



Network for Studies on Pensions, Aging and Retirement

Parameter Uncertainty

The Missing Piece of the Liquidity Premium Puzzle?

Ferenc Horvath

NETSPAR ACADEMIC SERIES

DP 11/2016-056

Parameter Uncertainty: The Missing Piece of the Liquidity Premium Puzzle?

Ferenc Horvath*

Tilburg University

November 14, 2016

Abstract

I analyze a dynamic investment problem with stochastic transaction cost and parameter uncertainty. I solve the problem numerically and obtain the optimal consumption and investment policy and the least-favorable transaction cost process. Using reasonable parameter values, I confirm the liquidity premium puzzle, i.e., the representative agent model (without robustness) produces a liquidity premium which is by a magnitude lower than the empirically observed value. I show that my model with robust investors generates an additional liquidity premium component of 0.05%-0.10% (depending on the level of robustness) for the first 1% proportional transaction cost, and thus it provides a partial explanation to the liquidity premium puzzle. Additionally, I provide a novel non-recursive representation of discrete-time robust dynamic asset allocation problems with transaction cost, and I develop a numerical technique to efficiently solve such investment problems.

JEL classification: C61, G11, G12

Keywords: liquidity, dynamic asset allocation, robustness, uncertainty, ambiguity, liquidity premium puzzle

*Tilburg University, Department of Finance, P.O. Box 90153, 5000 LE Tilburg, the Netherlands. Tel.: +31 13 466 4056. E-mail address: f.horvath@tilburguniversity.edu

1 Introduction

One of the many asset pricing puzzles in the finance literature is the liquidity premium puzzle. Theoretical work finds that transaction cost only has a second-order effect on expected returns, while empirical research documents a first-order effect. For example, the seminal work of Constantinides (1986) finds that a 1% proportional transaction cost on stocks increases the expected stock return only by 0.2%. On the other hand, Amihud and Mendelson (1986) empirically find that for a 1% proportional transaction cost, the liquidity premium is about 2%, based on NYSE data.

In this paper, I approach the liquidity premium puzzle from a new angle. A key feature of dynamic asset allocation problems with transaction cost is the set of assumptions about the transaction cost. In my model I not only allow the transaction cost to be stochastic, but I also relax the assumption that the investor knows the precise probability distribution of future transaction costs. She takes this uncertainty about the transaction cost process into account when she solves a dynamic investment problem, and she makes robust decisions with respect to consumption and asset allocation.

A robust investor is uncertain about one or several parameters of the underlying model. Acknowledging this uncertainty, she makes investment decisions that work well not only if the model that she had in mind is correct, but which work reasonably well even if her model turns out to be misspecified. In this paper I use the minimax approach of Anderson, Hansen, and Sargent (2003). The investor is uncertain about the (future) transaction costs in the underlying model, but she has a *base transaction cost level* in mind, which she considers to be the most reasonable. Since she is uncertain about the true future transaction costs, she considers other *alternative transaction cost levels* as well. But instead of setting an explicit constraint on the transaction costs to be considered, I introduce a penalty term in the goal function. If the transaction cost level under consideration is very different from the base transaction cost level, the investor is penalized through this function. Since she wants to make robust decisions, she prepares for the worst-case scenario, i.e., she uses the

alternative transaction cost level which results in the lowest value function (including the penalty term). The investor's tolerance towards uncertainty is expressed by the uncertainty-tolerance parameter, which multiplies the penalty term.

I solve the robust dynamic optimization problem numerically, using parameter estimates of the existing literature. I compare the optimal consumption and asset allocation decision of a robust investor – using several different levels of uncertainty aversion – to the decisions of an otherwise equivalent non-robust investor, and I determine the effects of robustness on asset allocation. Uncertainty-aversion affects the investor's decision about asset allocation through two channels, and these influence the investor's decision in opposite directions. On the one hand, a robust investor is induced to buy more of the risky asset now, so that she will not have to buy it later at an unknown (and potentially high) transaction cost. At the same time, she is also induced to buy less risky asset now, to avoid having to sell these asset in the future for consumption purposes – again, at an unknown and potentially high transaction cost. The argument is the same if the investor wants to sell some of her risky asset, instead of buying. Besides the parameter values of the model, the current value of the transaction cost and the inherited investment ratio jointly determine which of the two effects will dominate.

After showing the effects of robustness on the optimal asset allocation, I introduce the definitions of liquidity premium and its several components, and I study the asset pricing implications of robustness in my model. Concretely, I show that even a moderate level of robustness can generate a modest additional liquidity premium (0.05% for the first 1% proportional transaction cost), and this additional liquidity premium can be even higher if I assume higher levels of robustness. Thus, parameter uncertainty can provide a partial resolution to the liquidity premium puzzle.

Besides these economic contributions, my paper also provides two technical contributions. First, I provide a novel, non-recursive representation of a robust, discrete-time, dynamic asset allocation problem. In its original form, this problem is of non-standard recursive nature, and practically impossible to solve numerically. My novel, non-recursive

representation of the problem not only gives more economic insight into the effects of robustness on the investor's decisions, but it makes it possible to derive the Bellman equation, which is already in standard recursive form, and thus it is possible to solve the problem within a reasonable amount of time¹.

As a second technical contribution, I provide a novel approach to efficiently solve discrete-time robust dynamic asset allocation problems involving transaction cost. Given the robust nature of the problem, one cannot make use of the standard optimization techniques which work well for maximization and minimization problems. Moreover, the value function is of class C^0 , i.e., its first partial derivative is non-continuous, due to the presence of a no-trade region, as documented in Constantinides (1986). To circumvent these difficulties, I develop a semi-analytical approach to solve the optimization problem. First I derive the first-order conditions analytically, then I solve the resulting non-linear equation system numerically. Since these equations are the partial derivatives of the value function, they contain non-continuous functions. These non-continuities prevent me from making use of the Gaussian quadrature rule to evaluate numerically the arising double integrals². So I determine the no-trade boundaries explicitly, and I calculate the double integrals part-wise numerically, applying the Gaussian quadrature rule.

My paper relates to the literature on the liquidity premium puzzle. The first paper documenting the puzzle is by Constantinides (1986). Constantinides repeats the calculation including quasi-fixed transaction cost³, and he concludes that its effect is still of second order. Liu (2004) not only extends the model with a fixed transaction cost, but he also proposes a different definition of the liquidity premium: instead of matching the expected utility, he matches the average holding of the risky asset. The generated liquidity premium is, however, lower than in Constantinides (1986). Vayanos (1998) proposes a general equilib-

¹Using a computer with 16 threads and an Intel Xeon E5-2650 v2 Processor, it takes approximately 4 hours to solve a robust dynamic optimization problem with an investment horizon of 10 years and annual discretization.

²These double integrals are a result of having to evaluate a double expectation due to the two sources of risk: a stochastic stock return and a stochastic transaction cost.

³In the case of quasi-fixed transaction cost, each trade incurs a transaction cost that is proportional to the investor's current wealth, regardless of the value of the trade.

rium model instead of the partial equilibrium model of Constantinides (1986), but he finds that general equilibrium models can generate even lower liquidity premiums than partial equilibrium models. Buss, Uppal, and Vilkov (2014) propose a model with an Epstein-Zin utility function instead of the CRRA utility function of Constantinides (1986), but they find that there are no substantial differences in the generated liquidity premiums. Jang, Koo, Liu, and Loewenstein (2007) introduce a stochastic investment opportunity set into the model, and they find that this more than doubles the generated liquidity premium to 0.45% for the first 1% proportional transaction cost from the original 0.20%. The model of Lynch and Tan (2011) features three novel elements, which can all generate additional reasonable liquidity premium, but still not enough to match empirical facts. Their model features unhedgeable labor income, return predictability, and countercyclical stochastic transaction cost. Besides Lynch and Tan (2011) and the present paper, to the best of my knowledge only three other papers assume stochastic transaction cost: Driessen and Xing (2015) focus on quantifying the liquidity risk premium, while Garleanu and Pedersen (2013) and Glasserman and Xu (2013) analyze the effects of stochastic transaction cost on portfolio allocation, but due to the unconventional utility function which they use, they cannot derive conclusions regarding the level of the liquidity premium.

My paper also relates to the literature on robust dynamic asset allocation. The seminal paper introducing the penalty approach into the robust dynamic asset allocation and asset pricing literature is Anderson, Hansen, and Sargent (2003). Then Maenhout (2004) applies this approach to analyze equilibrium stock prices. In a follow-up paper, Maenhout (2006) allows for a stochastic investment opportunity set, and he finds that robustness increases the relative importance of the intertemporal hedging demand, compared to the non-robust case. Other papers analyzing different aspects of the effects of robustness on dynamic asset allocation include Branger, Larsen, and Munk (2013), Flor and Larsen (2014), and Munk and Rubtsov (2014).

This paper is organized as follows. Section 2 introduces my model, i.e., the financial market and the robust dynamic asset allocation problem. Section 2 also provides the least-

favorable transaction cost, and the optimal consumption and investment policy. Section 3 introduces the definitions of liquidity premium and its components, and it provides the model-implied liquidity premiums. Section 4 concludes. Appendix A provides the numerical procedure that I used to solve the robust optimization problem. Appendix B contains the proofs of theorems and lemmas.

2 Robust Investment Problem

The financial market consists of a money-market account with constant continuously compounded risk-free rate r_f and a risky security (a stock) with continuously compounded return r_t , which follows an i.i.d. normal distribution with mean μ_r and variance σ_r^2 . Buying and selling the money-market account is free. On the other hand, when the investor buys or sells the stock, she encounters a transaction cost, which is proportional to the traded dollar amount. This transaction cost is denoted by Φ_t , and it follows an i.i.d. log-normal distribution with parameters μ_ϕ and σ_ϕ^2 . I assume that Φ_t and r_t are uncorrelated⁴.

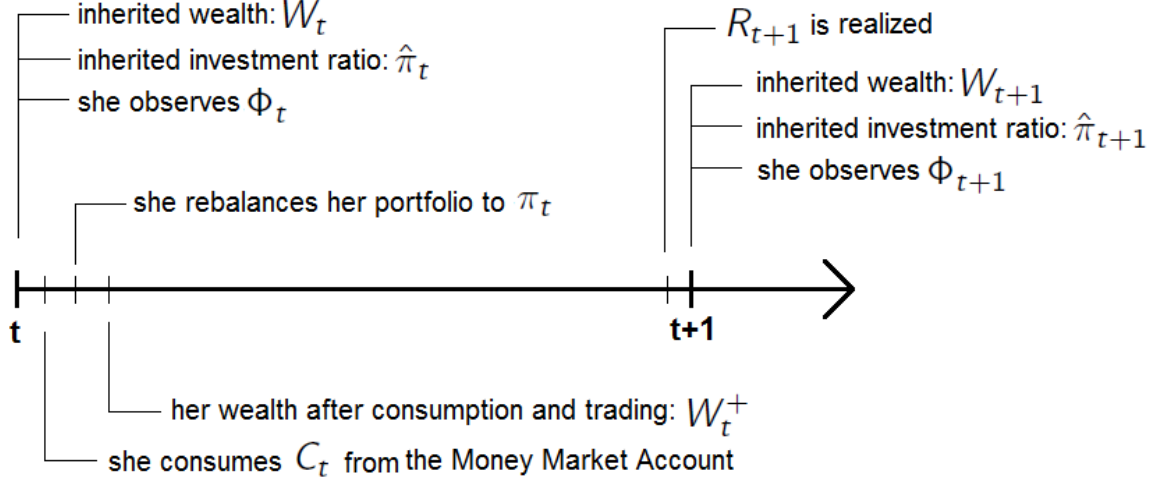
The order of decisions is shown in Figure 1. At each time t the investor inherits wealth W_t . A proportion of this wealth, $\hat{\pi}_t$ is inherited in the risky asset. This proportion I call the *inherited investment ratio*. The investor first learns about the proportional transaction cost Φ_t , then she consumes C_t . I assume that she finances her consumption from the riskless asset. This assumption is common in the literature, and it is in line with economic intuition, allowing one to think about the riskless asset as a money market account, which can be directly used to buy any goods for consumption purposes⁵. After consumption, she decides about her portfolio weight π_t , and then she trades in the securities (and pays the transaction

⁴Empirically, transaction costs are countercyclical, i.e., there is a negative correlation between current transaction costs and expected future returns. Driessen and Xing (2015) show that this results in a negative liquidity risk premium, however, its magnitude is very small, approximately 0.03%. Thus, assuming zero correlation between the current transaction cost and both current and future stock returns in my model does not influence my results significantly.

⁵In contrast to this assumption, Lynch and Tan (2011) assume that the investor finances her consumption by costlessly liquidating her risky and riskless assets in proportions $\hat{\pi}_t$ and $1 - \hat{\pi}_t$. They justify this by the fact that equities pay dividend, and this justification is reasonable in an infinite-horizon setup. However, in the finite-horizon setup of my model the optimal consumption ratio becomes higher and higher as the investor approaches the end of her investment horizon (and eventually becomes equal to one when $t = T$), making my assumption about financing consumption from the riskless asset more realistic.

Figure 1. Order of the representative investor's decisions

At time t the investor inherits wealth W_t . A proportion of this wealth, $\hat{\pi}_t$ is inherited in the risky asset. The investor learns about the proportional transaction cost Φ_t , then she consumes C_t . After consumption, she decides about her portfolio weight π_t , and then she trades in the securities (and pays the transaction cost) to obtain this portfolio weight. Then one time period elapses, and at time $t + 1$ the investor observes the return R_{t+1} , and the series of decisions starts over again.



cost) to obtain this portfolio weight.

The investor's wealth process is thus

$$W_{t+1} = W_t^+ [R_f + \pi_t (R_{t+1} - R_f)], \quad (1)$$

where $R_f = \exp(r_f)$, $R_{t+1} = \exp(r_{t+1})$, and W_t^+ denotes the investor's wealth at time t , after she has consumed, and rebalanced her portfolio (and paid the transaction cost). I.e.,

$$W_t^+ = W_t - C_t - \Phi_t |W_t^+ \pi_t - W_t \hat{\pi}_t|. \quad (2)$$

Since W_t^+ appears on both sides of equation (2), it is convenient to express it in a form which does not contain W_t^+ on the right-hand side. I.e.,

$$W_t^+ = \frac{W_t (1 + I_t \Phi_t \hat{\pi}_t) - C_t}{1 + I_t \Phi_t \pi_t}, \quad (3)$$

where I_t is an indicator function, which is equal to 1 if the investor is buying additional risky assets when rebalancing her portfolio (i.e., if $W_t^+ \pi_t > W_t \hat{\pi}_t$), -1 if the investor is selling part of her risky assets when rebalancing her portfolio (i.e., if $W_t^+ \pi_t < W_t \hat{\pi}_t$), and 0 if the investor is not trading to rebalance her portfolio (i.e., if $W_t^+ \pi_t = W_t \hat{\pi}_t$). At time $t + 1$ the investor learns about the outcome of R_{t+1} , and her investment in the risky security as a fraction of her total wealth becomes

$$\hat{\pi}_{t+1} = \frac{\pi_t R_{t+1}}{R_f + \pi_t (R_{t+1} - R_f)}. \quad (4)$$

This $\hat{\pi}_{t+1}$ I call the *inherited investment ratio at time $t+1$* .

Now I consider an investor with a finite investment horizon T . She derives utility from consumption, and she has a CRRA utility function with relative risk aversion γ . Her goal is to maximize her expected utility, but she is uncertain about the mean of the stochastic log-transaction cost process, μ_ϕ . She has a base parameter value in mind, which she considers to be the most likely. This parameter value is denoted by $\mu_\phi^{\mathbb{B}}$. But she is uncertain about the true value of μ_ϕ , so she considers other (alternative) parameter values as well. These alternative parameter values are denoted by $\mu_\phi^{\mathbb{U}}$. I formalize the relationship between $\mu_\phi^{\mathbb{B}}$ and $\mu_\phi^{\mathbb{U}}$ as

$$\mu_\phi^{\mathbb{U}} = \mu_\phi^{\mathbb{B}} + u_t, \quad (5)$$

where u_t is a stochastic decision variable, just as C_t and π_t are.

The investor wants to protect herself against unfavorable outcomes, so she makes robust investment decisions. Now I formalize the investor's robust optimization problem.

Problem 1. *Given W_0 , $\hat{\pi}_0$, and Φ_0 , find an optimal triple $\{C_t, \pi_t, u_t\} \forall t \in [0, T - 1]$ for the robust utility maximization problem*

$$V_0(W_0, \hat{\pi}_0, \Phi_0) = \inf_{u_t} \sup_{\{C_t, \pi_t\}} \mathbb{E}_0^{\mathbb{U}} \sum_{t=0}^{T-1} \left[\exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} + \Upsilon_t \frac{u_t^2}{2} \right], \quad (6)$$

subject to the budget constraints (1) and (2), to the terminal condition $C_T = W_T(1 - \hat{\pi}_T\Phi_T)$, and where $E_0^{\mathbb{U}}$ means that the expectation is calculated assuming $\mu_\phi^{\mathbb{U}}$, conditional on all information available up to time 0.

To ensure homotheticity of the optimal consumption ratio, the optimal investment policy, and the least-favorable distortion⁶, I scale the uncertainty tolerance parameter Υ_t (following Maenhout (2004)) as

$$\Upsilon_t = \frac{(1 - \gamma)V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})}{\theta}. \quad (7)$$

Substituting the parameterization of the uncertainty tolerance (7) into the value function (6), the value function becomes

$$V_0(W_0, \hat{\pi}_0, \Phi_0) = \inf_{u_t} \sup_{\{C_t, \pi_t\}} E_0^{\mathbb{U}} \sum_{t=0}^T \left[\exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)u_t^2 V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})}{2\theta} \right]. \quad (8)$$

This representation of the value function is recursive, i.e., the right-hand side of (8) contains future values of the value function itself. The following theorem (which I prove in Appendix B) gives a representation of the value function which is not recursive.

Theorem 1. *The solution to (8) with initial wealth x is given by*

$$V_0 = \inf_{u_t} \sup_{\{C_t, \pi_t\}} E_0^{\mathbb{U}} \sum_{t=0}^T \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} \prod_{s=0}^{t-1} \left[1 + \frac{u_s^2(1-\gamma)}{2\theta} \right] \right\}, \quad (9)$$

subject to the budget constraint (1) and (2), and to the terminal condition

$$C_T = W_T(1 - \hat{\pi}_T\Phi_T).$$

In line with Horvath, de Jong, and Werker (2016), equation (9) shows that introducing robustness effectively increases the subjective discount rate δ . I.e., a robust investor is effective-

⁶Homotheticity means that the optimal consumption ratio, the optimal investment policy, and the least-favorable distortion will all be independent of the wealth level.

tively more impatient than an otherwise equal non-robust investor. However, this is not the only place where robustness plays a role in (9), but it also affects the expectation operator by changing the probability density function of the transaction cost $\Phi_t \forall t \in \{1, \dots, T\}$. Actually, as it is shown in Merton (1969) and in Merton (1971), the change in the subjective discount factor affects only the optimal consumption policy, but not the optimal investment policy. The effect of robustness on the optimal investment policy is due to the change from $E_0^{\mathbb{B}}$ to $E_0^{\mathbb{U}}$.

As I mentioned previously, the optimal consumption ratio, the optimal investment rule, and the least-favorable distortion are homothetic, i.e., they do not depend on the wealth level. This is formally stated in the following theorem, which I again prove in Appendix B.

Theorem 2. *Denoting the consumption ratio by $c_t = C_t/W_t$, the optimal solution $\{c_t^*, \pi_t^*, u_t^*\}$ to the robust optimization problem (1) with parameterization (7) is independent of the wealth level W_t , $\forall t \in \{0, 1, \dots, T\}$. Moreover, the value function can be expressed in the form*

$$V_0(W_0, \hat{\pi}_0, \Phi_0) = \frac{W_0^{1-\gamma}}{1-\gamma} v_0(\hat{\pi}_0, \Phi_0). \quad (10)$$

To solve the optimization problem, I apply the principle of dynamic programming (Bellman (1957)). As a first step of this approach, I formulate the one-period optimization problem at time $T-1$ as

$$\begin{aligned} v_{T-1} = \inf_{u_{T-1}} \sup_{\{c_{T-1}, \pi_{T-1}\}} & \left\{ \exp(-\delta(T-1)) c_{T-1}^{1-\gamma} \right. \\ & + \exp(-\delta T) \left(1 + \frac{u_{T-1}^2(1-\gamma)}{2\theta} \right) \frac{(1 - c_{T-1} + I_{T-1}\Phi_{T-1}\hat{\pi}_{T-1})^{1-\gamma}}{(1 + I_{T-1}\Phi_{T-1}\pi_{T-1})^{1-\gamma}} \\ & \left. \times E_{T-1}^{\mathbb{U}} \left[(1 - \hat{\pi}_T \Phi_T)^{1-\gamma} (R_f + \pi_{T-1}(R_T - R_f))^{1-\gamma} \right] \right\}, \end{aligned} \quad (11)$$

the detailed derivation of which can be found in Appendix B, in the proof of Lemma 1. After obtaining the optimal $\{c_{T-1}, \pi_{T-1}, u_{T-1}\}$ triple, I apply backward induction to find the optimal $\{c_t, \pi_t, u_t\}$ triples for $t \in \{0, 1, \dots, T-2\}$. To this end, I formulate the Bellman

equation (the derivation of which can be found in Appendix B, in the proof of Theorem 2).

$$v_t(\hat{\pi}_t, \Phi_t) = \inf_{u_t} \sup_{\{c_t, \pi_t\}} \left\{ \exp(-\delta t) c_t^{1-\gamma} + \left[1 + \frac{u_t^2(1-\gamma)}{2\theta} \right] \left(\frac{1 + I_t \Phi_t \hat{\pi}_t - c_t}{1 + I_t \Phi_t \pi_t} \right)^{1-\gamma} \right. \\ \left. \times \mathbb{E}_t^\mathbb{U} \left[(R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right] \right\}. \quad (12)$$

A closed form solution to the robust utility maximization Problem 1 does not exist, therefore I solve the problem numerically. Still, obtaining the optimal consumption and investment policies and the least-favorable transaction cost $\mu_\phi^\mathbb{U}$ parameter is computationally challenging for several reasons. First, the robust nature of the problem (i.e., the minimax setup) results in a saddle-point solution, which prevents one from making use of standard numerical optimization techniques, that are otherwise well suited for maximization and minimization problems. Second, the value function is of class C^0 , since its first partial derivative with respect to the investment ratio, π_t , is not continuous due to the presence of a no-trade region. This implies that when I calculate the expected value of the first partial derivative of the value function with respect to π_t numerically, I cannot directly apply the Gaussian quadrature rule. Third, since my model contains two sources of risk (R_t and Φ_t), calculating the first partial derivative of the value function with respect to π_t involves the numerical approximation of a definite double-integral instead of the integral of a function of only one variable.

To circumvent these difficulties, I provide a semi-analytical approach: first I derive the first-order conditions on the Bellman equation (12) in closed form⁷, then I solve the resulting non-linear equation system numerically. To efficiently approximate the involved double integrals numerically, I first determine the boundaries of the no-trade region, then apply the Gaussian quadrature rule separately on the sell, no-trade, and buy regions, in which the first partial derivative of the value function with respect to the investment ratio is continuous. Using this approach, I can solve the robust optimization Problem 1 numerically

⁷Since the first-order condition on the Bellman equation (12) contains the partial derivative of the value function at time $t+1$ with respect to π_t , I apply the Benveniste-Scheinkman Condition (Envelope Theorem) to transform this first-order condition into a closed-form equation. The details can be found in Appendix A.

in a feasible time.

2.1 Model parameterization

Regarding the choice of the parameter values, I follow the literature to make my findings comparable to existing results. I assume that the investment horizon is 9 years, and the discretization frequency is annual. The effective annual risk-free rate is 3%. The gross stock returns, R_t , are i.i.d. log-normally distributed with parameters $\mu_r = 8\%$ and $\sigma_r = 20\%$. Following Constantinides (1986) and the estimates of Lesmond, Ogden, and Trzcinka (1999), I assume that the expected future transaction cost is 1%. About the standard deviation of the transaction cost, I assume it to be 0.5%. In their paper, Lynch and Tan (2011) use 0.76% standard deviation and 2% expected value for the transaction cost process. Since higher standard deviation of the transaction cost produces higher negative liquidity risk premium, I use 0.50% standard deviation, which is higher than half of the value used by Lynch and Tan (2011) for 2% transaction cost. This way I underestimate the total liquidity premium, which makes my results stronger⁸. The representative investor's relative risk-aversion parameter is $\gamma = 5$, her subjective discount factor is $\delta = 5\%$, and I vary her uncertainty-aversion parameter θ between 0 and 100, 0 corresponding to a non-robust investor, and 100 corresponding to a highly robust investor.

Determining a reasonable level of robustness is an important aspect of the model. Unfortunately, the uncertainty-aversion parameter θ does not have such a universal interpretation as the relative risk-aversion parameter γ , the reasonable value of which is between 1 and 5 according to the literature. The uncertainty-aversion parameter θ is always model specific.

To still give a general measure of uncertainty aversion, the dynamic asset allocation literature provides two approaches. A statistical approach is to make an additional assumption on the Detection Error Probability of the representative investor, as in, e.g., Anderson, Hansen, and Sargent (2003). The other approach relies more on economic intuition: it

⁸I model the logarithm of the transaction cost, instead of the transaction cost itself, to ensure that the realized transaction cost values are non-negative. Although the log-normal distribution theoretically allows the transaction cost to take values that are above 100%, given my parameter choices the probability of such an event is negligible.

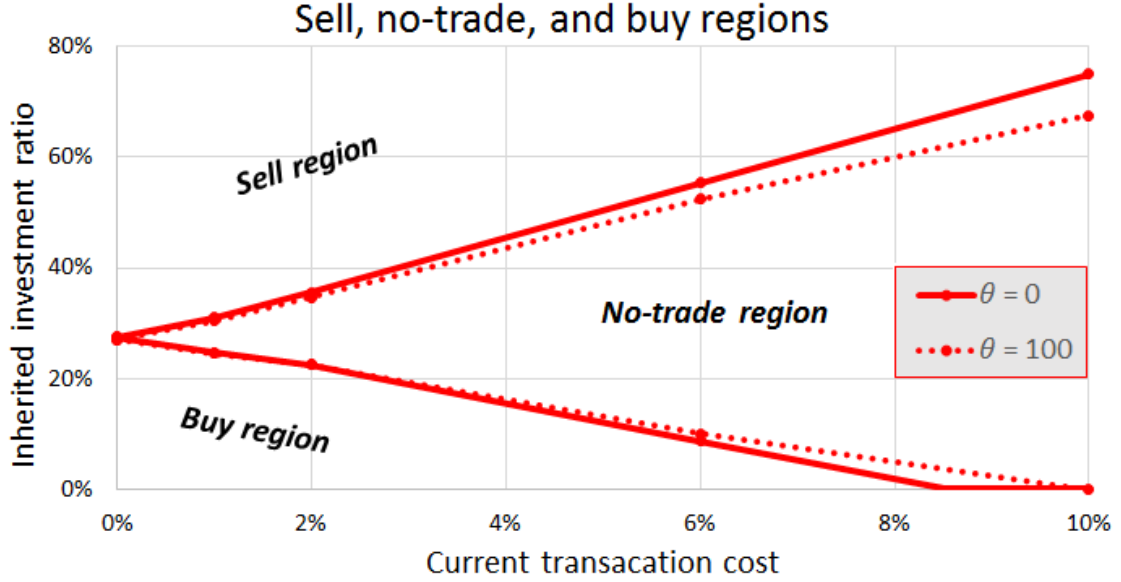
suggests deriving the worst-case scenario (e.g., worst-case mean transaction cost parameter in my model) explicitly, and the difference between the base parameter and the worst-case parameter gives an economic meaning to the uncertainty-aversion level of the particular investor. This approach is also followed by Maenhout (2004). In this paper, since calculating the exact detection-error probabilities to the various uncertainty-aversion parameter values would require computationally intensive recursive calculations, I use the second approach and provide the least-favorable expected future transaction cost levels for three – otherwise identical – investors with different levels of uncertainty aversion: $\theta = 0$ (no uncertainty aversion), $\theta = 50$ (moderate uncertainty aversion), and $\theta = 100$ (high uncertainty aversion). These are shown – as a function of the current transaction cost – in Figure 3, for three different inherited investment ratios.

As Constantinides (1986) showed, the optimal investment decision is determined by a single state variable, the inherited investment ratio. If this ratio is within given boundaries (determined by the model parameters), the optimal decision is to not trade. If the ratio is above the upper boundary, the investor optimally sells a fraction of her risky assets, while if it is below the lower boundary, she buys additional risky assets. The sell, no-trade, and buy regions in my calibrated model are shown in Figure 2.

The solid line represents the boundaries of the no-trade region for the non-robust investor. If her inherited investment ratio is above the upper boundary, she has to sell part of her risky assets, while if it is below the lower boundary, she has to buy additional risky assets. Allowing the investor to be uncertainty-averse shrinks the no-trade region. If she is in the sell region, an uncertainty-averse investor is supposed to sell more (so that she will have to sell less later at the anticipated higher expected transaction cost). On the other hand, if she is in the buy region, an uncertainty-averse investor is supposed to buy more, so that she will have to buy less in the future at the anticipated higher expected transaction cost. Let me emphasize that this conclusion about the shrinkage of the no-trade boundaries depends on the level of the investor’s uncertainty aversion. An investor with a different uncertainty-aversion parameter might buy less (if she is in the buy region), so that she

Figure 2. Sell, no-transaction, and buy zones

The investment horizon is 9 years, the discretization frequency is annual, the effective annual risk-free rate is 3%, the gross stock returns are IID and log-normally distributed with parameters $\mu_r = 8\%$ and $\sigma_r = 20\%$. The proportional transaction costs are IID and log-normally distributed with parameters $\mu_\phi^{\mathbb{B}} = -4.7167$ and $\sigma_\phi = 0.4724$, which correspond to a base expected transaction cost of 1% and standard deviation of 0.50%. The relative risk aversion is $\gamma = 5$, and the subjective discount factor is $\delta = 5\%$.



has to sell less later at the anticipated higher expected transaction cost. This is so because in the case of this other robust investor the “buy less, so that you have to sell less later” motive dominates, while in the case of the uncertainty-averse investor shown in Figure 2 the dominant motive is the “buy more, so that you will have to buy less later at the anticipated higher expected transaction cost”.

The effect of uncertainty on the consumption policy is quantitatively negligible, less than 0.04% on average. Moreover, the optimal consumption ratio is also very stable among the different states: it is around 11.85%.

2.2 Least-favorable transaction cost

The least-favorable future expected transaction cost is state-dependent, i.e., it varies among different combinations of the current values of the two state variables of my model: the

inherited investment ratio and the current transaction cost. In the top panel of Figure 3 the inherited investment ratio is set to $\hat{\pi}_0 = 0$, i.e., the investor inherits everything in the riskless asset. In the middle panel she inherits 31% (which is approximately equal to the zero-current-transaction-cost optimal investment ratio) of her wealth in the risky asset. In the bottom panel she inherits everything in the risky asset. Regardless of the current state of the system, higher robustness always means a higher expected future transaction cost. This is intuitive: a robust investor prepares for the worst-case scenario, thus she considers a higher expected transaction cost in the future.

On the other hand, it is not straightforward how the least-favorable expected future transaction cost changes if – *ceteris paribus* – we change one of the current state variables. If the investor has inherited everything in the riskless asset (top panel of Figure 3), then increasing the current transaction cost induces a higher least-favorable expected future transaction cost. The intuition behind this is that a higher current transaction cost will result in a lower optimal investment ratio, i.e., the investor will have less of her wealth invested in the risky security. Since uncertainty about the transaction cost shows up via having to sell the risky security in the future to finance consumption⁹, having a lower optimal investment ratio now means that a bad outcome will hurt the investor less. Thus, given the same level of uncertainty aversion, to prepare for the worst-case scenario, she will consider a higher future expected transaction cost.

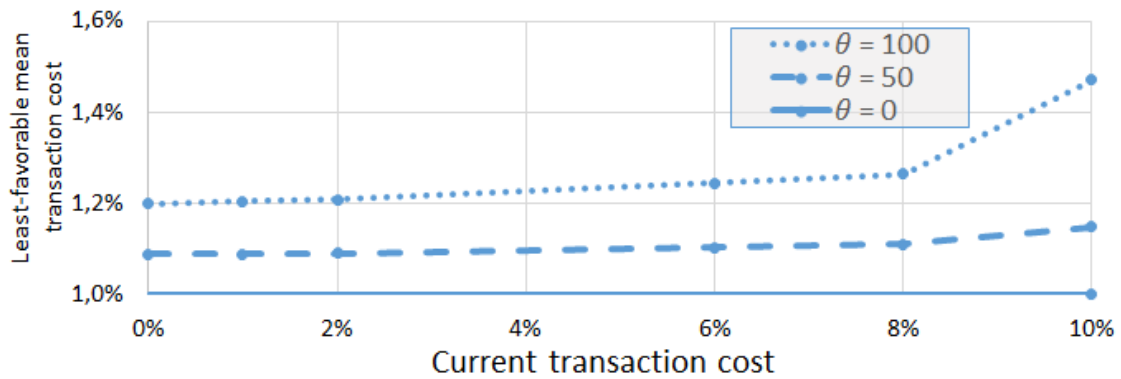
If the investor inherits part of her portfolio in the risky asset (middle panel of Figure 3), then the effect of increasing the current transaction cost is the opposite: a higher transaction cost induces a lower least-favorable expected future transaction cost. The intuition is that a higher current transaction cost will result in a higher optimal investment ratio (since the inherited investment ratio is above the no-transaction-cost optimum), thus the investor is exposed to more uncertainty. To compensate for this, she will consider a lower least-

⁹There is a second effect of a higher current transaction cost, which has the opposite direction: since the investor buys less risky asset now, she will have to buy more in the future, and if she considers a higher expected future transaction cost now, this additional demand will decrease her value function now. This effect is, however, of second order, and quantitatively it is dominated by the effect described in the main text.

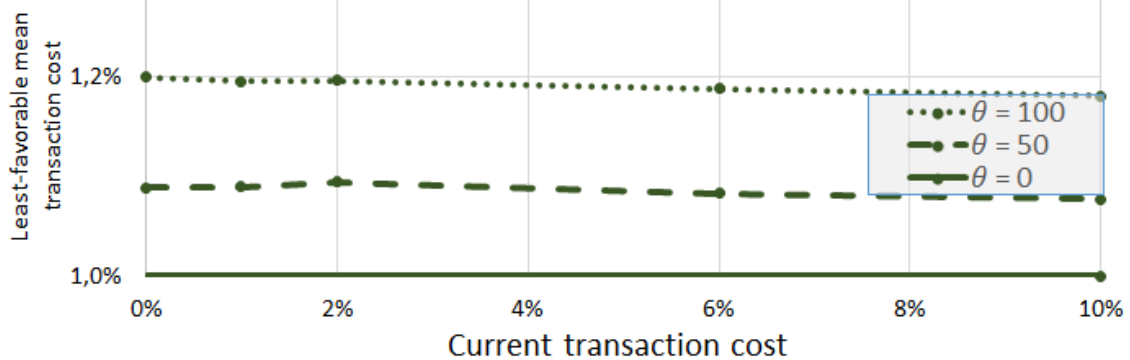
Figure 3. Least-favorable expected future transaction costs, for different inherited investment ratios.

The investment horizon is 9 years, the discretization frequency is annual, the effective annual risk-free rate is 3%, the gross stock returns are IID and log-normally distributed with parameters $\mu_r = 8\%$ and $\sigma_r = 20\%$. The proportional transaction costs are IID and log-normally distributed with parameters $\mu_\phi^{\mathbb{B}} = -4.7167$ and $\sigma_\phi = 0.4724$, which correspond to a base expected transaction cost of 1% and standard deviation of 0.50%. The relative risk aversion is $\gamma = 5$, and the subjective discount factor is $\delta = 5\%$. The inherited investment ratios in the three graphs are 0%, 31%, and 100%, respectively.

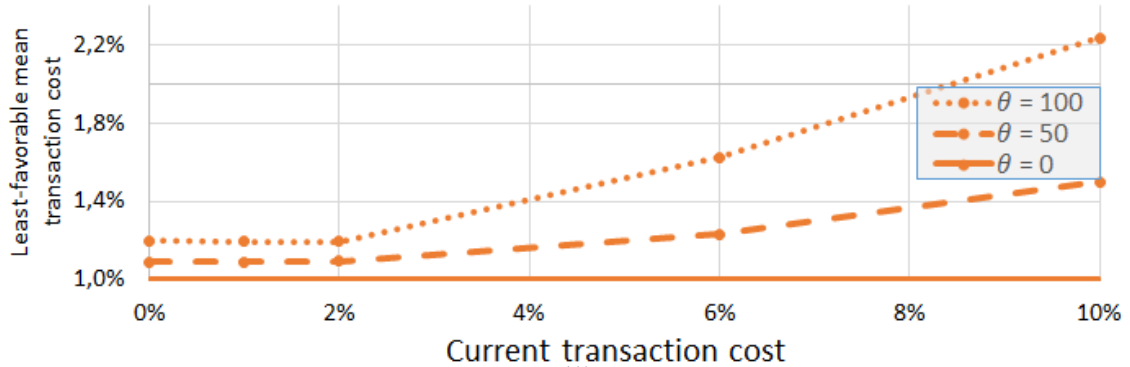
Least-favorable expected future transaction cost



Least-favorable expected future transaction cost



Least-favorable expected future transaction cost



favorable transaction cost.

Regarding the comparative statics changing the other state variable, the inherited investment ratio, we can observe that the intercepts in Figure 3 are the same. This means that if the current transaction cost is zero, the least-favorable expected future transaction cost is the same, regardless of the inherited investment ratio. Moreover, if the inherited investment ratio is the same as the zero-current-transaction-cost optimal investment ratio (post-consumption), then regardless of the current transaction cost, the investor does not have to trade, and the least-favorable transaction cost will be constant at the same level as the intercepts in Figure 3. This is intuitive, since in this particular case the current transaction cost does not have any effect on the probabilities of the future states of the system. If the inherited investment ratio is between zero and the zero-current-transaction-cost optimal investment ratio, then the least-favorable transaction cost will be an increasing function of the current transaction cost, and the higher the inherited investment ratio, the lower the slope of this function. The same is true if the inherited investment ratio is between the zero-current-transaction-cost optimal investment ratio and one: the least-favorable expected future transaction cost is an increasing function of the current transaction cost, and the higher the inherited investment ratio, the higher the slope of this function.

2.3 Optimal investment policy

Similarly to the least-favorable expected future transaction cost, the optimal investment ratio is also state-dependent, as it is shown in Figure 4. If the investor inherits everything in the risk-free asset, and she is non-robust, then she will buy the risky security to have an investment ratio of 31.21%. This is the intercept of the solid line in the top panel of Figure 4. If the current transaction cost is higher, the non-robust investor buys less of the risky asset, and at a current transaction cost of just above 8% she will not trade at all, but leave her entire wealth invested in the riskless asset.

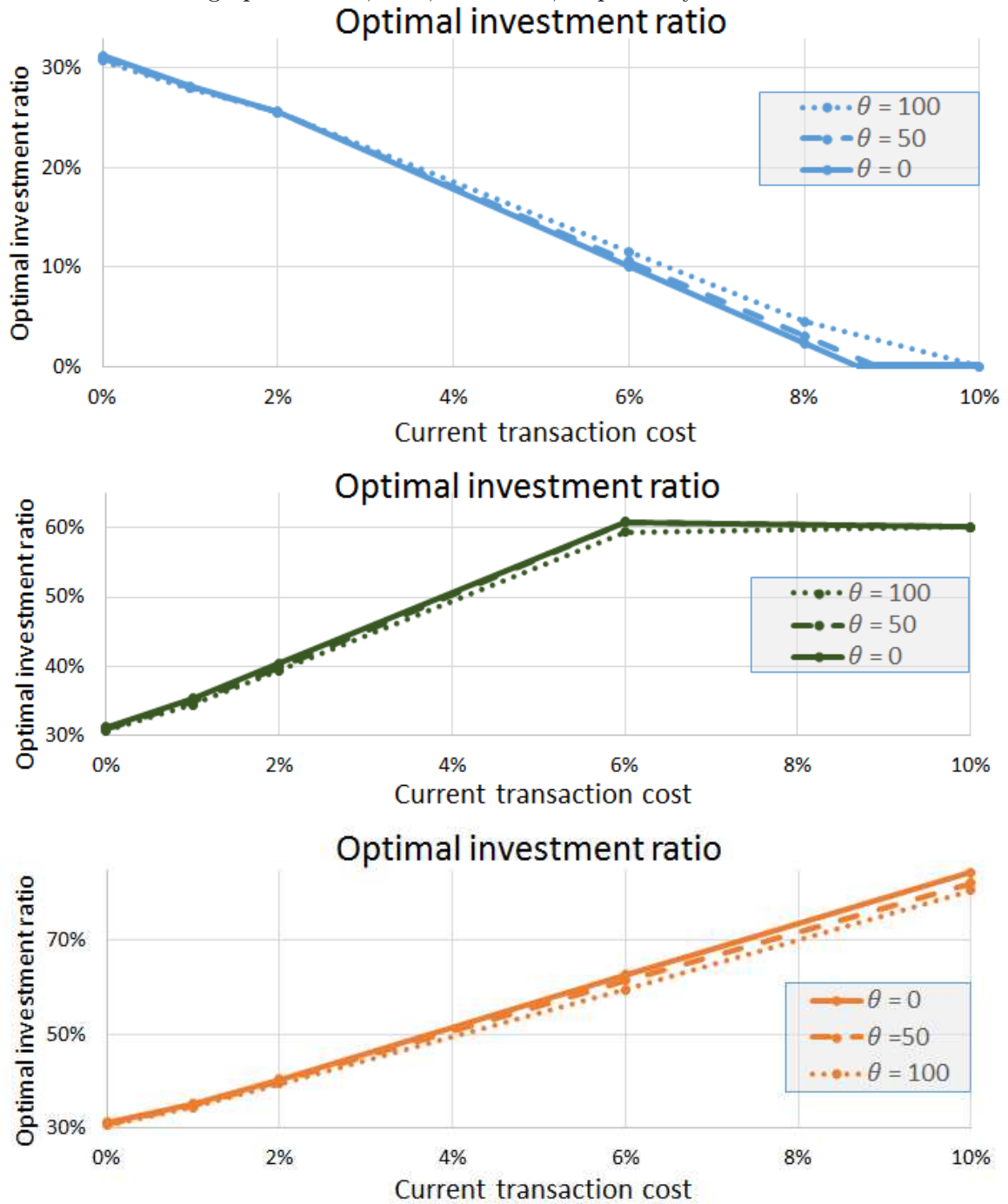
If the investor is uncertainty-averse, she will anticipate a higher expected transaction cost than a non-robust investor (see Figure 3). In the case of such an uncertainty-averse

investor, there are two effects working in the opposite directions. On the one hand, the investor wants to buy less risky asset than her non-robust counterpart, so that later she will have to sell less risky asset at the higher anticipated expected transaction cost. Instead, she will hold more of her wealth now in the riskless asset, which later she can use for consumption purposes without having to pay transaction cost to sell it. On the other hand, she wants to buy more risky asset now at the known transaction cost, so that later she will have to buy less risky asset at the higher anticipated expected transaction cost. As we can see in the top panel of Figure 4, in the case of an uncertainty-averse investor who inherited everything in the risk-free asset the first effect is the dominant: she will buy less risky security now than her non-robust counterpart.

If the investor inherits everything in the risky asset (bottom panel of Figure 4), she will have to sell part of her portfolio to achieve the optimal investment ratio. If the investor is non-robust, the zero-current-transaction-cost optimal investment ratio for her is 31.21% – the same is for the non-robust investor who inherited everything in the risk-free asset. This is intuitive: if the current transaction cost is zero, then the investor can rebalance her portfolio for free, hence the inherited portfolio allocation does not matter for her. But if the current transaction cost is non-zero, the optimal investment ratio becomes higher – i.e., she will sell less of her risky asset to save on the transaction cost. And as the current transaction cost is higher and higher, she will sell less and less, until she achieves the point where she will not trade any more and rather keep all of her wealth in the risky asset.

If this investor is uncertainty-averse, then again there will be two effects working in the opposite directions. On the one hand, the robust investor wants to sell more of the risky security so that she will have to sell less later at the higher anticipated expected transaction cost. On the other hand, she wants to sell less of the risky security to avoid having to buy additional risky security in the future for rebalance purposes. In the bottom panel of Figure 4, we can see that regardless of the current transaction cost, the second effect will be the dominant one, i.e., a robust investor will always sell more of the risky security now than a non-robust investor. This is due to the fact that her zero-current-

Figure 4. Optimal investment ratio, for different inherited investment ratios.
 The investment horizon is 9 years, the discretization frequency is annual, the effective annual risk-free rate is 3%, the gross stock returns are IID and log-normally distributed with parameters $\mu_r = 8\%$ and $\sigma_r = 20\%$. The proportional transaction costs are IID and log-normally distributed with parameters $\mu_\phi^{\mathbb{B}} = -4.7167$ and $\sigma_\phi = 0.4724$, which correspond to a base expected transaction cost of 1% and standard deviation of 0.50%. The relative risk aversion is $\gamma = 5$, and the subjective discount factor is $\delta = 5\%$. The inherited investment ratios in the three graphs are 0%, 31%, and 100%, respectively.



transaction-cost optimal portfolio is very different from her inherited portfolio. If these two portfolios were more similar in terms of asset allocation to each other – as it is the case in the top panel of Figure 4 –, then the first effect might be the dominant. But even now we can observe the fact that we encountered in the case of the investor who inherited everything in the risk-free asset that as the investor becomes more and more uncertainty-averse, the dominance of the second effect is less and less strong, especially if the current transaction cost is not too high (less than 2%). In the middle panel of Figure 4 this is reflected in the fact that below 2% current transaction cost the optimal investment ratio of a moderately uncertainty-averse investor is actually higher, than an otherwise identical, but more uncertainty-averse investor's.

3 Liquidity Premium

After showing the effects of uncertainty about future transaction cost on asset allocation in Section 2, now I go one step further and I analyze the effects of uncertainty about future transaction costs on asset pricing. To be more precise, I show that my model with a robust representative investor generates an additional liquidity premium of 0.05%-0.10% for the first 1% proportional transaction cost, depending on the level of uncertainty aversion of the investor.

To separate the effects of the level of the transaction cost, the volatility of the transaction cost, and the parameter uncertainty about the transaction cost process, I introduce the definition of the *liquidity-uncertainty premium*, the *liquidity-risk premium*, and the *liquidity-level premium*.¹⁰

Definition 1. *Let us consider a representative agent with uncertainty-aversion parameter θ solving the robust optimization Problem 1, and obtaining V_0 . If we impose the restriction $u_t = 0 \forall t \in \{0, 1, \dots, T - 1\}$, the continuously compounded expected return on the risky security, μ_r , has to be decreased to $\mu_r^{\theta=0}$ so that the investor achieves the same level of value*

¹⁰My liquidity premium definitions are in line with the majority of the literature, though some researchers use alternative definitions, e.g., Liu (2004).

V_0 as without this restriction. The difference $\mu_r - \mu_r^{\theta=0}$ is called the liquidity-uncertainty premium.

Definition 2. Let us consider the representative investor in Definition 1. If we impose the restriction $u_t = 0 \forall t \in \{0, 1, \dots, T-1\}$, and we change the volatility parameter of the transaction cost process from σ_ϕ to 0, the continuously compounded expected return on the risky security, μ_r , has to be decreased to $\mu_r^{\theta=0, \sigma_\phi=0}$ so that the investor achieves the same level of value V_0 as originally. The difference $\mu_r^{\theta=0} - \mu_r^{\theta=0, \sigma_\phi=0}$ is called the liquidity-risk premium.

Definition 3. Let us consider the representative investor in Definition 1. If we impose the restriction that $u_t = 0 \forall t \in \{0, 1, \dots, T-1\}$, we change the volatility parameter of the transaction cost process from σ_ϕ to 0, and we also change the base mean parameter of the transaction cost from $\mu_\phi^{\mathbb{B}}$ to $\tilde{\mu}_\phi^{\mathbb{B}}$ so that $\mathbb{E}_t^{\mathbb{B}} \{\Phi_{t+1}\} = 0 \forall t \in \{0, 1, \dots, T-1\}$, the continuously compounded expected return on the risky security, μ_r , has to be decreased to $\mu_r^{\theta=0, \sigma_\phi=0, \mu_\phi=0}$ so that the investor achieves the same level of value V_0 as originally. The difference $\mu_r^{\theta=0, \sigma_\phi=0} - \mu_r^{\theta=0, \sigma_\phi=0, \mu_\phi=0}$ is called the liquidity-level premium.

Using Definitions 1-3 and the parameter values described in Section 2, my model generates the liquidity-level premium, liquidity-risk premium, and liquidity-uncertainty premium values described in Table 1, and also shown in Figure 5 and Figure 6.

Table 1. Model implied liquidity premiums

Model implied liquidity-level premiums, liquidity-risk premiums, and liquidity-uncertainty premiums for different levels of uncertainty aversion. The investment horizon is $T=10$ years, the discretization is annual. The continuously compounded expected stock return is $\mu_r = 8\%$, the volatility of the return is $\sigma_r = 20\%$, the risk-free rate is 3%, the base transaction cost mean parameter is $\mu_\phi^{\mathbb{B}} = 1\%$, the transaction cost volatility parameter is $\sigma_\phi = 0.5\%$. The investor's risk aversion parameter is $\gamma = 5$, and her subjective discount rate is $\delta = 5\%$.

	$\theta = 0$	$\theta = 50$	$\theta = 100$
Liquidity uncertainty premium	0.00%	0.05%	0.10%
Liquidity risk premium	-0.03%	-0.02%	-0.04%
Liquidity level premium	0.83%	0.71%	0.72%
Total liquidity premium	0.80%	0.74%	0.78%

In the non-robust version of my model, the total liquidity premium generated is 0.80% for a proportional liquidity premium with expected value of 1% and standard deviation of 0.50%. The majority of this premium is due to the level of the transaction cost: the liquidity level premium is 0.83%. The liquidity-risk premium is negative, it is -0.03%. At first this might seem counterintuitive, but it is actually very logical. A rigorous demonstration of why the liquidity premium is negative if the correlation between the stock return and the transaction cost is zero can be found in Driessen and Xing (2015). The intuition is that the stochastic nature of the transaction cost gives additional opportunity to the investor to choose when to rebalance her portfolio. If market conditions are bad (i.e., the transaction cost is high), she can postpone rebalancing, while if market conditions are good (i.e., the transaction cost is low), she can trade more. Driessen and Xing (2015) call this the *Choice Effect*.

Introducing a moderate level of robustness decreases the total liquidity premium by 0.06%. Though the liquidity-level premium decreased, a liquidity-uncertainty premium showed up, which is 0.05%. The decrease in the liquidity level premium is due to the definition of the three components of the total liquidity premium: instead of taking $\mu_r = 8\%$ as the baseline non-robust value and increasing it when introducing robustness to calculate the liquidity-uncertainty premium, I take $\mu_r = 8\%$ as the robust baseline value. I do this because the μ_r value is observed/estimated independently of the model specification, and if the representative investor is assumed to be robust, then this representative investor generates $\mu_r = 8\%$ expected return, that we can observe on the market, and not a return which is higher than this by the liquidity uncertainty premium. Thus, the main message of the third column of Table 1 is: even a moderate level of uncertainty aversion can generate a liquidity uncertainty premium of 0.05%, which is 25% of the total liquidity premium generated in Constantinides (1986). Increasing the uncertainty aversion to a higher level, will produce a liquidity-uncertainty premium of 0.10%.

Figure 5. Decomposition of the liquidity premium.

The three components of the liquidity premium: the liquidity-level premium, the (negative) liquidity-risk premium, and the liquidity-uncertainty premium.

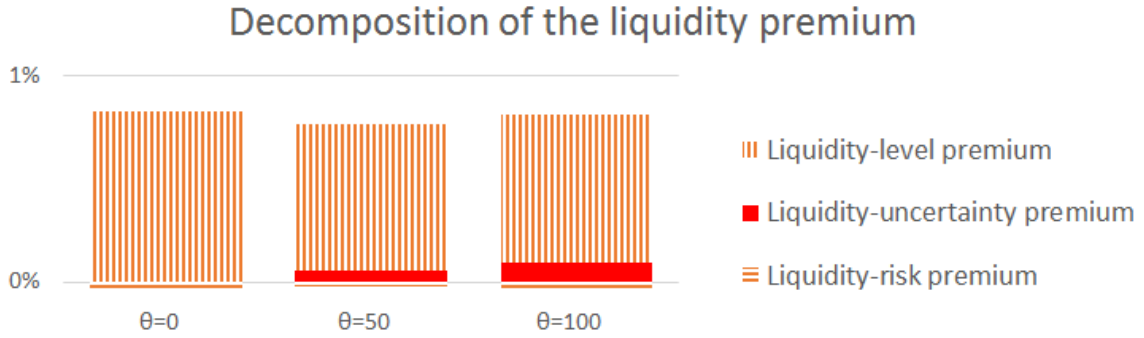
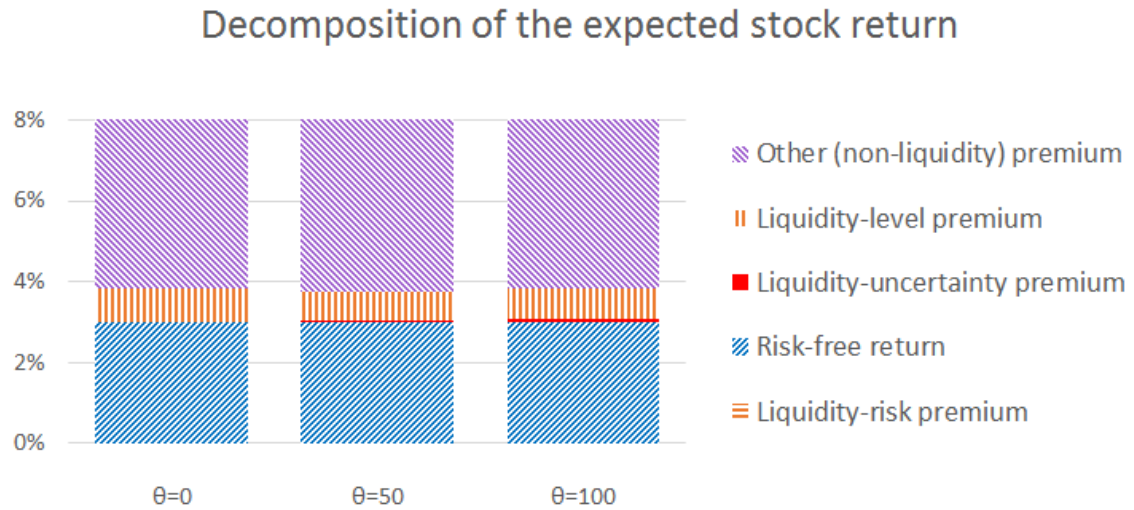


Figure 6. Decomposition of the expected stock return.

Decomposition of the expected stock return into the risk-free return, the (negative) liquidity-risk premium, the liquidity-uncertainty premium, the liquidity-level premium, and other (non-liquidity-related) premiums.



4 Conclusion

I have shown that uncertainty-aversion can explain a reasonable portion of the liquidity-premium puzzle. With a moderate level of uncertainty aversion my model can generate an additional 0.05% liquidity premium, which is even higher if I allow for a higher level of uncertainty aversion. I have also shown that uncertainty-aversion has two different channels through which it affects the optimal investment behavior, and the total effect depends on the current transaction cost and the inherited investment ratio. Besides these two economic contributions, I provided two technical contributions to efficiently solve robust dynamic asset allocation problems involving transaction cost.

When I introduced uncertainty aversion into my model, I relied on the magnitude of the least-favorable transaction cost values to assess the level of uncertainty aversion. This is an intuitive and appealing approach, however, more rigorous definitions of the level of uncertainty aversion, e.g., by using simulation-based detection error probabilities can be a fruitful line of further research.

Moreover, developing a continuous-time version of the robust model and using the relative entropy as a penalty term might provide opportunities for intuitive interpretation of the uncertainty-aversion parameter in the context of ambiguity about the future transaction costs, and it can also give additional insight into the working mechanism of the two channels through which robustness affects the investor's decision. This is also a promising area of future research.

Appendix A Numerical procedure

Solving the robust dynamic optimization Problem 1 numerically is computationally challenging for several reasons. First, due to the minimax setup of the problem, the solution will be a saddle point, thus I cannot use the standard numerical optimization techniques which can be used to efficiently solve maximization and minimization problems. Second, the value function is of class C^0 , i.e., its partial derivative with respect to the investment ratio is non-continuous due to the presence of the no-trade region. Third, my model contains two sources of uncertainty (both the stock return and the transaction cost are stochastic), thus calculating the expected value for the value function (and its partial derivatives) involves numerically approximating a double integral, where the function to be integrated is not necessarily continuous.

To solve the problem within a reasonable time, I take several measures. First, I show in Theorem 2 that the optimal consumption ratio, the optimal investment ratio and the least-favorable distortion at time t do not depend on the wealth level at time t . Since the wealth level is one of the state variables, this theorem substantially reduces the required computational time to solve the optimization problem. Second, I develop a novel technique to solve discrete-time robust dynamic asset allocation problems with transaction cost. This involves rewriting Problem 1 in a non-recursive formulation. I provide this reformulation in Theorem 1. Then I write down the first-order conditions in closed form, making use of the Benveniste-Scheinkman Condition (Envelope Theorem). This results in a non-linear equation system of three equations, with three variables. The equations themselves are the first partial derivatives of the value function, thus some of them are non-continuous. To evaluate the double expectations (which are equivalent to numerically evaluating double integrals), I explicitly determine the non-continuity lines, then I apply the Gaussian quadrature rule to calculate the numerical integral of the function part-wise. Moreover, I also parallelize the solution algorithm to distribute the computational workload among several working units.

The steps of the solution procedure are as follows.

1. First, I create grids for possible $\hat{\pi}_{T-1}$ and Φ_{T-1} values. The grids for $\hat{\pi}_{T-1}$ lie in $\{0, 0.5, 1\}$, while the grids for Φ_{T-1} lie in $\{0, 0.01, 0.02, 0.06, 0.1\}$. I obtain the optimal investment ratio π_{T-1} , the optimal consumption ratio c_{T-1} , and the least-favorable transaction cost parameter u_{T-1} for each $\{\hat{\pi}_{T-1}, \Phi_{T-1}\}$ pairs by numerically solving the first order conditions with respect to c_{T-1} , π_{T-1} , and u_{T-1} . These first-order conditions are (42), (43), and (44).
2. Now I go one period backwards. Since the solution procedure will be the same for all $t \in \{1, 2, \dots, T-2\}$, instead of $T-2$ I use the more general time period t in this step, keeping in mind that immediately after the above step I have $t = T-2$. Just as in the previous step, I create grids for possible $\hat{\pi}_t$ and Φ_t values. The grids for $\hat{\pi}_t$ lie in $\{0, 0.5, 1\}$, while the grids for Φ_t lie in $\{0, 0.01, 0.02, 0.06, 0.1\}$. The first order condition on the Bellman equation (12) with respect to c_t is

$$c_t = \left\{ \left[\exp(\delta t) \left[1 + \frac{u_t^2(1-\gamma)}{2\theta} \right] \frac{(1 + I_t \Phi_t \hat{\pi}_t)^{-\gamma}}{(1 + I_t \Phi_t \pi_t)^{1-\gamma}} \right. \right. \\ \left. \left. \times \mathbb{E}_t^{u_t} \left[(R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right] \right]^{\frac{1}{\gamma}} + (1 + I_t \Phi_t \hat{\pi}_t)^{-1} \right\}^{-1}, \quad (13)$$

the first order condition on the Bellman equation (12) with respect to π_t is

$$\mathbb{E}_t^{u_t} \left\{ (1-\gamma) v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) (R_f + \pi_t (R_{t+1} - R_f))^{-\gamma} \right. \\ \times \left[R_{t+1} - R_f - (R_f + \pi_t (R_{t+1} - R_f)) \frac{I_t \Phi_t}{1 + I_t \Phi_t \pi_t} \right] \\ \left. + R_f R_{t+1} (R_f + \pi_t (R_{t+1} - R_f))^{-1-\gamma} \frac{\partial v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1})}{\partial \hat{\pi}_{t+1}} \right\} = 0, \quad (14)$$

and the first order condition on the Bellman equation (12) with respect to u_t is

$$\mathbb{E}_t^{u_t} \left\{ (R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right. \\ \left. \times \left[\left(1 + \frac{u_t^2(1-\gamma)}{2\theta} \right) \frac{\phi_{t+1} - \mu_\phi - u_t}{\sigma_\phi^2} + \frac{u_t(1-\gamma)}{\theta} \right] \right\} = 0. \quad (15)$$

Condition (14) contains $\partial v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) / \partial \hat{\pi}_{t+1}$, which not only makes the evaluation of the expected value in (14) computationally intensive, but it also decreases the numerical accuracy of the obtained results. To circumvent this, I make use of the Benveniste-Scheinkman Condition (Envelope Theorem). Let us introduce the function

$$\begin{aligned} \bar{v}_t(c_t, \pi_t, u_t, \hat{\pi}_t, \Phi_t) &= \left(1 + \frac{u_t^2(1-\gamma)}{2\theta}\right) \left(\frac{1 + I_t \Phi_t \hat{\pi}_t - c_t}{1 + I_t \Phi_t \pi_t}\right)^{1-\gamma} \\ &\times \mathbb{E}_t^{u_t} \left[(R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right], \end{aligned} \quad (16)$$

the partial derivative of which with respect to $\hat{\pi}_t$ is

$$\begin{aligned} \frac{\partial \bar{v}_t(c_t, \pi_t, u_t, \hat{\pi}_t, \Phi_t)}{\partial \hat{\pi}_t} &= \left(1 + \frac{u_t^2(1-\gamma)}{2\theta}\right) (1-\gamma) \left(\frac{1 + I_t \Phi_t \hat{\pi}_t - c_t}{1 + I_t \Phi_t \pi_t}\right)^{-\gamma} \\ &\times \frac{I_t \Phi_t}{1 + I_t \Phi_t \pi_t} \mathbb{E}_t^{u_t} \left[(R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right]. \end{aligned} \quad (17)$$

Substituting (16) into (12), the Bellman equation becomes

$$v_t(\hat{\pi}_t, \Phi_t) = \inf_{u_t} \sup_{\{c_t, \pi_t\}} \left\{ \exp(-\delta t) c_t^{1-\gamma} + \bar{v}_t(c_t, \pi_t, u_t, \hat{\pi}_t, \Phi_t) \right\}. \quad (18)$$

Then the following theorem (which I prove in Appendix B) holds.

Theorem 3 (Benveniste-Scheinkman Condition (Envelope Theorem)). *If $c_t = c_t^*$, $\pi_t = \pi_t^*$, and $u_t = u_t^*$, then*

$$\frac{\partial v_t(\hat{\pi}_t, \Phi_t)}{\partial \hat{\pi}_t} = \frac{\partial \bar{v}_t(c_t^*, \pi_t^*, u_t^*, \hat{\pi}_t, \Phi_t)}{\partial \hat{\pi}_t}. \quad (19)$$

Using Theorem 3 and equation (17), I rewrite the first order condition with respect

to π_t , i.e., (14), as

$$\begin{aligned}
0 = & \mathbb{E}_t^{u_t} \left\{ (R_f + \pi_t (R_{t+1} - R_f))^{-\gamma} v_{t+1} (\hat{\pi}_{t+1}, \Phi_{t+1}) \right. \\
& \times \left[R_{t+1} - R_f - (R_f + \pi_t (R_{t+1} - R_f)) \frac{I_t \Phi_t}{1 + I_t \Phi_t \pi_t} \right] \\
& + R_f R_{t+1} (R_f + \pi_t (R_{t+1} - R_f))^{-1-\gamma} \left(1 + \frac{(u_{t+1}^*)^2 (1-\gamma)}{2\theta} \right) \\
& \times \frac{(1 + I_{t+1} \Phi_{t+1} \hat{\pi}_{t+1} - c_{t+1}^*)^{-\gamma}}{(1 + I_{t+1} \Phi_{t+1} \pi_{t+1}^*)^{1-\gamma}} I_{t+1} \Phi_{t+1} \\
& \left. \times \mathbb{E}_{t+1}^{u_{t+1}^*} \left[(R_f + \pi_{t+1}^* (R_{t+2} - R_f))^{1-\gamma} v_{t+2} (\hat{\pi}_{t+2}, \Phi_{t+2}) \right] \right\}. \tag{20}
\end{aligned}$$

Appendix B Proofs

Proof of Theorem 1. I will prove the theorem using backward induction. Throughout the proof, I assume that the choice variables, i.e., C_t , π_t , and u_t are always optimally chosen, thus I omit the inf and sup operators. Moreover, since it will not cause any confusion, instead of $V_t(W_t, \hat{\pi}_t, \Phi_t)$ I simply write V_t . At time T the value function is

$$\begin{aligned}
V_T &= \exp(-\delta T) \frac{C_T^{1-\gamma}}{1-\gamma} + \frac{u_T^2 (1-\gamma)}{2\theta} V_{T+1} \\
&= \exp(-\delta T) \frac{C_T^{1-\gamma}}{1-\gamma}, \tag{21}
\end{aligned}$$

since $V_{T+1} = 0$. Going one step backwards, the value function at time $T-1$ is by definition

$$\begin{aligned}
V_{T-1} &= \left\{ \exp\{-\delta(T-1)\} \frac{C_{T-1}^{1-\gamma}}{1-\gamma} + \frac{u_{T-1}^2 (1-\gamma)}{2\theta} V_T + \exp(-\delta T) \frac{C_T^{1-\gamma}}{1-\gamma} \right\} \\
&= \mathbb{E}_{T-1} \left\{ \exp\{-\delta(T-1)\} \frac{C_{T-1}^{1-\gamma}}{1-\gamma} + \exp(-\delta T) \frac{C_T^{1-\gamma}}{1-\gamma} \left(1 + \frac{u_{T-1}^2 (1-\gamma)}{2\theta} \right) \right\} \\
&= \mathbb{E}_{T-1} \sum_{t=T-1}^T \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} \prod_{s=T-1}^{t-1} \left[1 + \frac{u_s^2 (1-\gamma)}{2\theta} \right] \right\}. \tag{22}
\end{aligned}$$

Going one more step backwards, I can write down the value function V_{T-2} in the same way:

$$\begin{aligned}
V_{T-2} &= E_{T-2} \left\{ \exp(-\delta(T-2)) \frac{C_{T-2}^{1-\gamma}}{1-\gamma} + \exp(-\delta(T-1)) \frac{C_{T-1}^{1-\gamma}}{1-\gamma} \left(1 + \frac{u_{T-2}^2(1-\gamma)}{2\theta} \right) \right. \\
&\quad \left. + \exp(-\delta T) \frac{C_T^{1-\gamma}}{1-\gamma} \left(1 + \frac{u_{T-1}^2(1-\gamma)}{2\theta} \right) \left(1 + \frac{u_{T-2}^2(1-\gamma)}{2\theta} \right) \right\} \\
&= E_{T-2} \sum_{t=T-2}^T \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} \prod_{s=T-2}^{t-1} \left[1 + \frac{u_s^2(1-\gamma)}{2\theta} \right] \right\}. \tag{23}
\end{aligned}$$

Progressing backwards in the same way, I can write the value function at time t as

$$V_t = E_t \sum_{s=t}^T \left\{ \exp(-\delta s) \frac{C_s^{1-\gamma}}{1-\gamma} \prod_{m=t}^{s-1} \left[1 + \frac{u_m^2(1-\gamma)}{2\theta} \right] \right\}, \tag{24}$$

and the value function at time 0 as

$$V_0 = E_0 \sum_{t=0}^T \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} \prod_{s=0}^{t-1} \left[1 + \frac{u_s^2(1-\gamma)}{2\theta} \right] \right\}. \tag{25}$$

This is the same as in Theorem 1, and it completes the proof. \square

Proof of Theorem 2. I will prove the theorem using mathematical induction. Following Theorem 1, the value function at time t is

$$V_t(W_t, \hat{\pi}_t, \Phi_t) = \inf_{u_s} \sup_{\{C_s, \pi_s\}} E_t \sum_{s=t}^T \left\{ \exp(-\delta s) \frac{C_s^{1-\gamma}}{1-\gamma} \prod_{m=t}^{s-1} \left[1 + \frac{u_m^2(1-\gamma)}{2\theta} \right] \right\}. \tag{26}$$

From (26) follows that the Bellman equation is¹¹

$$\begin{aligned}
V_t(W_t, \hat{\pi}_t, \Phi_t) &= \inf_{u_t} \sup_{\{C_t, \pi_t\}} \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} + \left[1 + \frac{u_t^2(1-\gamma)}{2\theta} \right] \right. \\
&\quad \left. \times E_t^{u_t} [V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})] \right\}, \tag{27}
\end{aligned}$$

where $E_t^{u_t}$ means that the expectation is taken assuming μ_ϕ^U .

¹¹The Bellman equation can be also directly derived from the definition of the value function by rewriting (24).

The first order conditions on the right-hand side of the Bellman equation in (27) with respect to C_t , π_t , and u_t are

$$0 = \exp(-\delta t) C_t^{-\gamma} + \left[1 + \frac{u_t^2 (1 - \gamma)}{2\theta}\right] \mathbb{E}_t^{u_t} \left[\frac{\partial V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})}{\partial C_t} \right], \quad (28)$$

$$0 = \mathbb{E}_t^{u_t} \left[\frac{\partial V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})}{\partial \pi_t} \right], \quad (29)$$

and

$$\begin{aligned} & -\frac{u_t (1 - \gamma)}{\theta} \mathbb{E}_t^{u_t} [V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})] \\ & = \left[1 + \frac{u_t^2 (1 - \gamma)}{2\theta}\right] \mathbb{E}_t^{u_t} \left\{ \frac{\phi_{t+1} - \mu_\phi - u_t}{\sigma_\phi^2} V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1}) \right\}, \end{aligned} \quad (30)$$

respectively, where $\phi_{t+1} = \log(\Phi_{t+1})$.

Now let us assume (I will prove this in Lemma 1) that the value function $V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})$ can be expressed in the form

$$V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1}) = \frac{W_{t+1}^{1-\gamma}}{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}). \quad (31)$$

Then the first-order condition with respect to C_t , i.e., (28) becomes

$$\begin{aligned} c_t = & \left\{ \left[\exp(\delta t) \left[1 + \frac{u_t^2 (1 - \gamma)}{2\theta}\right] \frac{(1 + I_t \Phi_t \hat{\pi}_t)^{-\gamma}}{(1 + I_t \Phi_t \pi_t)^{1-\gamma}} \right. \right. \\ & \left. \left. \times \mathbb{E}_t^{u_t} \left[(R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right] \right]^{\frac{1}{\gamma}} + (1 + I_t \Phi_t \hat{\pi}_t)^{-1} \right\}^{-1} \end{aligned} \quad (32)$$

the first-order condition with respect to π_t , i.e., (29) becomes

$$\begin{aligned} \mathbb{E}_t^{u_t} \left\{ (1 - \gamma) v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) (R_f + \pi_t (R_{t+1} - R_f))^{-\gamma} \right. \\ \times \left[R_{t+1} - R_f - (R_f + \pi_t (R_{t+1} - R_f)) \frac{I_t \Phi_t}{1 + I_t \Phi_t \pi_t} \right] \\ \left. + R_f R_{t+1} (R_f + \pi_t (R_{t+1} - R_f))^{-1-\gamma} \frac{\partial v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1})}{\partial \hat{\pi}_{t+1}} \right\} = 0, \end{aligned} \quad (33)$$

and the first-order condition with respect to u_t , i.e., (30) becomes

$$\begin{aligned} \mathbb{E}_t^{u_t} \left\{ (R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right. \\ \times \left[\left(1 + \frac{u_t^2 (1 - \gamma)}{2\theta} \right) \frac{\phi_{t+1} - \mu_\phi - u_t}{\sigma_\phi^2} + \frac{u_t (1 - \gamma)}{\theta} \right] \left. \right\} = 0 \end{aligned} \quad (34)$$

Since none of the three first-order conditions contain W_t apart from the consumption ratio c_t (which is constant regardless of the level of W_t as (32) shows), I can conclude that if the value function at time $t+1$ can be expressed in the form (31), then the optimal consumption ratio c_t^* , the optimal investment ratio π_t^* , and the least-favorable distortion u_t^* will all be independent of the wealth level, W_t .

Now I prove the second part of Theorem 2, which states that the value function at time t can be expressed in the form

$$V_t(W_t, \hat{\pi}_t, \Phi_t) = \frac{W_t^{1-\gamma}}{1-\gamma} v_t(\hat{\pi}_t, \Phi_t). \quad (35)$$

Let us now again assume (I will prove this in Lemma 1) that the value function $V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})$ can be expressed in the form

$$V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1}) = \frac{W_{t+1}^{1-\gamma}}{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}). \quad (36)$$

Substituting this form of the value function (the right-hand side of (36)) into the value

function at time t , i.e., into (27), I obtain

$$\begin{aligned}
V_t(W_t, \hat{\pi}_t, \Phi_t) &= \inf_{u_t} \sup_{\{C_t, \pi_t\}} \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} \right. \\
&\quad \left. + \left[1 + \frac{u_t^2(1-\gamma)}{2\theta} \right] \mathbb{E}_t^{u_t} \left[\frac{W_{t+1}^{1-\gamma}}{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right] \right\} \\
&= \frac{W_t^{1-\gamma}}{1-\gamma} \inf_{u_t} \sup_{\pi_t} \left\{ \exp(-\delta t) c_t^{*1-\gamma} + \left[1 + \frac{u_t^2(1-\gamma)}{2\theta} \right] \right. \\
&\quad \times \left(\frac{1 + I_t \Phi_t \hat{\pi}_t - c_t^*}{1 + I_t \Phi_t \pi_t} \right)^{1-\gamma} \\
&\quad \left. \times \mathbb{E}_t^{u_t} \left[(R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right] \right\}. \quad (37)
\end{aligned}$$

In the last equation I already used the optimal consumption, C_t^* , thus I have omitted C_t below the sup operator; I substituted $c_t^* W_t$ in the place of C_t^* , since I showed in (32) that the optimal consumption ratio does not depend on the wealth level; and I also substituted the wealth dynamics (1) in the place of W_{t+1} . As we can see in (37), the value function at time t depends on the wealth level at time t only through the multiplicative factor $W_t^{1-\gamma}/(1-\gamma)$. Thus I can conclude that if the value function at time $t+1$ can be expressed in the form (36), then the value function at time t can be expressed in the form (35).

Following the logic of mathematical induction, the only thing left to prove is that the value function at time $T-1$ can be expressed in the form

$$V_{T-1}(W_{T-1}, \hat{\pi}_{T-1}, \Phi_{T-1}) = \frac{W_{T-1}^{1-\gamma}}{1-\gamma} v_{T-1}(\hat{\pi}_{T-1}, \Phi_{T-1}), \quad (38)$$

which is exactly what Lemma 1 contains. This completes the proof. \square

Lemma 1. *The value function at time $T-1$ can be expressed in the form*

$$V_{T-1}(W_{T-1}, \hat{\pi}_{T-1}, \Phi_{T-1}) = \frac{W_{T-1}^{1-\gamma}}{1-\gamma} v_{T-1}(\hat{\pi}_{T-1}, \Phi_{T-1}). \quad (39)$$

Proof of Lemma 1. First I show that both the optimal consumption ratio c_{T-1} and the optimal investment ratio π_{T-1} are independent of the wealth level W_{T-1} . Then, using the

fact of these independences, I show that the value function at time $T - 1$ can indeed be written in the form (39).

At time T , the investor consumes all of her wealth. The proportion of her wealth that she inherited in the risky asset, $\hat{\pi}_T$, she liquidates encountering a transaction cost of Φ_T , while the proportion which she inherited in the riskless asset she liquidates for free. Thus her consumption at time T is

$$C_T = W_T (1 - \hat{\pi}_T \Phi_T), \quad (40)$$

and her value function at time $T - 1$ is

$$V_{T-1} = \inf_{u_{T-1}} \sup_{\{C_{T-1}, \pi_{T-1}\}} \left\{ \exp(-\delta(T-1)) \frac{C_{T-1}^{1-\gamma}}{1-\gamma} + \exp(-\delta T) \left(1 + \frac{u_{T-1}^2 (1-\gamma)}{2\theta} \right) \mathbb{E}_{T-1}^{u_{T-1}} \left[\frac{W_T^{1-\gamma} (1 - \hat{\pi}_T \Phi_T)^{1-\gamma}}{1-\gamma} \right] \right\}. \quad (41)$$

To obtain the optimal consumption, the optimal investment ratio, and the least-favorable distortion at time $T - 1$, I write down the first-order conditions on the expression within the brackets on the right-hand side of (41) with respect to C_{T-1} , π_{T-1} , and u_{T-1} . The first-order condition with respect to C_{T-1} is

$$\begin{aligned} c_{T-1} &= (1 + I_{T-1} \Phi_{T-1} \hat{\pi}_{T-1}) \left\{ \exp\left(-\frac{\delta}{\gamma}\right) \left(1 + \frac{u_{T-1}^2 (1-\gamma)}{2\theta} \right)^{\frac{1}{\gamma}} \right. \\ &\quad \times \left\{ \mathbb{E}_{T-1}^{u_{T-1}} \left[(1 - \hat{\pi}_T \Phi_T)^{1-\gamma} (R_f + \pi_{T-1} (R_T - R_f))^{1-\gamma} \right] \right\}^{\frac{1}{\gamma}} \\ &\quad \left. \times (1 + I_{T-1} \Phi_{T-1} \pi_{T-1})^{\frac{\gamma-1}{\gamma}} + 1 \right\}^{-1}, \end{aligned} \quad (42)$$

the first-order condition with respect to π_{T-1} is

$$0 = \mathbb{E}_{T-1}^{u_{T-1}} \left\{ (R_f + \pi_{T-1} (R_T - R_f))^{-\gamma} (1 - \hat{\pi}_T \Phi_T)^{1-\gamma} \right. \\ \times \left[R_T - R_f - \frac{I_{T-1} \Phi_{T-1} (R_f + \pi_{T-1} (R_T - R_f))}{1 + I_{T-1} \Phi_{T-1} \pi_{T-1}} \right] \\ \left. - (R_f + \pi_{T-1} (R_T - R_f))^{-1-\gamma} (1 - \hat{\pi}_T \Phi_T)^{-\gamma} \Phi_T R_f R_T \right\}, \quad (43)$$

and the first-order condition with respect to u_{T-1} is

$$0 = \mathbb{E}_{T-1}^{u_{T-1}} \left\{ (R_f + \pi_{T-1} (R_T - R_f))^{1-\gamma} (1 - \hat{\pi}_T \Phi_T)^{1-\gamma} \right. \\ \times \left[\frac{u_{T-1} (1 - \gamma)}{\theta} + \left(1 + \frac{u_{T-1}^2 (1 - \gamma)}{2\theta} \right) \frac{\phi_T - \mu_\phi - u_{T-1}}{\sigma_\phi^2} \right] \left. \right\}. \quad (44)$$

This proves that at time $T-1$ each of the optimal consumption ratio, the optimal investment ratio, and the least-favorable distortion is independent of the wealth level W_{T-1} .

To show that the value function at $T-1$ can be expressed in the form (39), I substitute the wealth dynamics (1) into the original form of the value function at time $T-1$ (i.e., into (41)), and make use of the fact that none of the optimal consumption ratio c_{T-1}^* , the optimal investment ratio π_{T-1}^* , and the least-favorable distortion u_{T-1}^* depends on the wealth level W_{T-1} .

$$V_{T-1} = \frac{W_{T-1}^{1-\gamma}}{1-\gamma} \inf_{u_{T-1}} \sup_{\{c_{T-1}, \pi_{T-1}\}} \left\{ \exp(-\delta(T-1)) c_{T-1}^{1-\gamma} \right. \\ + \exp(-\delta T) \left(1 + \frac{u_{T-1}^2 (1-\gamma)}{2\theta} \right) \frac{(1 - c_{T-1} + I_{T-1} \Phi_{T-1} \hat{\pi}_{T-1})^{1-\gamma}}{(1 + I_{T-1} \Phi_{T-1} \pi_{T-1})^{1-\gamma}} \\ \left. \times \mathbb{E}_{T-1}^{u_{T-1}} \left[(1 - \hat{\pi}_T \Phi_T)^{1-\gamma} (R_f + \pi_{T-1} (R_T - R_f))^{1-\gamma} \right] \right\} \quad (45)$$

Remember, in the previous paragraph I showed that c_{T-1}^* , π_{T-1}^* , and u_{T-1}^* are all independent of W_{T-1} . Thus the right-hand side of (45) depends on the wealth level only through the multiplicative term $W_{T-1}^{1-\gamma}/(1-\gamma)$. This completes the proof. \square

Proof of Theorem 3. For the proof of the generalized version of the Envelope Theorem for

saddle-point problems, see Milgrom and Segal (2002).

□

References

- Amihud, Y., and H. Mendelson. 1986. Asset Pricing and the Bid-Ask Spread. *Journal of Financial Economics* 17:223–249.
- Anderson, E. W., L. P. Hansen, and T. J. Sargent. 2003. A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection. *Journal of the European Economic Association* 1:68–123.
- Bellman, R. E. 1957. *Dynamic Programming*. Princeton University Press.
- Branger, N., L. S. Larsen, and C. Munk. 2013. Robust Portfolio Choice with Ambiguity and Learning about Return Predictability. *Journal of Banking & Finance* 37:1397–1411.
- Buss, A., R. Uppal, and G. Vilkov. 2014. Asset Prices in General Equilibrium with Recursive Utility and Illiquidity Induced by Transactions Costs. *Working Paper*.
- Constantinides, G. M. 1986. Capital Market Equilibrium with Transaction Costs. *Journal of Political Economy* 94:842–862.
- Driessen, J., and R. Xing. 2015. The Liquidity Risk Premium for Long-Term Investors. *Working Paper*.
- Flor, C. R., and L. S. Larsen. 2014. Robust portfolio choice with stochastic interest rates. *Annals of Finance* 10:243–265.
- Garleanu, N., and L. H. Pedersen. 2013. Dynamic Trading with Predictable Returns and Transaction Costs. *The Journal of Finance* 68:2309–2340.
- Glasserman, P., and X. Xu. 2013. Robust Portfolio Control with Stochastic Factor Dynamics. *Operations Research* 61:874–893.
- Horvath, F., F. de Jong, and B. J. M. Werker. 2016. Robust Pricing of Fixed Income Securities. *Working Paper*.

- Jang, B.-G., H. K. Koo, H. Liu, and M. Loewenstein. 2007. Liquidity Premia and Transaction Costs. *The Journal of Finance* 52:2329–2366.
- Lesmond, D. A., J. P. Ogden, and C. A. Trzcinka. 1999. A New Estimate of Transaction Costs. *The Review of Financial Studies* 12:1113–1141.
- Liu, H. 2004. Optimal Consumption and Investment with Transaction Costs and Multiple Risky Assets. *The Journal of Finance* 59:289–338.
- Lynch, A. W., and S. Tan. 2011. Explaining the Magnitude of Liquidity Premia: The Roles of Return Predictability, Wealth Shocks, and State-Dependent Transaction Costs. *The Journal of Finance* 66:1329–1368.
- Maenhout, P. J. 2004. Robust Portfolio Rules and Asset Pricing. *The Review of Financial Studies* 17:951–983.
- Maenhout, P. J. 2006. Robust portfolio rules and detection-error probabilities for a mean-reverting risk premium. *Journal of Economic Theory* 128:136–163.
- Merton, R. C. 1969. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics* 51:247–257.
- Merton, R. C. 1971. Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory* 3:373–413.
- Milgrom, P., and I. Segal. 2002. Envelope Theorems for Arbitrary Choice Sets. *Econometrica* 70:583–601.
- Munk, C., and A. Rubtsov. 2014. Portfolio management with stochastic interest rates and inflation ambiguity. *Annals of Finance* 10:419–455.
- Vayanos, D. 1998. Transaction Costs and Asset Prices: A Dynamic Equilibrium Model. *The Review of Financial Studies* 11:1–58.