma = 1, 2$ the one bond SW-, BBB-, and MSV-strategy lead to similar CE-values. However, for the hedge investor with $\gamma = 25$, the BBB-strategy appears to be slightly more (less) robust than the other one bond SW-strategy to parameter uncertainty about $\kappa_R(\kappa_\pi)$. This is in in line with the results of the BX-strategy with respect to the two bond SW-strategy.

A final interesting aspect to consider is the worst-case scenario for the hedge investor $\gamma = 25$. Over all sensitivity sets, the two bond strategies lead to a CE of 0 in the worst cases, due to the large sensitivity to parameter changes. The MSV-strategy appears to have a low CE in the worst case as well, namely a CE of 8. However, the one bond SW-strategy performs better with a worst-case value of 70 and the BBB-strategy appears to be have the best worst-case CE of 128.

Hence, although the two bond strategies lead to considerable higher CE-values when the investor knows the correct parameters, they are very sensitive to parameter uncertainty. Especially when the risk aversion of the investor increases, incorrect beliefs about the mean-reversion parameters of R_t and π_t lead to large losses in CE. The advantage of the one bond strategies is that they are more robust for this parameter uncertainty. Moreover, for the hedge investor the BBB-strategy leads to the 'best worst case' CE among all sensitivity sets and strategies.

6 Conclusion

For long-term investors it is important to hedge both the real interest rate risk and inflation risk. Since Index Linked Bonds are illiquid in practice, and their supply is limited, we consider the optimal hedging strategy of Sangvinatsos and Wachter (2005) (SW-strategy). This strategy hedges the real terminal wealth by using a short-long composition of two nominal bonds. The bond demands of the SW-strategy are typically extreme: the absolute value of the portfolio weights typically exceed 100%. This happens especially when the time to maturity of the two bonds lay close to each other. To minimise the sum of absolute values of the bond demands, we define a robust combination of the bond maturities. This results in a robust choice of a 10-year and a 50-year bond. However, even with this specific maturity combination the strategy contains extreme demands.

Our main finding is that the optimal nominal bond investment strategy is heavily dependent on the parameters of the mean reversion matrix of the process driving the short-term nominal interest rate and expected inflation. When the mean reversion coefficients of nominal interest rate and expected inflation process are close to each other, or when the feedback from expected inflation to the nominal interest rate is close to zero, we find more extreme demands. We find that incorrect investor beliefs about these parameters can lead to large losses in the utility of her terminal real wealth, compared to knowing the correct parameters. The utility losses are particularly high when the investor has incorrect beliefs about the feedback parameter or when she overestimates the mean reversion of the nominal interest rate process. As it is difficult to estimate these parameter values precisely, parameter uncertainty is relevant.

To explore possible solutions for the extreme bond demands and the SW-strategy's sensitivity to parameter values, we study four alternative strategies based on restricted versions of the general Sangvinatsos and Wachter (2005) model. The first alternative is based on the model of Brennan and Xia (2002) (BX), which restricts the feedback from expected inflation to real interest rates to zero. Moreover, we study three one bond strategies: the one bond SW-strategy, and two alternatives based on the studies of Bilsen et al. (2020) (BBB) and Munk et al. (2004) (MSV). The one bond strategies lead to less extreme bond positions than the two bond strategies. As a result, these are more robust to parameter uncertainty. Overall, applying the BBB-strategy leads to smaller utility losses compared to correct beliefs about mean-reversion parameters than the other strategies. However, the one bond strategies fail to fully hedge the inflation risk which causes the utility to be low even in the case of correct beliefs, especially for a very risk averse investor. If the investor therefore wants to apply a two bond strategy, it is not straightforward whether the BX- or SW-strategy is more robust against parameter uncertainty, as this depends on the interplay of uncertainty of the mean-reversion parameters and other parameters.

In future research we plan to formalize the trade-off between robustness against parameter uncertainty and the ability to hedge inflation risk, depending on the investor's ambiguity aversion. Moreover, the results regarding the hedging ability and parameter uncertainty are highly dependent on the choice of the time to maturity of the bonds. Therefore, another relevant topic for future research is to let the investor also optimally choose the time to maturity of the bonds.

References

Baker, L., Haynes, R., Roberts, J., Sharma, R., and Tuckman, B. (2020). Risk transfer with interest rate swaps. *Working paper.*

Beetsma, R., Chen, D., and van Wijnbergen, S. (2020). Unhedgeable inflation risk within pension schemes. *Insurance: Mathematics and Economics*, 90:7–24.

Benzoni, L., Collin-Dufresne, P., and Goldstein, R. (2007). Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. *The Journal of Finance*, 62(5):2123–2167.

Bilsen, S. V., Boelaars, I. A., and Bovenberg, A. (2020). The duration puzzle in life-cycle investment. *Review of Finance*, pages 1–41.

Blommestein, H. (2007). Pension funds and the evolving market for (ultra-)long governement bonds. *Pensions: An International Journal*, 12:175–184.

Brennan, M. and Xia, Y. (2002). Dynamic asset allocation under inflation. *The Journal of Finance*, 57(3).

Ciocyte, O. and Westerhout, E. (2017). The role of inflation-linked bonds. increasing, but still modest. *CPB Discussion Paper*, 344.

Feldhütter, P., Larsen, L. S., Munk, C., and Trolle, A. (2012). Keep it simple: Dynamic bond portfolios under parameter uncertainty. *working paper.*

Flor, C. and Larsen, L. (2011). Robust portfolio choice with stochastic interest rates. *working paper.*

Garlappi, L., W. T. and Uppal, R. (2007). Portfolio selection with parameter and model uncertainty: A multi-prior approach. *The Review of Financial Studies*, 20:41–81.

Martellini, L., Milhau, V., and Tarelli, A. (2015). Hedging inflation-linked liabilities without inflation-linked instruments through long/short investments in nominal bonds. *The Journal of Fixed Income*, 24(3):5–29.

Mkaouar, F., Prigent, J., and Abid, I. (2017). Long-term investment with stochastic interest and inflation rates:the need for inflation-indexed bonds. *Economic Modeling*, 67:228–247.

Munk, C., Sørensen, C., and Vinther, T. N. (2004). Dynamic asset allocation under mean-reverting returns, stochastic interest rates, and inflation uncertainty: Are popular recommendations consistent with rational behavior? *International Review of Economics and Finance*, 13:141–166.

Quaedvlieg, R. and Schotman, P. (2020). Hedging long-term liabilities. *Journal of Financial Econometrics*, pages 1–34.

Sangvinatsos, A. and Wachter, J. (2005). Does the failure of the expectations hypothesis matter for long-term investors? *Journal of Finance*, 60:179–230.

Van den End, J. (2011). Statistical evidence on the mean reversion of interest rates. *DNB Working paper*.

Table 1. Relevant definitions regarding the optimal bond allocations. The third column shows the size in case of 2 risk factors.

Variable	Definition	Size
x_t^*	Optimal nominal bond allocations at time t	2x1
λ	Market prices of risk of nominal asset returns	2x1
$\Omega = \sigma\rho\sigma'$	Variance-covariance matrix of risks	2x2
$\Lambda = \sigma\lambda$	Risk premium vector	2x1
$\tilde{\lambda} = \rho^{-1}\lambda$	Factor loadings of nominal pricing kernel [30]	2x1

Table 2. The risk factor characteristics for the initial and several sensitivity parameter sets applied in this study. Set 0 corresponds to the initial set Θ_R, sets $k \in 1, 2, .., 8$ represent the sensitivity sets $\hat{\Theta}_R^k$. The bold numbers of the parameters correspond to changes with respect to Θ.

k	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
κ_R	0.105	**0.210**	0.105	**0.053**	0.105	**0.210**	**0.053**	0.105	0.105
σ_R	0.019	**0.027**	0.019	**0.014**	0.019	**0.027**	**0.014**	0.019	0.019
λ_R	-0.219	-0.219	-0.219	-0.219	-0.219	-0.219	-0.219	-0.219	-0.219
κ_π	0.027	0.027	**0.054**	0.027	**0.014**	**0.054**	**0.014**	0.027	0.027
σ_π	0.014	0.014	**0.020**	0.014	**0.010**	**0.020**	**0.010**	0.014	0.014
λ_π	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105
$\rho_{R\pi}$	0.733	0.733	0.733	0.733	0.733	0.733	0.733	**0.600**	**0.900**

Table 3. For several values of a fixed τ_1, the resulted τ_2^{min} and minimised sum of absolute bond allocations at $t = 0$ are shown under Θ_R. Recall that τ_2^{min} is computed according to the definition of a robust bond maturity as in equation (22). Furthermore, the allowed range for the second bond maturity is $\tau_2 \in \{1, 2, ..., 50\}$. τ_2^{min} appears to be independent of T and γ. Except for $\kappa_{R\pi} = -0.117$ and $\gamma = 1$, the robust choice for τ_2 does not change among $\kappa_{R\pi} \in \{-0.117, -0.078, -0.039, -0.001\}$. To compute the example of $|x_{0,1}^*| + |x_{0,2}^*|$, we assume $\kappa_{R\pi} = -0.078, \gamma = 2$ and $T = 30$.

τ_1 (fixed)	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
τ_2^{min}	50	50	50	50	50	8	9	10	10	11
$\lvert x_{0,1}^*\rvert + \lvert x_{0,2}^*\rvert$	5.7	4.6	4.8	5.7	7.2	6.5	5.7	5.2	4.8	4.6

[30] As the formula shows, these take into account the correlation between the risk factors, included in ρ^R. Recall that the definition of ρ^R is given in equation (6).

Table 4. The optimal bond allocations at $t = 0$ under Θ_R (given in Table 2), for different values for $\kappa_{R\pi}$, and $(\tau_1, \tau_2) = (10, 50)$. For readability, we suppress the subscript of $t = 0$. The allocations in the two bond SW-strategy are denoted by x_1^* and x_2^* respectively. The allocation in the one bond SW-strategy is denoted by $\tilde{x}^*$.

$\kappa_{R\pi}$	-0.117			-0.078			-0.039			-0.001			0		
γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
x_1^*	395%	304%	221%	426%	357%	294%	520%	516%	513%	7,632%	12,597%	17,165%			
x_2^*	-88%	-66%	-45%	-109%	-100%	-93%	-170%	-204%	-236%	-4,818%	-8,099%	-11,118%			
$\tilde{x}^*$	120%	100%	81%	138%	91%	48%	160%	83%	13%	185%	77%	-22%	185%	77%	-22%

Table 5. The Certainty Equivalent (CE) when the SW-strategy is applied, under Θ_R (given in Table 2), the actual feedback parameter $\kappa_{R\pi}$ of the GDP (given in the second row), the feedback parameter belief $\hat{\kappa}_{R\pi}$ (given in the first column), and $(\tau_1, \tau_2) = (10, 50)$. Recall that the definition of the CE is given in equation (19). The upper and lower table show the CE corresponding to the two and one bond SW-strategy respectively. The results on the diagonal correspond to the optimal strategies and are written in bold.

γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
$\hat{\kappa}_{R\pi}$ \ $\kappa_{R\pi}$	-0.117			-0.078			-0.039			-0.001			0		
-0.117	**3,718**	**3,165**	**2,722**	3,520	2,694	153	1,565	227	0	0	0	0	3,012	1,371	0
-0.078	3,639	2,835	649	**3,725**	**3,041**	**2,517**	2,998	1,592	0	0	0	0	3,012	1,371	0
-0.039	3,392	2,146	11	3,537	2,430	109	**3,730**	**2,843**	**2,204**	0	0	0	3,012	1,371	0
-0.001	3,023	1,390	0	3,029	1,398	0	3,045	1,421	0	**3,735**	**2,593**	**1,835**	3,012	1,371	0
0															

γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
$\hat{\kappa}_{R\pi}$ \ $\kappa_{R\pi}$	-0.117			-0.078			-0.039			-0.001			0		
-0.117	**2,912**	**2,442**	**620**	3,017	2,319	203	3,083	2,127	29	3,105	1,892	2	3,105	1,885	2
-0.078	2,876	2,434	487	**3,048**	**2,324**	**251**	3,171	2,142	50	3,236	1,911	4	3,237	1,904	4
-0.039	2,739	2,406	217	3,002	2,316	200	**3,209**	**2,148**	**60**	3,342	1,926	6	3,345	1,920	6
-0.001	2,470	2,347	60	2,840	2,285	105	3,156	2,139	51	**3,386**	**1,933**	**7**	3,391	1,927	7
0	2,461	2,344	57	2,833	2,283	103	3,153	2,139	51	3,386	1,933	7	**3,391**	**1,927**	**7**

Table 6. The optimal bond allocations at $t = 0$ under $\hat{\Theta}_R^k$ for every $k \in \{1, 2, ..., 8\}$ (given in Table 2), $\kappa_{R\pi} = -0.078$, and $(\tau_1, \tau_2) = (10, 50)$. For readability, we suppress the subscript of $t = 0$. The allocations to bond 1 and bond 2 in the two bond strategies are denoted by x_1 and x_2 respectively. The investment strategies correspond to the two bond SW-strategy, the BX-strategy, the one bond SW-strategy, the BB-strategy, and the MSV-strategy. The SW-strategies in case of two bond and one bond investments are based on the correct feedback parameter belief $\hat{\kappa}_{R\pi} = \kappa_{R\pi}$ and written in bold. The BX- and BBB-strategy are based on $\hat{\kappa}_{R\pi} = \hat{\kappa}_\pi - \hat{\kappa}_R$, and the MSV-strategy is based on $\hat{\kappa}_{R\pi} = 0$.

k	0			1			2			3			4		
	(Initial)			$(\kappa_R, \sigma_R \uparrow)$			$(\kappa_\pi, \sigma_\pi \uparrow)$			$(\kappa_R, \sigma_R \downarrow)$			$(\kappa_\pi, \sigma_\pi \downarrow)$		
γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
x_1^*	**426%**	**357%**	**294%**	**485%**	**422%**	**365%**	**483%**	**403%**	**329%**	**463%**	**377%**	**297%**	**415%**	**344%**	**278%**
x_2^*	**-109%**	**-100%**	**-93%**	**-187%**	**-201%**	**-215%**	**-146%**	**-130%**	**-116%**	**-75%**	**-59%**	**-43%**	**-102%**	**-92%**	**-83%**
x_1^{BX}	426%	357%	294%	410%	294%	188%	542%	504%	470%	637%	672%	704%	401%	322%	249%
x_2^{BX}	-109%	-100%	-93%	-121%	-89%	-60%	-184%	-197%	-208%	-152%	-189%	-223%	-92%	-77%	-64%
$\tilde{x}^*$	**138%**	**91%**	**48%**	**153%**	**65%**	**-16%**	**127%**	**85%**	**47%**	**142%**	**126%**	**112%**	**149%**	**103%**	**62%**
x^{BBB}	138%	91%	48%	115%	77%	42%	144%	80%	22%	183%	108%	38%	143%	106%	71%
x^{MSV}	185%	77%	-22%	194%	61%	-62%	185%	75%	-26%	209%	101%	2%	185%	91%	4%

k	0			5			6			7			8		
	(Initial)			$(\kappa_R, \sigma_R, \kappa_\pi, \sigma_\pi \uparrow)$			$(\kappa_R, \sigma_R, \kappa_\pi, \sigma_\pi \downarrow)$			$(\rho_{R\pi} \downarrow)$			$(\rho_{R\pi} \uparrow)$		
γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
x_1^*	**426%**	**357%**	**294%**	**565%**	**498%**	**437%**	**452%**	**367%**	**289%**	**296%**	**292%**	**288%**	**1089%**	**688%**	**320%**
x_2^*	**-109%**	**-100%**	**-93%**	**-257%**	**-268%**	**-278%**	**-71%**	**-55%**	**-40%**	**-59%**	**-75%**	**-91%**	**-349%**	**-221%**	**-102%**
x_1^{BX}	426%	357%	294%	484%	358%	242%	544%	505%	470%	296%	292%	288%	1089%	688%	320%
x_2^{BX}	-109%	-100%	-93%	-186%	-145%	-107%	-111%	-115%	-120%	-59%	-75%	-91%	-349%	-221%	-102%
$\tilde{x}^*$	**138%**	**91%**	**48%**	**144%**	**59%**	**-19%**	**156%**	**138%**	**121%**	**146%**	**99%**	**56%**	**129%**	**82%**	**39%**
x^{BBB}	138%	91%	48%	110%	66%	26%	180%	126%	77%	146%	99%	56%	129%	82%	39%
x^{MSV}	185%	77%	-22%	194%	58%	-66%	209%	116%	31%	185%	93%	9%	185%	57%	-62%

Table 7. The (changes in) the Certainty Equivalent (CE) for five different strategies when the GDP is based on $\Theta = \{\kappa_{R\pi}, \Theta_R\}$, $\kappa_{R\pi} = -0.078$, and the investment strategy is based on $\hat{\Theta}_R^k$ (given in Table 2) and a $\hat{\kappa}_{R\pi}$ depending on the strategy. Furthermore, $(\tau_1, \tau_2) = (10, 50)$. Recall that the definition of the CE is given in equation (19). For readability, we suppress the superscripts of indicator δ and parameter set k. The variable Δ corresponds to the relative change in CE under the belief set $\hat{\Theta}^k$ compared to the CE under the actual parameters of Θ. For $k = 0$, the investor has correct parameter beliefs about Θ_R. The underlying strategies are given in Table 6 and correspond to the two bond SW-strategy the BX-strategy, the one bond SW-strategy, the BB-strategy, and the MSV-strategy. The CE corresponding to the SW-strategies are based on the correct feedback parameter belief ($\hat{\kappa}_{R\pi} = \kappa_{R\pi}$) and are written in bold.

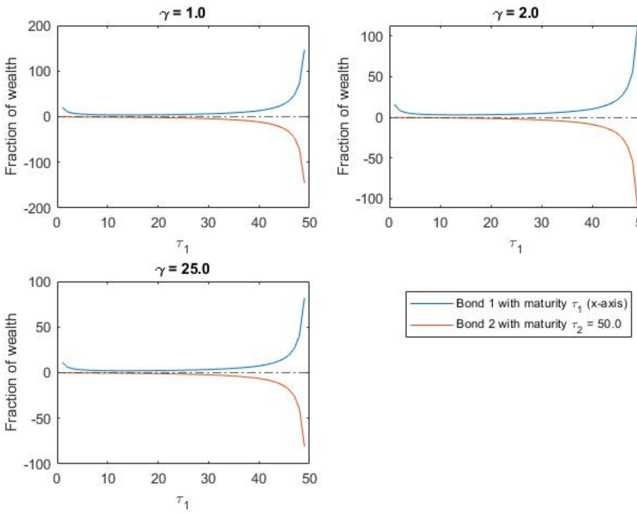
k	0			1			2			3			4		
	(Initial Θ)			$(\hat{\kappa}_R, \hat{\sigma}_R \uparrow)$			$(\hat{\kappa}_\pi, \hat{\sigma}_\pi \uparrow)$			$(\hat{\kappa}_R, \hat{\sigma}_R \downarrow)$			$(\hat{\kappa}_\pi, \hat{\sigma}_\pi \downarrow)$		
γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
CE^*	**3,725**	**3,041**	**2,517**	**1,657**	**571**	**0**	**3,436**	**2,712**	**937**	**2,171**	**1,501**	**4**	**3,715**	**3,013**	**2,061**
CE^{BX}	3,725	3,041	2,517	3,449	2,882	1,849	2,673	1,409	0	2,638	1,981	30	3,669	2,916	1,185
$CE^{*'}$	**3,048**	**2,324**	**251**	**3,025**	**2,297**	**106**	**3,037**	**2,309**	**232**	**3,046**	**2,291**	**139**	**3,037**	**2,302**	**212**
CE^{BBB}	3,048	2,324	251	2,995	2,304	249	3,044	2,315	182	2,851	2,284	232	3,045	2,304	193
CE^{MSV}	2,833	2,283	103	2,750	2,247	20	2,833	2,299	57	2,590	2,239	155	2,833	2,248	190

Δ^*				**-56%**	**-81%**	**-100%**	**-8%**	**-11%**	**-63%**	**-42%**	**-51%**	**-100%**	**0%**	**-1%**	**-18%**
Δ^{BX}				-7%	-5%	-27%	-28%	-54%	-100%	-29%	-35%	-99%	-2%	-4%	-53%
$\Delta^{*'}$				**-1%**	**-1%**	**-58%**	**0%**	**-1%**	**-8%**	**0%**	**-1%**	**-45%**	**0%**	**-1%**	**-15%**
Δ^{BBB}				-2%	-1%	-1%	0%	0%	-28%	-6%	-2%	-8%	0%	-1%	-23%
Δ^{MSV}				-3%	-2%	-80%	0%	1%	-44%	-9%	-2%	51%	0%	-2%	85%

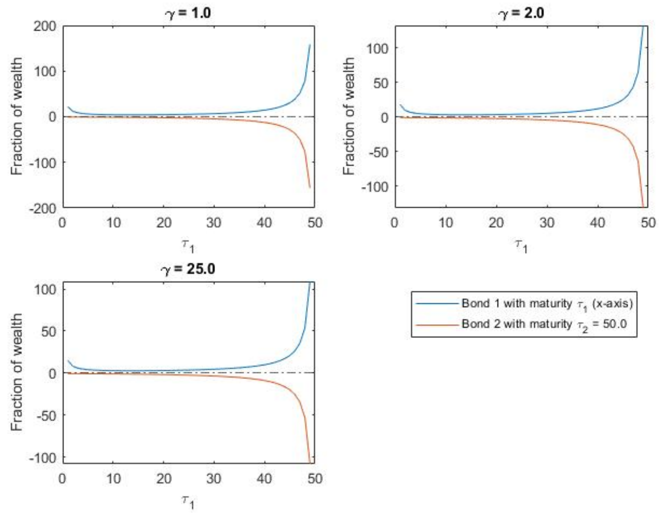
k	0			5			6			7			8		
	(Initial Θ)			$(\hat{\kappa}_R, \hat{\sigma}_R, \hat{\kappa}_\pi, \hat{\sigma}_\pi \uparrow)$			$(\hat{\kappa}_R, \hat{\sigma}_R, \hat{\kappa}_\pi, \hat{\sigma}_\pi \downarrow)$			$(\hat{\rho}_{R\pi} \downarrow)$			$(\hat{\rho}_{R\pi} \uparrow)$		
γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
CE^*	**3,725**	**3,041**	**2,517**	**313**	**24**	**0**	**2,126**	**1,445**	**3**	**3,574**	**2,979**	**2,511**	**1,291**	**1,802**	**2,425**
CE^{BX}	3,725	3,041	2,517	1,692	1,330	26	2,454	1,847	48	3,574	2,979	2,511	1,291	1,802	2,425
$CE^{*'}$	**3,048**	**2,324**	**251**	**3,045**	**2,271**	**70**	**3,014**	**2,226**	**87**	**3,042**	**2,318**	**243**	**3,041**	**2,315**	**242**
CE^{BBB}	3,048	2,324	251	2,970	2,259	211	2,876	2,212	200	3,042	2,318	243	3,041	2,315	242
CE^{MSV}	2,833	2,283	103	2,750	2,257	8	2,590	2,153	242	2,833	2,257	200	2,833	2,273	18

Δ^*				**-92%**	**-99%**	**-100%**	**-43%**	**-52%**	**-100%**	**-4%**	**-2%**	**0%**	**-65%**	**-41%**	**-4%**
Δ^{BX}				-55%	-56%	-99%	-34%	-39%	-98%	-4%	-2%	0%	-65%	-41%	-4%
$\Delta^{*'}$				**0%**	**-2%**	**-72%**	**-1%**	**-4%**	**-65%**	**0%**	**0%**	**-3%**	**0%**	**0%**	**-4%**
Δ^{BBB}				-3%	-3%	-16%	-6%	-5%	-20%	0%	0%	-3%	0%	0%	-4%
Δ^{MSV}				-3%	-1%	-92%	-9%	-6%	135%	0%	-1%	95%	0%	0%	-82%

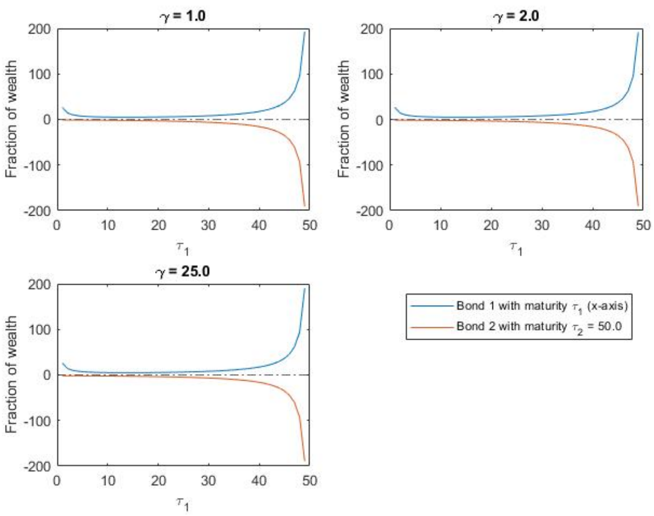
Figure 1. The optimal bond allocations at $t = 0$ under Θ given in the first column of Table 2 and four values of $\kappa_{R\pi}$. The blue (orange) line shows the long (short) positions in bond 1 (2) for $T = 30, \tau_2 = T_2 = 50$, and all $\tau_1 = T_1 \in \{1, 2, ..., 49\}$.



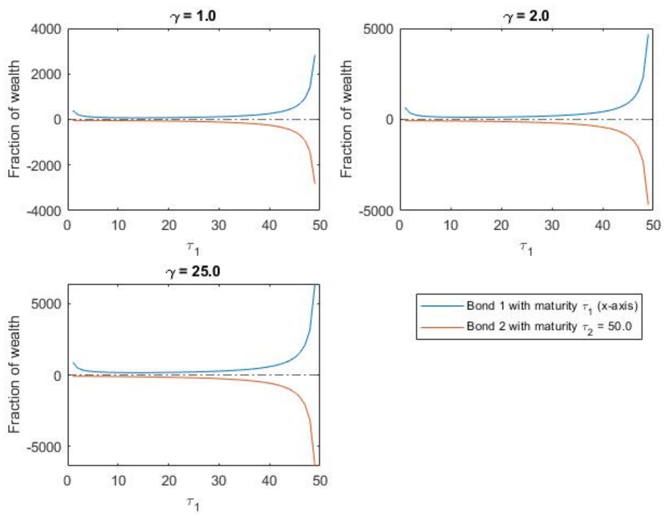
(a) $\kappa_{R\pi} = -0.117$



(b) $\kappa_{R\pi} = -0.087$



(c) $\kappa_{R\pi} = -0.039$



(d) $\kappa_{R\pi} = -0.001$

A Definitions of matrices A and B

Sangvinatsos and Wachter (2005) derive the optimal nominal bond strategy to hedge the nominal interest rate risk and inflation rate risk, given in equation (11). The solution in the paper of Sangvinatsos and Wachter solution contains row vectors $A_2(\tau)$ and $B_2(\tau)$, which we will suppress to $A(\tau_i)$ and $B(\tau)$ for convenience. These vectors are found by solving the Ordinary Differential Equation (ODE) below in case of constant risk premiums[31] (Sangvinatsos and Wachter, 2005)[eq.(A3) and (B8)]:

$$\begin{aligned} A'(\tau_i) &= -A(\tau_i) \cdot K - \delta_R \\ &= -A(\tau_i) \cdot K - 1 \\ B'(\tau) &= -B(\tau) \cdot K + (\delta_R - \delta_\pi) \\ &= -B(\tau) \cdot K + \begin{bmatrix} 1 & -1 \end{bmatrix} \end{aligned}$$

Which results in the solutions

$$\begin{aligned} A(\tau_i) &= \delta_R(e^{-K\tau_i} - I)K^{-1} \\ B(\tau) &= (-\delta_R + \delta_\pi)(e^{-K\tau} - I)K^{-1} \end{aligned}$$

To express these vectors in the mean-reversion parameters and bond maturities, let V be a matrix that contains the column vectors of the right eigenvectors of mean-reversion parameter matrix K from equation (5), and let K_d be a diagonal matrix with the eigenvalues of K corresponding to these eigenvectors on the diagonal entries. Since K is a diagonalizable matrix, we know by linear algebra that we can express K in these matrices:

$$\begin{aligned} K &= V K_d V^{-1} \\ \rightarrow -\tau K &= V \cdot -\tau K_d V^{-1} \\ \rightarrow \exp(-\tau K) &= V \cdot \exp(-\tau K_d) V^{-1} \end{aligned}$$

[31] Note that we leave out the term '$(1-\gamma)$' that is given in the equation of Sangvinatsos and Wachter (2005) in front of $(\delta_R - \delta_\pi)$. This is because of convenience. Including this term increases $B(\tau)$ by factor $(1-\gamma)$. Because Sangvinatsos and Wachter multiply the hedge term with $\frac{1}{\gamma}$ in the optimal asset allocations (see their equation (44)), this results in a total factor of $\frac{1}{\gamma} \cdot (1-\gamma) = -(1-\frac{1}{\gamma})$ which we recognize from the literature, e.g. in the hedge allocation in the BX-model, $-(1-\frac{1}{\gamma})b(T-t)\sigma_r$.)

Furthermore, since K is upper triangular, the eigenvalues are κ_R and κ_π. Therefore:

$$K_d = \begin{bmatrix} \kappa_R & 0 \\ 0 & \kappa_\pi \end{bmatrix}; V = \begin{bmatrix} 1 & -\frac{\kappa_{R\pi}}{\kappa_R - \kappa_\pi} \\ 0 & 1 \end{bmatrix}$$

And the matrix $\exp(-\tau K_d)$ equals

$$\exp(-\tau K_d) = \begin{bmatrix} e^{-\tau\kappa_R} & 0 \\ 0 & e^{-\tau\kappa_\pi} \end{bmatrix}$$

This results in:

$$\begin{aligned}
A(\tau_i) &= \delta_R(e^{-K\tau_i} - I)K^{-1} \\
&= \left(V \cdot \exp(-\tau_i K_d)V^{-1} - I\right) K^{-1} \\
&= \left[-\tfrac{1}{\kappa_R}(1 - e^{-\tau_i\kappa_R}); \quad \tfrac{\kappa_{R\pi}}{\kappa_R\kappa_\pi}(1 - e^{-\tau_i\kappa_R}) + \tfrac{\kappa_{R\pi}}{\kappa_\pi(\kappa_R - \kappa_\pi)}(e^{-\tau_i\kappa_R} - e^{-\tau_i\kappa_\pi})\right] \\
B(\tau) &= (-\delta_R + \delta_\pi)(e^{-K\tau} - I)K^{-1} \\
&= \begin{bmatrix} -1 & 1 \end{bmatrix} \left(V \cdot \exp(-\tau K_d)V^{-1} - I\right) K^{-1} \\
&= \left[\tfrac{1}{\kappa_R}(1 - e^{-\tau\kappa_R}); \quad -\tfrac{\kappa_{R\pi}}{\kappa_R\kappa_\pi}(1 - e^{-\tau\kappa_R}) - \tfrac{1}{\kappa_\pi}(1 - e^{-\tau\kappa_\pi}) - \tfrac{\kappa_{R\pi}}{\kappa_\pi(\kappa_R - \kappa_\pi)}(e^{-\tau\kappa_R} - e^{-\tau\kappa_\pi})\right]
\end{aligned}$$

In the main text of this paper, we use the notation of matrix A which corresponds to $A \equiv [A(\tau_1); A(\tau_2)]$ in case of the two bond strategy, and $A \equiv A(\tau_1)$ in case of the one bond strategy.

B Conversion r_t-parameters to R_t-parameters

For the initial parameter set 0 in Table 2 we use the estimates of Brennan and Xia (2002). However, these estimates are based on the real interest rate characteristics $\kappa_r; \sigma_r; \rho_{r\pi}$; and λ_r for $r_t = R_t - \pi_t$. This section shows how we convert these values to risk characteristics for $R_t : \kappa_R; \sigma_R; \rho_{R\pi}$; and λ_R. Because the nominal interest rate equals the sum of the real interest rate and the expected inflation, the parameters σ_R and $\rho_{R\pi}$ are given by:

$$
\begin{aligned}
\sigma_R &= \sqrt{\mathrm{Var}(r_t + \pi_t)} = \sqrt{\sigma_r^2 + \sigma_\pi^2 + 2\rho_{r\pi}\sigma_r\sigma_\pi} \\
\sigma_{R\pi} &= \rho_{R\pi}\sigma_R\sigma_\pi = \mathrm{Cov}(R, \pi) = \mathrm{Cov}(r + \pi, \pi) = \rho_{r\pi}\sigma_r\sigma_\pi + \ \sigma_\pi^2 \\
\to \rho_{R\pi} &= \frac{\rho_{r\pi}\sigma_r + \sigma_\pi}{\sigma_R}
\end{aligned}
$$

Consecutively, we determine κ_R by matching the approximated unconditional variances:

$$
\begin{aligned}
\frac{\sigma_R^2}{2\kappa_R} &= \frac{\sigma_r^2}{2\kappa_r} + \frac{\sigma_\pi^2}{2\kappa_\pi} + 2\rho_{r\pi}\frac{\sigma_r}{\sqrt{2\kappa_r}}\frac{\sigma_\pi}{\sqrt{2\kappa_\pi}} \\
\to \kappa_R &= \frac{\sigma_R^2}{2 * \left(\dfrac{\sigma_r^2}{2\kappa_r} + \dfrac{\sigma_\pi^2}{2\kappa_\pi} + \dfrac{\rho_{r\pi}\sigma_r\sigma_\pi}{\sqrt{\kappa_r\kappa_\pi}}\right)}
\end{aligned}
$$

Finally, we match the market prices of risk to compute λ_R:

$$
\begin{aligned}
\lambda_R\sigma_R &= \lambda_r\sigma_r + \lambda_\pi\sigma_\pi \\
\to \lambda_R &= \frac{\lambda_r\sigma_r + \lambda_\pi\sigma_\pi}{\sigma_R}
\end{aligned}
$$

Using these converted values of $\sigma_R; \rho_{R\pi}; \kappa_R$; and λ_R as input values leads to the same wealth process simulations based on shocks in R_t and π_t as using the original $\sigma_r; \rho_{r\pi}; \kappa_r$; and λ_r in simulations based on shocks in r_t and π_t.