Network for Studies on Pensions, Aging and Retirement

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An Application to the KNW Model

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NETSPAR ACADEMIC SERIES

DP 04/2016-018

An Overview of Derivative Pricing in Gaussian Affine Asset Pricing Models
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April 5, 2016

Abstract

We present a comprehensive overview of derivative pricing in Gaussian affine asset pricing models. Gaussian affine asset pricing models are widely used in practice for pricing and scenario analysis due to their tractable pricing implications and easy estimation. This tractability is essential to efficiently evaluate portfolios of derivatives within many scenarios and time periods. We present efficient closed-form pricing formulas for the most common derivative instruments used by pension funds and insurance companies, such as interest rate swaps, swaptions, inflation-linked swaps, equity options, based on results from the literature. The pricing formulas are presented in a comprehensive computable form by utilising results based on the matrix exponential. Next, we show how some models commonly used in practice fit in the Gaussian affine framework, so that the pricing formulas can be applied to these cases. In particular, we discuss the KNW model by Koijen, Nijman and Werker (2010), which is widely used in the pension industry. Finally we discuss how our results can be applied to a time-inhomogeneous extension of the model that allows perfect calibration to the observed yield curve.

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1 Introduction

Gaussian affine model are widely using in finance to model interest rates, inflation, equities and other assets. The Gaussian affine framework is the workhorse in asset pricing as it presents a modelling framework with tractable pricing implications and estimation and inference methodologies. From an empirical viewpoint, the Gaussian affine framework implies a standard Gaussian Vector AutoRegressive (VAR) model for the factors driving the economy, and hence can be easily estimated. But on top of that, it delivers a consistent arbitrage-free continuous-time pricing framework.

Banks, insurers and pension fund therefore widely use Gaussian affine models in scenario analysis to evaluate balance sheet risks. It is important to be able to efficiently value portfolios of derivative instruments over all scenarios and time periods. The Gaussian affine framework offers the tractability to yield efficient closed-form prices of common derivative instruments such as interest rate swaps, swaptions, inflation-linked swaps and equity options.

Examples of widely-used Gaussian affine asset pricing models are the Brennan and Xia (2002) model and the Koijen, Nijman and Werker (KNW) (2010) model. The Brennan-Xia model combines a two-factor affine yield model with a log-normal equity and inflation process. The KNW model is an extension of the Brennan-Xia model that allows for time-varying bond risk premia. The KNW model is widely used by Dutch pension fund industry as the standard model for Asset Liability Management (ALM) analysis. Scenario analyses are strongly influenced by modelling assumptions and parameter estimates. For that reason, the "Commissie Parameters" (Langejan *et al.*, 2014) promotes the use of a uniform scenario set generated by the KNW model as calibrated by Draper (2014). Moreover, the Dutch pension regulator (DNB) prescribed the uniform KNW scenario set for a regulatory scenario analysis exercise.[1]

[1] Pension funds have to perform the scenario analysis exercise known as the "haalbaarheidstoets" on an

The literature offers a diverse set of pricing results for Gaussian affine models, but these are often not presented in a uniform way as they often depend on specific restricted versions of the Gaussian affine model or are derived under alternative parameterisations. It therefore often requires extra analysis to cast these pricing results to the model at hand and to make expressions fully explicit. In this paper, we collect the pricing results for common derivative instruments in the literature and present them in a uniform and explicit way, such that they can be applied straightforwardly. In particular, we show that the pricing results depend on three key model quantities and use results on the matrix exponential to calculate these efficiently.

The outline of the paper is as follows. Section 2 we set out a canonical form of the general Gaussian affine asset pricing model and derive computational results for three key quantities that feature in all pricing formulae. Section 3 presents the pricing expressions for the most common financial instruments: nominal bonds, swaps, swaptions, inflation-indexed bonds, inflation swaps and equity options. Section 4 discusses the KNW model and demonstrates how this model is cast in the canonical form. Furthermore, we discuss the time-inhomogeneous short rate extension of Brigo and Mercurio (2007) and show how the pricing formulas can be adapted to this extension.

2 Gaussian affine asset pricing model

2.1 Model definition

The Gaussian affine asset pricing model is based on factors $\boldsymbol{Y}_t$ with Gaussian dynamics that drive all dynamics in the economy. In particular, the n factors $\boldsymbol{Y}_t$ follow a Gaussian

annual basis.

diffusion process given by

$$\mathrm{d}\boldsymbol{Y}_t = [\boldsymbol{c}_{\mathbb{P}} + \boldsymbol{D}_{\mathbb{P}}\boldsymbol{Y}_t]\,\mathrm{d}t + \boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{P}}, \tag{1}$$

where $\mathbb{P}$ denotes the objective probability measure and $\boldsymbol{W}_t^{\mathbb{P}}$ is a m-vector of standard (uncorrelated) Brownian motions under $\mathbb{P}$. The dependence on the probability measure $\mathbb{P}$ is emphasised to distinguish it from the risk-neutral dynamics.

Furthermore we consider the following indexes: equity total return index S_t, price index Π_t and nominal bank account $B_t \equiv e^{\int^t r_s\,\mathrm{d}s}$ earning instantaneous nominal short rate r_t:

$$\Pi_t = e^{\boldsymbol{\nu}_{\Pi}{}'\boldsymbol{Y}_t}, \qquad S_t = e^{\boldsymbol{\nu}_S'\boldsymbol{Y}_t}, \qquad B_t = e^{\boldsymbol{\nu}_B{}'\boldsymbol{Y}_t}. \tag{2}$$

Finally the market prices of risk $\boldsymbol{\lambda}_t$ is an m-dimensional vector driving the risk premia and is assumed to be a linear function of the factors:

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\Lambda}_1\boldsymbol{Y}_t. \tag{3}$$

The linear form of the market prices of risk implies that the risk-neutral dynamics of the factors $\boldsymbol{Y}_t$ also follows a Gaussian diffusion process. In particular, let $\mathbb{Q}$ denote the risk-neutral probability measure, then the risk-neutral dynamics are given by

$$\mathrm{d}\boldsymbol{Y}_t = [\boldsymbol{c}_{\mathbb{Q}} + \boldsymbol{D}_{\mathbb{Q}}\boldsymbol{Y}_t]\,\mathrm{d}t + \boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}$$

where $\boldsymbol{W}_t^{\mathbb{Q}}$ denotes an m-vector of standard (uncorrelated) Brownian motions under $\mathbb{Q}$ and

$$\boldsymbol{c}_{\mathbb{Q}} = \boldsymbol{c}_{\mathbb{P}} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_0 \qquad \text{and} \qquad \boldsymbol{D}_{\mathbb{Q}} = \boldsymbol{D}_{\mathbb{P}} - \boldsymbol{\Sigma}\boldsymbol{\Lambda}_1.$$

This is useful for pricing purposes since derivatives prices are governed by the risk-neutral dynamics.

We need to impose restrictions on the parameters to ensure the model is arbitrage free. In particular, the bank account B_t and equity total return index S_t are tradeable assets and hence should impose a drift equal to the instantaneous nominal short rate. Second, the bank account should be instantaneously risk free and hence its instantaneous volatility should be zero. Applying Itô's Lemma to B_t and S_t under the risk-neutral measure $\mathbb{Q}$, we obtain:

$$\frac{\mathrm{d}B_t}{B_t} = \underbrace{\left[\boldsymbol{\nu}_B'\boldsymbol{c}_{\mathbb{Q}} + \boldsymbol{\nu}_B'\boldsymbol{D}_{\mathbb{Q}}\boldsymbol{Y}_t + \tfrac{1}{2}\boldsymbol{\nu}_B'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\nu}_B\right]}_{\mu_B^{\mathbb{Q}}(\boldsymbol{Y}_t)}\mathrm{d}t + \underbrace{\boldsymbol{\nu}_B'\boldsymbol{\Sigma}}_{\boldsymbol{\sigma}_B'}\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}},$$

$$\frac{\mathrm{d}S_t}{S_t} = \underbrace{\left[\boldsymbol{\nu}_S'\boldsymbol{c}_{\mathbb{P}} + \boldsymbol{\nu}_S'\boldsymbol{D}_{\mathbb{P}}\boldsymbol{Y}_t + \tfrac{1}{2}\boldsymbol{\nu}_S'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\nu}_S\right]}_{\mu_S^{\mathbb{Q}}(\boldsymbol{Y}_t)}\mathrm{d}t + \underbrace{\boldsymbol{\nu}_S'\boldsymbol{\Sigma}}_{\boldsymbol{\sigma}_S'}\mathrm{d}\boldsymbol{W}_t^{\mathbb{P}}.$$

The requirement of zero instantaneous volatility of the bank account implies the following restriction

$$\boldsymbol{\sigma}_B \equiv \boldsymbol{\Sigma}'\boldsymbol{\nu}_B = \mathbf{0}. \tag{4}$$

Furthermore, no arbitrage requires $r_t \equiv \mu_B^{\mathbb{Q}}(\boldsymbol{Y}_t) = \mu_S^{\mathbb{Q}}(\boldsymbol{Y}_t)$ and since the drifts are linear in the factors $\boldsymbol{Y}_t$, this implies

$$\delta_{0,r} \equiv \boldsymbol{\nu}_B'\boldsymbol{c}_{\mathbb{Q}} = \boldsymbol{\nu}_S'\boldsymbol{c}_{\mathbb{Q}} + \tfrac{1}{2}\boldsymbol{\nu}_S'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\nu}_S \qquad \text{and} \qquad \boldsymbol{\delta}_{1,r} \equiv \boldsymbol{D}_{\mathbb{Q}}{}'\boldsymbol{\nu}_B = \boldsymbol{D}_{\mathbb{Q}}{}'\boldsymbol{\nu}_S, \tag{5}$$

and the instantaneous nominal short rate r_t is defined by $r_t = \delta_{0,r} + \boldsymbol{\delta}_{1,r}{}'\boldsymbol{Y}_t$.

Note that the real-world $\mathbb{P}$-dynamics of the bank account B_t and equity total return index S_t are given by

$$\frac{\mathrm{d}B_t}{B_t} = r_t\,\mathrm{d}t, \qquad \frac{\mathrm{d}S_t}{S_t} = [r_t + \boldsymbol{\sigma}_S'\boldsymbol{\lambda}_t]\,\mathrm{d}t + \boldsymbol{\sigma}_S'\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{P}}.$$

and hence under the real-world measure $\mathbb{P}$, the equity total return index earns an equity risk premium of $\boldsymbol{\sigma}_S'\boldsymbol{\lambda}_t$ on top of the short rate.

Applying Ito's Lemma to the price index under the real-world measure $\mathbb{P}$

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = \underbrace{\left[\boldsymbol{\nu}_\Pi'\boldsymbol{c}_{\mathbb{P}} + \boldsymbol{\nu}_\Pi'\boldsymbol{D}_{\mathbb{P}}\boldsymbol{Y}_t + \tfrac{1}{2}\boldsymbol{\nu}_\Pi'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\nu}_\Pi\right]}_{\pi_t}\mathrm{d}t + \underbrace{\boldsymbol{\nu}_\Pi'\boldsymbol{\Sigma}}_{\boldsymbol{\sigma}_\pi'}\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{P}}$$

π_t represents the instantaneous ($\mathbb{P}$-)expected inflation rate and is linear in the factors, i.e.

$$\pi_t = \delta_{0,\pi} + \boldsymbol{\delta}_{1,\pi}{}'\boldsymbol{Y}_t.$$

with $\delta_{0,\pi} = \boldsymbol{\nu}_\Pi'\boldsymbol{c}_{\mathbb{P}} + \frac{1}{2}\boldsymbol{\nu}_\Pi'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\nu}_\Pi$ and $\boldsymbol{\delta}_{1,\pi} = \boldsymbol{D}_{\mathbb{P}}{}'\boldsymbol{\nu}_\Pi$. The price index grows with expected inflation π_t plus unexpected inflation $\boldsymbol{\sigma}_\pi'\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{P}}$.

The model can be summarised by model equations (1), (2) and (3) together with the restrictions (4) and (5) and parameters $\Theta = (\boldsymbol{c}_{\mathbb{Q}}, \boldsymbol{D}_{\mathbb{Q}}, \boldsymbol{\Sigma}, \boldsymbol{\nu}_\Pi, \boldsymbol{\nu}_S, \boldsymbol{\nu}_B, \boldsymbol{\lambda}_0, \boldsymbol{\Lambda}_1)$.

2.2 Conditional moments of the affine model

The Gaussian affine model implies that the factors $\boldsymbol{Y}_t$ are conditionally normally distributed, i.e. $\boldsymbol{Y}_{t+\tau}|\boldsymbol{Y}_t$ is normally distributed under $\mathbb{P}$ and $\mathbb{Q}$. The conditional moments of $\boldsymbol{Y}_t$ under probability measure $\mathbb{M} = \mathbb{P}, \mathbb{Q}$ are found by solving a system of linear ordinary differential equations (see Fackler (2000)). First define the following quantities

$$\boldsymbol{A}_{\mathbb{M}}(\Delta) \equiv e^{\boldsymbol{D}_{\mathbb{M}}\Delta}, \qquad \boldsymbol{B}_{\mathbb{M}}(\Delta) \equiv \int_0^\Delta e^{\boldsymbol{D}_{\mathbb{M}}s}\,\mathrm{d}s, \qquad \boldsymbol{C}_{\mathbb{M}}(\Delta) \equiv \int_0^\Delta e^{\boldsymbol{D}_{\mathbb{M}}s}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'e^{\boldsymbol{D}_{\mathbb{M}}'s}\,\mathrm{d}s.$$

The conditional moments of $\boldsymbol{Y}_t$ are given by

$$\mathrm{E}_t^{\mathbb{M}}\left[\boldsymbol{Y}_{t+\Delta}\right] = \boldsymbol{A}_{\mathbb{M}}(\Delta)\boldsymbol{Y}_t + \boldsymbol{B}_{\mathbb{M}}(\Delta)\boldsymbol{c}_{\mathbb{M}},$$
$$\mathrm{var}_t^{\mathbb{M}}\left[\boldsymbol{Y}_{t+\Delta}\right] = \boldsymbol{C}_{\mathbb{M}}(\Delta),$$

where the exponential of a matrix denotes the matrix exponential. Secondly any exponentially affine function of $\boldsymbol{Y}_t$ follows a conditionally lognormal distributions. Hence

$$\mathrm{E}_t^{\mathbb{M}}\left[e^{u_0+\boldsymbol{u}_1'\boldsymbol{Y}_{t+\Delta}}\right] = e^{u_0+\boldsymbol{u}_1'\,\mathrm{E}_t^{\mathbb{M}}[\boldsymbol{Y}_{t+\Delta}]+\frac{1}{2}\boldsymbol{u}_1'\,\mathrm{var}_t^{\mathbb{M}}[\boldsymbol{Y}_{t+\Delta}]\boldsymbol{u}_1} \equiv e^{a_{\{u_0,\boldsymbol{u}_1,\mathbb{M}\}}(\Delta)+\boldsymbol{b}_{\{\boldsymbol{u}_1,\mathbb{M}\}}(\Delta)'\boldsymbol{Y}_t}, \tag{6}$$

where

$$a_{\{u_0,\boldsymbol{u}_1,\mathbb{M}\}}(\Delta) = u_0 + \boldsymbol{c}_{\mathbb{M}}{}'\boldsymbol{B}_{\mathbb{M}}(\Delta)\boldsymbol{u}_1 + \tfrac{1}{2}\boldsymbol{u}_1{}'\boldsymbol{C}_{\mathbb{M}}(\Delta)\boldsymbol{u}_1,$$
$$\boldsymbol{b}_{\{\boldsymbol{u}_1,\mathbb{M}\}}(\Delta) = \boldsymbol{A}_{\mathbb{M}}(\Delta)'\boldsymbol{u}_1.$$

The computationally challenging parts are the quantities $\boldsymbol{A}_{\mathbb{M}}(\Delta)$, $\boldsymbol{B}_{\mathbb{M}}(\Delta)$ and $\boldsymbol{C}_{\mathbb{M}}(\Delta)$. These can however be easily calculated using the results of Van Loan (1978) and are obtained from the following matrix exponential

$$\exp\left(\begin{pmatrix} -\boldsymbol{D}_{\mathbb{M}} & \boldsymbol{\Sigma}\boldsymbol{\Sigma}' & \mathbf{O} \\ \mathbf{O} & \boldsymbol{D}_{\mathbb{M}}{}' & \boldsymbol{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}\Delta\right) = \begin{pmatrix} * & \boldsymbol{G}(\Delta) & * \\ \mathbf{O} & \boldsymbol{A}_{\mathbb{M}}'(\Delta) & \boldsymbol{B}_{\mathbb{M}}'(\Delta) \\ \mathbf{O} & \mathbf{O} & * \end{pmatrix}.$$

and with $\boldsymbol{C}_{\mathbb{M}}(\Delta) = \boldsymbol{A}_{\mathbb{M}}(\Delta)\boldsymbol{G}(\Delta)$.

3 Asset prices

We use the closed-form expression for (6) in Section 2 to derive closed-form expressions for all major asset types.

3.1 Interest rates

3.1.1 Nominal bond prices

Nominal zero-coupon bond prices are found by

$$\begin{aligned} P_t(T) &= \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s \, \mathrm{d}s}\right] = \mathrm{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_T}\right] = B_t \, \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\boldsymbol{\nu}_B{}'\boldsymbol{Y}_T}\right] \\ &\equiv \exp\left[a^{\mathrm{Nom}}(T-t) + \boldsymbol{b}^{\mathrm{Nom}}(T-t)'\boldsymbol{Y}_t\right] \end{aligned} \tag{7}$$

where $a^{\mathrm{Nom}}(\tau) = a_{\{0,-\boldsymbol{\nu}_B,\mathbb{Q}\}}(\tau)$ and $\boldsymbol{b}^{\mathrm{Nom}}(\tau) = \boldsymbol{\nu}_B + \boldsymbol{b}_{\{-\boldsymbol{\nu}_B,\mathbb{Q}\}}(\tau)$.

Coupon bonds are just portfolios of zero-coupon bonds. Hence a coupon bond paying coupons C_i at T_i and paying N at T_N:

$$V_t = \sum_{i=1}^{N} C_i P_t(T_i) + N P_t(T_N)$$

3.1.2 Interest rate swap

Consider a standard interest rate swap with notional equal to 1 and (forward) starting at T_n with fixed and floating payments on T_{n+1} to T_N. The swap is assumed to be priced based on the yield curve defined by the model. Hence the effect of OIS versus LIBOR risk is ignored. The present value of a basis point (PV01) is defined by

$$PV01_t^{n+1,N} = \sum_{i=n+1}^{N} \Delta_{i-1} P_t(T_i).$$

The value of a payer swap (paying fixed rate and receiving floating rate) with fixed rate K and notional 1 is given by

$$V_t^{IR,P}(K) = \{P_t(T_n) - P_t(T_N)\} - K \cdot PV01_t^{n+1,N}.$$

The par swap rate is the fixed rate at with the contract can be initiated with zero market value and is given by

$$S_t^{n,N} = \frac{P_t(T_n) - P_t(T_N)}{PV01_t^{n+1,N}}. \tag{8}$$

3.1.3 Swaption

A swaption is an option to enter a swap and can be viewed as a call or put option on the par swap rate $S_t^{n,N}$. In particular a payer/receiver swaption with maturity T_n and strike K gives the right to at time T_n to enter a payer/receiver swap at time T_n with swap rate K, respectively. Hence its value at time T_n is given by $\left[V_{T_n}^{IR,P}(K)\right]^+$ and $\left[V_{T_n}^{IR,R}(K)\right]^+$, respectively.

As can be seen from (8), the swap rate has a non-linear dependence on the factors and hence does not follow a Gaussian process. The swap rate can therefore not be easily calculated in closed-form and we therefore use the approach by Schrager and Pelsser (2006) to obtain a good closed-form approximation to the swaption price.[2] Schrager and Pelsser (2006) propose an approximation to the swap rate process to bring it to an affine form, such that swaption prices can be derived in closed form. The swaption price follows a Gaussian option pricing formula based on the approximated swap rate dynamics. In particular, the

[2]In the KNW model, we can use Jamshidian's "trick" for 1-factor Gaussian models (Jamshidian, 1989) to price a swaption in semi-closed form. Our purpose however is to derive expressions that can easily be calculated by end-users, and since the accuracy of Schrager and Pelsser's approach is found to be satisfactory, we will only use their approximation here.

swaption prices at time t for a payer and receiver swaption are given by

$$PS_t^{n,N}(K) = PV01_t^{n+1,N}\left[(S_t^{n,N} - K)\Phi\left(\frac{S_t^{n,N} - K}{\sigma_{S^{n,N}}(\tau)}\right) + \sigma_{S^{n,N}}(\tau)\phi\left(\frac{S_t^{n,N} - K}{\sigma_{S^{n,N}}(\tau)}\right)\right], \quad (9)$$

$$RS_t^{n,N}(K) = PV01_t^{n+1,N}\left[(K - S_t^{n,N})\Phi\left(\frac{K - S_t^{n,N}}{\sigma_{S^{n,N}}(\tau)}\right) + \sigma_{S^{n,N}}(\tau)\phi\left(\frac{K - S_t^{n,N}}{\sigma_{S^{n,N}}(\tau)}\right)\right], \quad (10)$$

respectively, with $\tau \equiv T_n - t$. The swaption normal volatility $\sigma_{S^{n,N}}(\tau)$ is given by

$$\sigma_{S^{n,N}}(\tau) = \sqrt{\boldsymbol{q}'\left[\int_0^\tau e^{\boldsymbol{D}_\mathbb{Q}s}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'e^{\boldsymbol{D}_\mathbb{Q}'s}\,\mathrm{d}s\right]\boldsymbol{q}} = \sqrt{\boldsymbol{q}'\boldsymbol{C}_\mathbb{Q}(\tau)\boldsymbol{q}}, \quad (11)$$

where

$$\boldsymbol{q} = -\frac{1}{PV01_t^{n+1,N}}\Bigg[P_t(T_n) - P_t(T_N)\boldsymbol{A}_\mathbb{Q}(T_N - T_n) - S_t^{n,N}\sum_{i=n+1}^{N}\Delta_{i-1}P_t(T_i)\boldsymbol{A}_\mathbb{Q}(T_i - T_n)\Bigg]'\boldsymbol{\nu}_B. \quad (12)$$

3.2 Inflation

Inflation instruments have payoffs that depend on the price index Π_t. We consider inflation-linked bonds and inflation swaps and assume there is no basis risk driving differences between these two instruments.

3.2.1 Inflation-Linked bond prices

Inflation-linked zero-coupon bonds form a basic building block in common inflation derivatives. Inflation-linked zero-coupon bonds are similar to nominal zero-coupon bonds, but have their payoff linked to inflation. In particular, the pay-off at maturity is scaled by the growth in the price index Π_t. A common complicating factor is that the indexing is based on a lagged value of the index to ensure it is observed. The pay-off at maturity date T

given the initial price index fixation Π^0 and indexation lag Δ_{lag} equals

$$\frac{\Pi_{T-\Delta_{lag}}}{\Pi^0}.$$

The final price index fixing $\Pi_{T-\Delta_{lag}}$ is generally observed before the maturity date. The model assumes no publication lag by assuming $\Pi_{T-\Delta_{lag}}$ is observed at time $T-\Delta_{lag}$. This implies that the inflation-linked bond is essentially a nominal bond over the period of observing the last price index fixing and maturity.

The bond price is given by

$$\begin{aligned}
P_t^{IL}(T;\Pi^0,\Delta_{lag}) &= \mathrm{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,\mathrm{d}s}\frac{\Pi_{T-\Delta_{lag}}}{\Pi^0}\right] = \mathrm{E}_t^{\mathbb{Q}}\left[\frac{B_t\Pi_{T-\Delta_{lag}}}{B_T\Pi^0}\right]\\
&= \frac{B_t}{\Pi^0}\,\mathrm{E}_t^{\mathbb{Q}}\left[\frac{\Pi_{T-\Delta_{lag}}P_{T-\Delta_{lag}}(\Delta_{lag})}{B_{T-\Delta_{lag}}}\right]\\
&= \frac{B_t}{\Pi^0}\,\mathrm{E}_t^{\mathbb{Q}}\left[e^{a^{\mathrm{Nom}}(\Delta_{lag})+\boldsymbol{w}'\boldsymbol{Y}_{T-\Delta_{lag}}}\right]\\
&\equiv \frac{1}{\Pi^0}e^{a^{IL}(T-t,\Delta_{lag})+\boldsymbol{b}^{IL}(T-t,\Delta_{lag})'\boldsymbol{Y}_t}
\end{aligned} \tag{13}$$

with

$$\begin{aligned}
\boldsymbol{w} &\equiv \boldsymbol{b}^{\mathrm{Nom}}(\Delta_{lag})+\boldsymbol{\nu}_{\Pi}-\boldsymbol{\nu}_B,\\
\boldsymbol{b}^{IL}(\tau,\Delta_{lag}) &= \boldsymbol{b}_{\{\boldsymbol{w},\mathbb{Q}\}}(T-\Delta_{lag}-t)+\boldsymbol{\nu}_B,\\
a^{IL}(\tau,\Delta_{lag}) &= a_{\{a^{\mathrm{Nom}}(\Delta_{lag}),\boldsymbol{w},\mathbb{Q}\}}(T-\Delta_{lag}-t).
\end{aligned} \tag{14}$$

3.2.2 Zero-Coupon Inflation-Indexed swap

A Zero-Coupon Inflation-Indexed (ZCII) swap exchanges the growth in the price index for a fixed rate K at maturity T. From the perspective of the inflation receiver, the pay off at maturity T equals

$$\underbrace{\left[\frac{\Pi_{T-\Delta_{lag}}}{\Pi^0} - 1\right]}_{\text{inflation leg}} - \underbrace{\left[(1+K)^{DCF(t,T)} - 1\right]}_{\text{fixed leg}},$$

where Π^0 is the initial price index fixing and $DCF(t,T)$ denotes the day count fraction between t and T.

The market value of the inflation swap is

$$V_t^{ZCIIS,P} = P_t^{IL}(T;\Pi^0,\Delta_{lag}) - P_t(T)(1+K)^{DCF(t,T)}$$

and the par swap rate is found for a swap initiated with price index fixation $\Pi^0 = \Pi_{t-\Delta_{lag}}$ and swap rate K_t^* such that $V_t^{ZCIIS,P} = 0$. This leads to:

$$K_t^* = \left[\frac{P_t^{IL}(T;\Pi_{t-\Delta_{lag}},\Delta_{lag})}{P_t(T)}\right]^{1/DCF(t,T)} - 1.$$

3.3 Equity option

The equity total return index is modelled as a Black-Scholes model extended with Gaussian stochastic interest rates. This is similar to e.g. the Black-Scholes Hull-White model considered in Brigo and Mercurio (2007, section B.1).[3]

The log of the equity total return index follows a Gaussian process and therefore the index itself is conditionally lognormally distributed. Hence the price of a call option and put option with strike K satisfy the following Black-Scholes pricing formulae. In particular the standard call and put option with strike K have $(S_t - K)^+$ and $(K - S_t)^+$ as payoff at

[3]In this case, we focus on a time-homogeneous version of the model, whereas the Black-Scholes-Hull-White model is based on a time-inhomogeneous version.

maturity T. Their time-t price is given by:

$$CO_t(K) = S_t\Phi\left(\frac{\log\frac{S_t}{KP_t(T)} + \frac{1}{2}s_S^2(\tau)}{s_S(\tau)}\right) - KP_t(T)\Phi\left(\frac{\log\frac{S_t}{KP_t(T)} - \frac{1}{2}s_S^2(\tau)}{s_S(\tau)}\right), \tag{15}$$

$$PO_t(K) = KP_t(T)\Phi\left(\frac{-\log\frac{S_t}{KP_t(T)} + \frac{1}{2}s_S^2(\tau)}{s_S(\tau)}\right) - S_t\Phi\left(\frac{-\log\frac{S_t}{KP_t(T)} - \frac{1}{2}s_S^2(\tau)}{s_S(\tau)}\right), \tag{16}$$

where $\tau = T - t$ and the volatility $s_S(\tau)$ is a maturity dependent function of the model parameters that acknowledges the effect of the interest rate volatility on the overall volatility of the equity index. The volatility $s_S(\tau)$ is given by

$$\begin{aligned}
s_S^2(\tau) &= \int_0^\tau [\boldsymbol{\nu}_S - \boldsymbol{b}(s)]' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' [\boldsymbol{\nu}_S - \boldsymbol{b}(s)]\,\mathrm{d}s \\
&= [\boldsymbol{\nu}_S - \boldsymbol{\nu}_B]' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' [\boldsymbol{\nu}_S - \boldsymbol{\nu}_B]\,\tau + 2\,[\boldsymbol{\nu}_S - \boldsymbol{\nu}_B]' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \left[\int_0^\tau e^{\boldsymbol{D}_{\mathbb{Q}}' s}\,\mathrm{d}s\right] \boldsymbol{\nu}_B \\
&\quad + \boldsymbol{\nu}_B' \left[\int_0^\tau e^{\boldsymbol{D}_{\mathbb{Q}} s} \boldsymbol{\Sigma}\boldsymbol{\Sigma}' e^{\boldsymbol{D}_{\mathbb{Q}}' s}\,\mathrm{d}s\right] \boldsymbol{\nu}_B, \\
&= [\boldsymbol{\nu}_S - \boldsymbol{\nu}_B]' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' [\boldsymbol{\nu}_S - \boldsymbol{\nu}_B]\,\tau + 2\,[\boldsymbol{\nu}_S - \boldsymbol{\nu}_B]' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{B}_{\mathbb{Q}}(\tau)' \boldsymbol{\nu}_B \\
&\quad + \boldsymbol{\nu}_B' \boldsymbol{C}_{\mathbb{Q}}(\tau) \boldsymbol{\nu}_B.
\end{aligned} \tag{17}$$

see Brigo and Mercurio (2007, p. 889). Appendix A.2 presents details on the derivation.

4 Models and extensions

In this section, we discuss the KNW model by Koijen *et al.* (2010) and show how it can be cast in the general Gaussian affine framework so that the pricing formulas can be applied. Hereafter, we use the results by Brigo and Mercurio (2007, Section 3.8) to extend the Gaussian affine model to have a time-inhomogeneous short rate process. This extension provides the model with extra degrees of freedom such that the model can be perfectly calibrated to the observed market interest rate curve.

4.1 KNW model definition

The KNW model by Koijen *et al.* (2010) is essentially a two-factor Gaussian affine model for interest rates and expected inflation rates, combined with a Black-Scholes (GBM) model for equity and the price index. The two factors $\boldsymbol{X}_t = (X_{1t}, X_{2t})'$ follow mean-reverting Gaussian dynamics under $\mathbb{P}$ given by

$$\mathrm{d}\boldsymbol{X}_t = -\boldsymbol{K}_X \boldsymbol{X}_t + \boldsymbol{\Sigma}_X \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{P}},$$

where $\boldsymbol{\Sigma}_X = \begin{bmatrix} \boldsymbol{I}_{2\times 2} & \mathbf{O}_{2\times 2} \end{bmatrix}$ and $\boldsymbol{W}_t^{\mathbb{P}}$ is a 4-dimensional standard Brownian motion under $\mathbb{P}$. The instantaneous nominal short rate r_t and instantaneous expected inflation π_t are given by

$$r_t = \delta_{0,r}^* + \boldsymbol{\delta}_{1,r}^*{}' \boldsymbol{X}_t, \qquad \pi_t = \delta_{0,\pi}^* + \boldsymbol{\delta}_{1,\pi}^*{}' \boldsymbol{X}_t.$$

The dynamics of the stock index and price index are given by

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = \pi_t \, \mathrm{d}t + \boldsymbol{\sigma}_\pi{}' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{P}} \tag{18}$$

$$\frac{\mathrm{d}S_t}{S_t} = (r_t + \eta_S) \, \mathrm{d}t + \boldsymbol{\sigma}_S{}' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{P}} \tag{19}$$

The normalisation $\sigma_{\pi,(4)} = 0$ is imposed to ensure that the model is well-identified (Koijen *et al.*, 2010). Finally the market prices of risk are parametrized as

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0^* + \boldsymbol{\Lambda}_1^* \boldsymbol{X}_t.$$

Further restrictions are imposed on the market prices of risk. First, (19) requires that $\boldsymbol{\sigma}_S{}'\boldsymbol{\lambda}_t = \eta_S$, implying with the following restrictions imposed

$$\boldsymbol{\sigma}_S{}'\boldsymbol{\Lambda}_1^* = \mathbf{0}', \qquad\qquad \boldsymbol{\sigma}_S{}'\boldsymbol{\lambda}_0 = \eta_S.$$

Secondly, the price of unexpected inflation risk unrelated to bond and stock market risk is assumed zero as there is insufficient data available to estimate it accurately (Koijen *et al.*, 2010). Therefore, the following restrictions are imposed

$$\boldsymbol{e}_3{}'\boldsymbol{\Lambda}_1^* = \mathbf{0}', \qquad\qquad \boldsymbol{e}_3{}'\boldsymbol{\lambda}_0 = 0,$$

where $\boldsymbol{e}_3{}' = \begin{pmatrix} 0, & 0, & 1, & 0 \end{pmatrix}'$. The parameters of the model are given by $\boldsymbol{\Theta}_{KNW} = \left(\boldsymbol{K}_X, \delta_{0,r}^*, \boldsymbol{\delta}_{1,r}^*, \delta_{0,\pi}^*, \boldsymbol{\delta}_{1,\pi}^*, \boldsymbol{\sigma}_\pi, \boldsymbol{\sigma}_S, \boldsymbol{\lambda}_0, \boldsymbol{\Lambda}_1\right)$.

4.2 KNW model as a special case of Gaussian affine model

The KNW model can be cast in the standard form of the Gaussian affine asset pricing model of Section 2. Define the factors by extending the KNW factors with the logs of the price index, stock index and bank account as

$$\boldsymbol{Y}_t = \begin{pmatrix} \boldsymbol{X}_t{}' & \log \Pi_t & \log S_t & \log B_t \end{pmatrix}'.$$

The KNW model expressed in standard form is given by model equations (1), (2) and (3), with the corresponding parameters given by

$$\boldsymbol{\nu}_\Pi = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}', \qquad \boldsymbol{\nu}_S = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}', \qquad \boldsymbol{\nu}_B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}',$$

$$
\boldsymbol{c}^{\mathbb{Q}} = \begin{bmatrix} -\boldsymbol{\Sigma}_{X^*}\boldsymbol{\lambda}_0 \\ \delta^*_{0,\pi} - \frac{1}{2}\boldsymbol{\sigma}'_\pi\boldsymbol{\sigma}_\pi - \boldsymbol{\sigma}'_\pi\boldsymbol{\lambda}_0 \\ \delta^*_{0,r} - \frac{1}{2}\boldsymbol{\sigma}'_S\boldsymbol{\sigma}_S \\ \delta^*_{0,r} \end{bmatrix}, \quad \boldsymbol{D}^{\mathbb{Q}} = \begin{bmatrix} -(\boldsymbol{K}_{X^*} + \boldsymbol{\Sigma}_{X^*}\boldsymbol{\Lambda}^*_1) & \mathbf{O}_{2\times 3} \\ {\boldsymbol{\delta}^*_{1,\pi}}' - \boldsymbol{\sigma}'_\pi\boldsymbol{\Lambda}^*_1 & \mathbf{0}_3{}' \\ {\boldsymbol{\delta}^*_{1,r}}' & \mathbf{0}_3{}' \\ {\boldsymbol{\delta}^*_{1,r}}' & \mathbf{0}_3{}' \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_X \\ {\boldsymbol{\sigma}_\pi}' \\ {\boldsymbol{\sigma}_S}' \\ \mathbf{0}_4{}' \end{bmatrix},
$$

$$
\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}^*_0, \qquad \boldsymbol{\Lambda}_1 = \begin{bmatrix} \boldsymbol{\Lambda}^*_1, & \mathbf{O}_{4\times 3} \end{bmatrix}.
$$

It is straightforward to verify that the KNW model expressed in the standard form indeed satisfies the restrictions (4) and (5).

4.3 Time-inhomogeneous short rate extension

For many derivative applications, the standard Gaussian affine model is not suitable, because of an insufficient fit to market rates. In particular, the model-implied yield curve defined by (7) generally does not produce an accurate enough fit of observed market rates when the number of factors is much smaller than the number of observed market rates. For some applications this can be sufficient, but when one needs to price derivatives such an approach is not advisable. Certainly in the larger currencies the swap market is sufficiently liquid to necessitate a better fit than would be obtained from a one or two-factor model.

A solution is to extend the model to a time-inhomogeneous model by allowing some coefficients to be time-varying. This extension was initially proposed for the Vašíček model by Hull and White (1990), but has subsequently been generalised by Brigo and Mercurio (2007, Section 3.8) among others. This extension increases the degrees of freedom of the model such that it can be perfectly calibrated to observed market rates.

Let us start with a base model as defined above and denote all prices and rates with a tilde to differentiate them from the extended version. We now define the time inhomogeneous extensions by adding a time-dependent intercept to the short rate

$$r_t = \tilde{r}_t + \varphi(t).$$

The prices of the extended model can be obtained by appropriately adjusting the base-model prices. In particular, we have

$$P_t(T) = \exp\left\{-\int_t^T \varphi(s)\,\mathrm{d}s\right\}\tilde{P}_t(T) = \Phi(t,T)\tilde{P}_t(T), \tag{20}$$

and setting the equity index equal at time 0: $S_0 = \tilde{S}_0$, we have

$$S_t = \frac{\tilde{S}_t}{\Phi(0,t)}. \tag{21}$$

All other model quantities are not changed.

The extended model is flexible enough to perfectly fit an initial term structure of market observed rates. In particular, if we observe market rates $P_0^M(T)$ at time 0, then we achieve a perfect fit at time 0

$$P_0(T) = P_0^M(T)$$

by choosing

$$\varphi(t) = f_0^M(t) - \tilde{f}_t(T), \tag{22}$$

$$\Phi(t,T) = \exp\left\{-\int_t^T \varphi(s)\,\mathrm{d}s\right\} = \frac{P_0^M(T)}{\tilde{P}_0(T)}\frac{\tilde{P}_0(t)}{P_0^M(t)}, \tag{23}$$

where $f_t(T)$ denotes the (instantaneous) forward curve defined by

$$f_t(T) = -\frac{\partial \log P_t(T)}{\partial T}.$$

Using this, we need to make the following adjustments to our pricing results above

- Nominal bond prices. The base-model bond price $\tilde{P}_t(T)$ satisfies (7) and the extended-model price $P_t(T)$ is then given by (20).

- Interest rate swap. All expressions in Section 3.1.2 apply, but with $P_t(T)$ denoting the *extended-model* bond prices.

- Swaption. All expressions in Section 3.1.3 apply, but with $P_t(T)$, $S_t^{n,N}$ and $PV01_t$ denoting the *extended-model* versions. This is the result from the fact that the par swap rate remains a martingale under the swap measure in the extended model. Since the extension only affects the drift and not the volatility of the swap rate dynamics all steps of the derivation in Section A.1 continue to hold for the extended model, but then based on extended-model quantities.

- Inflation-Linked bond prices. The base-model bond price $\tilde{P}_t^{IL}(T;\Pi_0,\Delta_{lag})$ is given by (13). The extended-model bond prices $P_t^{IL}(T;\Pi_0,\Delta_{lag})$ is given by

$$P_t^{IL}(T;\Pi_0,\Delta_{lag}) = \Phi(t,T)\tilde{P}_t^{IL}(T;\Pi_0,\Delta_{lag}).$$

- Zero-Coupon Inflation-Indexed Swap. All expressions in Section 3.2.2 apply, but with $P_t(T)$ and $P_t^{IL}(T;\Pi_0,\Delta_{lag})$ denoting the *extended-model* prices.

- Equity option. All expressions in Section 3.3 apply, but with $P_t(T)$ and S_t denoting the *extended-model* prices. This results from the fact that the equity price index discounted by the bond should be a martingale under the forward measure, also in the extended model.

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A Details of derivation of asset prices

A.1 Swaption

We follow the derivation in Schrager and Pelsser (2006) and consider a change of measure to the swap measure $\mathbb{Q}^{n+1,N}$. This probability measure uses the swap annuity $PV01_t^{n+1,N}$ as a numeraire and is equivalent to $\mathbb{P}$. Consequently, the par swap rate $S_t^{n,M}$ is a martingale under $\mathbb{Q}^{n+1,T}$:

$$\mathrm{d}S_t^{n,N} = \frac{\partial S_t^{n,N}}{\partial \boldsymbol{Y}_t'} \boldsymbol{\Sigma} \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}^{n+1,N}}$$

with

$$\frac{\partial S_t^{n,N}}{\partial \boldsymbol{Y}_t'} = \frac{1}{PV01_t^{n+1,N}} \Bigg[P_t(T_n)\boldsymbol{b}^{\mathrm{Nom}}(T_n - t) - P_t(T_N)\boldsymbol{b}^{\mathrm{Nom}}(T_N - t) \\ - S_t^{n,N} \sum_{i=n+1}^{N} \Delta_{i-1} P_t(T_i) \boldsymbol{b}^{\mathrm{Nom}}(T_i - t) \Bigg]'.$$

The pricing formula for payer and receiver swaption are

$$PS_{t_0}^{n,N}(K) = PV01_{t_0}^{n+1,N} \, \mathrm{E}_{t_0}^{\mathbb{Q}^{n+1,N}} \left\{ \left[S_{T_n}^{n,N} - K \right]^+ \right\}, \tag{24}$$

$$RS_{t_0}^{n,N}(K) = PV01_{t_0}^{n+1,N} \, \mathrm{E}_{t_0}^{\mathbb{Q}^{n+1,N}} \left\{ \left[K - S_{T_n}^{n,N} \right]^+ \right\}, \tag{25}$$

where $\mathbb{Q}^{n+1,N}$ is the corresponding swap measure (with numeraire $PV01_t^{n+1,N}$).

The swap rate does not follow an affine process under the swap measure and therefore closed-form swaption prices are not available. Schrager and Pelsser (2006) propose an approximation to the swap rate process to bring it to an affine form, such that swaption prices can be derived in closed form. The swap rate dynamics for $t \geq t_0$ is approximated by

$$\mathrm{d}\widetilde{S_t^{n,N}} = \widetilde{\frac{\partial S_t^{n,N}}{\partial \boldsymbol{Y}_t'}} \boldsymbol{\Sigma}\, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}^{n+1,N}}$$

with

$$\widetilde{\frac{\partial S_t^{n,N}}{\partial \boldsymbol{Y}_t'}} = \frac{1}{PV01_{t_0}^{n+1,N}} \Bigg[P_{t_0}(T_n)\boldsymbol{b}^{\text{Nom}}(T_n - t) - P_{t_0}(T_N)\boldsymbol{b}^{\text{Nom}}(T_N - t) \\ - S_{t_0}^{n,N} \sum_{i=n+1}^{N} \Delta_{i-1} P_{t_0}(T_i)\boldsymbol{b}^{\text{Nom}}(T_i - t) \Bigg]'.$$

Comparing the original swap rate dynamics with the approximating dynamics, we see that the approximation freezes bond prices, the swap annuity and the par swap rate at its initial value t_0. The swap rate volatility only depends on time via the bond coefficients $\boldsymbol{b}^N(\cdot)$ and is hence non-stochastic. Consequently, the approximating swap rate dynamics is a Gaussian process. In particular, using the fact that $P_t(T_n) - P_t(T_N) = S_t^{n,N} \sum_{i=n+1}^{N} \Delta_{i-1} P_t(T_i)$, the approximating swap dynamics simplifies to

$$\widetilde{\frac{\partial S_t^{n,N}}{\partial \boldsymbol{Y}_t'}} = \boldsymbol{q}' e^{\boldsymbol{D}_{\mathbb{Q}}(T_n - t)}.$$

with $\boldsymbol{q}$ as defined in (12). Since the approximated swap rate dynamics is a Gaussian process, we obtain its conditional variance by integrating:

$$\begin{aligned}
\operatorname{var}_{t_0}^{\mathbb{Q}^{n+1,N}}(\widetilde{S_{T_n}^{n,N}}) &= \int_{t_0}^{T_n} \operatorname{var}_0 \left[\widetilde{\frac{\partial S_t^{n,N}}{\partial \boldsymbol{Y}_t'}} \boldsymbol{\Sigma}\, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}^{n+1,N}} \right] \\
&= \int_t^{T_n} \widetilde{\frac{\partial S_s^{n,N}}{\partial \boldsymbol{Y}_s'}} \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \widetilde{\frac{\partial S_s^{n,N}}{\partial \boldsymbol{Y}_s}}\, \mathrm{d}s = \boldsymbol{q}' \left[\int_0^{T_n - t_0} e^{\boldsymbol{D}_{\mathbb{Q}} s} \boldsymbol{\Sigma}\boldsymbol{\Sigma}' e^{\boldsymbol{D}_{\mathbb{Q}}' s}\, \mathrm{d}s \right] \boldsymbol{q} \\
&= \boldsymbol{q}' \boldsymbol{C}_{\mathbb{Q}}(T_n - t_0) \boldsymbol{q} \equiv \sigma^2_{S^{n,N}}(T_n - t_0).
\end{aligned}$$

The conditional mean is zero due to the martingale property. Using the fact that the swap rate is conditionally normally distributed, the pricing expressions (24, 25) are explicitly solved by (9, 10).

A.2 Equity option

The most straightforward way to derive the option price is to consider a change of measure to the T-forward measure $\mathbb{Q}^T$. This probability measure uses the bond price $P_t(T)$ a numeraire and is equivalent to $\mathbb{P}$ and such that $S_t/P_t(T)$ is a martingale under $\mathbb{Q}^T$. This implies the following pricing relations

$$CO_t(K) = P_t(T)\,\mathrm{E}_t^{\mathbb{Q}_T}\left\{(S_T - K)^+\right\}, \qquad PO_t(K) = P_t(T)\,\mathrm{E}_t^{\mathbb{Q}_T}\left\{(K - S_T)^+\right\}. \quad (26)$$

Furthermore, by applying Ito's formula to $S_t/P_t(T)$ and using the fact that it is a martingale under $\mathbb{Q}^T$, we have

$$\mathrm{d}\left(\frac{S_t}{P_t(T)}\right) = \left(\frac{S_t}{P_t(T)}\right)\boldsymbol{\sigma}_{S/P}\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}^T}$$
$$\boldsymbol{\sigma}_{S/P} = \boldsymbol{\Sigma}'\left[\boldsymbol{\nu}_S - \boldsymbol{b}^{\mathrm{Nom}}(T-t)\right].$$

$S_t/P_t(T)$ is therefore a lognormal process with $\mathrm{d}\log\left(\frac{S_t}{P_t(T)}\right) = -\frac{1}{2}\boldsymbol{\sigma}_{S/P}{}'\boldsymbol{\sigma}_{S/P}\,\mathrm{d}t + \boldsymbol{\sigma}_{S/P}\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}^T}$. Using the fact that $P_T(T) = 1$, we obtain the conditional mean and variance of $\log(S_T)$ by

$$\mathrm{E}_t^{\mathbb{Q}^T}(\log(S_T)) = -\tfrac{1}{2}s_S^2(T-t), \qquad \mathrm{var}_t^{\mathbb{Q}^T}(\log(S_T)) = s_S^2(T-t),$$

with $s_S^2(\tau)$ as defined in (17). Applying well-know results on the lognormal distribution (see e.g. Brigo and Mercurio (2007, section D)), the option price in (26) is solved by (15, 16).

B KNW explicit solution

Since KNW is essentially a two-factor model, we can exploit its structure to make the matrix exponetial calculations more explicit. In particular assume that the 2×2 matrix

$$-(\boldsymbol{K}_{X^*} + \boldsymbol{\Sigma}_{X^*}\boldsymbol{\Lambda}_1^*)$$

has two distinct real non-zero eigenvalues ω_1 and ω_2. Define $\boldsymbol{\omega} = \begin{pmatrix} \omega_1, & \omega_2, & 0, & 0, & 0 \end{pmatrix}'$ and denote by $\mathrm{diag}(\boldsymbol{\omega})$ a diagonal matrix with $\boldsymbol{\omega}$ on the diagonal. Then we have the following eigenvalue decomposition for $\boldsymbol{D}_{\mathbb{Q}}$ in the KNW model

$$\boldsymbol{D}_{\mathbb{Q}} = \boldsymbol{U}\,\mathrm{diag}(\boldsymbol{\omega})\boldsymbol{U}^{-1},$$

with $\boldsymbol{U}'\boldsymbol{U} = \boldsymbol{I}$. Hence we have

$$\boldsymbol{A}_{\mathbb{Q}}(\Delta) \equiv e^{\boldsymbol{D}_{\mathbb{Q}}\Delta} = \boldsymbol{U}\,\mathrm{diag}(e^{\boldsymbol{\omega}\Delta})\boldsymbol{U}^{-1}$$

and

$$\begin{aligned}
\boldsymbol{B}_{\mathbb{Q}}(\Delta) \equiv e^{\boldsymbol{D}_{\mathbb{Q}}\Delta} &= \boldsymbol{U}\,\mathrm{diag}\left(\int_0^{\Delta} e^{\boldsymbol{\omega}s}\,\mathrm{d}s\right)\boldsymbol{U}^{-1} \\
&= \boldsymbol{U}\begin{pmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & 0 & \boldsymbol{0}_{3\times1}{}' \\ 0 & \frac{e^{\lambda_2\Delta}-1}{\lambda_2} & \boldsymbol{0}_{3\times1}{}' \\ \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times1} & \Delta\boldsymbol{I}_{3\times3} \end{pmatrix}\boldsymbol{U}^{-1}
\end{aligned}$$

and

$$
\begin{aligned}
\boldsymbol{C}_{\mathbb{Q}}(\Delta) &\equiv \int_0^\Delta e^{\boldsymbol{D}_{\mathbb{Q}} s} \boldsymbol{\Sigma}\boldsymbol{\Sigma}' e^{\boldsymbol{D}_{\mathbb{Q}}' s} \, \mathrm{d}s \\
&= \boldsymbol{U} \left[\int_0^\Delta \operatorname{diag}(e^{\boldsymbol{\omega} s}) \boldsymbol{U}^{-1} \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{U}^{-1} \operatorname{diag}(e^{\boldsymbol{\omega} s}) \, \mathrm{d}s \right] \boldsymbol{U} \\
&= \boldsymbol{U} \begin{pmatrix}
\frac{e^{2\omega_1 \Delta}-1}{2\omega_1} w_{11} & \frac{e^{(\omega_1+\omega_2)\Delta}-1}{\omega_1+\omega_2} w_{12} & \frac{e^{\omega_1 \Delta}-1}{\omega_1} w_{13} & \frac{e^{\omega_1 \Delta}-1}{\omega_1} w_{14} & \frac{e^{\omega_1 \Delta}-1}{\omega_1} w_{15} \\
\frac{e^{(\omega_1+\omega_2)\Delta}-1}{\omega_1+\omega_2} w_{21} & \frac{e^{2\omega_2 \Delta}-1}{2\omega_2} w_{22} & \frac{e^{\omega_2 \Delta}-1}{\omega_2} w_{23} & \frac{e^{\omega_2 \Delta}-1}{\omega_2} w_{24} & \frac{e^{\omega_2 \Delta}-1}{\omega_2} w_{25} \\
\frac{e^{\omega_1 \Delta}-1}{\omega_1} w_{31} & \frac{e^{\omega_2 \Delta}-1}{\omega_2} w_{32} & \Delta w_{33} & \Delta w_{34} & \Delta w_{35} \\
\frac{e^{\omega_1 \Delta}-1}{\omega_1} w_{41} & \frac{e^{\omega_2 \Delta}-1}{\omega_2} w_{42} & \Delta w_{43} & \Delta w_{44} & \Delta w_{45} \\
\frac{e^{\omega_1 \Delta}-1}{\omega_1} w_{51} & \frac{e^{\omega_2 \Delta}-1}{\omega_2} w_{52} & \Delta w_{53} & \Delta w_{54} & \Delta w_{55}
\end{pmatrix} \boldsymbol{U}'
\end{aligned}
$$

where w_{ij} denotes the i, j-th element of the matrix $\boldsymbol{U}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{U}^{-1\prime}$.