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Dynamic Asset Allocation and Consumption under Time-inconsistent Preferences

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Abstract

Experimental evidence suggests that time preferences play an important role in intertemporal decisions. In a stochastic economy with two fundamental sources of risk, using the martingale method, I present closed-form solutions for optimal consumption and investment for a CRRA agent under any time preference. I introduce a generic discount function, with exponential and quasi-hyperbolic discounting as well-known special forms of time-consistent and time-inconsistent behaviour, respectively. Time-inconsistent preferences increase the consumption rate due to a desire for immediate gratification. There is no effect on the proportions of wealth invested in each asset, but invested monetary amounts are lower for time-inconsistent individuals. I identify similarities and differences with a deterministic economy. Special attention is devoted to a pension context.

1 Introduction

"Therefore, take me and bind me to the crosspiece half way up the mast; bind me as I stand upright, with a bond so fast that I cannot possibly break away, and lash the rope's ends to the mast itself. If I beg and pray you to set me free, then bind me more tightly still." — *The Odyssey, Homer.*

Figure 1: The Sirens and Odysseus, by *William Etty, 1837.*

How do optimal consumption and investment depend on individual time preferences in a financial market with two sources of risk? Alternatively, how do delays in consumption and the valuation of the future affect such decisions? Despite the simplicity of this issue, there is still no framework for analysing it. Namely, if time preferences may change throughout time, then we are required to step down from the rational intertemporal choice theory. This paper provides answers to these questions within a dynamic context based on Merton's (1969) continuous-time formulation of the intertemporal investment problem, where the dynamics of the term structure of interest rates are governed by a Vasicek (1977) one-factor model. Besides, I allow a general discount function that captures behavioural findings and, as such, allows for empirically observed discount structures. Overall, I find that time preferences affect the consumption rate and the monetary amounts invested, but the proportions of invested wealth remain unaffected.

Since its introduction by Samuelson in 1937, the exponential discounted-utility (DU) model has dominated economic and financial analyses of rational intertemporal choice, although Samuelson warned us about the normative and descriptive validity of the formulation. Over the last two decades researchers analysing empirically the behaviour of individuals have uncovered a wide range of phenomena that violate the assumptions of the DU model and, thereby, justified Samuelson's reservations. The key assumption of the DU model is that all of the motives underlying intertemporal choice can be condensed into a single parameter, the discount rate, which is moreover assumed to be independent of time. The best documented DU anomaly is hyperbolic discounting: empirically observed discount rates depend on time and appear to decline over time.

Rational choice theory assumes exponential discounting and yields time-consistent preferences, since any decision remains an optimal choice as time advances, because the discounting is independent of time. However, empirical evidence has been documented in psychology and in behavioural science that time inconsistency is standard in human and animal preferences (see the review article of Frederick et al., 2002, or see e.g. Thaler, 1981; Loewenstein and Prelec, 1992). For example, this year I may desire to start an aggressive savings plan for next year, but when next year actually comes, my taste will be to postpone any sacrifices another year and enjoy consumption now. Hyperbolic discount functions capture such behaviour, since hyperbolic discounting is characterised by a high discount rate in the short-run, while having a low discount rate over longer periods. Hence, this discount structure sets up a conflict between today's preferences and the preferences that the individual will have in the future.

In this paper, I argue that it may be possible to improve our understanding of such behaviour in consumption and investment decisions by refining the way we model time preferences. For guidance as to what kind of refinements might be important, I turn to the experimental evidence that has been accumulated on how people choose among delayed rewards (such as consumption). But, why does time inconsistency matters? Many of the studies in this literature suggest that three findings are usually interpreted as evidence for hyperbolic discounting and, thereby, for time inconsistency.

Firstly, preferences between two delayed rewards can reverse (Thaler, 1981). The driver is the desire for immediate gratification (as in the savings example), also known as a present bias. Secondly, what should a discount function exhibit to capture this reversal? When subjects are asked to compare a smaller-sooner reward to a larger-later reward, the implicit discount rate over longer time horizons is lower than the implicit discount rate of shorter time horizons. Hence, this implies that discount rates should decrease as a function of the time delay (Thaler, 1981). Thirdly, when mathematical functions are explicitly fit to such data, a hyperbolic functional form, which imposes declining discount rates, fits the data better than the exponential functional form, which imposes constant discount rates (Frederick et al., 2002). Overall, the pattern of declining discount rates suggested by many studies in this literature is also evident across many studies (Frederick et al., 2002).

In my analysis, I study two economies where I devote special attention to a pension setting. In the first economy, risks and uncertainties are absent; therefore, this economy is completely deterministic. The individual gets direct utility from consumption and cares only about delays in his consumption. I refer to this as the "deterministic economy". In the second economy, the individual faces risks and has to make his consumption and portfolio decisions under uncertainty. The individual receives direct utility from consumption, and cares about the time-delay of consumption and shocks to his wealth. This economy has two (correlated) fundamental sources of risk, and to ease interpretation, think of these sources of risk as shocks to a debt security and to a portfolio of equity securities. The economy is a combination of the Merton (1969) and Vasicek (1977) models. I refer to this as the "stochastic economy".

In both economies, I study four individuals that annually plan their consumption and investment strategies starting from retirement. I consider three individuals that have the same risk aversion, but differ in their time preferences, whereas the fourth individual differs in risk appetite and time preferences. I call the four types of individuals, respectively: time-consistent, time-inconsistent, present-biased and present-biased risk averse. The time-consistent individual has the (typical) exponential discount function with long-term discount factor δ. The time-inconsistent and present-biased individuals have a hyperbolic discount structure with an additional present-bias factor β. The present-biased individual has a lower β than the time-inconsistent individual and, therefore, puts even less weight on future consumption. The fourth present-biased risk averse individual has the same time preferences as the present-biased individual, but differs in the sense that he is more risk averse.

In my first set of results, I show that the deterministic economy is helpful for understanding time preferences. I formally define time-consistent behaviour and its relationship with exponential discounting. Preferences are said to be time consistent if a person's relative preference for well-being at an earlier date over a later date is the same no matter when he is asked. On the other hand, preferences are time inconsistent if a person's relative preference for well-being at an earlier date over a later date gets stronger as the earlier date gets closer (O'Donoghue and Rabin, 2001). Mathematically, time-consistent preferences are equivalent to exponential discounting, while any discount function other than exponential reflects time-inconsistent preferences.

I give special attention to the case of hyperbolic discounting, implying declining discount rates. But, how to maintain analytical tractability under hyperbolic discounting? To this end, I introduce quasi-hyperbolic discounting as a particularly simple functional form which follows true hyperbolic discounting closely, but maintains the analytical tractability of exponential discounting. With a cake-eating model, I demonstrate that time-inconsistent preferences, as represented by quasi-hyperbolic discounting, lead to a reversal in preferences since actual consumption differs from ex-ante planned consumption.[1] Particularly, I find that hyperbolic discounting leads a person to consume more today than he would like from a prior perspective and, therefore, increases the consumption rate compared to time-consistent preferences (or, equivalently, to under-save). Definitions, propositions and corollaries from the deterministic economy are directly ap-

[1]A cake-eating model considers how quickly and evenly one should eat a cake.

plicable in the stochastic economy. The deterministic economy does, however, not add conceptually new insights to the literature, while the next economy does.

The second set of results presents the optimal consumption decision and the optimal investment strategy for a retiree in the stochastic economy. The solution technique I use, is the martingale method of Cox and Huang (1989). I allow individuals to exhibit several forms of time discounting by means of a general discount factor. As special cases, I again consider exponential discounting and quasi-hyperbolic discounting as representations of time-consistent and time-inconsistent behaviours, respectively. Quasi-hyperbolic discounters prefer to consume more now, since they prefer immediate gratification due to the present-biased behaviour. As a result, remaining wealth levels are lower and consumption is less smoothed. Overall, this yields a higher consumption rate than under time-consistent behaviour. Specifically, individuals plan consumption paths such that these are proportional to their discount function, the bond prices in the financial market and a third correction term (specifically, the variance of the pricing kernel).

In general, individuals in this economy invest in stocks, bonds and cash through hedging and speculative demands. Remarkably, time preferences have no effect on the proportions of wealth invested in these three assets. However, I find that the monetary amounts of invested wealth are lower for quasi-hyperbolic discounters compared to time-consistent behaviour. Namely, due to the present-biased behaviour of quasi-hyperbolic discounters, remaining wealth levels are lower such that less wealth can be invested. The present-biased risk averse agent shows a particular effect: he prefers to smooth consumption over his desire for immediate current consumption and only invests according to his hedging demand.

To understand where these results come from, consider first the case of exponential discounting. The exponential discount function assumes that all motives underlying intertemporal choices can be condensed into a single parameter δ — the long-run discount rate. Namely, the individual's preferences will be the same no matter when he is asked. However, there is "no reason why an individual should have such a special discount function" (Strotz, 2016), except for its simplicity and its resemblance to the familiar compound interest formula (Frederick et al., 2002), but not as a result of empirical research demonstrating its validity. Quasi-hyperbolic discounting relaxes this assumption and introduces an additional short-term parameter β, also known as the present-bias factor. Many of the consumption and investment effects derive from this single additional parameter. Namely, a declining discount rate that induces the individual to value the present relatively more than the future.

Research with respect to the optimal asset allocation and optimal consumption decision under time-inconsistent preferences is scarce. However, my paper is not the only paper to address these issues. The paper of Zou et al. (2014) is one of the few closest to mine, and they find that the consumption rate increases under stochastic hyperbolic discounting, whereby they obtain closed-form expressions for log-utility individuals by using a dynamic programming approach. However, they consider only one fundamental source of risk as in the Merton (1969) model.

I contribute in four ways to the behavioural theoretical finance literature. Firstly, I extend the Merton (1969) and Zou et al. (2014) framework by allowing for a second source of risk through the interest rates by a Vasicek (1977) one-factor model. This extension introduces hedging demands in the economy through the availability of bonds. Secondly, I am the first to use the martingale method for deriving optimality conditions in a framework of time-inconsistent preferences — as opposed to the dynamic programming approach. Thirdly, I am able to find closed-form expressions for the optimal asset allocation and consumption strategy under CRRA-utility preferences for quasi-hyperbolic discounting, incorporating log-utility preferences as a special case. Fourthly, I completely generalise the closed-form solutions by allowing for a general discount function, having exponential and hyperbolic discounting as special cases.

My paper suggest that using experimental evidence to refine the way we model investor preferences may be a promising avenue for further research to understand empirically observed behaviour. Besides the best documented DU anomaly of hyperbolic discounting, there are DU anomalies that relate to mental accounting and the size of gains or losses. In response to these various anomalies, there is also a variety of theoretical models under development besides the well-known hyperbolic discount structure such as self-awareness, enriched utility functions and complete departures from the DU model. Another avenue for further research are the welfare implications for the individual and the society as a result of time-inconsistent preferences. Finally, my analysis has also policy implications, specifically for the pension and insurance industry. Namely, pension participants might benefit from ex-ante commitment to consumption and investment plans, and should be guided when given freedom of choice.[2]

The rest of the paper is organised as follows. In Section 2, within the deterministic economy, I state the definitions, propositions and corollaries regarding time-consistency and discount functions. Section 3 derives the optimal asset allocation and optimal consumption decision in the stochastic economy under power-utility preferences for a general discount function, incorporating exponential and quasi-hyperbolic discounting as special cases. Section 4 concludes and gives possibilities for further research as well as policy implications.

[2]The painting in Figure 1 illustrates this idea by depicting the scene from Homer's Odyssey in which Odysseus resists the bewitching song of the Sirens by having himself ex-ante tied up to the ship's mast.

2 Deterministic economy

I start with a deterministic economy with no risks and with no uncertainties. I discuss three main building blocks that are helpful in understanding the stochastic economy. Firstly, I define time consistency and I proof its relation with exponential discounting by considering the DU model. Secondly, I introduce the concept of quasi-hyperbolic discounting and I visualise its structure with respect to exponential and hyperbolic discounting. Finally, I conclude with a cake-eating model to illustrate the basic insights behind time-inconsistent behaviour, resulting from the additional present-bias factor β. Notably, I find that actual consumption differs from ex-ante planned consumption and time-inconsistent preferences yield less smoothed consumption paths. Definitions, propositions and the key insights are directly applicable in the stochastic economy.

To formalise one and another, I define $C^0 = (c_0, ..., c_T)$ as a consumption path starting at current time zero with consumption c_t in every period t until some terminal time T. Moreover, the utility of getting c_t at time t as perceived at time $s \leq t$ is $\phi_t(s)u(c_t)$. Here $u(.)$ is a concave utility function and $\phi_t(s)$ is a discount function. Then, in its most restrictive form, the (discrete) DU model of Samuelson states that a consumption path C^s is preferred over an alternative consumption path $C^{s'}$ at current time s if and only if

$$V(\phi_t(s), C^s) = \sum_{t=s}^{T} \phi_t(s)u(c_t) > \sum_{t=s}^{T} \phi_t(s)u(c_t') = V(\phi_t(s), C^{s'}),$$

where the discount function is exponential such that $\phi_t(s) = \delta^{t-s}$. Before proceeding, it is helpful to have a definition of time consistency and, therefore, I state the following along the lines of Frederick et al. (2002).

Definition 1. *Preferences are time consistent if* $V(\phi_t(s), C^s) \geq V(\phi_t(s), C^{s'})$ *if and only if* $V(\phi_t(s+1), C^{s+1}) \geq V(\phi_t(s+1), C^{s+1'})$.

The definition states that if the value of consuming the stream $C^{s'}$ is inferior to consuming stream C^s at time s, then the same should be true when time advances to $s + 1$. Equivalently, preferences are time consistent if and only if the trade-off between utility is the same when evaluated at different times. In other words, preferences are time consistent if a person's preference for well-being at an earlier date over a later date is the same no matter when (s)he is asked.

Any other preference pattern is, thus, defined as time inconsistent. For example, typical time-inconsistent preferences have a short-term bias such that intertemporal preferences are time inconsistent if the preferences of the selve at time s $\phi_s(s)u(c_s) \geq \phi_{s+1}(s)u(c'_{s+1})$ and $\phi_{t+s}(s)u(c_{t+s}) \leq \phi_{t+s+1}(s)u(c'_{t+s+1})$ imply that the selve at time $t+s$ prefers $\phi_{t+s}(t+s)u(c_{t+s}) \geq \phi_{t+s+1}(t+s)u(c'_{t+s+1})$ for any $t > s \geq 0$. In words, planned consumption at time s for future dates $t+s$ and $t+s+1$ differs from the actual preferred consumption at date $t + s$. Equivalently, reversal of preferences occurs such that immediate consumption is preferred, while delayed consumption is preferred when all choices are deferred by t. Alternatively, intertemporal preferences are time inconsistent if a person's relative preference for well-being at an earlier date over a later date gets stronger as the earlier date gets closer.

The best documented DU anomaly is time-inconsistent preferences, and the most adapted theoretical model is hyperbolic discounting. The term hyperbolic discounting often means that a person has a declining instantaneous rate of time preference, a formalisation I also adopt. But, why does time inconsistency matters? Three results are usually interpreted as evidence for time-inconsistent preferences and, thereby, for hyperbolic discounting (Frederick et al., 2002).

First of all, research shows that preferences between delayed consumption can reverse. The underlying discount factor is key and leads to the famous simple example of preference reversal by Thaler (1981). The reversal may be demonstrated as follows and, intuitively, shows the idea behind Definition 1:

(A)	Choose between:	(A.1)	One apple today
		(A.2)	Two apples tomorrow
(B)	Choose between:	(B.1)	One apple in 50 days
		(B.2)	Two apples in 51 days

People tend to prefer (A.1) over (A.2), but at the same time prefer (B.2) over (B.1). However, if the discount function is exponential, then the choices (A) and (B) are formally identical. Time inconsistency arises if (B.2) is preferred now, but when the choice is reconsidered in 49 days (B.1) is preferred. Note that the time delay of 49 days is the key driver of time inconsistency. I formally capture this above by using the general discount function and a utility function.

Another well-known experiment, described by Kahneman (2011), demonstrates preference reversal through a short-term bias: 63% of researched students preferred to receive \$3400 this month rather than \$3800 next month. The reversal in preferences yields evidence for hyperbolic discounting behaviour, since the preferences are time-inconsistent. Individuals with time-consistent preferences would never reverse their intertemporal choices.

Secondly, the reversal in preferences implies that discount rates should depend on the time delay. One may wonder when preferences over consumption are time consistent with respect to the underlying discount function. Alternatively, the main question is: what should the discount rate exhibit to capture time-inconsistency? To provide formally an answer, consider an individual at time s who is indifferent between adding c' units (one apple) to consumption at future time $t > s$ and $c > c'$ units (two apples) at an even later time T:

$$u(x+c')\phi_t(s) + u(x)_T\phi(s) = u(x)\phi_t(s) + u(x+c)\phi_T(s),$$

given a constant baseline consumption level x in all time periods. Dividing by $\phi_t(s)$ yields

$$u(x+c') - u(x) = (u(x+c) - u(x))\frac{\phi_T(s)}{\phi_t(s)},$$

which shows that the preference between the two consumption options depends only on the time (delay) interval $\tau = T - t$ separating them. For example, under exponential discounting, if $\phi_t(s) = \delta^{t-s}$, then $\frac{\phi_T(s)}{\phi_t(s)} = \delta^{T-t} = \delta^{\tau}$. This is the stationarity property underlying the exponential utility framework, which plays a critical role in the DU model (Loewenstein and Prelec, 1992). Any other discount function represents time-inconsistent preferences, since the ratio of discount functions (depending on the time delay) will not be a constant anymore.

However, when individuals are asked to compare a smaller-sooner reward to a larger-later reward, the implicit discount rate over longer time horizons is lower than the implicit discount rate over shorter time horizons. The same study of Thaler (1981) asked subjects to specify an amount of money they would require over various horizons to make them indifferent to receiving money now. He states that "the results imply that the subjects have a discount function which is non-exponential..., and ... to allow for the possibility that individual discount rates ... tend to vary with the length of the delay". Psychologists and economists — for example Loewenstein and Prelec (1992) and Laibson (1997) — have argued that such declining discount rates play an important role in representing human behaviour and in generating problems of self-regulation. So, discount rates should change depending on the length of delay and, more specifically, they should decrease as a function of the time delay.

Thirdly, when mathematical functions are explicitly fit to such data, a hyperbolic functional form, which imposes declining discount rates, fits the data better than the exponential functional form, imposing constant discount rates (Frederick et al., 2002). Therefore, the term hyperbolic discounting is used, since discount functions appear to be (mathematically) of a hyperbolic curvature instead of a exponential and, thereby, reflect the idea that a person has a declining rate of time preference. Furthermore, the pattern of declining discount rates suggested by many studies in this literature is also evident across these studies (Frederick et al., 2002).

To formalise these observations, I present the following proposition, which is proven via an axiomatic way (along the lines of Loewenstein and Prelec, 1992) and follows the intuition of the apple example.

Proposition 1. *Preferences are time consistent if and only if the discount function is* *exponential, implying that the instantaneous discount rate is independent of time t.*

Proof. See Appendix A.1. □

The proof is formal, but the intuition follows the apple example of Thaler (1981). I consider an individual who is indifferent between receiving one apple today and two apples tomorrow. Then, using the empirical evidence on the reversal in preferences, an individual will prefer two apples if both choices are postponed by a delay of, say, 50 days. In order to maintain indifference under a time delay of 50 days, I show that the discount function must be a generalised hyperbola, capturing exponential and hyperbolic discounting. Finally, I compute the exponential instantaneous discount rate, which turns out to be independent of time.

The proposition shows that exponential discount functions imply constant discount rates which induce time-consistent preferences. The exponential discount function

$$\phi_t(s) = \delta^{t-s}$$

is characterised by a constant discount rate

$$\rho = -\log(1/\delta),$$

which is independent of time.[3] By assuming that the discount function is exponential, the framework of Samuelson (1937) assumes a constant per-period discount rate $\rho_t(s) = \rho \ \forall t$, which violates the empirical evidence.

On the other hand, the proposition shows that hyperbolic discount functions imply discount rates that decrease as a function of the time delay and, therefore, may represent time-inconsistent behaviour. Namely, the discount structure is non-exponential and the discount rate is time-delay dependent. The hyperbolic discount function

$$\phi_t(s) = (1 + \alpha(t-s))^{-\beta/\alpha}, \quad \alpha, \beta > 0, \tag{1}$$

is characterised by a declining instantaneous discount rate as the time delay $\tau = t - s$ increases:

$$\rho_t(s) = \frac{\beta}{1 + \alpha(t-s)},$$

which depends on time. Note that the hyperbolic discount function captures time-consistent discounting if $\alpha \to 0$, and as such exponential discounting is a special case of hyperbolic discounting. α determines how far hyperbolic discounting deviates from exponential discounting. Since the discount rate is decreasing in the time delay τ, it captures

[3]Note that I denote the natural logarithm with log.

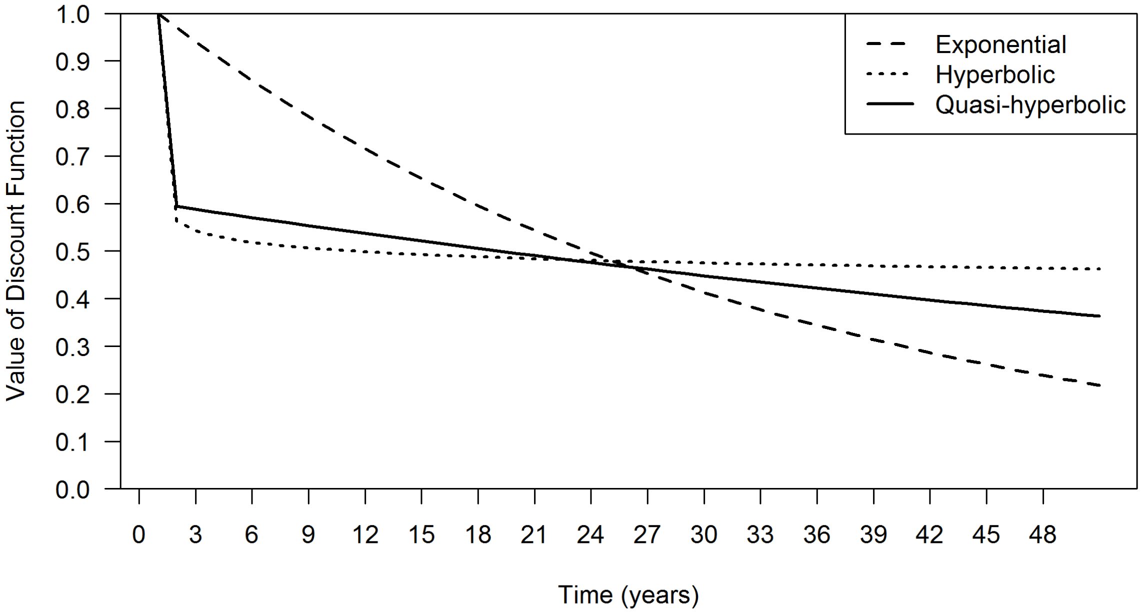

Figure 2: **Three discount functions for a period of 50 years.** This figure graphs the exponential discount function (assuming that $\delta = 0.97$), the generalised hyperbolic discount function (assuming that $\alpha = 10^5$ and $\beta = 5 \cdot 10^3$) and the quasi-hyperbolic discount function (assuming that $\beta = 0.6$ and $\delta = 0.99$). The points of the (annual) discrete-time quasi hyperbolic function have been connected to generate its curve.

decreasing impatience. In other words, time-inconsistent discounting yields non-constant discount rates which decrease as the discounted event is moved further away. Events in the near future are discounted at a higher implicit discount rate than events in the distant future. These properties match with the empirical evidence.

Figure 2 shows graphically the difference between exponential and hyperbolic discounting. Exponential discounting yields a constant decrease in the value of the discount function, whereas the value of the hyperbolic discount function depends strongly on the time-delay. The hyperbolic curve demonstrates that valuation of consumption falls relatively rapidly during the early years, but then falls more slowly for longer delayed years. In sum, the common difference effect reveals that the exponential utility framework is unable to explain time-inconsistent preferences and, moreover, that time-inconsistent preferences may be represented mathematically by hyperbolic discounting.

2.1 Quasi-hyperbolic discounting

But, how to maintain analytical tractability under the hyperbolic discount structure? To this end, I introduce a quasi-hyperbolic discount structure that mimics the properties of the generalised hyperbolic discount function as specified in (1), but maintains the analytical tractability of the exponential discount function. Figure 2 shows that quasi-hyperbolic discounting follows true hyperbolic discounting closely, but differs from (constant) exponential discounting. Following Laibson (1997), I formally define quasi-hyperbolic discounting as follows.

Definition 2. *A discount function $\phi_t(s)$ represents quasi-hyperbolic time discounting if*

$$\begin{aligned} \phi_t(s) &= 1 \text{ if } t = s \\ \phi_t(s) &= \beta\delta^{t-s} \text{ if } t > s, \end{aligned} \tag{2}$$

with $\beta \in (0,1)$ and $\delta \in (0,1]$.

Quasi-hyperbolic discounting has two determining parameters. The parameter δ is the standard discount factor and captures the long-run, time-consistent impatience. The parameter β is called the present-bias factor and represents a time-inconsistent preference for immediate gratification (i.e. short-term impatience). For this reason, quasi-hyperbolic time preferences are also sometimes referred to as "beta-delta" preferences. The condition $\phi_t(s) = 1$ states that rewards taken at the present time are not discounted. The quasi-hyperbolic discount structure compares future periods with each other using exponential discounting (the terms δ^{t-s}), but hits all future periods with an additional β. Exponential discounting (in discrete time) is nested as a special form when I allow $\beta = 1$. Using Proposition 1 and Definition 2, I directly obtain the following corollary.[4]

Corollary 1. *Preferences are time-inconsistent if the discount function is quasi-hyperbolic such that $\beta \in (0,1)$, while preferences are time-consistent if the discount function is exponential such that $\beta = 1$.*

The quasi-hyperbolic discount function in equation (2) captures the key qualitative properties of discounting with true hyperbolas as in equation (1), while retaining much of the analytical tractability of exponential discounting. From today's perspective, the discount rate between two long-run periods t and $t+1$ is the long-term (low) discount rate. However, when the individual arrives at time t, the discount rate between t and $t+1$ is the short-term (high) discount rate. So, quasi-hyperbolic discount functions imply discount rates that decrease as the discounted event is moved further away in time.

2.2 Cake-eating model

I conclude with a toy model to illustrate the main insights of time-inconsistent behaviour.[5] For example, imagine that you have a cake (that never decays) and you need to decide

[4]Note that for the quasi-hyperbolic preferences $U_t = u(c_t) + \beta \sum_{\tau=1}^{T-t} \delta^\tau u(c_{t+\tau})$ indeed the marginal rates of substitution between periods $t+1$ and $t+2$ from the perspective of the agent at time t and time $t+1$ differ, demonstrating the time-inconsistent reversal in preferences.

[5]The conceptual idea of the toy model stems from Zimmerman (2010).

how to optimally spread this consumption over three periods. I call this a cake-eating problem. In this setting, I derive planned and actual consumption levels in a three period model under time-inconsistent preferences, represented by quasi-hyperbolic discounting. I find that planned and actual consumption levels differ. Specifically, a young individual that planned consumption for his future, is unable to stick to his plan and prefers another (present-biased) actual consumption path when he arrives at the future. Consequently, the consumption rate under time-inconsistent preferences is larger than under time-consistent preferences, because the individual prefers immediate consumption over delayed consumption.

Formally, consider an agent with CRRA preferences and a concave utility function of the form

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where γ is the index of constant relative risk aversion (CRRA), and $\gamma = 1$ represents logarithmic preferences of the form $u(c) = \ln(c)$. Since this is a deterministic economy with no risks, γ should be interpreted as a preference for consumption smoothing and, therefore, should be seen as the marginal rate of substitution. Quasi-hyperbolic utility over consumption c_t at time t takes the following form in a three period model

$$U_t = u(c_t) + \beta\delta u(c_{t+1}) + \beta\delta^2 u(c_{t+2}), \tag{3}$$

where the three consecutive periods can be thought of as being young, middle-aged and retired. The budget constraint for each period t states that wages w_t and savings must add up to consumption:

$$\begin{aligned} w_t &= c_t + s_t \\ w_{t+1} + s_t &= c_{t+1} + s_{t+1} \\ w_{t+2} + s_{t+1} &= c_{t+2}, \end{aligned}$$

where s_t are the savings in period t and I assume that $w_{t+2} = 0$, which implies that there is no labour income during the retirement phase such that only savings are consumed. Then, the lifetime budget constraint becomes

$$W_t = w_t + w_{t+1} \geq c_t + c_{t+1} + c_{t+2}. \tag{4}$$

This states that consumption over the lifetime can never be bigger than the accumulated wealth over the lifetime — in optimum, equality will hold.

Planned consumption when young

At date t, the young individual maximises consumption over lifetime utility subject to the lifetime budget constraint

$$\max_{c_t, c_{t+1}, c_{t+2}} U_t = u(c_t) + \beta\delta u(c_{t+1}) + \beta\delta^2 u(c_{t+2})$$
$$\text{s.t. } W_t \geq c_t + c_{t+1} + c_{t+2}.$$

I set up the Lagrangian function to find optimal consumption levels:

$$L = u(c_t) + \beta\delta u(c_{t+1}) + \beta\delta^2 u(c_{t+2}) + \eta\left(w_t + w_{t+1} - c_t - c_{t+1} - c_{t+2}\right),$$

here η denotes the Lagrange multiplier for the budget constraint. Taking the first order conditions yields

$$\frac{\partial L}{\partial c_t} = u'(c_t) - \eta = 0, \tag{5}$$
$$\frac{\partial L}{\partial c_{t+1}} = \beta\delta u'(c_{t+1}) - \eta = 0, \tag{6}$$
$$\frac{\partial L}{\partial c_{t+2}} = \beta\delta^2 u'(c_{t+2}) - \eta = 0. \tag{7}$$

Note that $u'(c) = c^{-\gamma}$, such that combining equations (5) and (6) yields the optimal consumption level at time t for future period $t+1$

$$c^*_{t+1}(t) = (\beta\delta)^{1/\gamma} c^*_t(t), \tag{8}$$

while combining equations (5) and (7) yields

$$c^*_{t+2}(t) = \left(\beta\delta^2\right)^{1/\gamma} c^*_t(t). \tag{9}$$

Using the lifetime budget constraint, which holds with equality in optimum, I find that optimal current consumption is

$$c^*_t(t) = \frac{W_t}{1 + (\beta\delta)^{1/\gamma} + (\beta\delta^2)^{1/\gamma}}. \tag{10}$$

Finally, by rewriting the first order conditions, I find that the individual at time t plans consumption $c^P_{t+1}(t)$ during middle-age $t+1$ as (with respect to consumption during retirement c_{t+2})

$$c^P_{t+1}(t) = \frac{c_{t+2}}{\delta^{1/\gamma}}. \tag{11}$$

Actual consumption at middle-age

At date $t+1$, the middle-aged individual maximises lifetime utility again

$$\max_{c_{t+1},c_{t+2}} U_{t+1} = u(c_{t+1}) + \beta\delta u(c_{t+2})$$
$$\text{s.t. } W_{t+1} = w_{t+1} + s_t \geq c_{t+1} + c_{t+2}.$$

Setting up the Lagrangian function and taking the partial derivatives, yields the following first order conditions

$$\frac{\partial L}{\partial c_{t+1}} = u'(c_{t+1}) - \eta = 0,$$
$$\frac{\partial L}{\partial c_{t+2}} = \beta\delta u'(c_{t+2}) - \eta\frac{1}{R_{t+2}} = 0.$$

Solving the system and using the (remaining) life-time budget constraint W_{t+1}, I find that optimal current consumption equals

$$c^*_{t+1}(t+1) = \frac{W_{t+1}}{1+(\beta\delta)^{1/\gamma}},$$

whereas consumption during retirement is given by

$$c^*_{t+2}(t+1) = (\beta\delta)^{1/\gamma} c^*_{t+1}(t+1).$$

Finally, note that the actual consumption $c^A_{t+1}(t+1)$ at middle-age (with respect to consumption during c_{t+2}) equals

$$c^A_{t+1}(t+1) = \frac{c_{t+2}}{(\beta\delta)^{1/\gamma}}. \tag{12}$$

Comparing this result with planned consumption $c^P_{t+1}(t)$ in (11) demonstrates perfectly time-inconsistent behaviour. Namely, actual consumption $c^A_{t+1}(t+1)$ at middle-age is different from what the individual planned when he was young at date t. Notably, actual consumption is larger than the individual planned ex-ante: $c^A_{t+1}(t+1) > c^P_{t+1}(t)$. Mathematically, this inconsistency stems from the additional short-run discount factor β in expression (12). Intuitively, time-inconsistent individuals have a short-term bias and, therefore, prefer to consume more immediately (at $t+1$) than to delay consumption (for retirement at date $t+2$).

At date $t+2$, there is no optimisation, because the retired individual consumes what is left.

2.2.1 Evaluating four individuals

I consider three individuals that differ in their time-preferences, and a fourth one that differs also in risk appetite. I consider two ways of expressing consumption. On the one hand, I numerically compute planned lifetime consumption levels c^* (as defined above) for the young individual, while on the other hand I calculate the consumption rate over the individual's lifetime. By definition, the consumption rate per period equals current consumption divided by remaining total wealth.

I call the three types of individuals that differ in time preferences: time-consistent, time-inconsistent and present-biased. The time-consistent individual has the (typical) exponential discount function with long-term discount factor $\delta = 0.97$. I take this value from the review article of Frederick et al. (2002), since they find it across many studies for a time-range of at least 1 year up to 57 years. Note that exponential discounting is a special case of hyperbolic discounting when $\beta = 1$. The time-inconsistent individual has a quasi-hyperbolic discount function with present-bias factor $\beta = 0.6$ and long-run discount factor $\delta = 0.82$. These parameter values come from the large international survey of Wang et al. (2016) and correspond to the median time-preferences of their 53 researched countries. The present-biased individual also exhibits quasi-hyperbolic discounting, but with $\beta = 0.05$ and $\delta = 0.77$, where the values correspond to the highly present-biased (East-European) countries in the study of Wang et al. (2016). These three individuals have a (typical) risk aversion level of $\gamma = 2$, which is in line with e.g. Campbell and Cochrane (1999). Table 1 summarises these three individuals. I discuss the fourth individual below, but I first discus the behaviour of these three individuals.

Figure 3 illustrates planned lifetime-consumption paths following from $c_t^*(t), c_{t+1}^*(t)$ and $c_{t+2}^*(t)$ as given in equations (8)-(10) respectively. I set total wealth $W_t = 1$, but it may freely be scaled to any desirable number. The young individual plans his lifetime consumption at date $t = 1$. The time-consistent individual shows clear consumption smoothing and consumes (the cake) constantly with a fraction of approximately 1/3 per period. The time-inconsistent individual plans his consumption path closely to the time-consistent individual, but prefers to consume more at current date $t = 1$. However, the present-biased individual clearly prefers immediate consumption at the expense of lower consumption levels during middle-aged and retirement. Since the young present-biased individual prefers to consume more immediately, he consequently undersaves for future periods and, for this reason, has relatively lower consumption levels during middle-age and retirement. So, the consumption levels show clearly present-biased behaviour under quasi-hyperbolic discounting, while exponential discounting shows clear consumption smoothing.

Figure 4 shows the optimal consumption levels $c_t^*(t), c_{t+1}^*(t+1)$ and $c_{t+2}^*(t+2)$ as a

Table 1: **Four types of individuals**

This table reports the benchmark parameter values for the individual's preferences. *[a] From Campbell and Cochrane (1999); *[b] from Binswanger and Schunk (2012); **[a] from Frederick et al. (2002), and **[b] from Wang et al. (2016).

Parameter	Variable	Value
Time-consistent:		
Long-term discount factor**[a]	δ	0.97
Risk aversion*[a]	γ	2
Time-inconsistent:		
Long-term discount factor**[b]	δ	0.82
Present bias**[b]	β	0.6
Risk aversion*[a]	γ	2
Present-biased:		
Long-term discount factor**[b]	δ	0.77
Present bias**[b]	β	0.05
Risk aversion*[a]	γ	2
Present-biased risk averse:		
Long-term discount factor**[b]	δ	0.77
Present bias**[b]	β	0.05
Risk aversion*[b]	γ	12

fraction of total remaining wealth. I call this the consumption rate

$$\frac{c_t^*(t)}{W_t}.$$

Because quasi-hyperbolic discounters prefer immediate consumption, the current consumption levels are high, such that the remaining wealth levels decrease.[6] Hence, $c_t(t)^*$ is relatively higher under quasi-hyperbolic discounting and, consequently, the associated remaining wealth levels W_t are simultaneously lower. For this reason, quasi-hyperbolic preferences increase the consumption rate compared to time-consistent discounting. The effect is especially pronounced for the present-biased individual, having a consumption rate substantially higher than the other individuals. Again, the present-bias factor β is key to this result: since current consumption levels are higher than delayed consumption levels, the remaining total wealth is lower and, thus, the consumption rate is higher. The lower wealth levels indicate under-savings behaviour. Overall, this simple, yet illustrative, cake-eating model yields intuitive predictions that are identical to the findings of Zou et al. (2014).

The fourth individual differs from the former three types in the sense that addi-

[6] I assume that the individual uses wealth to finance consumption.

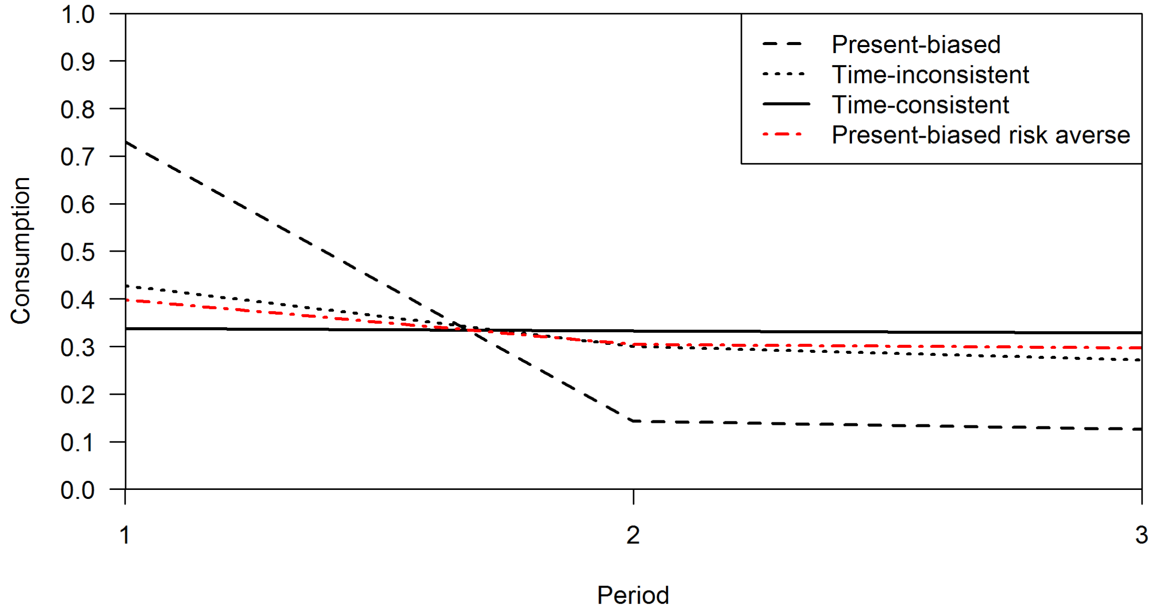

Figure 3: **Planned consumption, when young, over individual's lifetime.** The young individual plans consumption at date $t = 1$. The figure shows planned consumption levels for three individuals having different time-preferences, and a fourth individual having a different risk aversion level as well. The preference-parameters follow from Table 1.

tionally the risk-preference parameter γ varies. I call the fourth a present-biased risk averse individual. This individual is present-biased, but simultaneously also severely risk averse. The present-biased risk averse individual exhibits quasi hyperbolic-discounting with present-bias factor $\beta = 0.05$ and long-term discount factor $\delta = 0.77$, as the aforementioned present-biased individual. But, the risk preference is given by a risk aversion level of $\gamma = 12$ instead of the former $\gamma = 2$. I take this risk appetite from the survey of Binswanger and Schunk (2012) conducted in the U.S. and the Netherlands. Table 1 summarises this individual.

Figures 3 and 4 illustrate the planned consumption path and consumption rate over the lifetime for the present-biased risk averse individual. Because this individual is (more) risk averse (than the other types), he prefers to consume less at the current period — see Figure 3. Instead of consuming for immediate gratification, the individual prefers to save for future times and, thereby, acts less present biased. Consequently, the individual smooths consumption more evenly and plans consumption similar to a time-consistent individual. Figure 4 captures both the effect of less consumption and higher savings (i.e. less depletion of wealth). Namely, the consumption rate over the lifetime for a present-biased risk averse individual is lower than for a present-biased and time-inconsistent

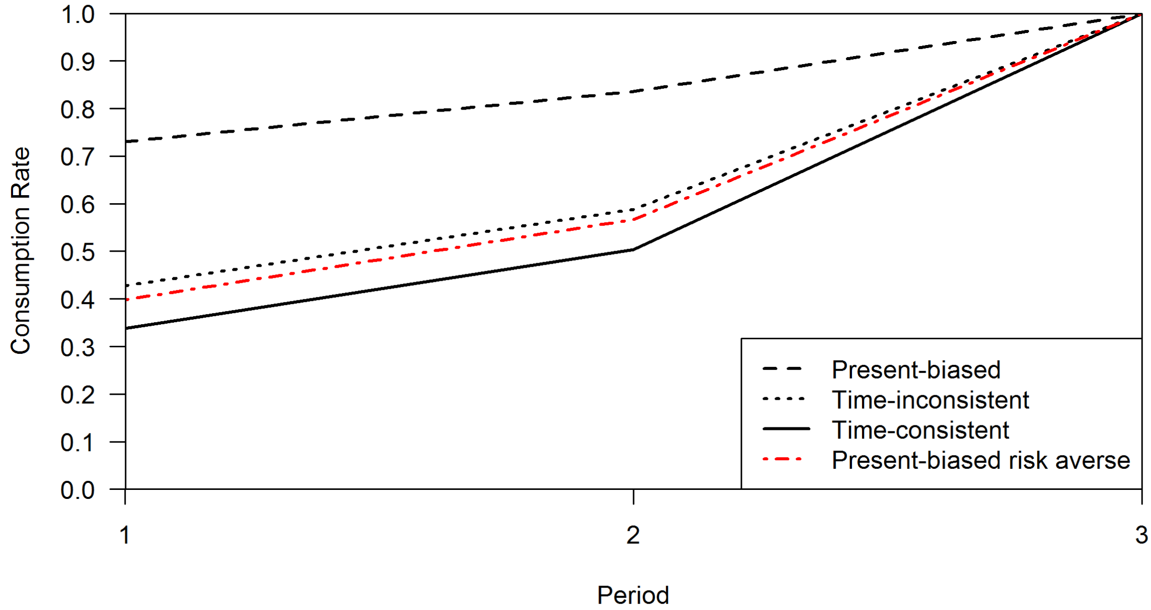

Figure 4: **Consumption rate $\frac{c_t^*(t)}{W_t}$ over individual's lifetime.** This figure shows the consumption rate over the lifetime for three individuals having different time-preferences. The preference-parameters follow from Table 1.

individual, but approaches the time-consistent consumption rate. I want to point out that the opposite effect occurs when an individual is present biased, but risk seeking: a present-biased risk seeking individual prefers extremely high amounts of current consumption (seeking immediate gratification) and, for this reason, has substantially lower wealth levels.

In summary, the widely used DU framework of Samuelson (1937) is only able to capture the special case of time-consistent behaviour, since the discount function is exponential. However, experimental evidence shows that time-inconsistent behaviour is common in human nature. For reasons of tractability, I consider the representation of quasi-hyperbolic discounting, implying declining discount rates that depend on the delay of consumption. A cake-eating model illustrates that quasi-hyperbolic discounters are unable to stick to ex-ante planned consumption, and prefer higher current consumption levels. Such behaviour causes present-biasedness and, therefore, higher consumption rates than under time-consistent preferences. However, if a present-biased individual is risk averse, then the individual smooths consumption more evenly over the lifetime. Definitions, propositions and the key insights from the deterministic economy are directly applicable in the stochastic economy. The set of results in this deterministic economy, however, do not add conceptually new insights to the literature, while the stochastic economy will do.

3 Stochastic economy

I now show how time preferences can be incorporated into a traditional dynamic asset-pricing framework of Merton (1969), whereby the dynamics of the term structure of interest rates are governed by a Vasicek (1977) one-factor model with constant risk premia. I introduce a general discount function that allows any time preference, having exponential and quasi-hyperbolic discounting as special forms. In this stochastic economy with shocks to a debt security and to a portfolio of equity securities, I want to the derive optimal consumption path as well as the optimal portfolio allocation that maximises investor's utility. To this end, I apply the martingale method from Cox and Huang (1989) as solution technique which I augment with an idea from Balter and Werker (2016) and Grebenchtchikova et al. (2017). Specifically, I implement the idea that an investor reserves a pot of wealth for each (retirement) year, which is used for consumption and investment purposes.

3.1 Martingale method

To solve the consumption and investment problems, I use the martingale method as introduced by Cox and Huang (1989). Additionally, I structure my derivations along the lines of Grebenchtchikova et al. (2017). The martingale method translates the investor's dynamic problem into a static final wealth problem, by replacing the dynamic budget constraint with a static budget constraint. In the static problem the investor chooses final wealth directly, whereas in the dynamic formulation final wealth follows from the consumption path and portfolio allocation (Munk, 2012). To use the martingale method, it must be assumed that the financial market is complete and, therefore, there exists a unique stochastic discount factor (SDF) process M_t. Since it will be needed in the derivations, please recall the asset pricing equation, stating that an asset with payoff X_T at terminal time T has price

$$X_t = \frac{1}{M_t} E_t \left[X_T M_T \right]. \tag{13}$$

In what follows, I explain generally the martingale method as solution technique, starting with the static problem that the individual wants to solve. Then, I characterise the financial market with its two fundamental sources of risk, which should be thought of as shocks to a debt security and to a portfolio of equity securities. Finally, I present an expression for the conditional expectation of the stochastic discount factor. These three ingrediënts form the building blocks for the closed-form solutions of the optimal consumption path and optimal investment strategy.

Static problem

The individual wants to choose an optimal consumption path and optimal portfolio allocation under a finite horizon T to maximise expected utility of future wealth. Given initial wealth W_0 and risk aversion level γ, the agent at current time $t = 0$ wishes to maximise final wealth W_T at horizon T:

$$\max E_0 \left[u(W_T) \right],$$

where I assume power utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Under the martingale method, the individual solves the following static problem

$$\begin{aligned} &\max_{W_T} E_0 \left[\frac{W_T^{1-\gamma}}{1-\gamma} \right] \\ &\text{s.t. } E_0 \left[W_T M_T \right] = W_0, \quad M_0 = 1. \end{aligned} \tag{14}$$

Note that the individual directly maximises final wealth W_T and, subsequently, the agent determines the optimal consumption path and portfolio allocation to attain this final wealth. The dynamic programming approach does the reverse by first determining the consumption and investment strategies, leading to the final wealth.

Following a similar approach as in the cake-eating model, I set up the Lagrangian function

$$L = E_0 \left[\frac{W_T^{1-\gamma}}{1-\gamma} \right] + \eta (W_0 - E_0 \left[W_T M T \right]),$$

where η is the Lagrange multiplier for the (static) budget constraint. Taking the first order condition gives

$$\frac{\partial L}{\partial W_T} = W_T^{-\gamma} - \eta M_T = 0,$$

which yields optimal final wealth

$$W_T^* = (\eta M_T)^{-1/\gamma}. \tag{15}$$

Substituting this result in the (static) budget constraint of equation (14) yields

$$E_0 \left[\eta^{-1/\gamma} M_T^{-1/\gamma} M_T \right] = W_0, \tag{16}$$

and, therefore, expresses the Lagrange multiplier as

$$\eta^{-1/\gamma} = \frac{W_0}{E_0\left[M_T^{1-1/\gamma}\right]}.$$

From equations (15) and (16) I find that the optimal final wealth of the investor is given by

$$W_T^* = \frac{W_0}{E_0\left[M_T^{1-1/\gamma}\right]} M_T^{-1/\gamma}. \tag{17}$$

Now, the optimal wealth path W_t^* follows from pricing equation (13) and optimal final wealth W_T^* (17) as

$$W_t^* = \frac{1}{M_t} E_t\left[W_T^* M_T\right] = \frac{W_0}{M_t} \frac{E_t\left[M_T^{1-1/\gamma}\right]}{E_0\left[M_T^{1-1/\gamma}\right]}. \tag{18}$$

Finally, I obtain the optimal indirect utility level, also known as the value function $J(.)$

$$\begin{aligned} J(M_T; W_0, \gamma) &= E_0\left[\frac{(W_T^*)^{1-\gamma}}{1-\gamma}\right] && (19)\\ &= E_0\left[\frac{\left(\frac{W_0}{E_0\left[M_T^{1-1/\gamma}\right]} M_T^{-1/\gamma}\right)^{1-\gamma}}{1-\gamma}\right] \\ &= \frac{W_0^{1-\gamma}}{1-\gamma} \frac{E_0\left[M_T^{1-1/\gamma}\right]}{E_0\left[M_T^{1-1/\gamma}\right]^{1-\gamma}} \\ &= \frac{W_0^{1-\gamma}}{1-\gamma} E_0\left[M_T^{1-1/\gamma}\right]^{\gamma}. && (20) \end{aligned}$$

Financial market

The individual operates in a complete market. In general, markets are complete and the optimal investment strategy can be implemented whenever the investor is able to trade continuously in only a few securities (see Sorensen, 1999). I assume that there are two sources of risk in the financial market, denoted by Z_S and Z_r. For now, to ease interpretation, think of these sources of risk as shocks to a portfolio of equity securities and to a debt security, respectively. I denote the corresponding constant prices of risk (or the Sharpe ratios) as λ_S and λ_r, respectively. I allow for correlation between both securities.

Following the approach of Vasicek (1977), I assume that the instantaneous interest rate dynamics evolve according to an Ornstein-Uhlenbeck process

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r dZ_{r,t}, \tag{21}$$

where the parameter θ describes the long-run mean of the instantaneous interest rate r_t, κ describes the degree of mean reversion, σ_r is the instantaneous interest rate volatility and $Z_{r,t}$ is a standard Brownian motion (also known as Wiener processes).

Stochastic discount factor

Recall the assumption of absence of arbitrage and, thus, the existence of a stochastic discount factor (also known as pricing kernel). Mathematically, the dynamics of the SDF in a financial market with two sources of risk are given by the following stochastic differential equation (SDE)

$$\frac{dM_t}{M_t} = -r_t dt - \lambda_S dZ_{S,t} - \lambda_r dZ_{r,t}, \quad M_0 = 1$$

implying the solution

$$M_t = M_0 \exp\left(-\int_0^t r_u du - \frac{1}{2}(\lambda_S^2 + \lambda_r^2)t - \lambda_S Z_{S,t} - \lambda_r Z_{r,t}\right), \tag{22}$$

which is log-normally distributed. This will be a convenient fact in the upcoming derivations. Equivalently, one can state the solution to the SDE as

$$\log[M_T/M_t] = -\int_t^T r_u du - \frac{1}{2}(\lambda_S^2 + \lambda_r^2)(T-t) - \int_t^T \lambda_S dZ_{S,u} - \int_t^T \lambda_r dZ_{r,u}, \tag{23}$$

which is normally distributed.

Now, the idea is to find an explicit form for the optimal wealth path in equation (18). Please notice that this expression depends on W_0, M_t and $E_t[M_T^\alpha]$ with $\alpha = 1 - 1/\gamma$, only. Besides, W_0 and γ are just parameters, while M_t is known through equation (22). Hence, the only unknown is $E[M_T^\alpha]$. For this reason, I state the following lemma.

Lemma 1. *The conditional expectation at time t of the stochastic discount factor at time $T > t$ for positive α is*

$$E_t\left[M_T^\alpha\right] = M_t^\alpha e^{\frac{1}{2}\alpha(\alpha-1)v^2(T-t)} \left(P(r_t; t, T)\right)^\alpha, \tag{24}$$

where

$$
\begin{aligned}
P(r_t;t,T) &= \exp\left(A(\tau)-B(\tau)r_t\right), \quad \textit{with } \tau = T-t \\
A(\tau) &= (B(\tau)-\tau)R_\infty - \frac{\sigma_r^2}{4\kappa}B(\tau)^2, \quad \textit{with } R_\infty = \theta - \frac{\lambda_r\sigma_r}{\kappa} - \frac{1}{2}\frac{\sigma_r^2}{\kappa^2} \\
B(\tau) &= \frac{1}{\kappa}\left(1-\exp\left(-\kappa\tau\right)\right),
\end{aligned}
$$

and where

$$
v^2(t,T) = \frac{\sigma_r^2}{\kappa^2}\left[\tau - B(\tau) - \frac{\kappa}{2}B(\tau)^2\right] + (\lambda_S^2+\lambda_r^2)\tau + \frac{2\sigma_r}{\kappa}\lambda_r\left[\tau - B(\tau)\right].
$$

Proof. See Appendix A.2. □

At a less formal level, equation (24) follows directly from the properties of a log-normal distribution, by using the expectation and variance over $\int_t^T r_u du$. Here, $P(r_t;t,T)$ is just the price on a zero-coupon bond with maturity T. Moreover, R_∞ is the limit of the term structure as T approaches infinity, such that this term describes the yield to maturity for a very long bond. Besides, $v^2(t,T)$ is the variance of the logarithm of the SDF.

Model calibration

In the next section, I solve for the optimal consumption path and optimal portfolio allocation. Herein, I use numerical examples to study consumption and investment behaviours for the four types of individuals. Construction of the optimised consumption path and the optimised portfolio allocation requires assumptions on the number of available assets, and their risk and return parameters. For this reason, I present in Table 2 my choice of benchmark parameters for the financial market. I divide the table into two panels to separate the two types of processes: the one that determines the dynamics related to the portfolio of equity securities and the other that relates to a debt security (to be discussed below).

3.2 Optimal consumption and optimal investment strategy

I consider a setting where individuals plan consumption and investment during the retirement phase. I assume each individual enjoys a retirement period of 20 years and then dies. However, I abstain from any analysis of the periods before the retirement age T. So, I take investment results, contributions and labour supply as exogenous during the pre-retirement phase. For this reason, it is convenient to take retirement year T as current time 0. Importantly, I assume the individual plans annually his consumption as well

Table 2: **Financial market**

This table reports the benchmark parameter values for the financial market, which I take from Brennan and Xia (2002) (using nominal values).

Parameter	Variable	Value
Risky investment process:		
$dS_t/S_t = (r_t + \lambda_S\sigma_S)dt + \sigma_S\left(\rho dZ_{r,t} + \sqrt{1-\rho^2}dZ_{S,t}\right)$		
Price of risk	λ_S	0.343
Volatility	σ_S	0.158
Correlation coefficient	ρ	-0.129
Interest rate process:		
$dr_t = \kappa(\theta - r_t)dt + \sigma_r dZ_{r,t}$		
Price of risk	λ_r	-0.209
Volatility	σ_r	0.013
Mean reversion speed	κ	0.105
Long-run mean	$\theta = r_0$	0.017

annually his investments. Thus, the individual makes 20 consumption and investment decisions during retirement.

The optimal consumption problem in the retirement phase consists of two independent problems. On the one hand, the individual has to determine the optimal allocation of initial retirement wealth to each of the pension payments and, on the other hand, the individual has to decide on the optimal investment strategy. I still assume that the individual has CCRA preferences over pension payments during the retirement phase, with risk aversion parameter γ and time-preference discount function $\phi_t(s)$, for consumption to be received at date t but perceived at current time s. This section consists of three (independent) parts.

Firstly, I derive the optimal consumption strategy for an individual with generic time-preference function $\phi_t(s)$. As special cases of this general discount function, I again consider four types of individuals: time-consistent (exponential discounting), time-inconsistent, present-biased and present-biased risk averse (quasi-hyperbolic discounting). As in the cake-eating model, I evaluate the consumption behaviours with each other whereby I compute a planned consumption path at retirement date T and the consumption rate over the individual's remaining (retired) lifetime. To derive the optimal consumption strategy, it will come in handy to have an explicit form for the value function $J(.)$. The value function from equation (20) can explicitly be written as, using Lemma 1 with $\alpha = 1 - /\gamma$,

$$
\begin{aligned}
J(M_T; W_0, \gamma) &= E_0 \left[\frac{(W_T^*)^{1-\gamma}}{1-\gamma} \right] \\
&= \frac{W_0^{1-\gamma}}{1-\gamma} E_0 \left[M_T^{1-1/\gamma} \right]^{\gamma} \\
&= \frac{W_0^{1-\gamma}}{1-\gamma} \left(\exp\left(-\frac{1}{2}(\frac{1}{\gamma} - \frac{1}{\gamma^2}) v^2(0,T) \right) (P(r_0; 0, T))^{1-1/\gamma} \right)^{\gamma} \\
&= \frac{W_0^{1-\gamma}}{1-\gamma} \left(\exp\left(-\frac{1}{2}\frac{\gamma - 1}{\gamma} v^2(0,T) \right) (P(r_0; 0, T))^{\gamma-1} \right) \qquad (25)
\end{aligned}
$$

which is again in the form of power utility with respect to initial wealth W_0.

Secondly, I determine the optimal investment strategy for the individual during retirement. The optimal portfolio allocation can be expressed in two ways. One, by decomposing the portfolio into hedging and speculative demands and, second, by decomposing the portfolio in terms of the available assets. Under both expressions I evaluate the investment behaviours for the three individuals that differ in their time preferences. I derive the optimal allocation by constructing a portfolio that replicates the desired path to attain the individual's optimal final wealth.

Finally, I conclude with the fourth individual, who differs in his risk appetite from the former three. I present closed-form optimal consumption and investment strategies.

3.2.1 Optimal consumption

The conceptual idea behind finding the optimal consumption decision stems from Balter and Werker (2016) and Grebenchtchikova et al. (2017). The idea is to allocate the total available retirement wealth to h individual pension payments during the retirement phase, starting at retirement year T.

Consider the situation where the present value of all future pension payments equals the total available pension wealth. In other words, the total available pension wealth can be thought of as the present value of the first pension payment, plus the second pension payment, and so forth. Alternatively, one can split the total pension wealth in reserved amounts needed to finance the first pension payment, and an amount needed for the second payment, and so forth. Intuitively, this provides an allocation and smoothing of consumption over time.

Thus, I start with the allocation of total pension wealth in retirement year T over the pension payments during the retirement years $T, ..., T+h-1$. In other words, the total pension wealth at time T is used to finance the agent's pension payments for the years $T, ..., T+h-1$ for a fixed h. Think of h as the moment of death, or the moment when the fixed annuity stops its payments. I ignore longevity risk.[7] Besides, I assume that all

[7]Idiosyncratic longevity risk can easily be accounted for in my model setup, see Grebenchtchikova

pension payments are used for consumption such that the optimal consumption strategy is proportional to the optimal allocation of pension wealth.

Mathematically, the agent splits available initial pension wealth W_T into h portions that are used to finance retirement consumption in periods $T+j$, where $j = 0, ..., h-1$. Then, the individual solves the following maximisation problem

$$\begin{aligned} &\max_{W_{Tj}, j=0,\ldots,h-1} \sum_{j=0}^{h-1} \phi_{T+j}(T) E_T \left[\frac{(W^*_{T+j})^{1-\gamma}}{1-\gamma} \right] \\ \text{s.t. } &\sum_{j=0}^{h-1} W_{Tj} = W_T, \end{aligned} \tag{26}$$

where W^*_{T+j} denotes optimal achievable wealth at time $T+j$ given initial wealth W_{Tj}. Recall that $\phi_{T+j}(T)$ is a discount function valuing optimal achievable wealth at future date $T+j$ perceived at (current) retirement year T. Remark that the individual lacks any bequest motives. Solving the maximisation problem, leads to the following theorem.

Theorem 1. *The optimal allocation of wealth at date j for a general discount function $\phi_{T+j}(T)$ equals*

$$W^*_{Tj} \propto \phi_{T+j}(T)^{1/\gamma} \exp\left(-\frac{1}{2} \frac{\gamma-1}{\gamma^2} v^2(T, T+j) \right) \left(P(r_T; T, T+j) \right)^{\frac{\gamma-1}{\gamma}}, \quad \forall j = 0, ..., h-1.$$

Proof. See Appendix A.3. □

The theorem shows that the optimal allocation of wealth depends on three terms: the general discount function, the (exponential of the) variance of the SDF and the bond prices in the financial market. More explicitly, Theorem 1 leads directly to the optimised consumption paths for each of the four individuals. The time-consistent individual has an exponential discount function, while the time-inconsistent and present-biased individuals have quasi-hyperbolic discount functions. For this reason, I present two corollaries.

Firstly, if $\phi_{T+j}(T) = e^{-\delta j}$, where δ is the standard (long-run) discount factor, I obtain the following.

Corollary 2. *The optimal allocation of wealth at date j under time-consistent preferences equals*

$$W^*_{Tj} \propto \exp\left(-\frac{\delta}{\gamma} j - \frac{1}{2} \frac{\gamma-1}{\gamma^2} v^2(T, T+j) \right) \left(P(r_T; T, T+j) \right)^{\frac{\gamma-1}{\gamma}}, \quad \forall j = 0, ..., h-1.$$

et al. (2017) for an approach.

The corollary shows that time-consistent individuals plan their consumption with respect to the variance of the SDF and the bond prices, discounted at the constant long-term discount rate δ.

Secondly, when the individual is time-inconsistent or present-biased, quasi-hyperbolic discounting may be used to represent their time preferences. Then, $\phi_{T+j}(T)$ follows from Definition 2, such that the following holds true.

Corollary 3. *The optimal allocation of wealth at date j under time-inconsistent preferences, as represented by quasi-hyperbolic discounting, equals*

$$W_{T0}^* \propto \exp\left(-\frac{1}{2}\frac{\gamma-1}{\gamma^2}v^2(T,T)\right)\left(P(r_T;T,T)\right)^{\frac{\gamma-1}{\gamma}} \quad \textit{and}$$
$$W_{Tj}^* \propto \beta^{1/\gamma}\exp\left(-\delta j-\frac{1}{2}\frac{\gamma-1}{\gamma^2}v^2(T,T+j)\right)\left(P(r_T;T,T+j)\right)^{\frac{\gamma-1}{\gamma}}, \quad \forall j=1,...,h-1,$$

where the proportionality signs are the same.

The corollary shows that time-inconsistent individuals do not discount current consumption (as for exponential discounting at horizon $j=0$), but future periods are discounted by the additional present-bias factor $\beta^{1/\gamma}$.

In response to the theoretical expressions, I visualise the consumption behaviour for each individual. Similar to the cake-eating model, I present planned consumption and consumption rates during the remaining lifetime. As before, I define the consumption rate as current consumption divided by total remaining wealth such that the consumption rate equals

$$\frac{c_t^*(t)}{W_t} \equiv \frac{W_{Tj}^*}{\sum_{i=0}^{h-j-1} W_{Ti}^*}. \tag{27}$$

In words, I divide the pension payment to be received at date j (i.e. the optimal allocation of wealth at date j) by the remaining total amount of optimal pension payments to be received. This formulation expresses the pension payment at date j, used for consumption, as a fraction of the remaining total wealth — the formulation is identical to the cake-eating model. But, planned consumption is now also expressed as a fraction of remaining wealth, because I am working with proportionality conditions. Planned consumption always satisfies the budget constraint and, therefore, sums up to 1. Notice that the expressions may be scaled by any desirable initial wealth.

Figure 5 presents the planned consumption paths over the remaining lifetime for the four individuals. In the figure, they plan their consumption at current retirement date T, corresponding to year 1 in the figure. As time advances one year, the individuals plan consumption again for their remaining $T-1$ years (not shown in the figure) and they do this for each of the remaining years. This figure is the stochastic counterpart

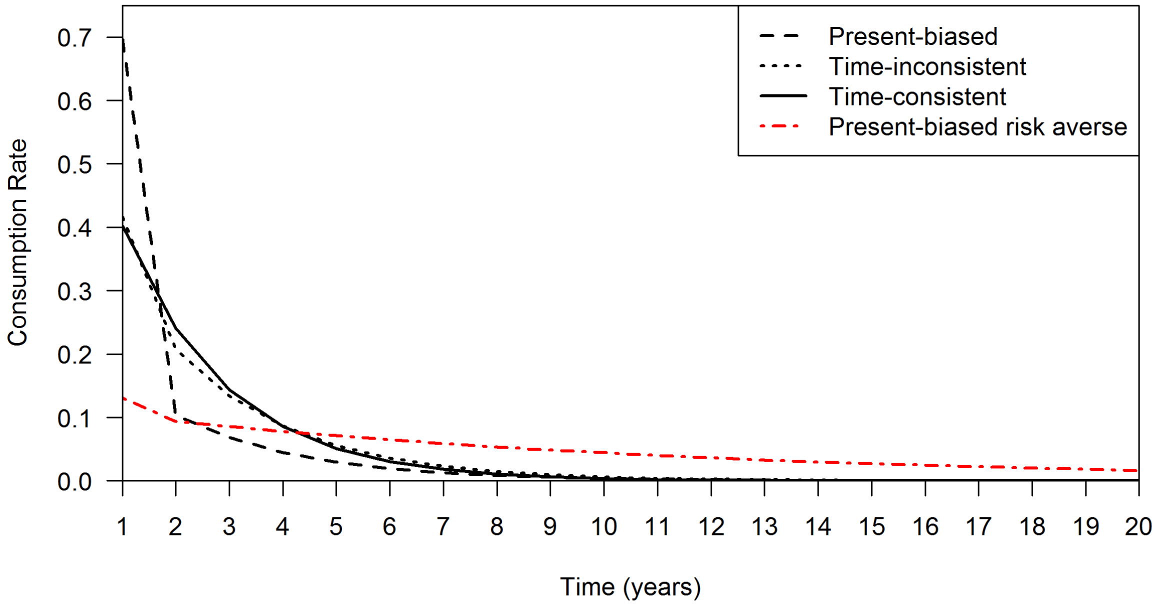

Figure 5: **Planned consumption at retirement.** The figure shows planned initial consumption rates over the lifetime for the four types of individuals. Preferences and financial market circumstances follow from Tables 1 and 2.

of Figure 3 from the cake-eating problem. The actual realisations of consumption are shown in Figure 6 by means of the consumption rate; note that indeed the consumption rates start at the planned consumption levels from year 1 and eventually the consumption rate equals 1 at year 20, such that all wealth is consumed. This figure is the stochastic counterpart of Figure 4.

With respect to planned consumption in Figure 5, I observe five things. First of all, the paths of the time-inconsistent and present-biased individual cross the time-consistent path, since the quasi-hyperbolic discounters prefer immediate consumption over delayed smoothed consumption. Secondly, the time-consistent path is less flattened than in the cake-eating model. Due to uncertainties in the stochastic economy, the time-consistent individual prefers to consume somewhat more during the first few years, but his path remains more smoothed than the time-inconsistent and present-biased paths since these have clear kinks. Thirdly, consumption is relatively low when the horizon is longer, because any individual consumes what is left when approaching end of life. Namely, anything left at death is meaningless to the investor, since he neglects any bequest motives. Fourthly, the present-biased risk averse agent clearly prefers less consumption at the present, since he is averse to changes in his consumption path and, thereby, saves for the uncertain future. In other words, the substitution effect dominates. For this reason, he smooths his pension payments more evenly and keeps a stable higher consumption path at longer horizons. Finally, the time-consistent and time-inconsistent individuals have planned consumption paths relatively close to each other. This indicates that the exponential and $\beta - \delta$ models, using their empirically frequently observed parameters, capture each other's features closely.

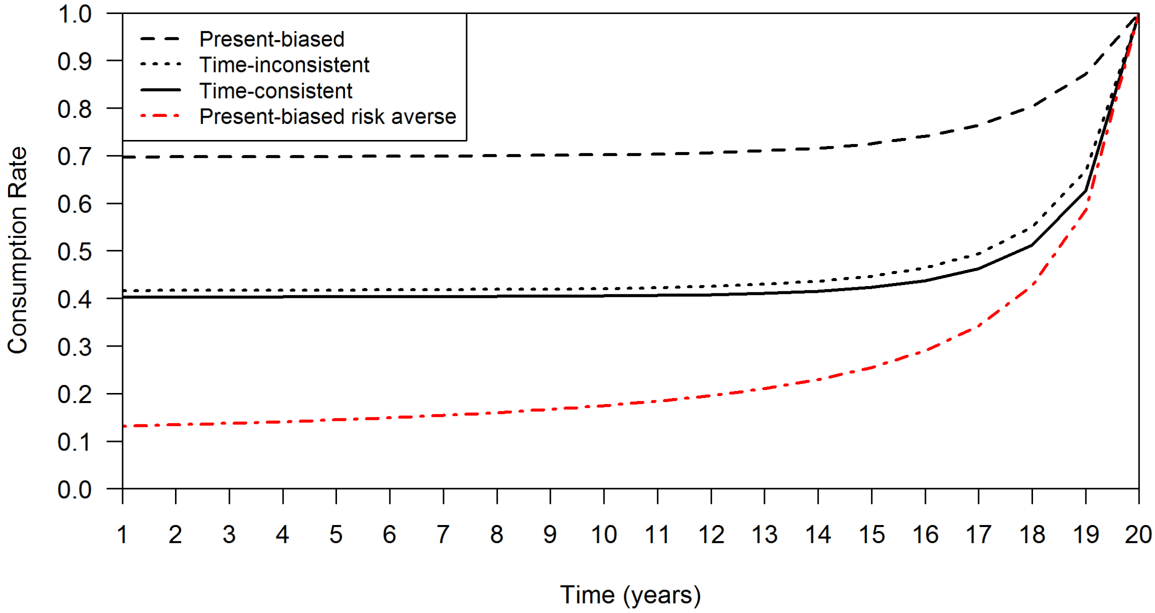

Figure 6: **Consumption rate during retirement.** This figure shows the consumption rates during a retirement period of 20 years for four individuals. Preferences and the financial market follow from Tables 1 and 2.

Figure 6 presents the actual realisations of the 20 planned consumption paths (of which one path is shown in Figure 6) by means of the consumption rates during retirement for the four individuals. I make two observations. The main observation is that time-inconsistent preferences increase the consumption rate (when risk preferences stay equal). Namely, quasi-hyperbolic discounters prefer current consumption over delayed consumption, such that current consumption levels are high while remaining wealth levels are low compared to exponential discounters. The present-biased and time-inconsistent individuals display clearly this behaviour.

The second observation is that a present-biased risk averse individual has a consumption rate lower than the other three individuals. Although the individual is present-biased, the risk aversion dominates. On the one hand, current consumption levels are lower than under quasi-hyperbolic discounting to maintain a smoothed non-kinked consumption path, while at the same time the individual saves more wealth for the uncertain future. Both effects emerge from the desire to prevent any changes in planned consumption paths, thereby fulfilling the desire for a smooth consumption path.

In summary, I derived the optimal allocation of wealth for a general discount function. This optimal allocation of wealth implies directly the optimal consumption strategy for an individual. As special cases, I consider exponential discounting and quasi-hyperbolic discounting. Quasi-hyperbolic discounters have a larger consumption rate than time-consistent discounters, since they prefer immediate gratification such that remaining wealth levels are lower. The reason that the results are different between the special cases of exponential discounting and quasi-hyperbolic discounting is due to the additional short-run parameter β, which is key to many of the time-inconsistent results, since the individual values the present more than the future. For present-biased risk averse individuals, the risk aversion tends to dominate the present bias such that these individuals prefer smooth consumption paths and, thus, retain higher wealth levels.

3.2.2 Optimal investment strategy

I now derive the optimal investment strategy in four steps. The framework is the continuous-time dynamic setting of Merton (1969), where the term structure of interest rates is governed by a Vasicek (1977) type model. In the first step I derive the optimal wealth path that maximises the individual's utility. Secondly, I introduce the three available assets in the financial market. Thirdly, I consider how the individual's wealth evolves when exposed to these three assets. Finally, the optimal portfolio follows from a replication recipe: individual wealth (from step 3) should be exposed such the optimal wealth path (from step 1) is followed, in order to achieve optimal utility.

Step 1: Optimal wealth process

I use the characteristics from the SDF to obtain an explicit form for the optimal final wealth path W_t^*. To this end, using (18) for the optimal wealth path and Lemma 1, I find that

$$\begin{aligned}
W_t^* &= \frac{W_0}{M_t}\frac{E_t\left[M_T^\alpha\right]}{E_0\left[M_T^\alpha\right]} \\
&= \frac{W_0}{M_t}\frac{M_t^\alpha \exp\left(\frac{1}{2}\alpha(\alpha-1)v^2(t,T)\left(P(r_t;t,T)\right)^\alpha\right)}{\exp\left(\frac{1}{2}\alpha(\alpha-1)v^2(0,T)\left(P(r_0;0,T)\right)^\alpha\right)} \\
&= W_0 M_t^{\alpha-1}\exp\left(\frac{1}{2}\alpha(\alpha-1)v^2(t,T)-v^2(0,T)\right) \\
&\times \exp\left(\alpha(A(T-t)-A(T))-\alpha(B(T-t)r_t-B(T)r_0)\right) \\
&= W_0\exp\left(-\int_0^t r_u du-\frac{1}{2}(\lambda_S^2+\lambda_r^2)t-\lambda_S Z_{S,t}-\lambda_r Z_{r,t}\right)^{\alpha-1} \\
&\times \exp\left(\frac{1}{2}\alpha(\alpha-1)v^2(t,T)-v^2(0,T)\right) \\
&\times \exp\left(\alpha(A(T-t)-A(T))-\alpha(B(T-t)r_t-B(T)r_0)\right).
\end{aligned} \tag{28}$$

The diffusion terms in W_t^* come from $-(\alpha-1)(\lambda_S Z_{S,t}+\lambda_r Z_{r,t})$ and from $-\alpha B(T-t)r_t$, while the other terms are deterministic functions of time t or the horizon $T-t$. Therefore, using Itô's lemma and plugging in $\alpha = 1-1/\gamma$ (due to the CRRA assumption), I know that optimal wealth evolves as

$$d\log W_t^* = g_1(r_t, T-t) + \frac{\lambda_S}{\gamma} dZ_{S,t} + \left(\frac{\lambda_r}{\gamma} - (1-1/\gamma)B(T-t)\sigma_r\right) dZ_{r,t}, \qquad (29)$$

where the drift term $g_1(r_t, T-t)$ is a function of the current interest rate r_t, current time t and the remaining investment horizon $T-t$. Clearly, the dynamics of wealth depend on the two sources of fundamental risk in the financial market, denoted by $Z_{S,t}$ and $Z_{r,t}$. The exposure to the interest rate risk consists of a speculative term and a hedging term. The speculative term is characterised by

$$\frac{\lambda_r}{\gamma},$$

while the hedging term is given by

$$(1-1/\gamma)B(T-t)\sigma_r.$$

The speculative term shows how the optimal portfolio choice depends on the prices of risk for the available assets in the financial market. The hedging term describes how the investor hedges changes in the investment opportunity set. Changes in the opportunity set are reflected through changes in the term structures of interest rate.

Step 2: Available assets

By investing in a menu of financial securities, the individual can transfer current wealth into future wealth (such as pension payments). Two sources of risk, Z_S and Z_r, characterise the financial market. Firstly, Z_S leads to a single risky investment opportunity, which is usually referred to as a stock index or a portfolio of (liquid) equity securities. For notational convenience, I just call this available asset a stock. Secondly, the instantaneous interest rate is risk-free and, therefore, the second asset can be referred to as a bank account with interest accrued according to the short interest rate (less formally also called cash). Thirdly, the long-run interest rate is risky due to Z_r. Hence, one can introduce a debt security by a zero-coupon bond paying 1 at maturity $h = 5$ years. Let $P_t(h)$ denote the bond price at time t maturing at time h.

Mathematically, the dynamics of the three assets are

$$\begin{aligned}
\frac{dB_t}{B_t} &= rdt, \quad B_0 = 1 \\
\frac{dS_t}{S_t} &= (r_t + \lambda_S\sigma_S)dt + \sigma_S\left(\rho dZ_{r,t} + \sqrt{1-\rho^2}dZ_{S,t}\right), \quad S_0 = 1 \\
\frac{dP(r_t;t,h)}{P(r_t;t,h)} &= (r_t - \lambda_r\sigma_r B(h-t))dt - \sigma_r B(h-t)dZ_{r,t}, \quad P(r_0;0,h) = 1
\end{aligned} \tag{30}$$

where Z_S and Z_r are standard Brownian motions such that the stock index S_t is assumed to follow a geometric Brownian motion while the bond prices $P(r_t;t,h)$ follow a linear stochastic differential equation.[8,9] The prices of risk λ_S and λ_r are then respectively the prices of risk of the Brownian motions Z_S and Z_r as systematic risk factors. The evolution of the risky investment depends on both shocks in the financial market, and ρ determines the degree of correlation.[10]

To provide intuition for the above abstract financial market, consider the shocks Z_S and Z_r. Recall from Table 2 that the correlation coefficient ρ between stocks and bonds has the familiar negative sign. So, an individual likes positive Z_S shocks, since the stock price increases. This renders higher returns for the investor. On the other hand, an investor dislikes positive Z_r shocks, because the stock price decreases and at the same time the bond price decreases. This yields lower (or even negative) returns when the individual has invested in stocks or bonds.

To be complete, note that the amount of cash at time t in the bank account is given by the solution to its SDE:

$$B_t = B_0 \exp\left(\int_0^t r_s ds\right).$$

Moreover, the price of a of a zero-coupon bond paying 1 at maturity h is directly given by Lemma 1 for $\alpha = 1$, such that the bond price equals

$$P(r_t;t,h) = \exp\left(A(\tau) - B(\tau)r_t\right), \quad \text{with } \tau = h - t.$$

To see that the bond prices follow if $\alpha = 1$, note that I can always price a bond at time t with the familiar pricing equation (13) where the final payoff equals 1. Then, by using Itô's lemma and the expressions for $A(\tau)$ and $B(\tau)$, one can easily find the bond price dynamics as I presented in (30).

[8]Dividends from the risky investment are assumed to be reinvested.

[9]One could also use constant maturity bonds in the analysis, replacing $B(h-t)$ by $B(h)$, such that at each date t new bonds with maturity h are bought.

[10]This process has an interpretation which is sometimes referred to as factor investing or beta pricing: for each unit of risk σ_S, the investor receives a compensation λ_S in terms of the expected return on top of the risk-free rate.

Step 3: Individual wealth process

Consider the return on a portfolio that consists of the three available assets. Let $\pi^*(t) = (\pi_S^*(t), \pi_P^*(t), \pi_r^*(t))$ be the optimal proportion of wealth invested at time t in a stock, a bond with maturity h and in cash. Then, if the individual invests in such a portfolio, his total wealth $A(t)$ evolves according to the following dynamics

$$\frac{dA_t}{A_t} = \left(\pi_S(t)\frac{dS_t}{S_t} + \pi_P(t)\frac{dP_t}{P_t} + (1-\pi_S(t)-\pi_P(t))\frac{dB_t}{B_t}\right), \quad A_0 = 1$$

where $\pi(t)$ denotes the exposures towards the risk factors. Using the dynamics of the assets from equation (30), the individual's wealth dynamics may explicitly be written as

$$\begin{aligned} dA_t &= (r_t + \lambda_S\sigma_S\pi_S - \lambda_r\sigma_r B(h-t)\pi_P)A_t dt \\ &\quad + \sqrt{1-\rho^2}\sigma_S\pi_S A_t dZ_{S,t} - \sigma_r B(h-t)\pi_P A_t dZ_{r,t} + \rho\sigma_S\pi_S A_t dZ_{r,t}. \end{aligned}$$

Using Itô's lemma, I find that

$$d\log A_t = g_2(r_t, h-t)dt + \sqrt{1-\rho^2}\sigma_S\pi_S dZ_{S,t} + (\rho\sigma_S\pi_S - \sigma_r B(h-t)\pi_P)\, dZ_{r,t}, \tag{31}$$

where the drift term $g_2(r_t, h-t)$ is a function of the current short rate r_t and the remaining horizon $h-t$. Clearly, the individual's wealth depends on exposures to both risk factors in the financial market.

Step 4: Optimal portfolio

Finally, I am able to present the optimal asset allocation. Note that (for a single horizon T) the above derivations are independent of the discount function, such that the optimal portfolio weights are the same no matter the type of discounting. Since the optimal asset allocation $\pi^*(t)$ yields the optimal final wealth payoff W_T^*, it follows that the coefficients in equations (29) and (31) must be identical. So, equating these coefficients yields the optimal portfolio allocation. This analysis leads to the following theorem.

Theorem 2. *The optimal portfolio weights at time t for a stock and a bond with maturity h under investment horizon T equals*

$$\begin{aligned} \pi_S^* &= \frac{\lambda_S}{\gamma\sigma_S\sqrt{1-\rho^2}}, \\ \pi_P^*(t) &= \left(1-\frac{1}{\gamma}\right)\frac{B(T-t)}{B(h-t)} - \frac{\lambda_r}{\gamma\sigma_r B(h-t)} + \frac{\rho}{\sqrt{1-\rho^2}}\frac{\lambda_S}{\gamma\sigma_r B(h-t)}, \end{aligned}$$

while the remainder is invested in the risk-free asset

$$\pi_r^*(t) = 1 - \pi_S^* - \pi_B^*(t).$$

The result is identical to the findings of Sorensen (1999), and Brennan and Xia (2002).[11]

The theorem demonstrates that the optimal stock investment weight is independent of time t and horizon T, and equals the classical mean-variance allocation. On the other hand, the optimal bond investment weight depends on the remaining investment horizon $T - t$, and the speculative and hedge demands. The speculative term (depended on the prices of risk) is characterised by

$$\frac{\lambda_S}{\gamma \sigma_S \sqrt{1-\rho^2}} - \frac{\lambda_r}{\gamma \sigma_r B(h-t)} - \frac{\rho \lambda_S}{\gamma \sigma_r B(h-t)},$$

while the hedging term (depended on current time t and horizon T) is given by

$$\left(1 - \frac{1}{\gamma}\right) \frac{B(T-t)}{B(h-t)}.$$

Here, I assume that the individual rebalances each year to the available bond with maturity $h = 5$. Since there are 5-years bonds available in the financial market, the individual buys each year a new 5-year bond and sells his 4-year bond such that the maturity of the bond stays the same in the portfolio.

In response to this analysis, I present the optimal invest strategy. Firstly, I present the optimal portfolio weights throughout retirement, i.e. the optimal proportions of wealth to be invested in each asset. Then, I discuss the optimal invested wealth in terms of hedging and speculative demands. Thirdly, I briefly show the invested amounts in each of the three assets. It is helpful to recall that the optimal portfolio weights are given by $\pi^*(t)$, such that optimal invested wealth is given by

$$V^*(t) = \pi^*(t) W_t, \tag{32}$$

where W_t is total remaining wealth following directly from equation (27). So, the invested wealth follows directly from the chosen consumption path.

The optimal portfolio weights are shown in Figure 7.[12] The proportions of invested wealth throughout retirement are equal under any time preference, since the analysis for a single horizon T is independent of the discount function. The fraction invested in the

[11] Please note that this future date not necessarily equals the investment horizon and, therefore, should not be confused with the investment horizon T as introduced before.

[12] The numerical results are close to those of Brennan and Xia (2002).

stock is constant during retirement, since the allocation is independent of time t and horizon T. However, the allocation to bonds is time- and horizon-dependent. Therefore, the share of wealth in bonds declines as the retiree approaches end of life, being a result of a declining hedging demand. The individuals invest a higher proportion of wealth in bonds than in stocks, which is due to the favourable price of risk λ_r. The individuals borrow money (i.e. go short in cash) to finance their bond position, and to finance partially their stock position. So, time preferences have no effect on the proportion of wealth invested in the available assets, however time preferences do have an effect on the amount of wealth invested — as will be discussed now.

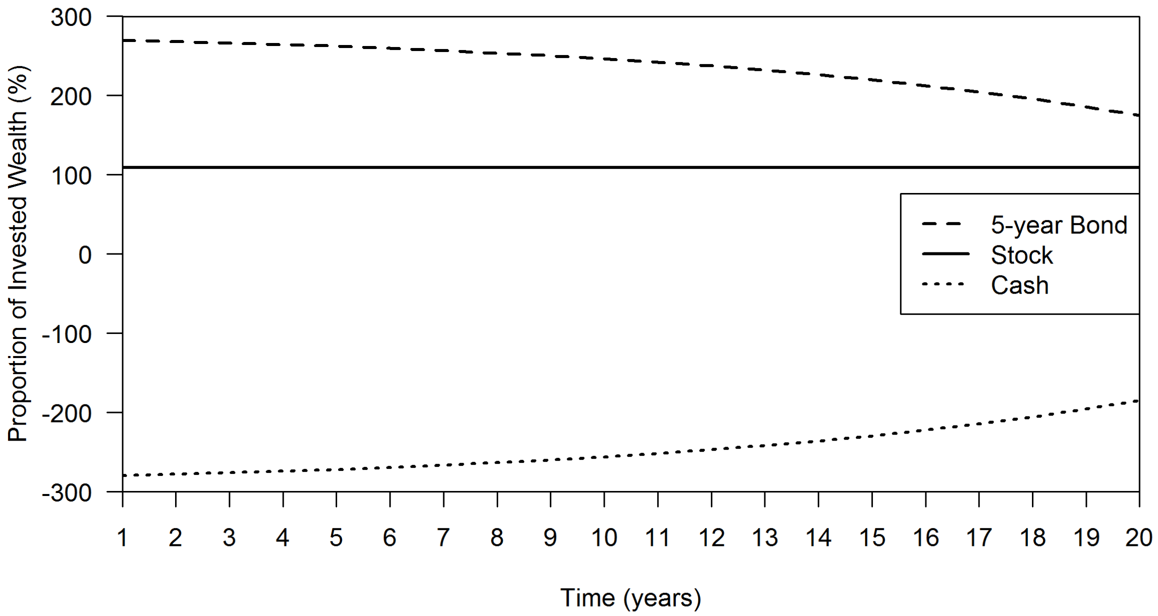

Figure 7: **Optimal portfolio asset allocation weights during retirement (annually revisited).** This figure plots the optimal proportion of invested wealth in terms of the available assets. Preferences and the financial market follow from Tables 1 and 2.

Figure 8 presents the invested wealth in terms of hedging and speculative demands during retirement for three individuals that differ in their time-preferences (the fourth individual is discussed separately). I notice two things. Firstly, the speculative demands are higher than the hedging demands, whereby both demands are decreasing over time. The individuals with risk aversion $\gamma = 2$ prefers to take some risks and, therefore, the speculative demand is higher than the hedging demand. Intuitively, the individuals are a bit risk seeking and have no desire to hedge any risks, but on the other hand prefer to speculate on their returns. The invested wealth is decreasing over the horizon, because total remaining wealth levels are lower towards the end of life.

Second, the time-inconsistent and present-biased individuals invest less than an exponential discounter. The result is especially pronounced for a present-biased individual. To see where this result comes from, reconsider the consumption strategies of the three individuals. The time-inconsistent and present-biased individuals prefer to consume more annually such that they save less at the expense of lower total remaining wealth levels. For this reason, quasi-hyperbolic discounters have less wealth to invest relative to the exponential discounter and, thus, follow a different monetary strategy.

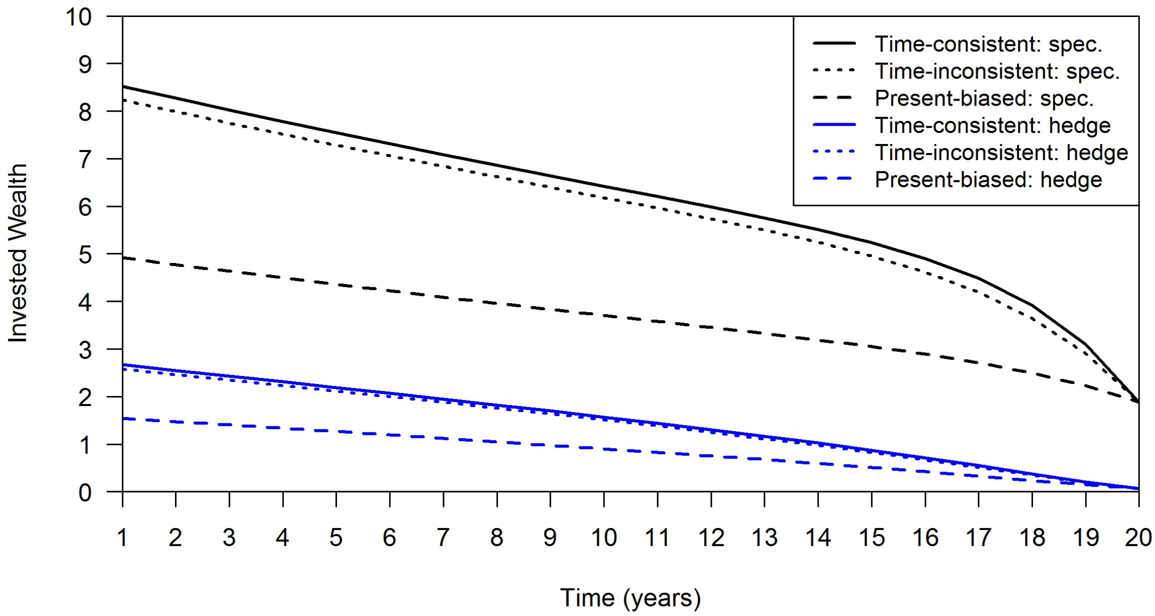

Figure 8: **Optimal portfolio allocation during retirement (annually revisited), expressed as invested remaining wealth, in terms of hedging and speculative demands.** The figure plots the optimal hedging and speculative exposures in terms of invested wealth against the year in retirement. Invested wealth follows from equation (32). The upper three lines correspond to the speculative demands, while the bottom three lines correspond to the hedging demands. Preferences and the financial market follow from Tables 1 and 2.

The hedging and speculative demands can be easily translated to demands for the three assets. Figure 9 presents the allocations for a time-consistent and present-biased individual.[13] I notice three things. Firstly, for the same reason as above, the invested wealth in the assets decreases over time. Note that the wealth allocation to stocks depends on time. Second, in line with Figure 7, both individuals invest more in bonds than in stocks due to the favourable price of risk λ_r. By going short in cash, the individual finances his bond position (and to a lesser extent his stock position). Thirdly, the time-consistent individual allocates more wealth to stocks and bonds relative to the present-

[13]Including the time-inconsistent individual as well blurs the economic insights and, therefore, is left out of the visualisation.

biased individual, since the consumption rate is lower for exponential discounting.

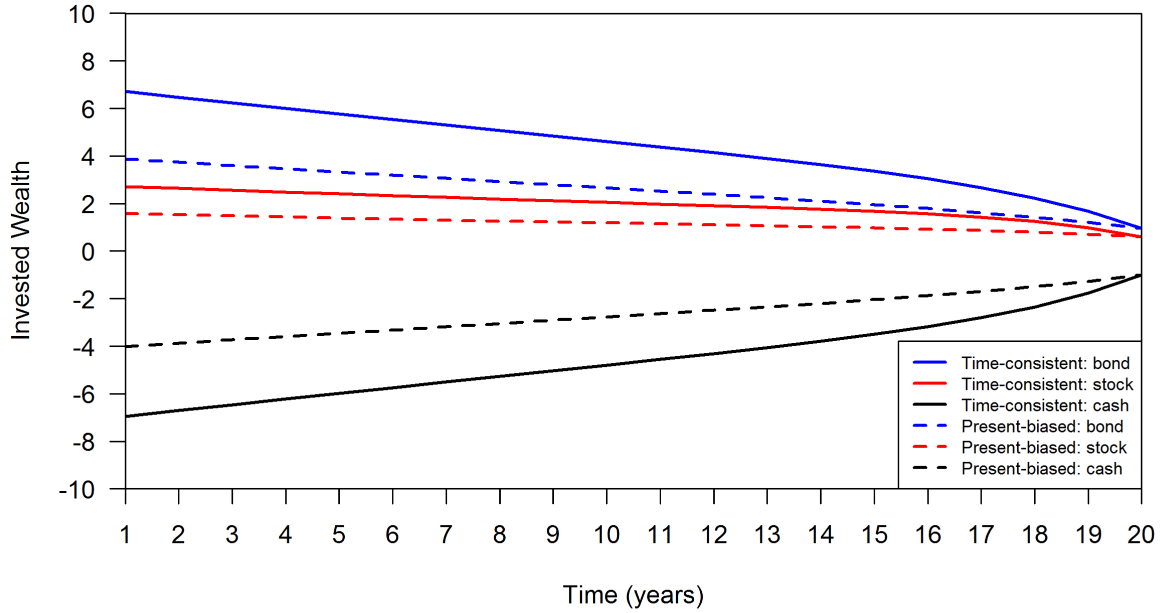

Figure 9: **Optimal portfolio allocation during retirement (annually revisited), expressed as invested remaining wealth, in terms of available assets.** The figure plots the optimal asset allocations for two individuals in terms of invested wealth against the year in retirement. Invested wealth follows from equation (32). The available assets are a (portfolio of) stock(s), a 5-year bond and cash. The upper two lines correspond to the bond, while the bottom two lines correspond to cash and, thus, the middle two lines correspond to the stock investment. Preferences and the financial market follow from Tables 1 and 2.

In summary, I derived the optimal investment strategy for a general discount function. As special cases, I consider exponential discounting and quasi-hyperbolic discounting. The discount function has no effect on the proportion of wealth invested in the available assets in a Merton (1969) and Vasicek (1977) type model. However, the amounts of invested wealth are lower for quasi-hyperbolic discounters compared to time-consistent behaviour. Due to the immediate gratification for consumption, remaining wealth levels are lower for quasi-hyperbolic discounters such that they invest less. Next, I consider the fourth individual.

Log utility and risk seekers

This final section considers the fourth individual, who differs in risk appetite compared to the former three individuals. I consider two limiting cases: an extremely risk-seeking individual and an extremely risk averse individual. The former follows when $\gamma = 1$, such that the individual has log-utility preferences. The latter follows if the risk aversion parameter γ goes to infinity, indicating extreme risk-seeking behaviour. For both, I present the optimal consumption path and the optimal investment strategy. The results follow directly from Theorems 1 and 2. The following corollary provides the result for a log-utility individual.

Corollary 4. *The optimal allocation of wealth at date j for a general discount function $\phi_{T+j}(j)$ for a log-utility individual equals*

$$W_{0j}^* \propto \phi_{T+j}(j),$$

whereas the optimal portfolio weights at time t for a stock, h-year bond and cash equal

$$\begin{aligned}
\pi_S^* &= \frac{\lambda_S}{\sigma_S \sqrt{1-\rho^2}}, \\
\pi_P^*(t) &= -\frac{\lambda_r}{\sigma_r B(h-t)} + \frac{\rho}{\sqrt{1-\rho^2}} \frac{\lambda_S}{\sigma_r B(h-t)}, \\
\pi_r^*(t) &= 1 - \pi_S^* - \pi_B^*(t).
\end{aligned}$$

The corollary shows that consumption is proportional to the discount function only, such that the variance of the SDF and the bond prices from Theorem 1 vanish. Equivalently, log-utility individuals, who are also quasi-hyperbolic discounters, prefer to consume even more immediately. Since they are risk seeking, they care less about a smooth consumption path such that they dare to consume more now and, as a consequence, dare to keep less remaining total wealth. Therefore, the consumption rate will be higher than those considered in Figure 6. Note that this special case coincides with the main finding of Zou et al. (2014) for quasi-hyperbolic discounting, and also with the classical consumption result of Merton (1969).[14]

The investment strategy for a risk-seeking individual depends only on speculative demand. The hedging term vanishes completely such that the investor is only concerned with the prices of risk. Intuitively, a risk-seeking investor cares less about changes in the investment opportunity set, so he has no desire to hedge. Therefore, compared to Figure 7, the individual prefers to invest more in stocks and borrows higher amounts of cash to finance this investment.

[14] Note that in the Merton (1969) setting, indicated by superscript M, the optimal consumption strategy under time-consistent preferences equals $W_{Tj}^{M*} \propto \exp\left(-\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma} r - \frac{1}{2}\frac{(1-1/\gamma)}{\gamma}\lambda_S^2\right)^j$, whereas under time-inconsistent preferences (represented by quasi-hyperbolic discounting and the superscript H) it follows that: $W_{T0}^{H*} \propto \exp\left(\frac{(1-\gamma)}{\gamma} rT - \frac{1}{2}\frac{(1-1/\gamma)}{\gamma}\lambda_S^2 T\right)$ and $W_{Tj}^{H*} \propto \beta^{1/\gamma} \exp\left(-\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma} r - \frac{1}{2}\frac{(1-1/\gamma)}{\gamma}\lambda_S^2\right)^j$. Moreover, the optimal portfolio allocation under time-(in)consistent preferences is given by $\pi_S^{M*} = \pi_S^{H*} = \frac{\lambda_S}{\gamma\sigma_S}$ and $\pi_r^{M*} = \pi_r^{H*} = 1 - \pi_S^{M*}$.

The second limiting case is an individual with infinite risk aversion $\gamma \to \infty$. The following corollary provides the consumption and portfolio strategies.

Corollary 5. *The optimal allocation of wealth at date j for a general discount function and an investor with infinite risk aversion γ equals*

$$W_{Tj} \propto P(r_T; T, T+j),$$

whereas the optimal portfolio weights for a stock, h-year bond and cash, at time t and horizon T, equal

$$\begin{aligned}\pi_S^* &= 0,\\ \pi_P^*(t) &= \frac{B(T-t)}{B(h-t)},\\ \pi_r^*(t) &= 1 - \pi_S - \pi_B(t).\end{aligned}$$

The corollary shows that a risk-averse individual follows a simple investment strategy. Namely, only invest in h-year bonds according to the hedging demand and put the rest in a risk-free asset. The reason is that risk averse investors dislike any risks such that they want to avoid investing in risky stocks and, therefore, take the safe bond investment according to his desirable hedging demand. Hence, the individual wishes to invest nothing in the stock. Consequently, the speculative demand is zero while the hedging demand becomes large, relative to Figure 8.

As a result, the optimal consumption path is proportional to the bond prices in the financial market. This path is lower than the consumption paths in Figure 5, but it is smoothed more evenly over the life cycle. Additionally, the consumption rate is lower than those in Figure 6. Because the individual is averse to shocks in his consumption pattern, he retains a higher amount of wealth for the uncertain future periods and, at the same time, consumes less but more evenly.

4 Conclusion

A substantial body of experimental evidence suggests that time-inconsistent preferences — a person's relative preference for well-being at an earlier date over a later date gets stronger as the earlier date gets closer — play an important role in determining how people evaluate intertemporal decisions. In this paper, I incorporate this idea into a behavioural finance framework, to improve our understanding of consumption behaviour and investment decisions by refining the way we model time preferences. In particular, I allow for a general discount structure that considers the well-known exponential and quasi-hyperbolic discount functions as special cases. I study the consumption and investment behaviours in two economies: one, labelled "deterministic economy", in which investors only care about the time delay to consumption, and another, labelled "stochastic economy", in which investors care about time delays and fundamental shocks to an interest rate and a stock.

Using the martingale method, I find closed-form solutions for optimal consumption and investment for CRRA utility. The main finding is that time-inconsistent preferences increase the consumption rate. A time-consistent individual prefers a smooth consumption path, while time-inconsistent and present-biased individuals prefer immediate consumption at the expense of lower remaining wealth levels. However, time-preferences have no effect on the proportions of wealth invested, but they lower the monetary amounts invested. The reason is that quasi-hyperbolic discounters prefer always to consume more today than in the future, they undersave and, therefore, invest less than time-consistent individuals. Present-biased risk averse individuals have a particular consumption path, since the risk aversion tends to dominate the desire for immediate gratification.

My paper suggests that using experimental evidence to refine the way we model investor preferences may be a promising avenue for further research — a statement also made by Barberis and Huang (2001). Besides hyperbolic discounting there are additional DU anomalies that may effect some of the phenomena discussed here. The DU model not only assumes that the discount rate should be constant throughout time, but it also assumes that the discount rate should be the same for all types of goods and categories of intertemporal decisions. Several empirical observations appear to contradict this assumption (Frederick et al., 2002): gains are discounted more than losses, small amounts are discounted more than large amounts and explicit sequences of multiple outcomes are discounted differently than outcomes considered singly. These phenomena may be seen as forms of mental accounting.

In response to the anomalies just enumerated, hyperbolic discounting is not the only theoretical model attempting to explain time preferences. Alternative models are under development that may also explain some of my findings, and these could be fruitful avenues for further research. Firstly, the extension closest to my research is that individuals may not be self-aware of their time-inconsistent behaviour (i.e. "naíve"), while others may correctly predict how their preferences change over time (i.e. "sophisticated"). Secondly, models that enrich the utility function may be promising, such as habit formation models, reference point models, anticipation models and models that incorporate emotions. Thirdly, there are models that not only affect the discount structure or utility functions, but take a more radical departure such as: projection bias, mental-accounting models, multiple-self models and temptation preferences. Another avenue for further research are the welfare implications for the individual and the society as a result of time-inconsistent preferences.

Finally, this paper has policy implications as well. The optimality conditions for consumption show that governments should be careful in offering lumpsums at retirement age, since time-inconsistent individuals have the tendency to consume immediately 70% of their pension wealth. Consequently, such individuals have substantially lower remaining wealth levels and may need to rely on social security of the state to maintain above poverty levels. On the other hand, governments might feel a tendency to offer freedom of choice to accommodate individuals with these preferences. Heterogeneity in risk preferences and time preferences is decisive here. The main message is then that it could be beneficial for some people to commit ex-ante to a pension plan, like Odysseus did when travelling across The Sirens.

A Proofs

A.1 Proof of Proposition 1

The proof consists of two steps. Firstly, I show that reversal in preferences may mathematically be achieved by a generalised hyperbolic discount function. Secondly, I show that exponential discounting is a special case of the generalised hyperbolic discount function and yields constant instantaneous discount rates, while the hyperbolic discounting yields time-dependent instantaneous discount rates.

Step 1: Consider an individual at current time zero who is indifferent between receiving $c' > 0$ immediately and $c > c'$ at some distant time t. Then, using the (empirical evidence on) reversal in preferences, he will strictly prefer the higher outcome if both outcomes are incremented (or postponed) by a constant amount of time T. Mathematically,

$$u(c') = u(c)\Phi(t), \quad \text{implies that} \quad u(c')\Phi(T) < u(c)\Phi(T+t),$$

where $\Phi(t)$ is the discount function at current time 0 for consumption to be received at time t. Please note that this notation is different from the main text and, therefore, I use the capital phi; I do this for notational convenience, as will become apparent soon. In order to maintain indifference, the later larger outcome would have to be delayed some period $t' > t$. To account for this phenomenon, I derive a general functional form for $\Phi(.)$, where I postulate that the delay $T + t'$ for the larger outcome is a linear function of the time T to the smaller, earlier outcome for some constant k. Mathematically,

$$u(c') = u(c)\Phi(t), \quad \text{implies that} \quad u(c')\Phi(T) = u(c)\Phi(kT+t), \tag{33}$$

This can be thought of as a more general form of stationarity, in which the "clocks" for the two outcomes being compared run at different speeds. From the above it follows that

$$u(c')\Phi(T') = u(c)\Phi(kT' + t) \tag{34}$$

and, thus,

$$\begin{aligned} u(c')\Phi(\lambda T + (1-\lambda)T') &= u(c)\Phi(k(\lambda T + (1-\lambda)T') + t) \\ &= u(c)\Phi(\lambda(kT + t) + (1-\lambda)(kT' + t)) \\ &= u(c)\Phi\left(\lambda\Phi^{-1}\left(\frac{u(c')\Phi(T)}{u(c)}\right) + (1-\lambda)\Phi^{-1}\left(\frac{u(c')\Phi(T')}{u(c)}\right)\right), \end{aligned}$$

where I substituted the values for $kT+t$ and $kT'+t$ by using (33) and (34), respectively. Now, let $r = \frac{u(c')}{u(c)}$, $w = \Phi(T)$, $z = \Phi(T')$ and $u = \Phi^{-1}$, which yields the equation

$$ru^{-1}(\lambda u(w) + (1-\lambda)u(z)) = u^{-1}(\lambda u(rw) + (1-\lambda)u(rz)).$$

The only solutions to this functional equation are the logarithmic and power functions, $u(T) = c\log(T) + d$ and $u(T) = cT^t + d$ respectively (Aczel, 1966). Since $\Phi(T) = u^{-1}(T)$, it can be easily found that the discount function is either exponential or hyperbolic. The generalised hyperbola

$$\Phi(T) = (1 + \alpha T)^{-\beta/\alpha}, \quad \alpha, \beta > 0,$$

captures both these discount forms. Namely, it incorporates the exponential discounting function $\Phi(T) = e^{-\beta T}$ if I let α go to zero (and use the Binomial theorem). α determines how far the hyperbolic discount function departs from exponential discounting.

Step 2: Definition 1 states that preferences are time consistent if and only if a person's relative preference for well-being at an earlier date over a later date is the same, no matter when (s)he is asked. For this reason, preferences are time consistent if the instantaneous discount rate is independent of time, because then the trade-off between utility is the same when evaluated at different times. Given the discount function $\Phi(T)$, the instantaneous discount rate at time T or the per-period discount rate for period T (that is, the discount rate applied between periods T and $T+1$) is defined as

$$\rho(T) = -\Phi'(T)/\Phi(T).$$

Hence, an exponential discount function δ^T is characterised by a constant discount rate

$$\rho = -\log(1/\delta),$$

which is independent of time. On the other hand, the generalised hyperbolic discount function is characterised by an instantaneous discount rate that falls as T rises

$$\rho(T) = \frac{\beta}{1 + \alpha T},$$

which depends on time. These results match with the empirical evidence. □

A.2 Proof of Lemma 1

The idea is as follows: derive an expression for $\int_t^T r_u du$, then compute the expectation of the logarithm of the SDF (23) and, finally, calculate the variance of the logarithm of the SDF (23). Consequently, using the computed first and second moment of the normal distribution of the logarithm of the SDF, the expectation of the log-normally distributed SDF follows directly.

Firstly, to find an expression for the integral of the instantaneous interest rate r_t, I must have an expression for r_t. To this end, I solve the SDE for the instantaneous interest rate in (21) given by the Ornstein-Uhlenbeck process. Solving the equation requires trick. Instead of thinking about the instantaneous interest rate alone, one needs to think about the differential of $(r_t e^{\kappa t})$, which equals

$$\begin{aligned} d(r_t e^{\kappa t}) &= e^{\kappa t} dr_t + r_t \kappa e^{\kappa t} dt \\ &= e^{\kappa t} \left(\kappa \theta dt + \sigma_r dZ_{r,t} \right), \end{aligned}$$

where I substituted in the second equality the dynamics of dr_t given by equation (21) — note that r_t cancelled out. Now, by straightforward algebra, integrating the left-hand side and right-hand side from time t until time T yields

$$[r_s e^{\kappa s}]_t^T = \int_t^T \kappa \theta e^{\kappa s} ds + \int_t^T \sigma_r e^{\kappa s} dZ_{r,s} \tag{35}$$

$$r_T - \theta = (r_t - \theta) e^{-\kappa(T-t)} + \sigma_r \int_t^T e^{-\kappa(T-t)} dZ_{r,s}. \tag{36}$$

Hence, integrating this result from time t until T gives

$$\int_t^T (r_s - \theta) ds = \int_t^T \left((r_t - \theta) e^{-\kappa(s-t)}) + \sigma_r \int_t^s e^{-\kappa(s-u)} dZ_{r,u} \right) ds,$$

which may be rewritten as

$$\int_t^T r_s ds = \int_t^T \left(r_t e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}) + \sigma_r \int_t^s e^{-\kappa(s-u)} dZ_{r,u} \right) ds.$$

This expression can be simplified by rearranging the equation to only one integral. Thus, I need to deal with the term

$$\int_t^T \sigma_r \int_t^s e^{-\kappa(s-u)} dZ_{r,u} ds.$$

I use (stochastic) integration by parts, and I find that

$$\begin{aligned}
\int_t^T \sigma_r \int_t^s e^{-\kappa(s-u)} dZ_{r,u} ds &= \sigma_r \int_t^T e^{-\kappa s} \int_t^s e^{\kappa u} dZ_{r,u} ds \\
&= \sigma_r \int_t^T \left(\int_t^s e^{\kappa u} dZ_{r,u} \right) d_t \left(\int_t^s e^{-\kappa v} dv \right) \\
&= \sigma_r \left[\int_t^s e^{\kappa u} dZ_{r,u} \int_t^s e^{-\kappa v} dv \right]_t^T - \sigma_r \int_t^T e^{\kappa s} dZ_{r,s} \int_t^s e^{-\kappa v} dv \\
&= \sigma_r \int_t^T e^{\kappa u} dZ_{r,u} \int_t^T e^{-\kappa v} dv - \sigma_r \int_t^T e^{\kappa s} dZ_{r,s} \left(\int_t^T e^{-\kappa v} dv - \int_s^T e^{-\kappa v} dv \right) \\
&= \sigma_r \int_t^T e^{\kappa s} dZ_{r,s} \int_s^T e^{-\kappa v} dv \\
&= \sigma_r \int_t^T dZ_{r,s} e^{\kappa s} \left(-\frac{1}{\kappa} e^{-\kappa T} + \frac{1}{\kappa} e^{-\kappa s} \right) \\
&= \sigma_r \int_t^T dZ_{r,s} \left(\frac{1}{\kappa} - \frac{1}{\kappa} e^{-\kappa(T-s)} \right) \\
&= \sigma_r \int_t^T \frac{1}{\kappa} (1 - e^{-\kappa(T-s)} dZ_{r,s} \\
&= \sigma_r \int_t^T B(T-s) dZ_{r,s}
\end{aligned}$$

So, the simplified expression is

$$\int_t^T r_s ds = \int_t^T \left(r_t e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}) \right) ds + \sigma_r \int_t^T B(T-s) dZ_{r,s} \tag{37}$$

I am now in the position to compute conditional expectations and conditional variances of $\int_t^T r_s ds$ and, then, the conditional expectation and variance of the SDF can be addressed. Splitting equation (37) into the terms that contribute to the expectation (recall the property of Wiener processes $Z_{r,T} - Z_{r,t} \sim N(0, T-t)$) and by using Fubini's theorem, I find after some lines of routine algebra that

$$\begin{aligned}
E_t \left[\int_s^T r_t dt \right] &= \int_t^T \left(r_t e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}) \right) ds \\
&= (r_s - \theta) B(T-s) + \theta(T-s)
\end{aligned} \tag{38}$$

Splitting equation (37) into the terms that contribute to the variance, it follows that

$$
\begin{aligned}
V_t\left[\int_s^T r_t dt\right] &= E_t\left[\left(\sigma_r \int_t^T B(T-s)dZ_{r,s}\right)^2\right] \\
&= E_t\left[\sigma_r^2 \int_t^T B(T-s)^2 ds\right] \\
&= \frac{\sigma_r^2}{\kappa^2}\left[(T-s) - B(T-s) - \frac{\kappa}{2}B(T-s)^2\right], \qquad (39)
\end{aligned}
$$

where I used Itô isometry in the second equality and I obtain the third equality by routine algebra of the remaining deterministic expression.

Secondly, I can now easily compute the first moment $m(r_t, T-t)$ of the logarithm of the SDF

$$
\begin{aligned}
m(r_t, T-t) &= E_t[\log[M_T/M_t] \\
&= E_t\left[-\int_t^T r_u du - \frac{1}{2}(\lambda_S^2 + \lambda_r^2)(T-t) - \int_t^T \lambda_S dZ_{S,u} - \int_t^T \lambda_r dZ_{r,u}\right] \\
&= -E_t\left[\int_t^T r_u du\right] - \frac{1}{2}(\lambda_S^2 + \lambda_r^2)(T-t)] \\
&= (\theta - r_t)B(T-t) - \theta(T-t) - \frac{1}{2}(\lambda_S^2 + \lambda_r^2)(T-t)], \qquad (40)
\end{aligned}
$$

where the second equality follows by substitution of the SDF from (23), the third by the properties of Brownian motions and the last equality follows by substituting the mean of integral over the interest rate from (38).

Finally, I compute the second moment $v^2(T-t)$ of the logarithm of the SDF. It follows that

$$
\begin{aligned}
v^2(T-t) &= V_t[\log[M_T/M_t] \\
&= V_t[-\int_t^T r_u du - \frac{1}{2}(\lambda_S^2 + \lambda_r^2)(T-t) - \int_t^T \lambda_S dZ_{S,u} - \int_t^T \lambda_r dZ_{r,u}] \\
&= V_t\left[\int_t^T r_u du\right] + (\lambda_S^2 + \lambda_r^2)(T-t) + 2\lambda_r E_t\left[\int_t^T r_u du \int_t^T dZ_{r,u}\right] \\
&= \frac{\sigma_r^2}{\kappa^2}\left[(T-t) - B(T-t) - \frac{\kappa}{2}B(T-t)^2\right] + (\lambda_S^2 + \lambda_r^2)(T-t) \\
&+ 2\lambda_r E_t\left[\int_t^T r_u du \int_t^T dZ_{r,u}\right], \qquad (41)
\end{aligned}
$$

where in the third equality I use that Z_S and Z_r are uncorrelated, and the fourth equality follows by using equation (39) for the variance of the integral over the interest rate. The final step is to compute the last term in the above expression with the expectation. Recalling the expression for the integral over r_u from equation (37), where only the stochastic part is relevant, I find that

$$
\begin{aligned}
E_t\left[\int_t^T r_u du \int_t^T dZ_{r,u}\right] &= E_t\left[\sigma_r \int_t^T B(T-s)dZ_{r,s}Z_{r,T}\right] \\
&= E_t\left[\sigma_r \int_t^T B(T-s)dZ_{r,s}\int_t^T dZ_{r,s}\right] \\
&= E_t\left[\sigma_r \int_t^T B(T-s)ds\right] \\
&= \frac{\sigma_r}{\kappa}[(T-t)-B(T-t)],
\end{aligned} \tag{42}
$$

where the last equality follows after some straightforward algebra. So, combining the results from equations (41) and (42), the second moment $v^2(T-t)$ of the logarithm of the SDF is given by

$$
\begin{aligned}
v^2(T-t) &= V_t[\log[M_T/M_t] \\
&= \frac{\sigma_r^2}{\kappa^2}\left[(T-t)-B(T-t)-\frac{\kappa}{2}B(T-t)^2\right] + (\lambda_S^2+\lambda_r^2)(T-t) \\
&+ \frac{2\sigma_r}{\kappa}\lambda_r\left[(T-t)-B(T-t)\right].
\end{aligned} \tag{43}
$$

In conclusion, the conditional expectation at time t of the stochastic discount factor at time $T > t$ for positive α follows by

$$
\begin{aligned}
E_t\left[\left(\frac{M_T}{M_t}\right)^\alpha\right] &= \exp\left(\alpha m(r_t, T-t) + \frac{1}{2}\alpha^2 v^2(T-t)\right) \\
&= \exp\left(\frac{1}{2}(\alpha-1)\alpha v^2(T-t) + \alpha\left(m(r_t,T-t)+\frac{1}{2}v^2(T-t)\right)\right) \\
&= e^{\frac{1}{2}(\alpha-1)\alpha v^2(T-t)}P(r_t;t,T)^\alpha,
\end{aligned}
$$

where I use that

$$
\begin{aligned}
&P(r_t;t,T) = \exp\left(A(T-t)-B(T-t)r_t\right) = \exp\left(m(r_t,T-t)+\frac{1}{2}v^2(T-t)\right), \quad \text{with} \\
&A(T-t) = (B(T-t)-(T-t))R_\infty - \frac{\sigma^2}{4\kappa}B(T-t)^2, \quad \text{with } R_\infty = \theta - \frac{\lambda_r\sigma_r}{\kappa} - \frac{1}{2}\frac{\sigma_r^2}{\kappa^2} \\
&B(T-t) = \frac{1}{\kappa}\left(1-\exp(-\kappa(T-t))\right),
\end{aligned}
$$

which can be shown by straightforward (tedious) algebra. □

A.3 Proof of Theorem 1

In view of the result for the value function in equation (25) and using Lemma 1 with $\alpha = 1 - 1/\gamma$ (due to CRRA utility), the individual maximises (subject to the budget constraint)

$$\max_{W_{Tj}, j=0,...,h-1} \sum_{j=0}^{h-1} \phi_{T+j}(T) \frac{W_{Tj}^{1-\gamma}}{1-\gamma} \exp\left(-\frac{1}{2}\frac{\gamma-1}{\gamma} v^2(T,T+j)\right) (P(r_T;T,T+j))^{\gamma-1}.$$

Setting up the Lagrangian function, with η as Lagrange multiplier for the budget constraint, I find that

$$L = \sum_{j=0}^{h-1} \phi_{T+j}(T) \frac{W_{Tj}^{1-\gamma}}{1-\gamma} \exp\left(-\frac{1}{2}\frac{\gamma-1}{\gamma} v^2(T,T+j)\right) (P(r_T;T,T+j))^{\gamma-1} + \eta \left(W_T - \sum_{j=0}^{h-1} W_{Tj}\right).$$

Taking the first order conditions with respect to initial pension wealth W_{Tj} yields for every $j = 0, ..., h-1$

$$\frac{\partial L}{\partial W_{Tj}} = \phi_{T+j}(T) W_{Tj}^{-\gamma} \exp\left(-\frac{1}{2}\frac{\gamma-1}{\gamma} v^2(T,T+j)\right) (P(r_T;T,T+j))^{\gamma-1} - \eta = 0$$

which may be rearranged as

$$W_{Tj} = \eta^{-1/\gamma} \phi_{T+j}(T)^{1/\gamma} \exp\left(-\frac{1}{2}\frac{\gamma-1}{\gamma} v^2(T,T+j)\right) (P(r_T;T,T+j))^{\frac{\gamma-1}{\gamma}}.$$

In conclusion, the optimal allocation of wealth over the payments at the various horizons is proportional to

$$W_{Tj} \propto \phi_{T+j}(T)^{1/\gamma} \exp\left(-\frac{1}{2}\frac{\gamma-1}{\gamma} v^2(T,T+j)\right) (P(r_T;T,T+j))^{\frac{\gamma-1}{\gamma}}.$$

The proportionality sign is determined by the fact that initial pension wealth must add up to the multiple consumption moments. □

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