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The liquidity premium in illiquid asset classes

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Abstract

Illiquid assets play an increasing role in institutional investors' portfolios, even though recent empirical studies have challenged the conventional wisdom of liquidity level premiums for these asset classes. Solving a model that captures both possible transactions costs and infrequencies of trading opportunities for illiquid assets, we show that short-term investors or investors that face substantial liquidity shocks demand a sizable liquidity premium. We model heterogeneous agents for four different asset classes (private equity, real estate, corporate bonds, and stocks) to explain the empirically observed differences in liquidity premiums using clientele effects. We find an average annual liquidity premium of 45 basis points for private equity, 65 basis points for real estate, 30 basis points for corporate bonds and 40 basis points for stocks.

Keywords: liquidity premium, illiquid assets, portfolio choice, institutional investors.

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1. Introduction

Illiquid assets play an increasingly key role in investors' portfolios. Illiquid assets of large U.S. endowment funds comprise 43 percent in 2011 (Goetzmann and Oster, 2012), and the seven largest pension funds in the world have increased their average illiquid asset allocation from 4 percent in 1997 to 25 percent in 2017 (Willis Towers Watson Global Pension Asset Study 2018). One potential reason to investing in illiquid assets is to capture liquidity premiums (OECD, 2014). In other words, investing in illiquid assets might compensate the investor for bearing the illiquidity of these assets. Yet, there is still no consensus in either the empirical or theoretical literature as to whether liquidity premiums can be justified. This paper studies liquidity premiums from a theoretical perspective to bridge this gap, by modeling heterogeneous investors in four asset classes: private equity, real estate, corporate bonds, and stocks. We show that only short-term investors or investors facing substantial liquidity shocks demand a first-order liquidity premium. Eventually, in equilibrium, the marginal investor determines the liquidity premium. Clientele effects enable us to relate our findings to the empirical evidence about liquidity premiums in these four asset classes.

The empirical literature finds significant effects of illiquidity on prices. For instance, Amihud and Mendelson (1986) and Brennan and Subrahmanyam (1996) showed that a 1% higher transaction cost implied 1.5% to 2% higher expected returns for stocks. Similar magnitudes were found for corporate bonds, e.g. (Bao et al., 2011) and (Bongaerts et al., 2017). Yet, some recent studies have challenged the empirical evidence for liquidity premium in these asset classes.[1] For instance, Ben-Rephael et al. (2015) claimed that liquidity premiums almost disappeared in recent decades for public U.S. equities, except for very small stocks. Palhares and Richardson (2018) found only limited evidence for liquidity premiums in the cross-section of corporate bonds using illiquidity-factor portfolios. Similarly, for private equity there is no clear consensus regarding the existence of a liquidity premium,

[1] Note that we refer here to the *level* liquidity premium. There also exists a literature on the liquidity *risk* premium.

although the evidence is more indirect. Franzoni et al. (2012) reported no out-performance of private equity relative to public equity, whereas Harris et al. (2014) found a substantial out-performance of 3% annually. The former is suggestive of no liquidity premium, whereas the latter leaves room for a liquidity premium. Finally, opposing indirect evidence also exists for real estate investments. Qian and Liu (2012) found a somewhat higher expected return for direct compared to indirect real estate, whereas Ang et al. (2013) find comparable performance for direct and indirect real estate investments.

Mixed evidence on liquidity premiums also emerges from the theoretical literature. Constantinides (1986) showed that transaction costs, or illiquidity, only have a second-order effect on prices. Vayanos (1998) confirmed this finding by studying a general equilibrium setting. After their work, a large literature developed studying illiquidity or transactions costs using assumptions more in line with real-world investment problems. This work found that illiquidity can have first-order effects on prices. For instance, theoretical work from Huang (2003) and Gârleanu (2009) showed that illiquidity may arise when investors face borrowing constraints. Jang et al. (2007) added return predictability to the investor's problem in a market with transactions costs, and found a slight increase in liquidity premiums. Lynch and Tan (2011) solved a model including labor income, wealth shocks, return predictability, and transaction costs and found liquidity premiums in the same order of magnitude as the early empirical literature. Nevertheless, its still unclear in which asset classes liquidity matters, to the extent that first-order liquidity premiums are likely be present. In this paper we aim to bridge this gap by modeling heterogeneous investor types and assets in four different asset classes: private equity, real estate, corporate bonds, and stocks.

Previous studies have either modeled illiquidity as proportional transactions costs, e.g. Constantinides (1986), or as the inability to trade illiquid assets for random points in time, e.g. Ang et al. (2014). We combine these two dimensions of illiquidity here for two reasons. First, several asset classes show both aspects of illiquidity simultaneously. For instance, traders of corporate bonds face transaction costs, yet sometimes specific bonds may not trade for several days. Similarly, selling real estate may on average take 110 to 250 days, and it obviously caries transaction costs as well. Second, disentangling both sources of illiquidity allows us to capture the heterogeneity between asset classes by adjusting the prominence of both effects. This, in turn, allows us to explain empirically-observed differences in liquidity premiums.

We solve for liquidity premiums in a partial equilibrium power utility framework. We propose that the cost of illiquidity involves two aspects: suboptimal asset allocation and suboptimal consumption. If the cost of liquidating the illiquid asset is too high and the investor prefers not to trade the illiquid asset, or the illiquid asset cannot be traded at all, illiquidity leads to suboptimal portfolio allocations. Yet, being off from the optimal asset allocation generally only leads to small utility costs (Constantinides, 1986). At the same time, illiquidity may hamper the possibility of smoothing consumption after negative wealth shocks. The investor may be unable to sell the illiquid asset, and thus will face a negative consumption shock. As such shocks to consumption generally carry a high utility cost, this consequence of illiquidity does generate significant liquidity premiums. Essentially, we show that only illiquidity resulting in suboptimal consumption is able to generate a substantial liquidity premium, while illiquidity leading to suboptimal asset allocation does not.

Our model's flexibility allows us to quantify the magnitude of the liquidity premium for four markets. For different investor types we calculate the premium they would be willing to pay for the illiquid asset to become liquid. We find that this willingness-to-pay is large only for short-term investors or investors facing large liquidity shocks. From this, we conclude that, in equilibrium, the liquidity premiums depend on the relative prevalence of different types of investors. For instance, Amihud and Mendelson (1986) showed clientele effects whereby short-term investors held the liquid assets and long-term investors the illiquid assets. Longstaff (2003) confirmed this by modeling the illiquid asset as an inability to trade. This enables us to relate the composition of market participants to the empirical evidence about liquidity premiums in that market. This turns out to explain several of the empirical findings.

We have several contributions. Although private equity is the most illiquid asset that we analyze, we find an average liquidity premium equal to 45 basis points only. Private equity investors have to lockup their money for a long period of time, and mainly for that reason only long-term investors are likely to be present in this market. For these investors illiquidity is unlikely to significantly harm investors' consumption patterns. Moreover, the non-trading period is fixed, knowing exactly when the position is liquidated. For real estate we find an average liquidity premium equal to 65 basis points. Real estate can often not be traded for a significant amount of time, and the timing of the trading opportunities are uncertain. Yet, the threat of illiquidity is dampen significantly because of the liquid return component (rents) of real estate investments. The average liquidity premium for corporate bonds equals 30 basis points. The transaction costs are small and trading occurs quite frequently. The liquidity premium does not disappear however, because the investor still faces the risk of insufficient liquid wealth to fulfill immediate consumption needs. For stocks we find an average liquidity premium equal to 40 basis points, mainly because of relatively high transaction costs.

Our paper is most closely related to the theoretical literature that aims to understand the gap between the low liquidity premium found in theoretical papers, versus the relative high liquidity premium found in some empirical papers. Huang (2003), who introduced liquidity shocks, found that illiquidity can have a first-order effect on prices in the case of borrowing constraints. Gârleanu (2009) confirmed the finding that borrowing constraint can amplify liquidity premiums. Both papers solved a general equilibrium model to study the effect of illiquidity. The main goal of these papers was to understand the mechanism behind the effect of illiquidity in equilibrium. By contrast, we focus on explaining the observed variation in the magnitude of the liquidity premium for different asset classes and markets using heterogeneity of investor preferences and liquidity shocks.

Our paper is also related to the literature that imposes trading restrictions on investors, e.g. Kahl et al. (2003). Examples are entrepreneurs, venture capitalists, and managers who face selling restrictions in the form of executive stock or stock-option based compensation contracts. Kahl et al. (2003) showed that liquidity restrictions can have substantial welfare effects. Although we model the source of illiquidity arising from the asset itself, as opposed to a restriction imposed on the investor, we also find large effects of illiquidity if a large fraction of investor wealth is invested in the illiquid asset.

The remainder of the paper is organized as follows. Section 2 gives an overview of the theoretical framework of the model, describes the corresponding optimal strategies, and the partial equilibrium implications for the liquidity premiums. In Section 2.5, we show the results of the baseline model. We link our theoretical findings to the empirical literature on liquidity premiums in Section 3. Section 4 concludes.

2. Optimal consumption and investment with illiquid assets

We model illiquidity as the inability to trade an asset frequently and/or by the cost that occurs when trading. This is formalized in Section 2.1. In Section 2.2 we describe the optimization problem of the investor. The solution of this optimization problem is presented in Section 2.3.

2.1. Financial market

The financial market consists of three assets: a risk-free asset B, a liquid risky asset S, and an illiquid risky asset denoted by X. The risk-free asset has a constant (annual) rate of return r_f. The liquid risky asset earns a nominal return r_t^S over the period $(t, t+1]$, while we denote the nominal return on the illiquid asset over the same period by r_t^X. All returns are continuously compounded. Note that we assume that the price of the illiquid asset is observed, even though it cannot be traded every period.

The prices of risk of the liquid and illiquid asset are denoted by λ_S and λ_X, respectively. Their volatilities are similarly denoted by σ_S and σ_X; their correlation by ρ_{SX}. The returns r_t^S and r_t^X are jointly normally distributed:

$$\begin{bmatrix} r_t^S \\ r_t^X \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} r_f + \lambda_S \sigma_S - \frac{1}{2}\sigma_S^2 \\ r_f + \lambda_X \sigma_X - \frac{1}{2}\sigma_X^2 \end{bmatrix}, \begin{bmatrix} \sigma_S^2 & \rho_{SX}\sigma_S\sigma_X \\ \rho_{SX}\sigma_S\sigma_X & \sigma_X^2 \end{bmatrix} \right). \tag{1}$$

The differences between the liquid and the illiquid risky asset include the trading opportunities and transaction costs. Whereas the investor can always trade the liquid risky asset S at no cost, the illiquid asset X can only be traded at random points in time and at a cost. We denote the trading indicator for time t by $\mathbf{1}_t$, with the interpretation that $\mathbf{1}_t = 1$ means that a trading opportunity arises in the illiquid asset X, while $\mathbf{1}_t = 0$ indicates that the illiquid asset cannot be traded that period. We assume throughout that the trading indicators are i.i.d. Bernoulli random variables with trading probability $p = \mathbb{P}\{\mathbf{1}_t = 1\}$. If a trading opportunity occurs ($\mathbf{1}_t = 1$) and the investor decides to trade, proportional transaction costs ϕ must be paid, $0 \leq \phi \leq 1$.

2.2. The investors' consumption and investment problem

We pose that different types of investors demand significantly different liquidity premiums. In Section 3, we link the prevalence of investor types in various markets to the empirical results on liquidity premiums in those markets. We first formally introduce investors and their characteristics.

The investor has an investment horizon equal to T. We assume that the illiquid asset can be traded (and thus liquidated) at cost ϕ at this final date T. Notice that under the assumption that the illiquid asset might not be liquidated at the final date T, liquidity premiums are obviously amplified. We do not consider this as, in that case, the investor might postpone liquidating the illiquid asset until a trading opportunity arises at cost ϕ. We introduce the following notation to distinguish liquid and illiquid wealth. We denote liquid wealth available at time t by W_t. This consists of both investment in the risk-free asset B and the risky liquid asset S. The value of the investment in the illiquid asset at time t is denoted by X_t. Therefore, total wealth equals $W_t + X_t$. We assume that the investor is endowed with initial wealth levels $W_0 > 0$ and $X_0 \geq 0$. The investor may face a liquidity shock assumed to be fraction l of total wealth $W_t + X_t$. The liquidity shock indicator is denoted by L_t and follows an i.i.d. Bernoulli process with probability $q = \mathbb{P}\{L_t = 1\}$. Various causes of such a liquidity shock arise. For individual investors, examples include health care costs, unforeseen expenditures, or extreme weather events. For institutional investors, another important source potentially leading to liquidity shocks are margin calls on derivative positions. For long-term investors such as pension funds and life insurers a shock to the mortality rate can increase the temporary need for liquidity.

We denote the fraction of liquid wealth W_t invested in the liquid risky asset S by θ_t, and $1 - \theta_t$ is invested in the risk-free asset B. Consumption at time t is denoted by C_t and must be financed from liquid wealth W_t. Preferences are represented by a standard constant relative risk aversion (CRRA) expected utility function with risk-aversion parameter γ.

Illiquid wealth X_t can only be converted into liquid wealth (and, if desired, immediately consumed) if a trading opportunity arises, i.e., if $\mathbf{1}_t = 1$. We denote the transfer from liquid to illiquid wealth by $\triangle X_t$. Thus, $\triangle X_t > 0$ means that at time t, an additional amount $\triangle X_t$ of the illiquid asset is bought, and thus liquid wealth W_t decreases by $\triangle X_t(1 + \phi)$. If no trading opportunity arises, $\mathbf{1}_t = 0$, we automatically have $\triangle X_t = 0$.

The investor optimizes its utility of a stream of consumption levels C_t over a horizon $t = 0, \ldots, T$. Thus, the criterion function is:

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right], \tag{2}$$

where β denotes the time-preference discount factor and $\gamma > 1$ is the risk-aversion parameter.

The investor faces two budget constraints: one for liquid wealth W_t and one for illiquid wealth X_t. Formally, for this consumption-stream investor we have:

$$
\begin{aligned}
W_t &= (W_{t-1} - \triangle X_{t-1} - \phi|\triangle X_{t-1}| - C_{t-1} - l_{t-1}) \left(\exp(r_f) + \theta_{t-1}\left[\exp(r_t^S) - \exp(r_f)\right]\right) && (3) \\
X_t &= (X_{t-1} + \triangle X_{t-1}) \exp(r_t^X). && (4)
\end{aligned}
$$

We assume that the investor cannot borrow against the illiquid investments. The effect of illiquidity would be much reduced if this were possible, as the investor could always undo the illiquidity be borrowing against the illiquid asset if needed. Alternatively stated, consumption should always be less or equal to liquid wealth W_t, thus:

$$
C_t \leq W_t, \quad t = 0, \ldots, T. \tag{5}
$$

Equation (3)-(5) do allow for negative liquid wealth due to the liquidity shock l. The optimal consumption problem can now be stated as follows:

Problem 2.1. *The consumption-stream investor maximizes*

$$
\max_{\{\theta_t, \triangle X_t, C_t\}_{t=0}^T} \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right] \tag{6}
$$

subject to the budget constraints (3) and (4) and the borrowing constraint (5). Moreover, when $\mathbf{1}_t = 0$*, we must have* $\triangle X_t = 0$*.*

The decision variables θ_t, $\triangle X_t$, and C_t are not anticipative. Formally, $\{\theta_t, \triangle X_t, C_t\}$ are adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0}^T$, where $\{\mathcal{F}_t\}$ is the filtration generated by $\{r_t^S, r_t^X, \mathbf{1}_t, L_t\}$.

To summarize, illiquidity limits the investor's consumption and investment decisions in three ways compared to the case where the illiquid asset were fully liquid. First, the inability to trade the illiquid assets for uncertain periods of time. Second, transaction costs of the illiquid asset when a trading opportunity arises. Third, the investor cannot borrow against the illiquid assets. All three assumptions are important characteristics of illiquid assets.

2.3. Optimal strategies

The optimization in Problem 2.1 cannot be solved analytically, as far as we know. So we resort to numerical methods. For these, it is known that the numerical complexity strongly increases with the dimension of the endogenous state variables. In the formulation of Problem 2.1 there would be two: W_t and X_t. Yet, in line with Ang et al. (2014), a simple transformation leads to a partly analytical result due to the homogeneity of the CRRA utility function we consider, see Theorem 2.2 below.

More precisely, we consider as endogenous state variables the total wealth $W_t + X_t$ and the fraction of total wealth invested in the illiquid asset, i.e.,

$$\xi_t = \frac{X_t}{W_t + X_t}. \tag{7}$$

With this reparametrization we define the value function using the Bellman principle as:

$$V_t\left(W_t + X_t, \xi_t\right) = \max_{\theta_t, \triangle X_t, C_t} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} + \mathbb{E}_t V_{t+1}(W_{t+1} + X_{t+1}, \xi_{t+1}), \tag{8}$$

with the boundary condition at time T given by:

$$V_T\left(W_T + X_T, \xi_T\right) = \beta^T \frac{\left(W_T + (1-\phi)X_T\right)^{1-\gamma}}{1-\gamma}. \tag{9}$$

The boundary condition implies that we assume that all assets can be traded (and thus liquidated) at time T.

With the above introduced change of variables, i.e., the pair (W_t, X_t) is replaced by the pair $(W_t + X_t, \xi_t)$, then the solution to the consumption-stream investor's problem satisfies the following theorem:

Theorem 2.2. *There exist time-dependent (deterministic) functions α_t, θ_t, and H_t such that the optimal solution $\{C_t^*, \triangle X_t^*, \theta_t^*\}$ to Problem 2.1 can be written*

$$V_t\left(W_t + X_t, \xi_t\right) = \beta^t \frac{\left(W_t + X_t\right)^{1-\gamma}}{1-\gamma} H_t\left(\xi_t\right), \tag{10}$$

$$C_t^* = \alpha_t(\xi_t)\left(W_t + X_t\right), \tag{11}$$

$$\theta_t^* = \theta_t\left(\xi_t\right), \tag{12}$$

$$\xi_t^* = \text{argmin}_{\xi_t} H_t(\xi_t). \tag{13}$$

The proof of Theorem 2.2 is provided in Appendix A.

Note that the optimal investment decision concerning illiquid wealth X, i.e., $\triangle X_t$, is determined by the choice ξ_t^*, but, of course, only when trading is allowed, i.e., when $\mathbf{1}_t = 1$.

The advantage of Theorem 2.2 is that the dependence of the value function on total wealth $W_t + X_t$ is known analytically. The fact that the value function is proportional to $\left(W_t + X_t\right)^{1-\gamma}$ simplifies the numerical optimization to a one dimensional grid search over ξ_t only. Details are provided in Appendix B.

Intuitively, the function $H_t(\xi_t)$ can be viewed as a penalty function which is minimized at the optimal fraction of total wealth invested in the illiquid asset, ξ_t^*. If the investor is able to trade the illiquid asset at time t, she will re-balance her portfolio towards the optimal ratio of illiquid wealth ξ_t^* to total wealth if the decrease in the penalty function is sufficient to out-weigh the transaction cost ϕ. Thus, in line with Constantinides (1986), there is a no-trading region where the investor will not re-balance her portfolio. Yet, the investor is sometimes unable to trade the illiquid asset at all, even at a cost.

Theorem 2.2 also implies that the optimal consumption choice and the optimal investment strategy in the liquid risky asset depend on the fraction of total wealth invested in the illiquid asset, ξ_t. Intuitively, if illiquid wealth is substantial relative to liquid wealth, for instance after a liquidity shock l_{t-1} occurs, the investor might have to cut her consumption relative to the case where the illiquid asset can always be traded. Moreover, to compensate for the relative high fraction invested in illiquid wealth, the investor will reduce her allocation to the liquid risky asset.

2.4. Willingness to pay for liquidity: theory

To understand why liquidity premiums exist in some asset classes but not in others, we analyze the willingness to pay for liquidity. In Section 3 we will use these results to explain some empirical findings in the literature of liquidity premiums in various asset classes.

We define the investor's willingness to pay, δ_t as the decrease in the expected return on the illiquid asset over period t that she is willing to pay, to convert the illiquid asset into a liquid one. In other words, δ_t is the compensation the investor demands for holding the illiquid asset. To formalize the willingness to pay, denote the value function for Problem 2.1 assuming that the asset X is actually also liquid by $V_t^{LIQ}(W_t + X_t)$. In other words, we solve Problem 2.1, subject to the budget constraints (3) and (4), where $p = 1$, $\phi = 0$. It is well known that this value function factorizes as:

$$V_t^{LIQ}(W_t + X_t) = \beta^t \frac{(W_t + X_t)^{1-\gamma}}{1-\gamma} H_t^{LIQ}, \tag{14}$$

for some constant H_t^{LIQ}. This H_t^{LIQ} no longer depends on ξ_t as the illiquid asset is tradeable as well.

The value function V_t^{LIQ} depends on the expected return $r_f + \lambda_X \sigma_X - 0.5\sigma_X^2$ of asset X. Subtracting δ from this expected return leads to a (lower) value function that we denote by $V_t^{LIQ}(W_t + X_t|\delta)$. We then define the willingness to pay δ_t as that value of δ_t that solves:

$$V_t^{LIQ}(W_t + X_t|\delta_t) = V_t(W_t + X_t, \xi_t). \tag{15}$$

Given (10) and (14), we can find δ_t by solving:

$$H_t^{LIQ}(\delta_t) = H_t(\xi_t), \tag{16}$$

where $H_t^{LIQ}(\delta_t)$ denotes the constant in the value function when the illiquid asset is actually liquid at a risk premium reduced by δ_t. Note that this willingness to pay depends on the actual allocation to the illiquid asset, i.e., on ξ_t. In the next section, we describe the baseline results of the model and explain in more detail how the liquidity premium depends on key assumptions.

2.5. Willingness to pay for liquidity: baseline results

Next we describe the model's baseline results, which we summarize as follows. Illiquidity can lead to two types of costs: suboptimal asset allocation, and suboptimal consumption. We show that only the latter is able to generate a substantial liquidity premium. This implies that, in equilibrium, only short-term investors or investors with high liquidity shocks demand first-order liquidity premiums.

2.5.1. Baseline parameter values

To calibrate the parameter values of the model, we use a range of reasonable values and show how the results change depending on assumptions. In the next section, when linking our model to empirical evidence on liquidity premiums in different asset classes, we motivate the choice of the parameter values more thoroughly. With respect to the investor's preferences we assume the investor faces a liquidity shock with probability $q = 10\%$ equal to $l_t = 20\%$ of total wealth, (i.e., the liquidity shock occurs on average once in 10 years), has a risk-aversion parameter equal to $\gamma = 3$, the time-preference discount factor equals $\beta = 0.91$, and investment horizon $T = 10$. With respect to the financial market, we assume the liquid asset has a price of risk $\lambda_S = 28\%$ and $\sigma_S = 18.5\%$, and the risk-free rate is $r_f = 2\%$. This results in an optimal risky asset allocation of approximately 50% and a risk-free bond allocation of 50%.

The parameter values of the illiquid asset are set equal to the parameter values of the liquid risky asset; $\lambda_X = 28\%$ and $\sigma_X = 18.5\%$. In this way, we isolate the effect of illiquidity instead of relying on a higher Sharp ratio for the illiquid asset. In line with this reasoning, we also assume no correlation between the liquid and illiquid risky asset in the baseline model; $\rho_{SX} = 0$. We assume that the investor can trade the illiquid asset on average once in two years, or in other words $p = 50\%$. If the investor decides to trade the illiquid asset when a trading opportunity arises, proportional transactions costs equal $\phi = 1\%$.

Table 1. **Parameter values**
This table summarizes the parameter values of our baseline model.

Parameters	*Symbol*	*Value*
Liquidity shock	l	0.20
Probability liquidity shock	q	0.10
Risk aversion parameter	γ	3
Time-preference discount factor	β	0.91
Investment horizon	T	10
Risk-free rate	r_f	0.02
Price of risk liquid risky asset	λ_S	0.28
Volatility liquid risky asset	σ_S	0.185
Price of risk illiquid asset	λ_X	0.28
Volatility illiquid asset	σ_X	0.185
Correlation coefficient	ρ	0.00
Trading probability illiquid asset	p	0.50
Transaction costs illiquid asset	ϕ	0.01

2.5.2. Liquidity premium and investment horizon

Perhaps the most straightforward, but nevertheless important result, is that the liquidity premium depends on the investment horizon. Figure 5 shows that the liquidity premium is amplified the shorter the investor's investment horizon. Illiquidity is a bigger threat for short term investors as they desire to consume a large part of their total wealth compared to longer term investors. Under the baseline parameters, the liquidity premium demanded by the short-term investors ($T = 1$) equals 63 basis points, whereas this converges to zero as the horizon T becomes large.

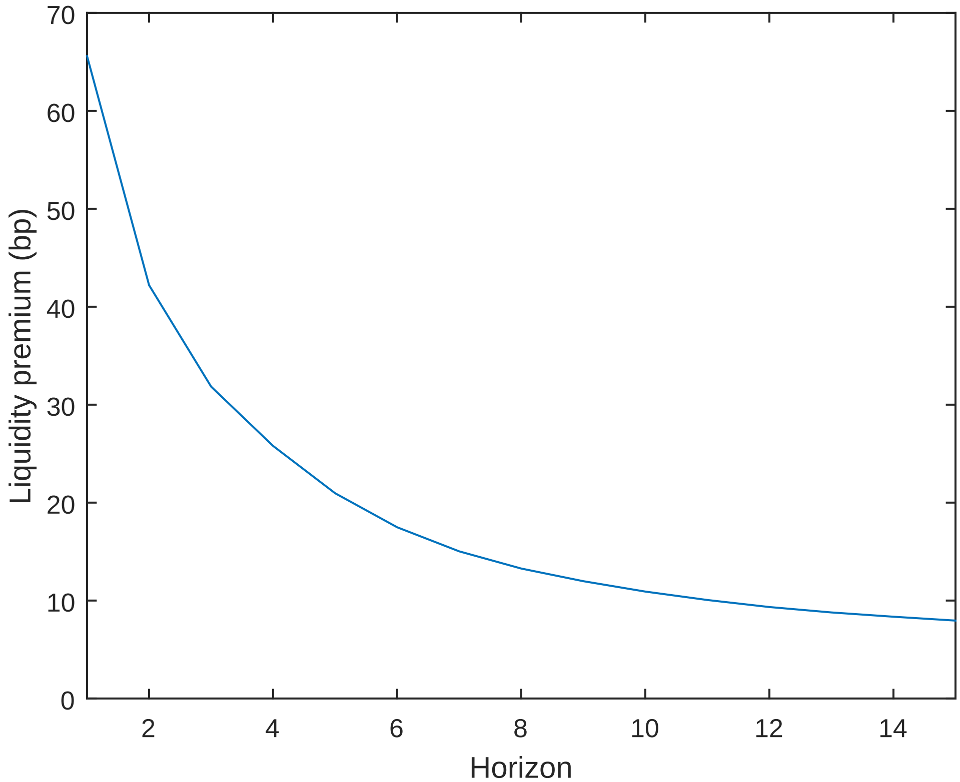


Figure 1. **The liquidity premium as a function of the investment horizon**
This graph shows the liquidity premium as a function of the investment horizon T, assuming the following parameter values. The risk-aversion parameter $\gamma = 3$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 20\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 7\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the trading probability of the illiquid asset $p = 50\%$, and transactions costs of the illiquid asset $\phi = 1\%$.

2.5.3. Liquidity premium and trading uncertainty

Next we show how the liquidity premium depends on times between trading opportunities. Figure 2 shows that for the long-term investor ($T = 10$), the liquidity premium equals 16 basis points if the investor is unable to trade the illiquid asset before the final date, and decreases to 10 basis points if the investor has a high probability of trading. Notice that the trading probabilities do not matter for investors with a horizon $T = 1$, as the illiquid asset can be liquidated in the next year with certainty.

The inability to trade has a small impact on the liquidity premium, because the investor's optimal consumption level is unlikely to be adversely affected under the baseline assumptions. An investment horizon of $T = 10$ years implies that the investor wants to consume approximately 10% of her wealth. In a scenario of a liquidity shock, an additional 20% of her wealth is lost. As the optimal allocation to the illiquid asset equals 50%, its very unlikely that her illiquid wealth is to large that she is unable to consume the desired level. Section 2.5.5 shows that inability to trade the illiquid asset significantly increases the liquidity premium if the level of liquidity shocks is higher. In that case liquid wealth may not be sufficient to consume the desired level.

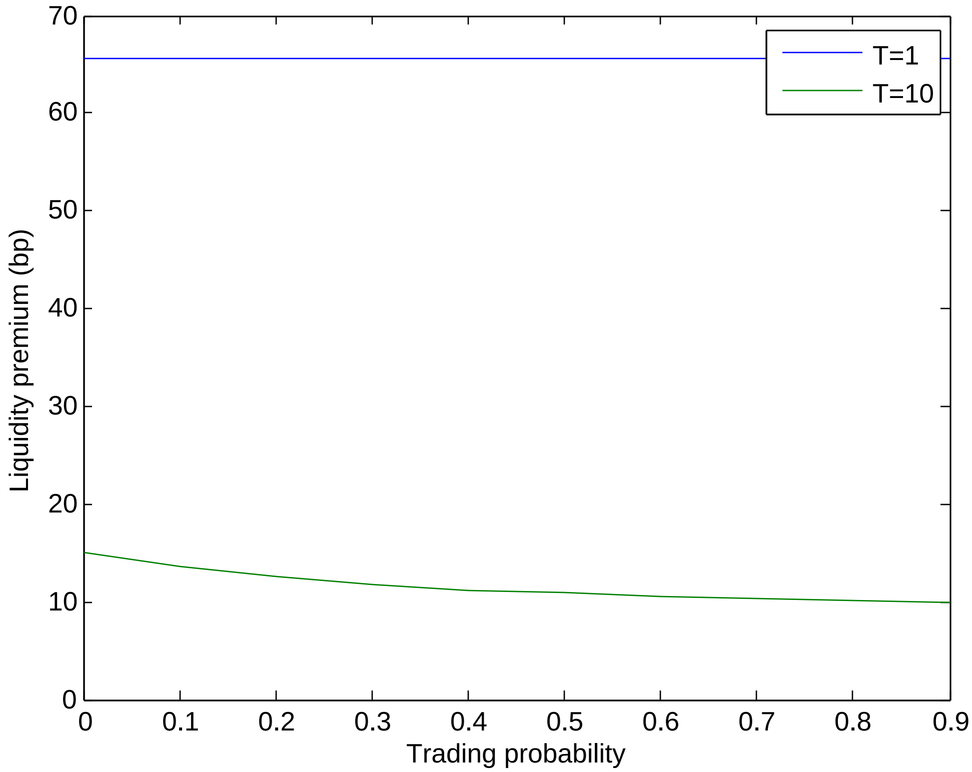


Figure 2. **The liquidity premium as a function of the trading probability**
This graph show the liquidity premium as a function of the trading probability p for the investor with horizon $T = 1$ and $T = 10$, assuming the following parameter values. The risk-aversion parameter $\gamma = 3$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 20\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 7\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, and transactions costs of the illiquid asset $\phi = 1\%$.

2.5.4. Liquidity premium and transaction costs

Figure 3 shows that rising transactions costs increase the liquidity premium for both short-term ($T = 1$) and long-term ($T = 10$) investors. Nevertheless, the increase is much more substantial for the short-term investor, such that a one percent increase in transaction costs increases the liquidity premium demanded by 0.50%. Short-term investors always liquidate their illiquid wealth at time $T = 1$, and they can only do so at cost ϕ. Long-term investors only liquidate their illiquid wealth at proportional cost ϕ if the realized illiquid asset allocation is too far from the optimal level. This confirms earlier results from Constantinides (1986), where the investor has an infinite horizon and transactions costs endogenously decrease the investor's trading frequency. The larger the transaction costs, the larger the investor's no-trading region. As the investors value function is insensitive to relatively large deviations from the optimal (non-transaction) portfolio allocation, transaction costs lead to second-order effects on prices.

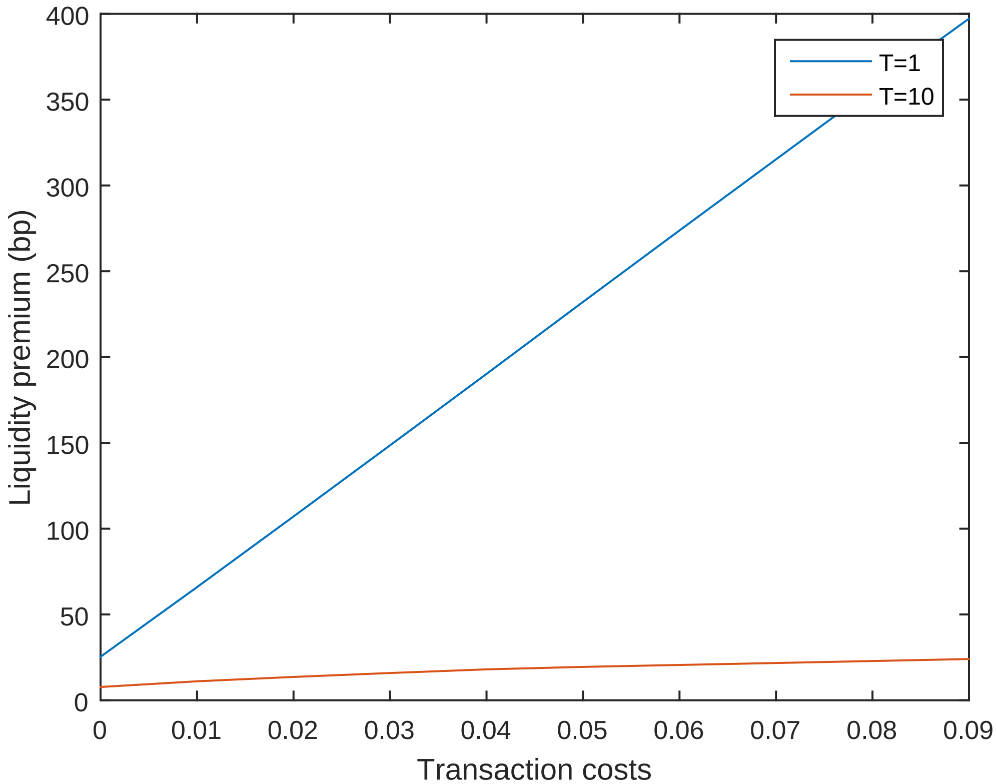


Figure 3. **The liquidity premium as a function of transaction costs**
This graph show the liquidity premium as a function of proportional transaction costs ϕ for the investor with horizon $T = 1$ and $T = 10$, assuming the following parameter values. The risk-aversion parameter $\gamma = 3$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l_t = 20\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 7\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the trading probability of the illiquid asset $p = 50\%$, and transactions costs of the illiquid asset $\phi = 1\%$.

2.5.5. Liquidity premium and liquidity shocks

Next we show how the liquidity premium depends on the level of the liquidity shock. Figure 4 shows that the liquidity premium is strongly amplified when the level of the liquidity shock increases. The driver of the liquidity premium here is the inability to trade, and not the transaction costs. Even a very small probability that the investor might be unable to trade the illiquid asset in case of large shocks leads to a substantial liquidity premium demanded by the investor. If the long-term investor ($T = 10$) faces a liquidity shock equal to 40% of total wealth and is unable to trade the illiquid asset, this could lead to a cut in her optimal consumption level. In order to prevent these states of the world, the investor reduces her optimal allocation to the illiquid asset substantially, as compared to the liquid case.

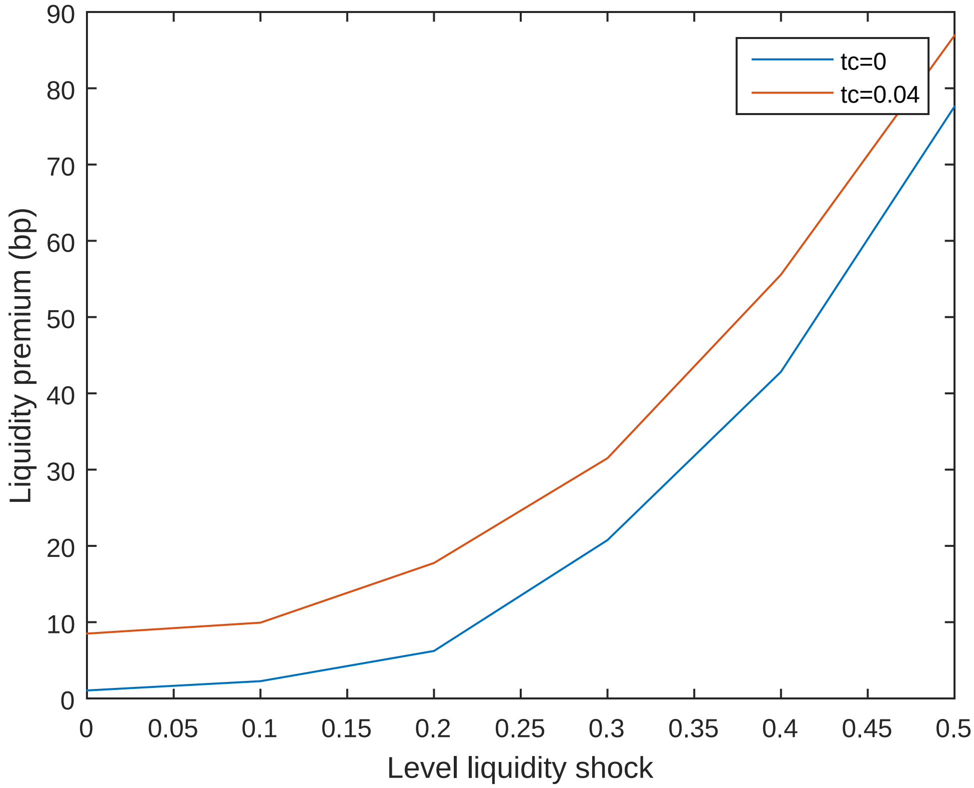


Figure 4. **The liquidity premium as a function of the level liquidity shock** This graph show the liquidity premium as a function of the level liquidity shock l for transaction costs equal to $\phi = 0$ and $\phi = 0.04$, assuming the following parameter values. The investment horizon $T = 10$, risk-aversion parameter $\gamma = 3$, the time-preference discount factor $\beta = 0.91$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid and illiquid risky asset $\mu_S = \mu_X = 7\%$, the volatility of the liquid and illiquid risky asset $\sigma_S = \sigma_X = 18.5\%$, the correlation coefficient $\rho_{SX} = 0$, the trading probability of the illiquid asset $p = 50\%$.

2.5.6. Risk premium

As a robustness check, we next illustrate how the liquidity premium depends on the price of risk λ_X of the illiquid asset. A higher price of risk increases the illiquid asset's attractiveness, so that optimally the investor is willing to have a higher fraction of her total wealth allocated to the illiquid asset. The higher the fraction of total wealth the investor optimally desires to invest in the illiquid asset, the stronger the threat of illiquidity. To remain able to smooth consumption, the investor must reduce her allocation to the illiquid asset further from the optimal asset allocation as the illiquid asset becomes more attractive. These findings are consistent with Kahl et al. (2003) and Longstaff (2003), who showed that the welfare effects of illiquidity are much larger the more wealth is tied up in the illiquid asset.

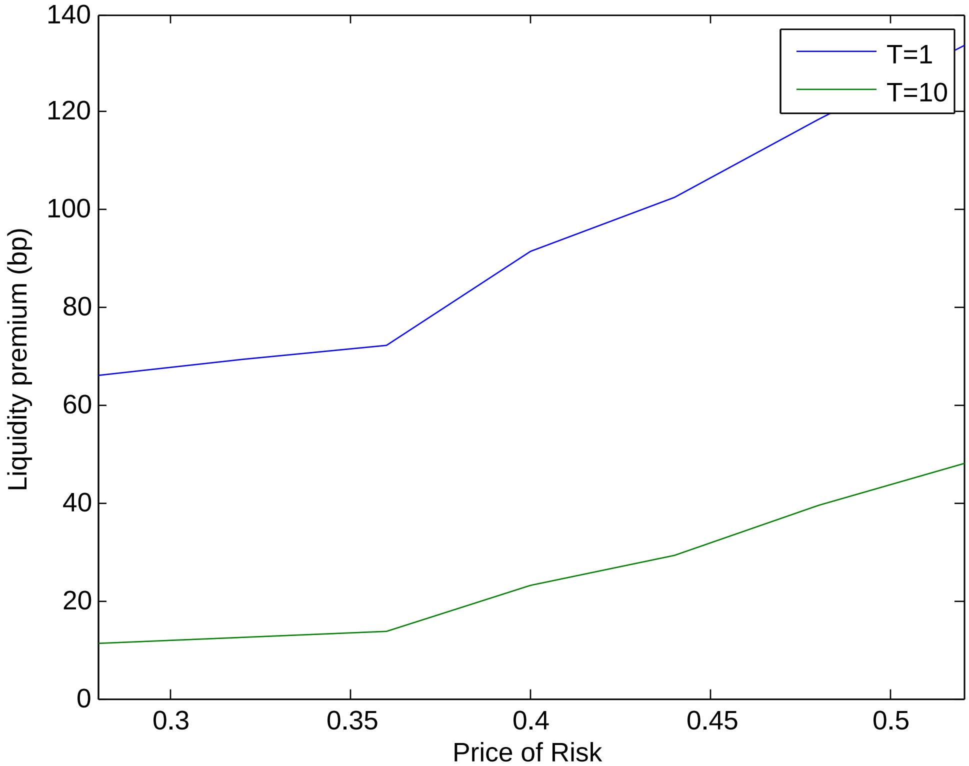


Figure 5. **The liquidity premium as a function of the price of risk**
This graph show the liquidity premium as a function of the price of risk of the illiquid asset, assuming the following parameter values. The investment horizon $T = 10$, risk-aversion parameter $\gamma = 3$, the time-preference discount factor $\beta = 0.91$, a liquidity shock $l = 20\%$, the return on the risk-free rate $r_f = 2\%$, the average return on the liquid risky asset $\mu_S = 7\%$, the volatility of the liquid risky asset $\sigma_S = 18.5\%$ (corresponds to a price of risk $\lambda_S = 0.28$), the correlation coefficient $\rho_{SX} = 0$, the trading probability of the illiquid asset $p = 50\%$, and transactions costs of the illiquid asset $\phi = 1\%$.

3. Liquidity premium in four asset classes

Next, we link our theoretical findings to the empirical evidence for liquidity premiums in several asset classes. Section 3.1 describes the implications of our theoretical findings in an equilibrium setting. In subsequent sections we relate these findings to the empirical evidence for liquidity premiums in four asset classes. The asset classes we consider are: private equity, real estate, corporate bonds, and stocks. We primarily focus on U.S. markets.

3.1. Implications Equilibrium

So far we have analyzed the liquidity premium demanded by a single investor. The next question to ask is: what does this imply for the liquidity premium in a general equilibrium setting? Under market clearing conditions the total amount of illiquid assets in the market should equal the sum of all investors' holdings of the illiquid asset:

$$X_{mt} = \sum_{i=1}^{N} X_{it}$$

The N individual investors will demand illiquid assets depending on their preferences and their total wealth. The liquidity premium in each asset class will depend on the composition of the type of investors in each particular asset class. Suppose all investors in a certain asset class have low liquidity needs, then a positive liquidity premium would generate excessive demand for the illiquid asset and therefore push down the liquidity premium. In that case, in equilibrium, the liquidity premium will converge to zero. On the other hand, if liquidity matters for a significant portion of the investors, the demand for investors who are less liquidity constraint might not be enough to push down the liquidity premium. Below we translate our theoretical findings and the market clearing conditions to the evidence for liquidity premiums in four asset classes.

To quantify the liquidity premium demanded by different types of investors for each asset class, we choose parameter values representative for that asset class. Throughout, we assume that the liquid risky asset S represents a liquid stock index. We use the mean and the standard deviation (annualized) of the S&P500 Index to model the diversified liquid stock index. Calibrated over the last 25 years, the average return is $\mu_S = 11.3\%$ and the standard deviation $\sigma_S = 17.8\%$. Moreover, we use as the risk-free rate the annualized 1-year Treasury yield over the last 25 years, which gives us $r_f = 2.8\%$. See Appendix C for details on the used indexes.

Preferences of investors in each market are less well known, as researchers only have a very rough idea about investors' investment horizons and their liquidity needs, usually proxied by holdings periods or investors' liability structures. These measures are generally incomplete as these proxies do not measure other liquidity risks such as margin calls on derivative positions or rare disasters that investors potentially face. For this reason, we do not model investors precisely, but rather provide qualitative indicators of investors preferences for each asset class. The main goal of this section is therefore not to model investors perfectly, but to give a rough indication of the average preferences of investors in each market.

3.2. Private equity

The evidence on liquidity premiums in private equity markets is mixed, mostly because there is only indirect evidence of potential existence or non-existence of liquidity premiums. For instance, Franzoni et al. (2012) showed that including a liquidity risk factor to the traditional three Fama French factors reduced alpha to zero. This is indirect evidence of no liquidity premium. By contrast, Harris et al. (2014) found out-performance of private equity versus the S&P500 Index of 3% annually. They used the compensation for illiquidity as potential explanation for this out-performance. Here we find an average liquidity premium of around 45 basis points on an annual basis using both calibrated parameters values and reasonable assumptions about investor preferences.

To assess the liquidity premium for private equity, we specify model parameters such that the illiquid asset has properties similar to private equity investments. We use the mean and standard deviation of the S&P500 Index to model the liquid counterpart of the illiquid private equity investment in our model, as the S&P500 Index is generally taken as the benchmark for private equity, e.g. Franzoni et al. (2012) and Harris et al. (2014): this implies $\mu_X = 11.3\%$ and $\sigma_X = 17.8\%$. We do not take a stand on the correlation between private equity and the S&P500 Index. The performance of private equity varies substantially across private equity investments, as noted by e.g. Phalippou and Gottschalg (2009), and pinning down the correlation coefficient is difficult. Although several studies claim the performance of private equity is comparable to the performance of public equities, this does not directly translate into the correlation coefficient. We therefore analyze results for correlation coefficients of $\rho_{SX} = 0$, $\rho_{SX} = 0.25$ and $\rho_{SX} = 0.5$.

Private equity investments have two destructive features. First, private equity investments generally run for 10 years and trading is unusual before a contract expires (Metrick and Yasuda, 2010). Therefore we set $p = 0$ over the first 10 years of the investment horizon. Second, private equity usually involves capital commitment agreements. The investor agrees to provide a preset amount of capital over the first three to five years of the project. Yet, the capital commitment is preset, so we treat the capital commitment as an upfront investment in our model. We also assume no transaction costs at exit, so $\phi = 0$. The private equity investment is sold at the end of the contract to the public (IPO) or to another private company, and the investor is not forced to trade to liquidate her position.

Turning to investors' preferences in the private equity market, we posit that clientele effects in private equity markets are likely to be strong, as the lock-up period of private equity is long and known beforehand. We therefore assume that investors with a horizon shorter than $T = 10$ years do not invest in illiquid assets at all. We then assess the liquidity premium for longer term investors facing different liquidity shocks.

Table 2 shows that liquidity premiums are significantly reduced for a positive correlation between the liquid and illiquid asset. The intuition is that the effect of illiquidity is smaller because of lower diversification benefits when the correlation coefficient is larger. The liquidity shock also plays a non-trivial role in the level of the liquidity premium. Taking the average of Table 2, these results together suggest an average liquidity premium of 45 basis points over the past 25 years.

Table 2. **Liquidity premiums for private equity**
This table shows the liquidity premium in percent demanded by investors facing liquidity shock l and correlation coefficient equal to ρ_{SX}, using the following parameter values: investment horizon $T = 10$, risk-aversion parameter $\gamma = 3$, time-preference discount factor $\beta = 0.91$, return on the risk-free rate $r_f = 2.8\%$, average and standard deviation of the return on the liquid risky asset $\mu_S = 11.3\%$ and $\sigma_S = 17.8\%$, average and standard deviation of the return on the illiquid asset $\mu_X = 11.3\%$ and $\sigma_X = 17.8\%$, dividend payments $d = 0$, and trading probability of the illiquid asset $p = 0$.

	Correlation coefficient ρ_{SX}		
	0.0	0.25	0.5
Liquidity shock l			
0.0	0.47	0.15	0.05
0.2	0.63	0.24	0.10
0.4	1.44	0.80	0.41

3.3. Real Estate

Similar mixed evidence exists for liquidity premiums in real estate markets. Benveniste et al. (2002) showed that creating liquidity by introducing REITS for illiquid properties increased the value of these assets by 12-22%. Qian and Liu (2012) also found a link between higher illiquidity and lower prices, though the effect on expected returns is fairly small. Ang et al. (2013) showed comparable performance of direct and indirect real estate investments, suggestive of no liquidity premium. We find an average liquidity premium around 60 basis points on an annual basis using both calibrated parameters values and reasonable assumptions about the investors' preferences.

To model a direct real estate investment, we use the first two moments of the S&P United States REIT Index to model the liquid counterpart of a direct real estate investment. We calibrate the annualized average return on the corporate bond index as $\mu_X = 12.22\%$ and the corresponding standard deviation as $\sigma_X = 18.31\%$. The correlation coefficient between the liquid and illiquid asset is calibrated as the correlation between the S&P500 Index and S&P United States REIT Index, which equals $\rho_{SX} = 0.4$.

The return on real estate includes both income return and capital gains. The income return refers to the rent on properties, so therefore real estate returns are partially liquid. In fact, the largest part of real estate returns consists of these income returns. In order to take into account the two return components, we separate the return on illiquid assets into two parts: the liquid part (income return), which we refer to as dividends d, and the illiquid part (capital gains) of the return $r_t^X - d$. The dividends d are added to liquid wealth every period, whereas illiquid wealth grows at a rate $r_t^X - d$. We set $d = \mu_X - r_f$. Holding d constant also implies that all volatility is in capital gains, which is generally true.

To describe illiquidity parameters for real estate, we note that transaction costs consist of various main components: registration costs, real estate agent fees, legal fees, and sales and transfer taxes. On average, total transaction costs lie in the range of 6%-10%. The typical time between transactions for residential housing is 4-5 years and 8-11 years for institutional real estate, e.g. Hansen (1998) and Miller et al. (2011). We thus assume trading probabilities p in the range of 10%-20%. For preferences, we again analyze long-term investors, i.e. $T = 10$, and different liquidity shocks as before.

Table 3. **Liquidity premiums for real estate**
This table shows the liquidity premium in percent demanded by investors facing liquidity shock l and trading probability p, using the following parameter values: investment horizon $T = 10$, risk-aversion parameter $\gamma = 3$, time-preference discount factor $\beta = 0.91$, return on the risk-free rate $r_f = 2.8\%$, average and standard deviation of the return on the liquid risky asset $\mu_S = 11.3\%$ and $\sigma_S = 17.8\%$, average and standard deviation of the return on the illiquid asset $\mu_X = 12.2\%$ and $\sigma_X = 18.3\%$, correlation coefficient $\rho_{SX} = 0.4$, dividend payments $d = 8.6\%$, and transaction costs $\phi = 6\%$.

	Trading probability p	
	10%	20%
Liquidity shock l		
0.0	0.42	0.36
0.2	0.46	0.42
0.4	1.04	1.01

The liquidity premum for real estate is diminished mainly by the liquid return component of real estate returns. The threat of low trading opportunities and the relatively high transaction costs are therefore partly compensated. As we assume the investor receives the dividend for sure, the dividend can be used to smooth consumption and dampen the effect of liquidity shocks. Taking the average of Table 3, we expect liquidity premiums to be on average 60 basis points over the past 25 years.

3.4. Corporate bonds

In the corporate bond market, several papers have found empirical evidence for a first-order level liquidity premium. For instance, Bongaerts et al. (2017) showed that transaction costs of 0.54% led to a 0.56% liquidity level premium. Yet, Palhares and Richardson (2018) found only limited evidence for liquidity premiums of corporate bonds using illiquidity-factor portfolios. We find an average liquidity premium around 30 basis points on an annual basis using both calibrated parameters values and reasonable assumptions about the investors' preferences.

To assess liquidity premiums of corporate bonds we again choose parameter values representative for the U.S. corporate bond market. Unless for private equity and real estate, there is no trivial liquid counterpart for a corporate bond. We construct the liquid counterpart as a combination of a government index, with a default probability. We use the Bloomberg Barclays US Treasury Index over the last 25 years, and assume a default probability equal to 1% per year, where returns equal $r_t^X = -40\%$ if default occurs. We calibrate the annualized average return on the Treasury Index as $\mu_X = 5.3\%$ and the corresponding standard deviation as $\sigma_X = 5.9\%$. We calibrate the correlation coefficient as the correlation between the Bloomberg Barclays U.S. Corporate Bond Index and the S&P500 Index, which equals $\rho_{SX} = 0.35$. Finally, we assume a fixed coupon payments, and as for real estate, we assume $d = \mu_X - r_f$.

Turning to the illiquidity parameters, we see that Bongaerts et al. (2017) found average transaction costs for corporate bonds of 0.52%, ranging between 0.46% and 0.58% from the most liquid to the least liquid corporate bond. As the spread in transaction costs is rather low, we use the average (proportional) transaction costs $\phi = 0.52\%$ in our model. Corporate bonds are not traded as often as public equities, and the period between trades for corporate bonds differs substantially across bonds. For instance, Bongaerts et al. (2017) showed that 15% to 25% of corporate bonds were not traded in a given week. This results in trading probabilities varying between $p = 85\%$ and $p = 95\%$, which implies trades occur on average once in every two months to once every half a month, so wee set $p = 0.90\%$.

Table 4 shows the liquidity premium demanded by both short-term and long-term investors. The relatively low transaction costs, high trading opportunities and coupon payments result in a small liquidity premium for corporate bonds. On average we expect a liquidity premium equal to 30 basis points over the past 25 years.

Table 4. **Liquidity premiums for corporate bonds**
This table shows the liquidity premium in percent demanded by investors having horizon T, facing liquidity shock l, using the following parameter values: risk-aversion parameter $\gamma = 3$, time-preference discount factor $\beta = 0.91$, return on the risk-free rate $r_f = 2.8\%$, average and standard deviation of the return on the liquid risky asset $\mu_S = 11.3\%$ and $\sigma_S = 17.8\%$, average and standard deviation of the return on the illiquid asset $\mu_X = 5.3\%$ and $\sigma_X = 5.9\%$, default probability 1% with $r_t^X = -0.4\%$, correlation coefficient $\rho_{SX} = 0.35$, dividend payments $d = 2.5\%$, trading probability of the illiquid asset $p = 0.9$, and transaction costs $\phi = 0.52\%$.

	Investment horizon T	
	1	10
Liquidity shock l		
0.0	0.25	0.05
0.2	0.43	0.15
0.4	0.63	0.40

3.5. *Stocks*

Several early literatures reported significant liquidity premiums for U.S. equities. For instance, Amihud and Mendelson (1986) found that a 1% increase in bid-ask spread increased the expected return with 2.4% per year. Acharya and Pedersen (2005) also showed a liquidity premium of 4.5% for the most illiquid stocks versus the most liquid stocks. The existence of the liquidity premium in U.S. stocks was subsequently challenged by Ben-Rephael et al. (2015), who noted that liquidity premium almost disappeared over the 1997-2008 period due to reduced transaction costs. For instance, for stocks listed on the NYSE, the liquidity premium for an average liquid stock was 16 basis points per month in the period 1964-1974, but was in fact zero for the period 1997-2008. They liquidity premium only survived for very small stocks. We find an average liquidity premium around 40 basis points on an annual basis using both calibrated parameters values and reasonable assumptions about the investors' preferences.

For our model we again use the first two moments of the S&P500 Index to model the liquid counterpart of the illiquid stock, so $\mu_X = 11.3\%$ and $\sigma_X = 17.8\%$. Equity asset classes, such as small-growth stocks, value stocks, and mid cap stocks, are highly correlated with the U.S. stock market.[2] For instance, the correlation between Small-Growth stocks and the S&P500 Index is approximately 0.9. We therefore set the correlation coefficient equal to $\rho_{SX} = 0.9$. Given the improved liquidity of U.S. equities we assume that the trading probability equals $p = 100\%$, in other words the investor can always trade the illiquid stock. The transaction costs for stocks ranges from 0.25% for the most liquid stocks, to 8% for the least liquid stocks (Beber et al., 2012). We therefore assess the liquidity premium for transaction costs ranging from 1%-8%.

As we do not know the investors' exact preferences, we again show how the liquidity

[2] Correlations between different equity asset classes can, for instance, be found on www.portfoliovisualizer.com.

premium depends on these preferences. Table 5 shows that the short-term investors demand substantial higher liquidity premiums than long-term investor. Taking the average of Table 5, we expect liquidity premiums to be on average 40 basis points over the past 25 years.

Table 5. **Liquidity premiums for stocks**
This table shows the liquidity premium in percent demanded by investors having horizon T, facing liquidity shock l, and transaction costs ϕ using the following parameter values: risk-aversion parameter $\gamma = 3$, time-preference discount factor $\beta = 0.91$, return on the risk-free rate $r_f = 2.8\%$, average and standard deviation of the return on the liquid risky asset $\mu_S = 11.3\%$ and $\sigma_S = 17.8\%$, average and standard deviation of the return on the illiquid asset $\mu_X = 11.3\%$ and $\sigma_X = 17.8\%$, correlation coefficient $\rho_{SX} = 0.9$, dividend payments $d = 0$, and trading probability of the illiquid asset $p = 1$.

Panel A: Investment horizon $T = 1$	Transaction costs ϕ							
	1%	2%	3%	4%	5%	6%	7%	8%
Liquidity shock l								
0.0	0.48	0.91	0.91	0.91	0.91	0.91	0.91	0.91
0.2	0.45	0.88	0.91	0.91	0.91	0.91	0.91	0.91
0.4	0.36	0.62	0.87	0.87	0.87	0.87	0.87	0.87

Panel B: Investment horizon $T = 10$	Transaction costs ϕ							
	1%	2%	3%	4%	5%	6%	7%	8%
Liquidity shock l								
0.0	0.02	0.03	0.04	0.05	0.06	0.08	0.09	0.10
0.2	0.04	0.05	0.06	0.08	0.09	0.10	0.11	0.12
0.4	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15

4. Conclusion

In this paper we study the liquidity premium demanded by different investor types. The cost of illiquidity may be twofold: suboptimal asset allocation and suboptimal consumption. Essentially, in this paper we show that only illiquidity resulting in suboptimal consumption is able to generate a substantial liquidity premium, while illiquidity leading to suboptimal asset allocations does not. From this we conclude that, in equilibrium, the liquidity premiums will depend on the relative prevalence of both types of investors in the financial market. This enables us to relate the composition of market participants to the empirical evidence about the liquidity premiums in that market.

We have several contributions. Although private equity is the most illiquid asset that we analyze, we find an average liquidity premium equal to 45 basis points only. Private equity investors have to lockup their money for a long period of time, and mainly for that reason only long-term investors are likely to be present in this market. For these investors illiquidity is unlikely to significantly harm investors' consumption patterns. Moreover, the non-trading period is fixed, knowing exactly when the position is liquidated. For real estate we find an average liquidity premium equal to 65 basis points. Real estate can often not be traded for a significant amount of time, and the timing of the trading opportunities are uncertain. Yet, the threat of illiquidity is dampen significantly because of the liquid return component (rents) of real estate investments. The average liquidity premium for corporate bonds equals 30 basis points. The transaction costs are small and trading occurs quite frequently. The liquidity premium does not disappear however, because the investor still faces the risk of insufficient liquid wealth to fulfill immediate consumption needs. For stocks we find an average liquidity premium equal to 40 basis points.

A. Proof of optimal consumption and investment strategies

Proof of Theorem 2.2. Instead of the endogenous variables (W_t, X_t), we use the pair $(W_t + X_t, \xi_t)$ as endogenous state variables. That is, we write:

$$C_t = \alpha_t(W_t + X_t, \xi_t)(W_t + X_t),$$

$$\theta_t = \theta_t(W_t + X_t, \xi_t).$$

Now, rewrite the evolution of total wealth $W_t + X_t$ using budget constraints (3)-(4) as:

$$\begin{aligned} W_t + X_t &= (W_{t-1} + X_{t-1}) \\ &\times [(1 - \xi_{t-1} - \alpha_{t-1} - l_{t-1} - \mathbf{1}_{t-1}\phi|\triangle\xi_{t-1}|) \times \big(\exp(r_f) + \theta_{t-1}(\exp(r_t^S) - \exp(r_f))\big) \\ &+ \xi_{t-1}\exp(r_t^X)], \end{aligned} \tag{17}$$

$$\xi_t = \frac{\xi_{t-1}\exp(r_t^X)}{(1 - \xi_{t-1} - \alpha_{t-1} - l_{t-1} - \mathbf{1}_{t-1}\phi|\triangle\xi_{t-1}|) \times (\exp(r_f) + \theta_{t-1}(\exp(r_t^S) - \exp(r_f))) + \xi_{t-1}\exp(r_t^X)}. \tag{18}$$

where $\triangle\xi_{t-1} = \xi_{t-1} - \xi^*_{t-1}$.

The proof is by backward induction. At the final horizon $t = T$, the claim is obviously correct with $\alpha_T \equiv 1$ and $H_T \equiv (1 - \phi\xi_T)^{1-\gamma}$. At time T, θ_T and ξ_T are irrelevant. Now, for the induction argument, assume that (10)-(13) holds at time t. Then, we need to show that the value function also holds at time $t-1$. From the value function (8), evaluated at time

$t-1$ and substituting (10), we find:

$$
\begin{aligned}
&V_{t-1}(W_{t-1}+X_{t-1},\xi_{t-1})\\
&\quad = \max_{\alpha_{t-1},\theta_{t-1},\xi_{t-1}} \beta^{t-1}\frac{C_{t-1}^{1-\gamma}}{1-\gamma} + \mathbb{E}_{t-1}V_t(W_t+X_t,\xi_t)\\
&\quad = \max_{\alpha_{t-1},\theta_{t-1},\xi_{t-1}} \beta^{t-1}\frac{C_{t-1}^{1-\gamma}}{1-\gamma} + \mathbb{E}_{t-1}[\beta^t\frac{(W_t+X_t)^{1-\gamma}}{1-\gamma}H_t(\xi_t)]\\
&\quad = \max_{\alpha_{t-1},\theta_{t-1},\xi_{t-1}} \beta^{t-1}\frac{(W_{t-1}+X_{t-1})^{1-\gamma}}{1-\gamma}\\
&\quad \times \Big(\alpha_{t-1}^{1-\gamma} + \beta\mathbb{E}_{t-1}[\{(1-\xi_{t-1}-\alpha_{t-1}-l_{t-1}-\mathbf{1}_{t-1}\phi|\triangle\xi_{t-1}|)\times\\
&\quad (\exp(r_f)+\theta_{t-1}(\exp(r_t^S)-\exp(r_f)))+\xi_{t-1}\exp(r_t^X)\}^{1-\gamma}H_t(\xi_t)]\Big)\\
&\quad = \max_{\alpha_{t-1},\theta_{t-1},\xi_{t-1}} \beta^{t-1}\frac{(W_{t-1}+X_{t-1})^{1-\gamma}}{1-\gamma}H_t(\xi_t)
\end{aligned}
$$

At time $t-1$, the penalty function $H_{t-1}(\xi_{t-1})$ equals

$$
\begin{aligned}
H_{t-1}(\xi_{t-1}) &= \alpha_{t-1}^{1-\gamma} + \beta\mathbb{E}_{t-1}[\{(1-\xi_{t-1}-\alpha_{t-1}-l_{t-1}-\mathbf{1}_{t-1}\phi|\triangle\xi_{t-1}|)\times\\
&\quad (\exp(r_f)+\theta_{t-1}(\exp(r_t^S)-\exp(r_f)))+\xi_{t-1}\exp(r_t^X)\}^{1-\gamma}H_t(\xi_t)] \qquad (19)
\end{aligned}
$$

Therefore, the function $H_{t-1}(\xi_{t-1})$ is a function of $t-1$ and ξ_{t-1} only and hence (10) from (10)-(13) holds for all t. We continue with proving (11)-(12) at time t. The first-order conditions of the decision variables α_{t-1} and θ_{t-1} equal:

$$
\alpha_{t-1}^{UC*} = \text{argmax}_{\alpha_{t-1}}\frac{(\alpha_{t-1}(W_{t-1}+X_{t-1}))^{1-\gamma}}{1-\gamma} + \mathbb{E}_{t-1}V_t(W_t+X_t,\xi_t), \qquad (20)
$$

$$
\theta_{t-1}^{*} = \text{argmax}_{\theta_{t-1}}\frac{(\alpha_{t-1}(W_{t-1}+X_{t-1}))^{1-\gamma}}{1-\gamma} + \mathbb{E}_{t-1}V_t(W_t+X_t,\xi_t). \qquad (21)
$$

where α_{t-1}^{UC*} is the solution if the investor were unconstrained, i.e. when constraint (5) does not bind. As we assume that the investor cannot borrow against the illiquid asset the constrained solution becomes:

$$\alpha_{t-1}^{C*} = \begin{cases} \alpha_{t-1}^{UC*} & \text{if} \quad \alpha_{t-1}^{UC*} \leq 1 - \xi_{t-1} \\ 1 - \xi_{t-1} & \text{if} \quad \alpha_{t-1}^{UC*} > 1 - \xi_{t-1} \end{cases} \tag{22}$$

We can now rewrite (20) and (21) as:

$$\begin{aligned} \frac{\partial V_{t-1}}{\partial \alpha_{t-1}} &= \beta^{t-1}(\alpha_{t-1}(W_{t-1} + X_{t-1}))^{-\gamma} \\ &- \mathbb{E}_{t-1}[\frac{\partial V_t}{\partial W_t + X_t}\Big(\exp(r_f) + \theta_{t-1}(\exp(r_t^S) - \exp(r_f))\Big)] \\ &+ \mathbb{E}_{t-1}[\frac{\partial V_t}{\partial \xi_t}\xi_t \frac{1}{W_t + X_t}\Big(\exp(r_f) + \theta_{t-1}(\exp(r_t^S) - \exp(r_f))\Big)] = 0, \\ \frac{\partial V_{t-1}}{\partial \theta_{t-1}} &= \mathbb{E}_{t-1}[\frac{\partial V_t}{\partial W_t + X_t}\Big(\exp(r_t^S) - \exp(r_f)\Big)] \\ &+ \mathbb{E}_{t-1}[\frac{\partial V_t}{\partial \xi_t}\xi_t \frac{1}{W_t + X_t}\Big(\exp(r_t^S) - \exp(r_f)\Big)] = 0. \end{aligned}$$

To see that both α_{t-1}^* and θ_{t-1}^* depend only on ξ_{t-1}, we substitute (8) and (17) into (20) and (21), we get:

$$\begin{aligned} \frac{\partial V_{t-1}}{\partial \alpha_{t-1}} &= \alpha_{t-1}^{-\gamma} \\ &+ \beta \mathbb{E}[\{(1 - \xi_{t-1} - \alpha_{t-1} - l_{t-1} - \mathbf{1}_{t-1}\phi|\triangle \xi_{t-1}|)(\exp(r_f) + \theta_{t-1}(\exp(r_t^S) - \exp(r_f))) \\ &+ \xi_{t-1}\exp(r_t^X)\}^{-\gamma} \times \Big(\frac{H_t'(\xi_t)}{1-\gamma}\xi_t - H_t(\xi_t)\Big)\exp(r_f)|\xi_{t-1}] = 0, \\ \frac{\partial V_{t-1}}{\partial \theta_{t-1}} &= \mathbb{E}[\{(1 - \xi_{t-1} - \alpha_{t-1} - l_{t-1} - \mathbf{1}_{t-1}\phi|\triangle \xi_{t-1}|)(\exp(r_f) + \theta_{t-1}(\exp(r_t^S) - \exp(r_f))) \\ &+ \xi_{t-1}\exp(r_t^X)\}^{-\gamma} \times \Big(H_t(\xi_t) - \frac{H_t'(\xi_t)}{1-\gamma}\xi_t\Big)\Big(\exp(r_t^S) - \exp(r_f)\Big)|\xi_{t-1}] = 0. \end{aligned}$$

The first-order conditions (23) and (23) depend only on time $t-1$ and the fraction invested in the illiquid asset ξ_{t-1}. In this way, the optimal consumption and the fraction invested in the liquid risky assets can indeed be written as in (11) and (12), so (11)-(12) from (10)-(13) holds for all t. We finish the proof by also showing that (13) also hold for all

t. When a trading opportunity arises at $t-1$, the investor chooses ξ_{t-1} such that the value function at $t-1$ is optimized:

$$\begin{aligned}
\xi^*_{t-1} &= \text{argmax}_{\xi_{t-1}} V_{t-1}(W_{t-1}+X_{t-1},\xi_{t-1}) = \text{argmax}_{\xi_{t-1}} \beta^{t-1}\frac{(W_{t-1}+X_{t-1})^{1-\gamma}}{1-\gamma}H_{t-1}(\xi_{t-1}) \\
&= \text{argmin}_{\xi_{t-1}} H_{t-1}(\xi_{t-1}). \qquad (23)
\end{aligned}$$

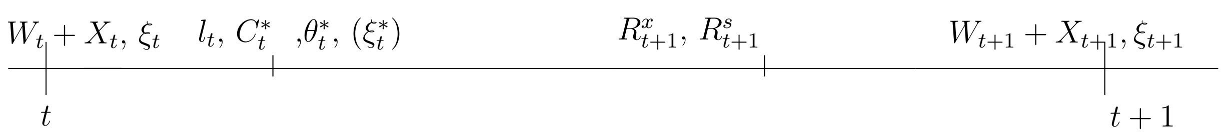
□

B. Solution technique

This appendix provides an outline of the numerical method to solve the baseline model. First, the sequence of decision making is described. Second, the numerical solution technique to solve for the decision variables is explained.

B.1. Sequence of decision making

Figure B.1 depicts the sequence of decision making. The endogenous variables, liquid wealth W_t and illiquid wealth X_t, are defined as total wealth before consumption and returns earned in period $(t, t+1]$. Based on the actual fraction allocated to the illiquid asset, ξ_t, the investor chooses the optimal fraction of total wealth to be consumed in period $(t, t+1]$, $\alpha_t(\xi_t)$, and the optimal allocation towards the liquid risky asset, $\theta_t(\xi_t)$. If a trading opportunity arises at time t, the investor chooses simultaneously ξ^*_t, $\alpha_t(\xi^*_t)$ and $\theta_t(\xi^*_t)$. Notice that by the assumption $X_t \geq 0$ and the inability to borrow against the illiquid asset, the possible values for ξ_t are restricted to the interval $[0, 1]$.



B.2. Numerical solution technique

The baseline model is solved by means of backward induction, where we start solving the problem at the final date $t = T$ and solve the model backwards for each period until arriving at time $t = 0$. At the final horizon $t = T$, we have $\alpha_T \equiv 1$ and $H_T \equiv (1 - \phi\xi_T)^{1-\gamma}$. To solve for ξ^*_{T-1}, $\alpha_t(\xi_{T-1})$ and $\theta_t(\xi_{T-1})$, we construct a grid for $\xi_{T-1} \in [0, 1)$. We simulate $M = 100000$ trajectories for the exogenous state variables, the returns on the liquid and illiquid risky asset in the period $(T-1, T]$, R^S_T and R^X_T, from a multi-normal distribution with means and variance-covariance matrix, and the liquidity shock l_T, as described in Section 2.1.

For each grid point by using non-linear least squares, we solve the first order conditions with respect to consumption (23) and the allocation towards the liquid risky asset (23) by using $H_T \equiv (1 - \phi\xi_T)^{1-\gamma}$, R^S_T, and R^X_T to find $\alpha_{T-1}(\xi_{T-1})$ and $\theta_{T-1}(\xi_{T-1})$. Then we are able to calculate H_{T-1} and solve for $\xi^*_{T-1} = argmin_{\xi} H_{T-1}(\xi_{T-1})$. We can continue this approach until we arrive at $t = 0$.

C. Market Indexes

Here we list the market indexes U.S.ed to calibrate the four different asset classes: private equity, real estate, corporate bonds, and stocks.

Table 6. **Market Indexes calibration**
This table lists the indexes used to calibrate the four different asset classes described in Section 3, as well as the source for the risk-free rate.

Risk-free rate	*1-year Treasury yield*
Stocks	*S&P500 Index* The index includes 500 leading companies publicly traded in the U.S. stock market.
Corporate bonds	*Bloomberg Barclays U.S. Treasury Index* The index measures U.S. dollar-denominated, fixed-rate, nominal debt issued by the U.S. Treasury. *Bloomberg Barclays U.S. Corporate Bond Index* The index measures the investment grade, fixed-rate taxable corporate bond market. It includes USD denominated securities publicly issued by U.S. and non-U.S. industrial, utility and financial issuers.
Real estate	*S&P United States REIT Index* The index includes the invest-able universe of publicly traded real estate investment trusts domiciled in the U.S..
Private Equity	*S&P500 Index*

References

Acharya, V. and Pedersen, L. (2005). Asset pricing with liquidity risk. *Journal of Financial Economics*, 77(2):375–410.

Amihud, Y. and Mendelson, H. (1986). Asset pricing and the bid-ask spread. *Journal of Financial Economics*, 17(2):223–249.

Ang, A., Nabar, N., and Wald, S. (2013). Searching for a common factor in public and private real estate returns. Available at SSRN.

Ang, A., Papanikolaou, D., and Westerfield, M. (2014). Portfolio choice with illiquid assets. *Management Science*, 60(11):2737–2761.

Bao, J., Pan, J., and Wang, J. (2011). The illiquidity of corporate bonds. *Journal of Finance*, 66(3):911–946.

Beber, A., Driessen, J., and Tuijp, P. (2012). Pricing liquidity risk with heterogeneous investment horizons.

Ben-Rephael, A., Kadan, O., and Wohl, A. (2015). The diminishing liquidity premium. *Journal of Financial and Quantitative Analysis*, 50(1-2):197–229.

Benveniste, L., Capozza, P., and Seguin, P. (2002). The value of liquidity. *Real Estate Economics*, 29(4):633–630.

Bongaerts, D., De Jong, F., and Driessen, J. (2017). An asset pricing approach to liquidity effects in corporate bond markets. *The Review of Financial Studies*, 30(4):1229–1269.

Brennan, M. and Subrahmanyam, A. (1996). Market microstructure and asset pricing: On the compensation for illiquidity in stock returns. *Journal of Financial Economics*, 41(3):441–464.

Constantinides, G. (1986). Capital market equilibrium with transaction costs. *Journal of Political Economy*, 94(4):842–862.

Franzoni, F., Nowak, E., and Phalippou, L. (2012). Private equity performance and liquidity risk. *Journal of Finance*, 67(6):2341–2373.

Gârleanu, N. (2009). Portfolio choice and pricing in illiquid markets. *Journal of Economic Theory*, 144:532–564.

Goetzmann, W. and Oster, S. (2012). Competition among university endowments. NBER working paper No. 18173.

Hansen, K. (1998). Seasonality of moves and duration of residence. Technical report, Bureau of Census, U.S. Department of Commerce.

Harris, R., Jenkinson, T., and Kaplan, S. (2014). Private equity performance: What do we know? *Journal of Finance*, 69(5):1851–1882.

Huang, M. (2003). Liquidity shocks and equilibrium liquidity premia. *Journal of Economic Theory*, 109:104–129.

Jang, B., Koo, H., Liu, H., and Loewenstein, M. (2007). Liquidity premia and transaction costs. *The Journal of Finance*, 62(5):2329–2366.

Kahl, M., Liu, J., and Longstaff, F. (2003). Paper millionaires: how valuable is stock to a stockholder who is restricted from selling it? *Journal of Financial Economics*, 67:385–410.

Longstaff, F. (2003). Portfolio claustrophobia: Asset pricing in markets with illiquid assets. *American Economic Review*, 99(4):1119–1144.

Lynch, A. and Tan, S. (2011). Explaining the magnitude of liquidity premia: The roles of return predictability, wealth shocks, and state-dependent transaction costs. *The Journal of Finance*, 66(4):1329–1368.

Metrick, A. and Yasuda, A. (2010). The economics of private equity funds. *Review of Financial Studies*, 23(6):2303–2341.

Miller, N., Peng, L., and Sklarz, M. (2011). The economic impact of anticipated house price changes - evidence from home sales. *Real Estate Economics*, 39(2):345–378.

Palhares, D. and Richardson, S. (2018). (il)liquidity premium in credit markets: A myth? AQR working paper.

Phalippou, L. and Gottschalg, O. (2009). The performance of private equity funds. *The Review of Financial Studies*, 22(4):1747–1776.

Qian, W. and Liu, P. (2012). Does (and what) illiquidity matter for real estate prices? Measure and evidence. SSRN Electronic Journal.

Vayanos, . (1998). Transaction costs and asset prices: A dynamic equilibrium model. *The Review of Financial Studies*, 11(1):1–58.