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Abstract

We analyze a dynamic Asset Liability Management problem with model uncertainty in a complete market. The fund manager acts in the best interest of the pension holders by maximizing the expected utility derived from the terminal funding ratio. We solve the robust multi-period Asset Liability Management problem in closed form, and identify two constituents of the optimal portfolio: the myopic demand, and the liability hedge demand. We find that even though the investment opportunity set is stochastic, the investor does not have intertemporal hedging demand. We also find that model uncertainty induces a more conservative investment policy regardless of the risk attitude of the fund manager, i.e., a robust investment strategy corresponds to risk exposures which provide a much stronger liability hedge.

JEL classification: C61, G11, G12
Keywords: asset liability management, liability-driven investment, robustness, uncertainty, ambiguity

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1 Introduction

Financial institutions, such as pension funds and insurance companies, are exposed to several sources of risk through their assets and their liabilities. During their decision-making process, they simultaneously consider the potential effects of their decisions on both the asset and the liability side of their balance sheet, hence the term Asset Liability Management. However, unless the decision maker (the financial institution)[1] knows the true model (the data-generating process driving the asset and liability values) precisely, it faces not only risk, but also uncertainty. Disregarding uncertainty can lead to suboptimal investment and Asset Liability Management decisions, thus financial institutions want to make decisions that *"not only work well when the underlying model for the state variables holds exactly, but also perform reasonably well if there is some form of model misspecification"* (Maenhout (2004)). In the literature these decisions are called *robust decisions.* Although the utility loss resulting from model misspecification can be substantial (Branger and Hansis (2012)), the majority of the literature still assumes perfect knowledge of the underlying model on the decision maker's side. Our aim with this paper is to fill this gap in the dynamic Asset Liability Management literature.

Our model features a complete financial market with stochastic interest rates governed by an N-factor Gaussian affine term structure model. The fund manager solves a dynamic Asset Liability Management problem under model uncertainty. Using the martingale method of Cox and Huang (1989), we provide the optimal terminal wealth, the least-favorable physical probability measure, and the optimal investment policy in closed form. We find that the optimal portfolio weights consist of two components: the myopic demand and the liability hedge demand, but notwithstanding the stochastic investment opportunity set, the fund manager does not have an intertemporal hedging demand component. We then use 42 years of U.S. data to calibrate our model. We show that robustness induces a more conservative investment policy: a robust fund manager's optimal risk exposures are closer

[1] In this paper we focus on pension funds, but our results can be interpreted in a more general sense, and they are valid for any financial institution which has to make Asset Liability Management decisions.

to the liability risk exposures, hence reducing the speculative demand and increasing the liability-hedging demand. In parallel with this, a robust fund manager invests less in the constant maturity bond fund with a relatively short maturity (in our numerical example 1 year) and also in the stock market index than an otherwise identical non-robust fund manager. The portfolio weight of the constant maturity bond fund with maturity equal to the investment horizon increases due to its strong liability hedge effect, and thus effectively reduces the exposure to the factor-specific risk sources.

Our paper relates to the literature on Asset Liability Management. The modern Asset Liability Management literature dates back to Leibowitz (1987), who introduces the concept of the *surplus function* (the excess of the plan's asset value over the value of its liabilities). Based on this notion, Sharpe and Tint (1990) extend the Mean-Variance portfolio allocation model of Markowitz (1952) to an Asset Liability Management model. The basic idea is that instead of asset returns, the investor cares about the surplus returns, where surplus means the value of assets minus the value of liabilities. Sharpe and Tint (1990) find that the optimal portfolio consists of two components: a speculative portfolio and a liability-hedge portfolio. Moreover, only the speculative portfolio depends on the investor's preferences, the liability-hedge portfolio is the same for each investor. More recently, Hoevenaars, Molenaar, Schotman, and Steenkamp (2008) extend the multi-period portfolio selection model of Campbell and Viceira (2005) into a multi-period Asset Liability Management model. They confirm the finding of Sharpe and Tint (1990) that the optimal portfolio consists of two parts: a speculative portfolio and a liability hedge portfolio. Instead of maximizing a subjective utility function, Shen, Pelsser, and Schotman (2014) assume that the fund manager minimizes the expected shortfall (i.e., the expected amount by which the value of liabilities exceeds the value of assets) at the terminal date. This assumption emphasizes that the fund manager acts in the best interest of the sponsoring firm, but does not consider the interest of the pension holders. In contrast to this, van Binsbergen and Brandt (2016) assume that the objective function is a sum of two parts: the first part expresses the (positive) utility of the pension holders, while the second part represents the (negative) utility of the sponsoring firm. To be more concrete, pension holders derive (positive) utility from a high funding ratio at the terminal date, while the sponsoring firm derives (negative) utility from having to provide additional contributions to the fund in order to keep the funding ratio above one throughout its life cycle. This model of van Binsbergen and Brandt (2016) nests the non-robust version of the model of Shen, Pelsser, and Schotman (2014), if the weight of the utility function of the pension holders is set to zero.

Our paper also relates to the literature on robust dynamic asset allocation. Anderson, Hansen, and Sargent (2003) in their seminal paper develop a framework for dynamic asset allocation models, which allows the investor to account for being uncertain about the physical probability measure. Within this framework, Maenhout (2004) provides an analytical and homothetic solution to the robust version of the Merton problem. Maenhout (2006) extends this model to incorporate a stochastic investment opportunity set. Branger, Larsen, and Munk (2013) solve a robust dynamic stock-cash allocation problem including return predictability, while Munk and Rubtsov (2014) also allow for ambiguity about the inflation process. Horvath, de Jong, and Werker (2016) provide a non-recursive formulation of the problem of Maenhout (2004), and also extend it to models featuring interest rate risk.

The paper is organized as follows. Section 2 introduces our model, i.e., the financial market and the fund manager's objective function. Moreover, Section 2 also provides the analytical solution of the robust dynamic Asset Liability Management problem. In Section 3 we calibrate our model to 42 years of U.S. market data using Maximum Likelihood and the Kalman filter. In Section 4 we link the level of uncertainty aversion to the theory of Detection Error Probabilities. In Section 5 we quantitatively analyze the effects of model uncertainty on the optimal Asset Liability Management decision. Section 6 concludes.

2 Robust Asset Liability Management Problem

Our model features a complete financial market and a robust fund manager. By robustness we mean that the fund manager is uncertain[2] about the underlying model. To be more precise, we assume that she is uncertain about the physical probability measure. She has a *base measure* $\mathbb{B}$ in mind, which she thinks to be the most reasonable probability measure. But she is uncertain about whether the base measure is indeed the true measure or not, so she considers other probability measures as well. We call these *alternative measures* and denote them by $\mathbb{U}$. We provide the exact relationship between $\mathbb{B}$ and $\mathbb{U}$, and also the restrictions on the set of $\mathbb{U}$ measures under consideration in Section 2.3.

2.1 Financial market

We consider pension funds which have access to a complete, arbitrage-free financial market consisting of a money-market account, N constant maturity bond funds, and a stock market index. The short rate is assumed to be affine in an N-dimensional factor $\boldsymbol{F}_t$, i.e.,

$$r_t = A_0 + \boldsymbol{\iota}' \boldsymbol{F}_t, \tag{1}$$

where $\boldsymbol{\iota}$ denotes a column vector of ones. The factor $\boldsymbol{F}_t$ follows an Ornstein-Uhlenbeck process under the base measure, i.e.,

$$\mathrm{d}\boldsymbol{F}_t = -\boldsymbol{\kappa}(\boldsymbol{F}_t - \boldsymbol{\mu}_F)\mathrm{d}t + \boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}^{\mathbb{B}}_{F,t}. \tag{2}$$

Here $\boldsymbol{\kappa}$ is an $N \times N$ diagonal matrix with the mean reversion parameters in its diagonal; $\boldsymbol{\mu}_F$ is an N-dimensional column vector containing the long-term means of the factors under the base measure $\mathbb{B}$; $\boldsymbol{\sigma}_F$ is an $N \times N$ lower triangular matrix, with strictly positive elements in

[2] The terms *uncertainty* and *ambiguity* have slightly different meanings in the behavioral finance literature. In the robust asset pricing and robust asset allocation literature, however, they are used interchangeably. Since our paper primarily belongs to this latter branch of the literature, we do not differentiate between the meaning of *uncertainty* and *ambiguity*, and use the two words interchangeably.

its diagonal; and $\boldsymbol{W}^{\mathbb{B}}_{F,t}$ is an N-dimensional column vector of independent standard Wiener processes under $\mathbb{B}$.

The stock market index can be correlated with the factor $\boldsymbol{F}_t$, i.e.,

$$\mathrm{d}S_t = S_t\left[r_t + \boldsymbol{\sigma}'_{F,S}\boldsymbol{\lambda}_F + \sigma_{N+1,S}\lambda_{N+1}\right]\mathrm{d}t + S_t\left(\boldsymbol{\sigma}'_{F,S}\mathrm{d}\boldsymbol{W}^{\mathbb{B}}_{F,t} + \sigma_{N+1,S}\mathrm{d}W^{\mathbb{B}}_{N+1,t}\right), \quad (3)$$

where $\boldsymbol{\lambda}_F$ and λ_{N+1} are the market prices of risk corresponding to the base measure $\mathbb{B}$, $\boldsymbol{\sigma}_{F,S}$ is an N-dimensional column vector governing the covariance between stock and bond returns, $\sigma_{N+1,S}$ is a strictly positive constant, and $W^{\mathbb{B}}_{N+1,t}$ is a standard Wiener process under the base measure $\mathbb{B}$, which is independent of $\boldsymbol{W}^{\mathbb{B}}_{F,t}$. The liability of the pension fund is assumed to evolve according to

$$\mathrm{d}L_t = L_t\left(r_t + \boldsymbol{\sigma}'_{F,L}\boldsymbol{\lambda}_F + \sigma_{N+1,L}\lambda_{N+1}\right)\mathrm{d}t + L_t\left(\boldsymbol{\sigma}'_{F,L}\mathrm{d}\boldsymbol{W}^{\mathbb{B}}_{F,t} + \sigma_{N+1,L}\mathrm{d}W^{\mathbb{B}}_{N+1,t}\right), \quad (4)$$

where $\boldsymbol{\sigma}_{F,L}$ is an N-dimensional column vector, and $\sigma_{N+1,L}$ is a scalar.

To simplify notation, we denote $\boldsymbol{W}^{\mathbb{B}}_{F,t}$ and $W^{\mathbb{B}}_{N+1,t}$ jointly as

$$\boldsymbol{W}^{\mathbb{B}}_t = \begin{bmatrix} \boldsymbol{W}^{\mathbb{B}}_{F,t} \\ W^{\mathbb{B}}_{N+1,t} \end{bmatrix}, \quad (5)$$

$\boldsymbol{\lambda}_F$ and λ_{N+1} jointly as

$$\boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_F \\ \lambda_{N+1} \end{bmatrix}, \quad (6)$$

$\boldsymbol{\sigma}_{F,S}$ and $\sigma_{N+1,S}$ jointly as

$$\boldsymbol{\sigma}_S = \begin{bmatrix} \boldsymbol{\sigma}_{F,S} \\ \sigma_{N+1,S} \end{bmatrix}, \quad (7)$$

and $\boldsymbol{\sigma}_{F,L}$ and $\sigma_{N+1,L}$ jointly as

$$\boldsymbol{\sigma}_L = \begin{bmatrix} \boldsymbol{\sigma}_{F,L} \\ \sigma_{N+1,L} \end{bmatrix}. \quad (8)$$

2.2 The liability-risk-neutral measure

Our fund manager, as we describe in more detail in Section 2.3, is optimizing over the terminal funding ratio. Therefore, to facilitate the problem solving process, throughout the paper we use the liability value as numeraire. Since the financial market is complete and free of arbitrage opportunities, there exists a unique probability measure under which the value of any traded asset scaled by the value of liability is a martingale.

Let X_t be the value of any traded asset, with

$$\mathrm{d}X_t = X_t \left[r_t + \boldsymbol{\sigma}_X' \boldsymbol{\lambda} \right] \mathrm{d}t + X_t \boldsymbol{\sigma}_X' \mathrm{d}\boldsymbol{W}_t^{\mathbb{B}}. \tag{9}$$

Applying Ito's lemma, we find the dynamics of the asset price scaled by the value of the liability as

$$\mathrm{d}\left(\frac{X}{L}\right)_t = \left(\frac{X}{L}\right)_t (\boldsymbol{\sigma}_X - \boldsymbol{\sigma}_L)' (\boldsymbol{\lambda} - \boldsymbol{\sigma}_L)\, \mathrm{d}t + \left(\frac{X}{L}\right)_t (\boldsymbol{\sigma}_X - \boldsymbol{\sigma}_L)'\, \mathrm{d}\boldsymbol{W}_t^{\mathbb{B}}. \tag{10}$$

Defining

$$\mathrm{d}\boldsymbol{W}_t^{\mathbb{B}} = \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}} - \boldsymbol{\lambda}^{\mathbb{L}} \mathrm{d}t, \tag{11}$$

with

$$\boldsymbol{\lambda}^{\mathbb{L}} = \boldsymbol{\lambda} - \boldsymbol{\sigma}_L, \tag{12}$$

the dynamics of the asset X_t scaled by the liability value can be rewritten as

$$\mathrm{d}\left(\frac{X}{L}\right)_t = \left(\frac{X}{L}\right)_t (\boldsymbol{\sigma}_X - \boldsymbol{\sigma}_L)'\, \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}. \tag{13}$$

Then (12), together with (11), uniquely determines the relationship between the liability-risk-neutral measure $\mathbb{L}$ and the base measure $\mathbb{B}$.

2.3 Preferences, beliefs, and problem formulation

We consider a pension fund manager who acts in the best interest of the pension holders. She is risk-averse, and she has CRRA preferences over the terminal funding ratio. The fund manager wants to maximize her expected utility, but she is uncertain about the physical probability measure under which the expectation is supposed to be calculated. She has a base measure ($\mathbb{B}$) in mind, but she considers other, alternative probability measures ($\mathbb{U}$) as well. We assume that the investor knows which events will happen with probability one and with probability zero, i.e., she considers only alternative probability measures which are equivalent to the base measure. We now formalize the relationship between the base measure $\mathbb{B}$ and the alternative measure $\mathbb{U}$ as

$$\mathrm{d}\boldsymbol{W}_t^{\mathbb{U}} = \mathrm{d}\boldsymbol{W}_t^{\mathbb{B}} - \boldsymbol{u}(t)\mathrm{d}t, \tag{14}$$

where $\boldsymbol{W}_t^{\mathbb{B}}$ and $\boldsymbol{W}_t^{\mathbb{U}}$ are $(N+1)$-dimensional standard Wiener processes under the measures $\mathbb{B}$ and $\mathbb{U}$, respectively. Similarly to identifying $\boldsymbol{\lambda}$ as the $(N+1)$-dimensional vector of prices of risks of the base measure $\mathbb{B}$, we can identify $\boldsymbol{u}(t)$ as the $(N+1)$-dimensional vector of prices of risks of $\mathbb{U}$.[3] We assume that $\boldsymbol{\lambda}$ is constant, while $\boldsymbol{u}(t)$ is assumed to be a deterministic function of time.[4]

We now formalize the robust optimization problem of the fund manager. Her investment horizon is T, she has a utility function with a constant relative risk aversion of $\gamma > 1$ over the terminal funding ratio,[5] and a subjective discount rate of $\delta > 0$. Her uncertainty-tolerance is determined by the parameter Υ_t, which is allowed to be stochastic.

[3] Throughout the paper we assume $\int_0^T \|\boldsymbol{u}(s)\|^2 \mathrm{d}s < \infty$.

[4] We could allow $\boldsymbol{\lambda}$ to be a deterministic function of time without much change in our conclusions, but it would result in more complex expressions due to time-integrals involving $\boldsymbol{\lambda}(t)$. Thus, since for our purposes a constant $\boldsymbol{\lambda}$ suffices, we throughout take $\boldsymbol{\lambda}$ to be constant.

[5] The case $\gamma = 1$ corresponds to the fund manager having log-utility. All of our results can be shown to hold for the log-utility case as well.

Problem 1. *Given initial funding ratio A_0/L_0, find an optimal pair $\{A_T, \mathbb{U}\}$ for the robust utility maximization problem*

$$V_0\left(\frac{A_0}{L_0}\right) = \inf_{\mathbb{U}} \sup_{A_T} \mathrm{E}^{\mathbb{U}} \left\{ \exp(-\delta T) \frac{\left(\frac{A_T}{L_T}\right)^{1-\gamma}}{1-\gamma} \right.$$
$$\left. + \int_0^T \Upsilon_s \exp(-\delta s) \frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_s\right]}{\partial s} \mathrm{d}s \right\}, \tag{15}$$

subject to the budget constraint

$$\mathrm{E}^{\mathbb{L}}\left(\frac{A_T}{L_T}\right) = \frac{A_0}{L_0}. \tag{16}$$

The formulation of Problem 1 follows the logic of the Martingale Method of Cox and Huang (1989): the fund manager optimizes over the terminal wealth A_T.[6] The first part of the objective function in Problem 1 expresses that the fund manager derives utility from the terminal funding ratio. The second part is a penalty term, which assures that the investor will use a pessimistic, but reasonable physical probability measure to calculate her expected utility. This penalty term – in line with Anderson, Hansen, and Sargent (2003) – is the integral of the discounted time-derivative of the Kullback-Leibler divergence (also known as the relative entropy) between the base measure $\mathbb{B}$ and the alternative measure $\mathbb{U}$, multiplied by the fund manager's uncertainty-tolerance parameter Υ_s. Intuitively, this penalty term is high if the alternative measure $\mathbb{U}$ and the base measure $\mathbb{B}$ are very different from each other, and low if they are similar to each other. If $\mathbb{U}$ and $\mathbb{B}$ coincide, the penalty term is zero. Using Girsanov's theorem, we can express the Kullback-Leibler divergence as

$$\begin{aligned}\frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_t\right]}{\partial t} &= \frac{\partial}{\partial t}\mathrm{E}^{\mathbb{U}}\left[\frac{1}{2}\int_0^t \|\boldsymbol{u}(s)\|^2 \mathrm{d}s - \int_0^t \boldsymbol{u}(s)\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}\right] \\ &= \frac{1}{2}\|\boldsymbol{u}(t)\|^2. \end{aligned} \tag{17}$$

To insure homotheticity of the solution, i.e., that the optimal portfolio weights do not

[6] Actually, the fund manager optimizes over the terminal funding ratio A_T/L_T. However, as we describe it in more detail in Section 2.1, the liability process L_t is assumed to be exogenous, hence choosing an optimal terminal funding ratio A_T/L_T is equivalent to choosing "only" an optimal terminal wealth A_T.

depend on the actual funding ratio, we – following Maenhout (2004) – express the manager's uncertainty-tolerance parameter as

$$\Upsilon_t = \exp(\delta t)\,\frac{1-\gamma}{\theta} V_t\left(\frac{A_t}{L_t}\right), \tag{18}$$

where $V_t\left(A_t/L_t\right)$ is the value function of the fund manager at time t, i.e.,

$$V_t\left(\frac{A_t}{L_t}\right) = \inf_{\mathbb{U}} \sup_{A_T} \mathrm{E}_t^{\mathbb{U}} \left\{ \exp(-\delta T)\frac{\left(\frac{A_T}{L_T}\right)^{1-\gamma}}{1-\gamma} \right. \\ \left. + \int_t^T \Upsilon_s \exp(-\delta s) \frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_s\right]}{\partial s} \mathrm{d}s \right\}, \tag{19}$$

subject to the budget constraint

$$\mathrm{E}_t^{\mathbb{L}}\left(\frac{A_T}{L_T}\right) = \frac{A_t}{L_t}. \tag{20}$$

Substituting (18) into the value function (15), and also making use of (17), we can rewrite Problem 1 in a form which has a non-recursive goal function. This is stated in the following theorem, the proof of which is provided in the Appendix.

Theorem 1. *If the fund manager's uncertainty-tolerance parameter* Υ_t *takes the form (18), then the value function in Problem 1 is equivalent to*

$$V_0\left(\frac{A_0}{L_0}\right) = \inf_{\mathbb{U}} \sup_{A_T} \mathrm{E}^{\mathbb{U}} \left\{ \exp\left(\frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T\right) \frac{\left(\frac{A_T}{L_T}\right)^{1-\gamma}}{1-\gamma} \right\}, \tag{21}$$

subject to the budget constraint

$$\mathrm{E}^{\mathbb{L}}\left(\frac{A_T}{L_T}\right) = \frac{A_0}{L_0}. \tag{22}$$

As noted by Horvath, de Jong, and Werker (2016), the expression in (21) provides an alternative interpretation of robustness: the goal function of a robust fund manager is equivalent to the goal function of a more impatient[7] non-robust fund manager. Besides

[7]By a fund manager being more impatient, we mean that her subjective discount rate is higher.

increasing the subjective discount rate, the other effect of robustness is a change in the physical probability measure from $\mathbb{B}$ to $\mathbb{U}$.

2.4 Optimal Terminal Funding Ratio

To solve the robust dynamic ALM problem, we apply the martingale method (developed by Cox and Huang (1989), and adapted to robust problems by Horvath, de Jong, and Werker (2016)). The next theorem – which we prove in the Appendix – provides the optimal terminal wealth and the least-favorable distortions.

Theorem 2. *The solution to Problem 1 under (18) is given by*

$$\hat{A}_T = L_T \frac{A_0}{L_0} \frac{\exp\left[\frac{1}{\gamma}\int_0^T \left(\boldsymbol{\lambda}^{\mathbb{L}} + \hat{\boldsymbol{u}}\left(t\right)\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{\gamma}\int_0^T \left(\boldsymbol{\lambda}^{\mathbb{L}} + \hat{\boldsymbol{u}}\left(t\right)\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}\right]}{\mathrm{E}^{\mathbb{L}} \exp\left[\frac{1}{\gamma}\int_0^T \left(\boldsymbol{\lambda}^{\mathbb{L}} + \hat{\boldsymbol{u}}\left(t\right)\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{\gamma}\int_0^T \left(\boldsymbol{\lambda}^{\mathbb{L}} + \hat{\boldsymbol{u}}\left(t\right)\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}\right]}, \tag{23}$$

with the least-favorable distortion

$$\hat{\boldsymbol{u}}\left(t\right) = -\frac{\theta}{\gamma + \theta}\boldsymbol{\lambda}^{\mathbb{L}}. \tag{24}$$

Using the martingale method to solve the robust dynamic ALM problem has the advantage of providing insight into the optimization process of the fund manager. The form of the optimal terminal wealth (23) suggests that the decision process of the fund manager can be separated into two parts. First, as a starting point, she wants to obtain a perfect hedge for the liabilities at time T, i.e., she wants a terminal wealth equal to L_T. Then, she modifies this terminal wealth based on her preferences to achieve the optimal terminal wealth.

The least-favorable distortion of the fund manager differs in two important aspects from the least-favorable distortion of an otherwise identical investor who optimizes over her terminal wealth, instead of the terminal funding ratio (see Horvath, de Jong, and Werker (2016)). First, the least-favorable distortion of the fund manager is independent of time, while the least-favorable distortion of an investor deriving utility from terminal wealth contains a time-dependent component. This time-dependent component is present due to the intertemporal hedging potential of the constant maturity bond funds, and it results in the investor having a "less severe" distortion. However – as we show in Section 2.5 –, deriving utility from the terminal funding ratio instead of the terminal wealth, the fund manager does not have an intertemporal hedging demand for the constant maturity bond funds. So, intuitively, regardless of how far away from the end of her investment horizon the fund manager is, she will distort her base measure to the same extent.

Another aspect in which the least-favorable distortion of the fund manager differs from the least-favorable distortion of an otherwise identical investor deriving utility from terminal wealth is that the market price of risk in which the least-favorable distortion is affine corresponds to the liability value as numeraire, instead of to the money market account.[8] Intuitively, this means that the magnitude of the distortion is reduced due to the fund manager deriving utility from the terminal funding ratio instead of the terminal wealth. This is true for both $\hat{\boldsymbol{u}}_F$ and for $\hat{u}_{N+1}$. Because the liability process behaves very similarly to a zero-coupon bond with approximately 15 years of maturity, we expect the elements of $\boldsymbol{\sigma}_{F,L}$ to be negative. Since the first N elements of $\boldsymbol{\lambda}$ are also negative, and the difference between the market price of risk corresponding to to the money market account as numeraire and the market price of risk corresponding to the liability as numeraire is $\boldsymbol{\sigma}_L$, deriving utility from the terminal funding ratio instead of the terminal wealth reduces the (positive) elements of $\hat{\boldsymbol{u}}_F$. The same logic applies to $\hat{u}_{N+1}$. The market price of risk using the money market account as numeraire, i.e., λ_{N+1} is positive, and intuition suggests $\sigma_{L,N+1}$ is also positive, therefore, optimizing over the terminal funding ratio instead of the terminal wealth will reduce the magnitude of the (negative) $\hat{u}_{N+1}$.

If the fund manager is not uncertainty averse at all, her θ parameter is equal to zero and her least-favorable distortion reduces to zero as well. In other words, she will use her base probability measure $\mathbb{B}$ to evaluate her expected utility. At the other extreme, if her

[8]That is, the fund manager's least-favorable distortion is affine in $\boldsymbol{\lambda}^{\mathbb{L}}$, while the least-favorable distortion of an otherwise identical investor who derives utility from terminal wealth is affine in $\boldsymbol{\lambda}$, i.e., the market price of risk of the base measure $\mathbb{B}$ over the risk-neutral measure with the money market account as numeraire.

uncertainty aversion (i.e., her θ parameter) is infinity, she uses the globally-least-favorable distortion

$$\tilde{\boldsymbol{u}} = -\boldsymbol{\lambda}^{\mathbb{L}}. \tag{25}$$

Optimizing over the terminal wealth, the globally-least-favorable distortion would be equal to the market price of risk using the money market account as numeraire, and the investor would consider the scenario when she receives no compensation above the risk-free rate for bearing any risk. For a fund manager optimizing over the funding ratio, however, this scenario would still be "of value" in the sense that she would still be willing to bear some risk, due to its hedging potential.[9] To achieve the least-favorable distortion, the fund manager has to correct for this and hence her least-favorable distortion becomes the market price of risk of the $\mathbb{L}$ measure over the base measure $\mathbb{B}$.

2.5 Optimal Portfolio Strategy

Since our financial market is complete, there exists a unique investment process which enables the fund manager to achieve the optimal terminal wealth (23). We provide the optimal risk exposure process corresponding to this optimal investment policy in Corollary 1, and the optimal investment process itself in Corollary 2. Both proofs are provided in the Appendix.

Corollary 1. *Under the conditions of Theorem 2, the optimal investment is a continuous re-balancing strategy where the exposures to the N+1 risk sources – as a fraction of wealth – are*

$$\hat{\boldsymbol{\Pi}}_t = \frac{1}{\gamma+\theta}\boldsymbol{\lambda}^{\mathbb{L}} + \boldsymbol{\sigma}_L \tag{26}$$

$$= \frac{1}{\gamma+\theta}\boldsymbol{\lambda} + \left(1 - \frac{1}{\gamma+\theta}\right)\boldsymbol{\sigma}_L. \tag{27}$$

[9] We would like to emphasize here that this hedging potential refers to the liability hedge, i.e., by being exposed to some risk in the above-mentioned scenario the investor can achieve a lower volatility of her terminal funding ratio than by investing everything in the money market account.

The form of the optimal exposure to the risk sources in (26) also reflects the separation of the investment decision into two parts which we described in Section 2.4, i.e., the fund manager first achieves a perfect hedge of the liability (second part of (26)), then she modifies her exposure according to her preferences (first part of (26)). If the correlation between the asset returns and the liability return is zero (i.e., if $\boldsymbol{\sigma}_L = \mathbf{0}$), then the optimal exposure to the risk sources is equal to the scaled market price of risk, i.e., $\boldsymbol{\lambda}^{\mathbb{L}} / (\gamma + \theta)$, which in that case coincides with $\boldsymbol{\lambda} / (\gamma + \theta)$.[10]

In the next corollary we provide the unique optimal investment process, with notation

$$\boldsymbol{\mathcal{B}}(\tau) = \left[\boldsymbol{B}(\tau_1)\boldsymbol{\iota}; \ldots; \boldsymbol{B}(\tau_N)\boldsymbol{\iota}\right], \tag{28}$$

where τ_j denotes the maturity of bond fund j, and $\boldsymbol{B}(t)$ is defined as

$$\boldsymbol{B}(t) = \left(\boldsymbol{I} - \exp\{-\boldsymbol{\kappa} t\}\right)\boldsymbol{\kappa}^{-1}. \tag{29}$$

Corollary 2. *Under the conditions of Theorem 2, the optimal investment is a continuous re-balancing strategy where the fraction of wealth invested in the constant maturity bond funds is*

$$\begin{aligned}\hat{\boldsymbol{\pi}}_{B,t} = &-\frac{1}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\left(\boldsymbol{\sigma}_F'\right)^{-1}\left(\boldsymbol{\lambda}_F - \frac{\lambda_{N+1}}{\sigma_{N+1,S}}\boldsymbol{\sigma}_{F,S}\right)\\ &+\frac{1-\gamma-\theta}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\left(\boldsymbol{\sigma}_F'\right)^{-1}\left(\boldsymbol{\sigma}_{F,L} - \frac{\sigma_{N+1,L}}{\sigma_{N+1,S}}\boldsymbol{\sigma}_{F,S}\right),\end{aligned} \tag{30}$$

and the fraction of wealth invested in the stock market index is

$$\hat{\pi}_{S,t} = \frac{\lambda_{N+1}}{(\gamma+\theta)\,\sigma_{N+1,S}} - \frac{(1-\gamma-\theta)}{(\gamma+\theta)}\frac{\sigma_{N+1,L}}{\sigma_{N+1,S}}. \tag{31}$$

[10] We want to stress that this does not mean that the optimal decision for an investor optimizing over terminal wealth only (instead of over the terminal funding ratio) is equal to $\boldsymbol{\lambda}^{\mathbb{L}} / (\gamma + \theta)$ or $\boldsymbol{\lambda} / (\gamma + \theta)$. The reason of the difference is that even if the liability process is a constant (i.e., $\boldsymbol{\sigma}_L = \mathbf{0}$), the fund manager still hedges against it, and the liability hedge demand is equal to the negative of the intertemporal hedge demand. Thus, the two latter demand components of the fund manager cancel out. In contrast with this, if the investor optimizes over her terminal wealth only, she still has a non-zero intertemporal hedging demand.

In line with Sharpe and Tint (1990) and Hoevenaars, Molenaar, Schotman, and Steenkamp (2008), we find that the optimal portfolio consists of two parts: a speculative portfolio (first line of (30) and first part of (31)), and a liability hedge portfolio (second line of (30) and second part of (31)). The source of the liability hedge demand is the covariance between the asset returns and the liability returns. The higher the covariance between the bond fund returns and the liability returns (i.e., the lower the elements of $\boldsymbol{\sigma}_{F,L}$), the higher the optimal portfolio weight of the constant maturity bond funds. Also: the higher the covariance between the stock market index return and the liability return (i.e., the higher $\sigma_{N+1,L}$), the higher the optimal portfolio weight of the stock market index. Intuitively, a higher covariance between the return of an asset and the liability induces a higher optimal investment in that particular asset, because a higher covariance provides a higher hedging potential and therefore makes the asset more desirable. The second terms within the brackets in both the first and the second line of (30) are correction terms to the speculative constant maturity bond fund demand and the liability hedge constant maturity bond fund demand, respectively. These two correction terms arise due to the covariance between the bond returns and the stock market index return. The higher this covariance (i.e., the lower the elements of $\boldsymbol{\sigma}_{F,S}$), the lower the correction term to both the speculative bond demand and the liability hedge bond demand. The intuition of this is that a higher covariance between the constant maturity bond fund returns and the stock market index return results in the same investment in the stock market index providing a higher exposure to the N factors, and hence to retain the optimal exposure to these factors, the constant maturity bond funds should have lower portfolio weights than with zero covariance.

We find that the optimal asset allocation is determined by the sum of the risk-aversion parameter and the uncertainty-aversion parameter, i.e., by $\gamma + \theta$. This is in line with, e.g., Maenhout (2004), Maenhout (2006), and Horvath, de Jong, and Werker (2016). Intuitively, a robust fund manager behaves the same way as a non-robust, but more risk-averse fund manager.

Table 1. Parameter estimates and standard errors
Estimated parameters and standard errors using Maximum Likelihood. We observed four points weekly on the U.S. zero-coupon, continuously compounded yield curve, corresponding to maturities of 3 months, 1 year, 5 years and 10 years; and the total return index of Datastream's US-DS Market. The observation period is from 5 January 1973 to 29 January 2016.

	Estimated parameter	*Standard error*
$\hat{\kappa}_1$	0.0763***	*0.0024*
$\hat{\kappa}_2$	0.3070***	*0.0108*
$\hat{A}_0$	0.0862***	*0.0013*
$\hat{\lambda}_{F,1}$	−0.1708	*0.1528*
$\hat{\lambda}_{F,2}$	−0.5899***	*0.1528*
$\hat{\lambda}_{N+1}$	0.3180**	*0.1528*
$\hat{\sigma}_{F,11}$	0.0208***	*0.0009*
$\hat{\sigma}_{F,21}$	−0.0204***	*0.0012*
$\hat{\sigma}_{F,22}$	0.0155***	*0.0003*
$\hat{\sigma}_{FS,1}$	−0.0035	*0.0038*
$\hat{\sigma}_{FS,2}$	−0.0121***	*0.0035*
$\hat{\sigma}_{N+1}$	0.1659***	*0.0025*

3 Model Calibration

The two-factor version of our model for the financial market is identical to the model of Horvath, de Jong, and Werker (2016). Hence, we directly adapt the estimates therein for our model parameters. For completeness, we briefly recall the estimation methodology followed by Horvath, de Jong, and Werker (2016). The model is calibrated to U.S. market data using the Kalman filter and Maximum Likelihood. The data consist of weekly observations of the 3-month, 1-year, 5-year, and 10-year points of the yield curve, and Datastream's U.S. Stock Market Index. The observation period is from 1 January 1973 to 29 January 2016. The starting values of the filtered factors are equal to their long-term means. The parameter estimates can be found in Table 1.

All model parameters are estimated with small standard errors, the only exception being the market price of risk. This confirms the validity of our model setup, namely, that the fund manager is uncertain about the physical probability measure, which – together with her considering only equivalent probability measures – is equivalent to saying that she is uncertain about the market price of risk.

As a proxy for the liability process, we follow van Binsbergen and Brandt (2016) and use the price of a zero-coupon bond. Intuitively, we think of the liability as a rolled-over asset with constant duration. As of the duration itself, we use 15 years, which is approximately the average duration of U.S. pension fund liabilities (van Binsbergen and Brandt (2016)). Then, the volatility parameters of the liabilities are

$$\boldsymbol{\sigma}_{F,L} = -\boldsymbol{\sigma}_F' \boldsymbol{B}\,(15)\,\boldsymbol{\iota} \tag{32}$$

and

$$\sigma_{N+1,L} = 0. \tag{33}$$

Using our parameter estimates in Table 1, the estimated volatility vector of the liability process is

$$\boldsymbol{\sigma}_L' = \begin{bmatrix} -0.1201 & -0.0501 & 0 \end{bmatrix}. \tag{34}$$

4 Detection Error Probabilities

In the previous section we estimated the model parameters related to the financial market, based on historical data. Calibrating the parameters related to the preferences, i.e., the risk-aversion parameter γ and the uncertainty-aversion parameter θ, is less straightforward.

There is no agreement in the literature about what the relative risk aversion of a representative investor precisely is, but the majority of the literature considers risk aversion parameters between 1 and 5 to be reasonable. Several studies attempt to estimate what a reasonable risk aversion value is, usually by using consumption data or by conducting experiments. Friend and Blume (1975) estimate the relative risk aversion parameter to be around 2; Weber (1975) and Szpiro (1986) estimate it to be between about 1.3 and 1.8; the estimates of Hansen and Singleton (1982) and Hansen and Singleton (1983) are 0.68–0.97 and 0.26-2.7, respectively; using nondurable consumption data, Mankiw (1985) estimates the relative risk aversion to be 2.44-5.26, and using durable goods consumption data it to be 1.79-3.21; Barsky, Juster, Kimball, and Shapiro (1997) use an experimental survey to estimate the relative risk aversion parameter of the subjects, the mean of which turns out to be 4.17; while in the study of Halek and Eisenhauer (2001) the mean relative risk aversion is 3.7. Later in this section we vary the risk-aversion parameter between 1 and 5 to see its effect on the optimal investment decision.

Calibrating the uncertainty-aversion parameter θ is even more complicated than the calibration of the risk aversion. Ever since the seminal paper of Anderson, Hansen, and Sargent (2003), the most puzzling questions in the robust asset pricing literature are related to how to quantify uncertainty aversion, and how much uncertainty is reasonable. Anderson, Hansen, and Sargent (2003) propose a theory to address these problems based on the Detection Error Probabilities. They assume that the investor can observe a sample of historical data, and she performs a likelihood ratio test to decide whether these data are generated by a data-generating process corresponding to the base measure $\mathbb{B}$, or by a data-generating process corresponding to the alternative measure $\mathbb{U}$. Based on this test, the investor is assumed to be able to correctly guess the true physical probability measure in $p\%$ of the cases, i.e., she is wrong in $(1-p)\,\%$ of the cases. Making this $(1-p)\,\%$ equal to the probability of making an error based on the likelihood ratio test, we can disentangle the risk aversion and the uncertainty aversion. The question of what a reasonable level of $(1-p)\%$, i.e., the Detection Error Probability, is, is the subject of an active line of research. Anderson, Hansen, and Sargent (2003) suggest that Detection Error Probabilities between 10% and 30% are plausible. Now we give the formal definition of the Detection Error Probability.

Definition 1. *The Detection Error Probability (DEP) is defined as*

$$DEP = \frac{1}{2}P^{\mathbb{B}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}} < 0\right) + \frac{1}{2}P^{\mathbb{U}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}} > 0\right). \tag{35}$$

Following the reasoning of Horvath, de Jong, and Werker (2016), we can express the

Detection Error Probability in closed form. This is stated in Theorem 3 and in Corollary 3.

Theorem 3. *Assume that the fund manager continuously observes the prices of N constant maturity bond funds, and the level of the stock market index. The observation period lasts from $t - H$ to the moment of observation, t. Then, the detection error probability of the fund manager for given $\mathbb{U}$ is*

$$DEP = 1 - \Phi\left(\frac{1}{2}\sqrt{\int_{t-H}^{t} \|\boldsymbol{u}(s)\|^2 \mathrm{d}s}\right), \tag{36}$$

where $\boldsymbol{u}(\cdot)$ is defined in (14).

Substituting the least-favorable distortion (24) into (36), we obtain the closed-form expression in Corollary 3.

Corollary 3. *Assume that the conditions of Theorem 3 hold. Then, the detection error probability of the fund manager for the least-favorable $\mathbb{U}$ is*

$$DEP = 1 - \Phi\left(\frac{\theta}{2(\gamma+\theta)}\sqrt{H}\|\boldsymbol{\lambda}^{\mathbb{L}}\|\right). \tag{37}$$

The Detection Error Probability used by a fund manager who is not uncertainty-averse at all (i.e., whose θ parameter is zero) is 0.5. That is, she might as well flip a coin to distinguish between two probability measures instead of performing a likelihood-ratio test on a sample of data. On the other hand, a fund manager with an uncertainty aversion parameter of infinity uses the lowest possible Detection Error Probability, which is $1 - \Phi\left((1/2)\sqrt{H}\|\boldsymbol{\lambda}^{\mathbb{L}}\|\right)$.[11]

We assume that the observation period of the investor is 42 years,[12] and that her Detection Error Probability is 10%. Given that she has access to a relatively long sample of

[11] One might expect that the lowest possible Detection Error Probability is zero, which would mean that the fund manager knows the physical probability measure precisely. However, as (37) also shows, this is only the case if the length of her observation period is infinity, i.e., $H = \infty$. If her observation period is finite, the limitation of available data will always result in the fund manager not being able to correctly tell apart two probability measures in 100% of the cases.

[12] We use 42 years of market data to estimate our model parameters, thus it is a reasonable assumption that the fund manager has access to the same length of data. Even though our observation frequency is weekly, assuming that the fund manager can observe data continuously does not cause a significant difference.

data, and that she can observe the prices continuously, our choice of 10% as the Detection Error Probability is justifiable.

5 Policy Evaluation

Now we use our parameter estimates from Section 3 to analyze the effects of robustness on the optimal ALM decision, if the fund manager has access to a money market account, to two constant maturity bond funds with 1 and 15 years of maturities, and to a stock market index. Her investment horizon is 15 years. We show the quantitative relationship between the level of uncertainty aversion and the optimal exposure to the different sources of risk, and also the optimal portfolio weights. We find that regardless of the risk attitude of the fund manager, robustness substantially changes the magnitude of her optimal portfolio weights. Generally speaking, robustness translates into making more conservative ALM decisions. More concretely, while investing significantly less in the stock market index and the constant maturity bond fund with the shorter (1 year) maturity, the fund manager increases her investments in the constant maturity bond fund with the same maturity as her investment horizon (15 years) and in the money market account.

Figure 1 shows the optimal exposure of the fund manager to the three risk sources for different levels of risk aversion and uncertainty aversion. Her uncertainty aversion is measured by the Detection Error Probability. If she uses a Detection Error Probability of 50%, then she is not uncertainty-averse at all, while if she uses the lowest possible level of Detection Error Probability (which in our case is 2.08%), her uncertainty aversion is infinitely high. Figure 2 shows the optimal portfolio weights, which enable the fund manager to achieve the optimal exposure to the risk sources. Due to the inherent nature of affine term structure models, our fund manager takes a high short position in the money market account, and she uses this money to obtain a highly leveraged long position in the 1-year constant maturity bond fund.[13] In Table 2 we provide the numerical values of the optimal

[13]If there are no constraints on the position which the fund manager can take in the different assets, it is a common finding in the literature that she takes extremely large short and longe positions to achieve the

exposures and the optimal portfolio weights for different levels of risk aversion of robust and non-robust fund managers. A non-robust fund manager applies a Detection Error Probability of 50%, while a robust fund manager assumes a Detection Error Probability of 10%.

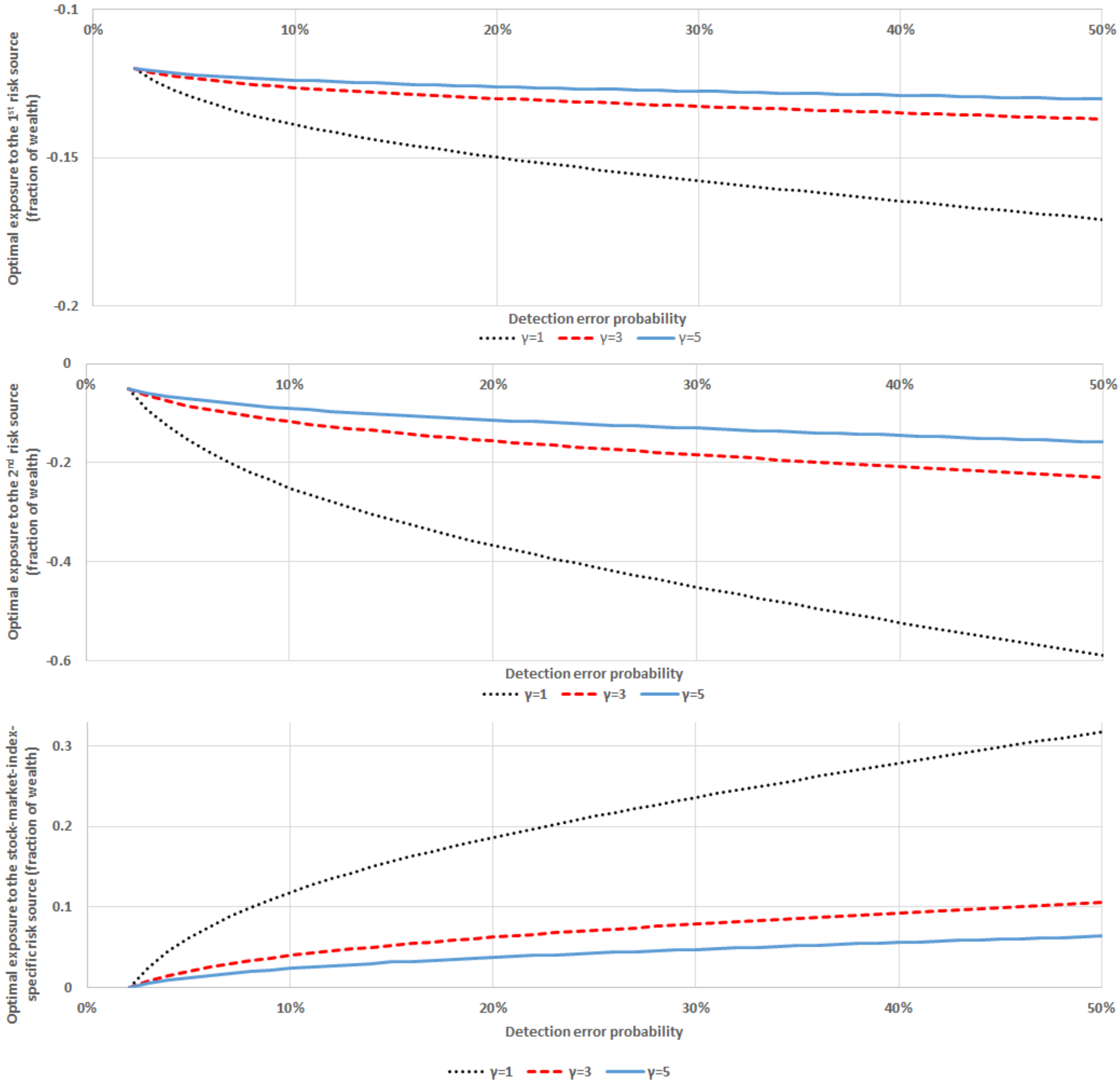
We find that the fund manager – regardless of her risk-aversion and uncertainty-aversion – always chooses a negative exposure to the first two risk sources. This is intuitive, since these two risk sources have negative market prices of risk. But it is less straightforward why an infinitely risk-averse or an infinitely uncertainty-averse fund manager decides to take a strictly negative exposure to these risk sources, instead of opting for zero exposure. Given an infinitely high risk or uncertainty aversion, the fund manager's optimal ALM decision is to obtain a perfect hedge for the liabilities. Since the liabilities are a linear combination of the first two factors, she will expose herself to these factors to an extent which is equal to the exposure of the liabilities to these factors. This can be shown analytically by taking the limit of the right-hand-side of (26) as $\theta \to \infty$. Were the fund manager optimizing over the terminal wealth instead of the terminal funding ratio, her optimal exposure to the first two factors would be also strictly negative, but for a different reason: in this case her goal would be to achieve an exposure equal to that of a zero-coupon bond with the maturity of her investment horizon, thus eliminating risk and uncertainty totally, since she will receive the face value of the zero-coupon bond at the end of her investment horizon for sure. Looking at her decision from a different angle: her myopic demand for the constant maturity bond funds would be zero, and the entire total (strictly positive) demand would be due to the intertemporal hedging demand. In our case, when the fund manager optimizes over the terminal funding ratio, and she is either infinitely risk-averse or infinitely uncertainty-averse, her myopic demand for the constant maturity bond funds is zero, and her total demand is due to the liability hedge demand.

The exposure of an infinitely risk-averse or infinitely uncertainty-averse fund manager to the stock-market-index-specific source of risk is zero, because we assumed that the liability

optimal risk exposures. See, e.g., Brennan and Xia (2002), Figure 4 and Figure 6.

Figure 1. Optimal exposure to the risk sources

Optimal exposure of the fund manager to the risk sources as a function of the Detection Error Probability (DEP), for different levels of relative risk aversion. We use our parameter estimates in Table 1, and assume that the liability value is always equal to a zero-coupon bond with 15 years of maturity. The fund manager's investment horizon is 15 years. A DEP of 50% corresponds to a non-uncertainty-averse fund manager, while a DEP of 2.08% corresponds to a fund manager with infinitely high uncertainty aversion.



value is always equal to the value of a zero-coupon bond with 15 years of maturity, and the value of such a bond is not influenced by the stock-market-index-specific risk source. In absence of this assumption, the optimal exposure of an infinitely risk-averse or an infinitely

Figure 2. Optimal portfolio weights

Optimal portfolio weights as a function of the Detection Error Probability (DEP), for different levels of relative risk aversion. We use our parameter estimates in Table 1, and assume that the liability value is always equal to a zero-coupon bond with 15 years of maturity. The fund manager's investment horizon is 15 years. A DEP of 50% corresponds to a non-uncertainty-averse fund manager, while a DEP of 2.08% corresponds to a fund manager with infinitely high uncertainty aversion.

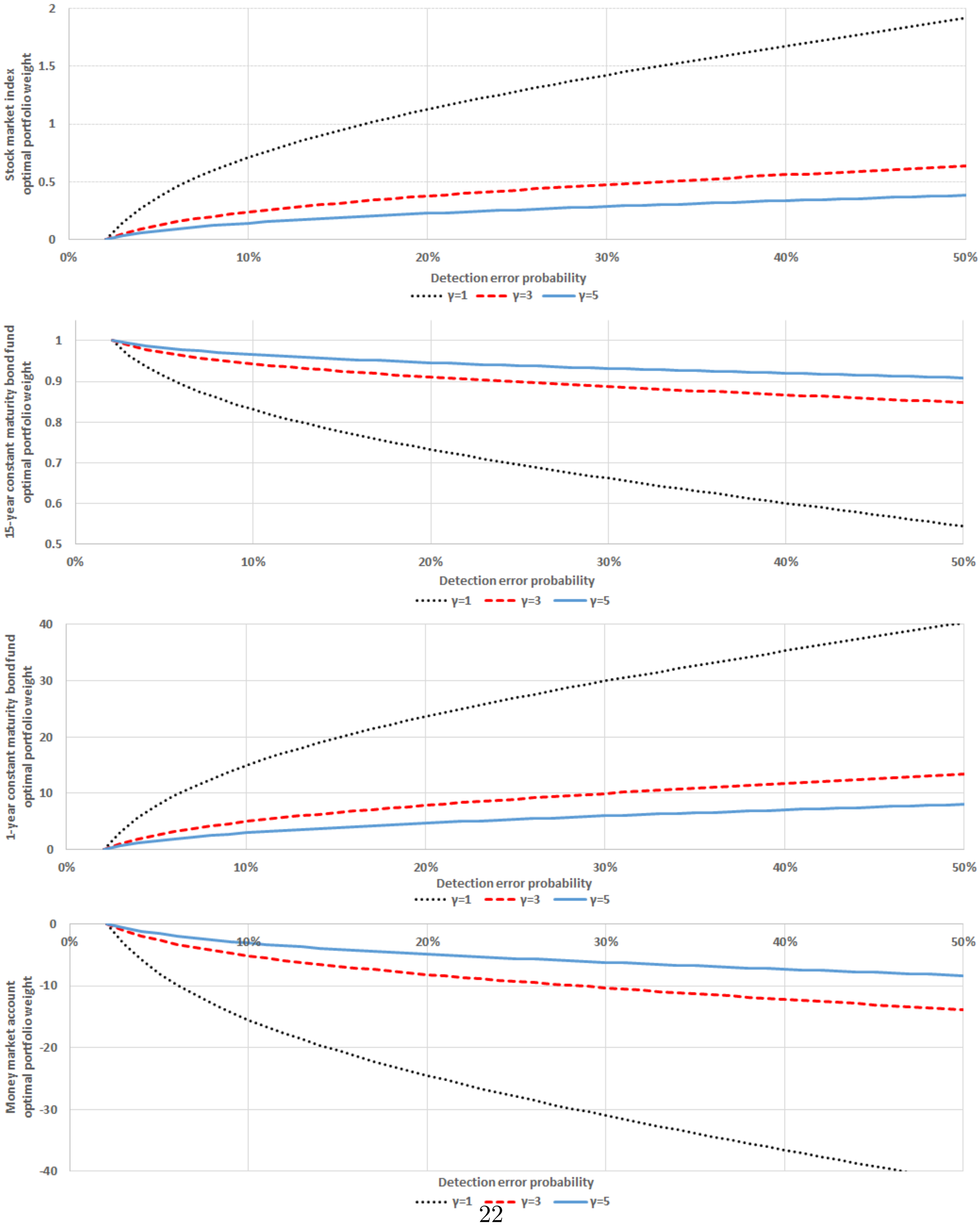


Table 2. Optimal risk exposures and portfolio weights
Optimal exposures and portfolio weights for different levels of risk aversion of robust and non-robust fund managers. The non-robust exposures and portfolio weights correspond to a Detection Error Probability of 50%, while their robust counterparts assume a Detection Error Probability of 10%. Optimal exposures and portfolio weights are calculated using the assumptions of Section 3.

	Optimal portfolio weights and exposures					
	$\gamma = 1$		$\gamma = 3$		$\gamma = 5$	
	Non-robust	Robust	Non-robust	Robust	Non-robust	Robust
Π_1	-0.17	-0.14	-0.14	-0.13	-0.13	-0.12
Π_2	-0.59	-0.25	-0.23	-0.12	-0.16	-0.09
Π_3	0.32	0.12	0.11	0.04	0.06	0.02
$\pi_{B(1)}$	4030%	1494%	1343%	498%	806%	299%
$\pi_{B(15)}$	54%	83%	85%	94%	91%	97%
π_S	192%	71%	64%	24%	38%	14%
π_{MMA}	-4176%	-1548%	-1392%	-516%	-835%	-310%

uncertainty-averse fund manager would be $\sigma_{N+1,L}$, due to the reasoning in the previous paragraph.[14] If the fund manager were optimizing over the terminal wealth instead of the terminal funding ratio, her optimal exposure to the stock-market-index-specific source of risk would be zero even without our previous assumption about the liabilities, because her total demand would be equal to the intertemporal hedging demand, and the stock-market-specific risk source cannot be hedged intertemporally.

We also find that both a higher risk aversion and a higher uncertainty aversion result in a lower optimal exposure in absolute value to the risk sources, and in order to achieve this lower exposure the fund manager has lower optimal portfolio weight (again, in absolute value) in the constant maturity bond fund with 1 year of maturity and in the stock market index. Her optimal portfolio weight for the constant maturity bond fund with 15 years of maturity is, on the other hand, an increasing function of both risk aversion and uncertainty aversion, due to its strong liability hedge potential.[15]

[14]The stock-market-index-specific risk source affects the liability of a pension fund if, e.g., the pension payout is linked to the industry wage level, and the industry wage level is affected by the stock-market-index-specific risk source via, e.g., performance-dependent wage schemes.

[15]The fact that the effects of risk aversion and uncertainty aversion have the same sign can directly be deduced from the risk-aversion parameter and the uncertainty-aversion parameter appearing only as a sum in the optimal portfolio weights in (30) and (31).

Accounting for uncertainty aversion has a substantial effect on both the optimal exposures to the risk sources and the optimal portfolio weights. We find that the optimal exposures (in absolute value) to each of the risk sources are a decreasing function of the level of robustness. I.e., the more uncertainty-averse the fund manager, the less exposure she finds optimal to each risk source. The decrease in the optimal exposure is especially substantial in the case of the stock-market-specific risk source (more than 60%) and the factor-specific risk source with a higher (absolute value of) market price of risk (more than 40%), while it is less significant in the case of the factor-specific risk source with a lower (absolute value of) market price of risk. The intuition behind this is that as the uncertainty aversion of the fund manager increases and approaches infinity, the optimal exposures approach the volatility loadings of the liability, concretely -0.1201, -0.0501, and 0. The optimal exposure of a non-robust fund manager with log-utility (i.e., $\gamma = 1$) to the first risk source is -0.17, hence there is not much scope for reduction in the magnitude of this exposure.[16] In contrast with this, the exposure of a non-robust fund manager with log-utility to the second and third risk sources is relatively higher in magnitude (-0.59 and 0.32) due to their higher market price of risk (in absolute terms). Moreover, the magnitudes of liability exposures to the second and third risk sources are lower than that of the first risk source (-0.0501 and 0, respectively), therefore there is more scope for reduction in the optimal exposure as the uncertainty aversion increases. The lower exposure levels due to robustness translate to a lower demand for the constant maturity bond fund with 1 year of maturity (more than 62% decrease) and for the stock-market-index (also more than 62%). The demand for the constant maturity bond fund with a maturity equal to the investment horizon of the fund manager, however, increases with the level of robustness, due to its liability hedging potential.

[16] I.e., even if her level of robustness is infinity, her optimal exposure would still be -0.1201, which is equal to the liability exposure to this risk source.

6 Conclusion

We have shown that model uncertainty has significant effects on Asset Liability Management decisions. A fund manager who derives utility from the terminal funding ratio and who accounts for model uncertainty does not necessarily have an intertemporal hedging demand component, even though the investment opportunity set is stochastic. Robustness substantially changes the optimal exposures to the risk sources: as the level of uncertainty aversion increases, the optimal exposures approach the liability exposures to the respective risk sources. In the case of a two-factor affine term structure model and an additional, stock-market-specific risk source, optimal exposures can change by more than 60% due to robustness. These changes in the risk exposures translate into substantial changes in the optimal portfolio weights as well: while a robust fund manager invests less in the constant maturity bond fund with a relatively short maturity and also in the stock market index, she increases her investment in the constant maturity bond fund with a maturity equal to her investment horizon to make use of its liability hedge potential.

In our model we assume that the fund manager acts in the best interest of the pension holders, and she does not consider the interest of the pension fund sponsors. Extending the model to include the negative utility derived by the pension fund sponsors from having to contribute to the fund can provide further insight into the effects of model uncertainty on more complex Asset Liability Management decisions.

We also assume that the fund manager's uncertainty tolerance parameter is linear in the value function, hence the solution of our robust dynamic Asset Liability Management problem is homothetic, and it can be obtained in closed form. There is, however, an active and current debate in the literature whether this functional form of the uncertainty tolerance is justifiable. Solving our robust dynamic Asset Liability Managent problem with a differently formulated uncertainty tolerance parameter is another fruitful line of further research.

Appendix

Proof of Theorem 1. Substituting (17) and (18) into (15), the value function at time t satisfies

$$\begin{aligned} V_t\left[\left(\frac{A}{L}\right)_t\right] &= \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}}{1-\gamma}+\int_t^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}V_s\left[\left(\frac{A}{L}\right)_s\right]\mathrm{d}s\right\} \\ &= \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}}{1-\gamma}\right\}+\mathrm{E}_t^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}V_s\left[\left(\frac{A}{L}\right)_s\right]\mathrm{d}s\right\} \\ &\quad -\int_0^t \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2 V_s\left[\left(\frac{A}{L}\right)_s\right]}{2\theta}\mathrm{d}s, \end{aligned} \tag{38}$$

where $\left(\frac{A}{L}\right)_T$ and $\mathbb{U}$ denote the optimal terminal wealth and least-favorable physical measure, respectively. Introduce the square-integrable martingales, under $\mathbb{U}$,

$$M_{1,t} = \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}}{1-\gamma}\right\}, \tag{39}$$

$$M_{2,t} = \mathrm{E}_t^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}V_s\left[\left(\frac{A}{L}\right)_s\right]\mathrm{d}s\right\}. \tag{40}$$

The martingale representation theorem (see, e.g., Karatzas and Shreve (1991), pp. 182, Theorem 3.4.15) states that there exist square-integrable stochastic processes $\boldsymbol{Z}_{1,t}$ and $\boldsymbol{Z}_{2,t}$ such that

$$M_{1,t} = \mathrm{E}_0^{\mathbb{U}}\left\{\exp(-\delta T)\frac{\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}}{1-\gamma}\right\}+\int_0^t \boldsymbol{Z}_{1,s}'\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}, \tag{41}$$

$$M_{2,t} = \mathrm{E}_0^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}V_s\left[\left(\frac{A}{L}\right)_s\right]\mathrm{d}s\right\}+\int_0^t \boldsymbol{Z}_{2,s}'\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}. \tag{42}$$

Substituting in (38), we can express the dynamics of the value function as

$$\mathrm{d}V_t\left[\left(\frac{A}{L}\right)_t\right] = -\frac{(1-\gamma)\,\|\boldsymbol{u}\,(t)\,\|^2}{2\theta}V_t\left[\left(\frac{A}{L}\right)_t\right]\mathrm{d}t+(Z_{1,t}+Z_{2,t})'\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{U}}. \tag{43}$$

This linear backward stochastic differential equation with the terminal condition $V_T\left[\left(\frac{A}{L}\right)_T\right] = \exp(-\delta T)\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}/(1-\gamma))$ has an explicit particular solution (see, e.g., Pham (2009), pp. 141-142). The unique solution to (43) is given by

$$\Gamma_t V_t\left[\left(\frac{A}{L}\right)_t\right] = \mathrm{E}_t^{\mathbb{U}}\left\{\Gamma_T \exp(-\delta T)\frac{\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}}{1-\gamma}\right\}, \tag{44}$$

where Γ_t solves the linear differential equation

$$\mathrm{d}\Gamma_t = \Gamma_t \frac{(1-\gamma)\,\|\boldsymbol{u}\,(t)\,\|^2}{2\theta}\mathrm{d}t; \qquad \Gamma_0 = 1, \tag{45}$$

i.e.,

$$\Gamma_t = \exp\left(\int_0^t \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}\mathrm{d}s\right). \tag{46}$$

Substituting into (44), we obtain the closed-form solution of the value function as

$$V_t(X_t) = \mathrm{E}_t^{\mathbb{U}}\left\{\exp\left(\int_t^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}\mathrm{d}s - \delta T\right)\frac{X_T^{1-\gamma}}{1-\gamma}\right\}, \tag{47}$$

with $\left[\left(\frac{A}{L}\right)_t\right]$ and $\mathbb{U}$ representing the optimal funding ratio and the least-favorable physical probability measure. As a result, we obtain (21). □

Proof of Theorem 2. The first step of the optimization is to determine the optimal terminal wealth, given the budget constraint. In order to determine the optimal terminal wealth, we form the Lagrangian from (21) and (22). This Lagrangian is

$$\begin{aligned}\mathcal{L}(A_0) &= \inf_{\mathbb{U}}\sup_{A_T}\left\{\mathrm{E}^{\mathbb{U}}\exp\left(\frac{1-\gamma}{2\theta}\int_0^T\|\boldsymbol{u}(t)\|^2\mathrm{d}t-\delta T\right)\frac{\left(\frac{A_T}{L_T}\right)^{1-\gamma}}{1-\gamma}\right.\\ &\quad\left.-y\left[\mathrm{E}^{\mathbb{L}}\left(\frac{A_T}{L_T}\right)-\frac{A_0}{L_0}\right]\right\}\\ &= \inf_{\mathbb{U}}\sup_{A_T}\left\{\mathrm{E}^{\mathbb{L}}\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{L}}\right)_T\exp\left[\frac{1-\gamma}{2\theta}\int_0^T\|\boldsymbol{u}(t)\|^2\mathrm{d}t-\delta T\right]\frac{\left(\frac{A_T}{L_T}\right)^{1-\gamma}}{1-\gamma}\right.\\ &\quad\left.-y\left[\mathrm{E}^{\mathbb{L}}\left(\frac{A_T}{L_T}\right)-\frac{A_0}{L_0}\right]\right\},\end{aligned}\tag{48}$$

where y is the Lagrange-multiplier. Now we solve the inner optimization, taken $\mathbb{U}$ as given. The first-order condition for the optimal terminal funding ratio, denoted by $\hat{A}_T/L_T$, is

$$\frac{\hat{A}_T}{L_T}=\frac{y^{-\frac{1}{\gamma}}}{\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{L}}\right)_T^{-\frac{1}{\gamma}}\exp\left\{-\frac{1}{\gamma}\left[\frac{1-\gamma}{2\theta}\int_0^T\|\boldsymbol{u}(t)\|^2\mathrm{d}t-\delta T\right]\right\}}.\tag{49}$$

After substituting the optimal terminal funding ratio into the budget constraint, we obtain the Lagrangian as

$$y^{-\frac{1}{\gamma}}=\frac{A_0}{L_0\mathrm{E}^{\mathbb{L}}\left\{\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{L}}\right)_T^{\frac{1}{\gamma}}\exp\left\{\frac{1}{\gamma}\left[\frac{1-\gamma}{2\theta}\int_0^T\|\boldsymbol{u}(t)\|^2\mathrm{d}t-\delta T\right]\right\}\right\}}.\tag{50}$$

Together with the Radon-Nikodym derivative

$$\begin{aligned}\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{L}}\right)_t =\exp\Bigg\{&\int_0^t\left(\boldsymbol{\lambda}^{\mathbb{L}}+\boldsymbol{u}(s)\right)'\mathrm{d}\boldsymbol{W}_s^{\mathbb{L}}\\ &+\int_0^t\left[\left(\boldsymbol{\lambda}^{\mathbb{L}}+\boldsymbol{u}(s)\right)'\boldsymbol{\sigma}_L-\frac{1}{2}\left(\|\boldsymbol{\lambda}^{\mathbb{L}}+\boldsymbol{\sigma}_L+\boldsymbol{u}(s)\|^2-\|\boldsymbol{\sigma}_L\|^2\right)\right]\mathrm{d}s\Bigg\},\end{aligned}\tag{51}$$

we substitute the Lagrangian back into (49) to determine the optimal terminal funding ratio as

$$\frac{\hat{A}_T}{L_T} = \frac{A_0}{L_0} \frac{\exp\left[\frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}\right]}{\mathrm{E}^{\mathbb{L}} \exp\left[\frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}\right]}. \tag{52}$$

Multiplying both sides by L_T, we obtain (23), and this proves the first part of Theorem 2. Now we solve the outer optimization problem. Substituting the optimal terminal wealth back into the value function, we obtain

$$\begin{aligned} V_0(A_0) =& \frac{\left(\frac{A_0}{L_0}\right)^{1-\gamma}}{1-\gamma} \exp\left(\frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T - \frac{1}{2}\int_0^T \left[\|\boldsymbol{\lambda}^{\mathbb{L}} + \boldsymbol{\sigma}_L + \boldsymbol{u}(t)\|^2 - \|\boldsymbol{\sigma}_L\|^2\right] \mathrm{d}t\right) \\ & \times \exp\left(\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{2\gamma}\int_0^T \|\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\|^2 \mathrm{d}t\right). \end{aligned} \tag{53}$$

Now we can write down the first-order condition for $\boldsymbol{u}(t)$ and we obtain

$$\boldsymbol{u}(t) = -\frac{\theta}{\gamma+\theta}\boldsymbol{\lambda}^{\mathbb{L}}, \tag{54}$$

which is indeed the same as (24). This completes the proof. □

Proof of Corollary 1. The optimal wealth process can be written as

$$\hat{A}_t = L_t \mathrm{E}_t^{\mathbb{L}}\left(\frac{\hat{A}_T}{L_T}\right). \tag{55}$$

Substituting the optimal terminal wealth (23) into (55), the optimal wealth at time t becomes

$$\begin{aligned} \hat{A}_t =& \frac{A_0}{L_0} L_t \exp\left[\frac{1}{\gamma}\int_0^t \left(\hat{\boldsymbol{u}}(s) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \mathrm{d}\boldsymbol{W}_s^{\mathbb{L}} + \frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(s) + \boldsymbol{\lambda}\right)' \boldsymbol{\sigma}_L \mathrm{d}s\right] \\ & \times \frac{\exp\left(\frac{1}{2\gamma^2}\int_t^T \|\hat{\boldsymbol{u}}(s) + \boldsymbol{\lambda}^{\mathbb{L}}\|^2 \mathrm{d}s\right)}{\mathrm{E}^{\mathbb{L}} \exp\left[\frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}} + \frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \boldsymbol{\sigma}_L \mathrm{d}t\right]}. \end{aligned} \tag{56}$$

Substituting the solution of the stochastic differential equation (4) for L_t, the optimal wealth at time t becomes

$$\begin{aligned}\hat{A}_t =& A_0 \exp\Bigg\{\int_0^t \left(\frac{1}{\gamma}\left(\hat{\boldsymbol{u}}_F + \boldsymbol{\lambda}_F^{\mathbb{L}}\right) + \boldsymbol{\sigma}_{F,L} + \boldsymbol{\sigma}_F' \boldsymbol{B}\,(t-s)\,\boldsymbol{\iota}\right)' \mathrm{d}\boldsymbol{W}_{F,s}^{\mathbb{L}} \\ &+ \int_0^t \left(\frac{1}{\gamma}\left(\hat{\boldsymbol{u}}_{N+1} + \boldsymbol{\lambda}_{N+1}^{\mathbb{L}}\right) + \boldsymbol{\sigma}_{N+1,L}\right)' \mathrm{d}\boldsymbol{W}_{N+1,s}^{\mathbb{L}} + \int_0^t \boldsymbol{\iota}'\boldsymbol{B}\,(t-s)\,\boldsymbol{\sigma}_F\boldsymbol{\sigma}_L \mathrm{d}s \\ &+ \left(A_0 + \boldsymbol{\iota}'\left(\boldsymbol{\mu}_F^{\mathbb{L}} - \boldsymbol{\sigma}_F\boldsymbol{\sigma}_L\right)\right)t + \boldsymbol{\iota}'\boldsymbol{B}\,(t)\left(\boldsymbol{F}_0 - \left(\boldsymbol{\mu}_F^{\mathbb{L}} - \boldsymbol{\sigma}_F\boldsymbol{\sigma}_L\right)\right) + \frac{1}{2}\|\boldsymbol{\sigma}_L\|^2 t \\ &+ \frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}\,(s) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \boldsymbol{\sigma}_L \mathrm{d}s\Bigg\} \times \frac{\exp\left(\frac{1}{2\gamma^2}\int_t^T \|\hat{\boldsymbol{u}}\,(s) + \boldsymbol{\lambda}^{\mathbb{L}}\|^2 \mathrm{d}s\right)}{\mathrm{E}^{\mathbb{L}} \exp\left[\frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}\,(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}} + \frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}\,(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \boldsymbol{\sigma}_L \mathrm{d}t\right]}.\end{aligned} \tag{57}$$

Applying Ito's lemma, the optimal wealth dynamics can be expressed as

$$\mathrm{d}\hat{A}_t = ...\mathrm{d}t + \frac{\hat{A}_t}{\gamma}\left(\hat{\boldsymbol{u}} + \boldsymbol{\lambda}^{\mathbb{L}} + \gamma\boldsymbol{\sigma}_L\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}. \tag{58}$$

Substituting the least-favorable distortion for $\hat{\boldsymbol{u}}$, we obtain

$$\mathrm{d}\hat{A}_t = ...\mathrm{d}t + \hat{A}_t\left(\frac{1}{\gamma+\theta}\boldsymbol{\lambda}^{\mathbb{L}} + \boldsymbol{\sigma}_L\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}. \tag{59}$$

From (59) the optimal risk exposures follow directly. This completes the proof. □

Proof of Corollary 2. Using the portfolio weights $\boldsymbol{\pi}_{B,t}$ and $\boldsymbol{\pi}_{\boldsymbol{S},\boldsymbol{t}}$, the optimal wealth dynamics can be expressed as

$$\mathrm{d}\hat{A}_t = ...\mathrm{d}t + \hat{A}_t\left(-\boldsymbol{\pi}_{B,t}'\mathcal{B}'\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,t}^{\mathbb{L}} + \pi_{S,t}\left[\boldsymbol{\sigma}_{F,S}'; \sigma_{N+1,S}\right]\mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}\right). \tag{60}$$

Moreover, the optimal wealth dynamics can equivalently be written as (59). Then, due to the martingale representation theorem we can write down a system of N+1 equations by making the exposures to the N+1 risk sources in (59) and (60) equal to each other. Solving this equation system, we indeed obtain the optimal portfolio weights (30) and (31). This completes the proof. □

Proof of Theorem 3. From (11) and (14) the Radon-Nikodym derivative $d\mathbb{B}/d\mathbb{U}$ follows directly. Substituting this into Definition 1 and evaluating the expectations, the closed-form solution in (36) is obtained. □

Proof of Corollary 3. Substituting the least-favorable distortions (24) into (36), (37) is immediately be obtained. □

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