Network for Studies on Pensions, Aging and Retirement

A note on the long rate in factor models of the term structure

J. de Kort

DP 03/2016-014

NETSPAR ACADEMIC SERIES

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March 10, 2016

Abstract

We show that, as a consequence of the Dybvig-Ingersoll-Ross theorem, the existence of a non-deterministic long rate in a factor model of the term structure implies that the model has an equivalent representation in which one of the state variables is nondecreasing. Moreover, for two-dimensional factor models, we prove that if the long rate is non-deterministic, the yield curve flattens out and the factor process is asymptotically non-deterministic, then the term structure is unbounded. Finally, following up on an open question in El Karoui et al. (1997), we provide an explicit example of a three-dimensional affine factor model with a non-deterministic yet finite long rate in which volatility of the factor process does not vanish over time.

Keywords: Long rate, factor model, term structure, Dybvig-Ingersoll-Ross Theorem

1 INTRODUCTION

A theorem by Dybvig, Ingersoll, and Ross (1996) states that the long-term interest rate[1] in an arbitrage-free term structure model cannot decrease over time. Nevertheless, the long rate can be a non-deterministic process. It was shown in El Karoui et al. (1997) that a Heath-Jarrow-Morton (HJM) model may generate a non-deterministic long rate if, for large expiries, the volatility of the forward rate vanishes sufficiently fast to make the long rate finite, and sufficiently slow to ensure that it is nonconstant. In this note we consider the question whether the subclass of factor models of the term structure is rich enough to admit a finite stochastic long rate if we additionally require that the volatility of the factor process does not vanish asymptotically over time. We show that this question can be answered affirmatively, but that at least three state variables are required.

The result of Dybvig, Ingersoll and Ross has been extended in a number of directions, for example for weaker notions of convergence of the long rate, see Goldammer and Schmock (2012). Others, see Kardaras and Platen (2012), considered the convergence speed of the zero-coupon yields at different times as the expiration date tends to infinity. Recently, Biagini et al. (2013) studied convergence of the long rate in HJM models where forward rates are driven by affine processes on a state space of symmetric positive semidefinite matrices. In another interesting paper, Biagini and Härtel (2014) look at interest rate models driven by Lévy processes and find that if the driving process has negative jumps only and paths of finite variation then, contrary to the diffusion case, volatility must not necessarily vanish for large maturities in order for the long rate to be finite. In their paper Dybvig et al. (1996) also proved that the long rate cannot increase almost surely if the sample space is finite. Schulze (2008) generalized this result to infinite state spaces by making use of a modified notion of arbitrage. In other recent work, Zhao (2009) and Bao and Yuan (2013) characterize almost sure convergence to a constant long rate in two-factor Cox-Ingersoll-Ross (CIR) models with Lévy jumps. To build a model in which the long Libor-rate can randomly move up and down, without violating no-arbitrage, Brody and Hughston (2013) use

*Netspar and Faculty of Economics and Business, University of Amsterdam, Roetersstraat 11, 1018 WB, Amsterdam, The Netherlands, E-mail: *j.dekort@uva.nl* .

[1] Let $P(t,t+x)$ denote the time t price of a zero-coupon bond paying 1 unit of currency at time $t+x$. The long forward rate is defined by $f_\infty(t) = \lim_{x\to\infty} -\partial_x \log P(t,t+x)$, whenever the limit exists, while the long zero-coupon rate is given by $z_\infty(t) = \lim_{x\to\infty} -x^{-1}\log P(t,t+x)$, provided that the limit exists.

discount functions of the generalized hyperbolic type; these are discount functions for which the relation between rates and bond prices is not asymptotically exponential.

Long-term returns in models that admit a factor representation have previously been studied by Deelstra and Delbaen (1995). They considered a single-factor generalized CIR process and proved that the long rate converges almost surely to a random variable which is proportional to the reversion level of the CIR process. Based on these results, Deelstra (2000) suggests a modification of a three-factor short rate model introduced by Tice and Webber (1997) such that the long rate converges to a stochastic mean-reversion level. Note that the present work is concerned with models in which not only the short rate but also the yield or the forward rate admit a factor representation.[2] Yao (1999), using a forward rate specification with separable volatility function and building on results in Ritchken and Sankarasubramanian (1995), constructs a factor model with two state variables in which the long rate is non-deterministic. Both the factor process and the yield curve parameterization in that model are time-inhomogeneous, and the volatility of the factor process vanishes over time. In the work of El Karoui et al. (1997) a series of two-dimensional affine factor models is discussed. They find that, in their examples, either the long zero-coupon rate is infinite or the model contains a nondecreasing state variable.

We extend these results by showing that a term structure model with three state variables, with one of the state variables having finite variation, is the most parsimonious model specification that can accommodate a stochastic long rate if we also require that the volatility of the factor process does not vanish over time. We thus show that the properties exemplified by the models in El Karoui et al. (1997) are general characteristics which are shared by all two-dimensional factor models. Finally, we provide an explicit example of an affine factor model in which the long rate converges almost surely to a nondecreasing stochastic process.

The paper is structured as follows. In Lemma 2.1 we prove that, up to a change of coordinates, any model in which the long-rate is not deterministic while the term structure of interest rates does not go to infinity must have a nondecreasing state variable, where we even allow the factor process to be time-inhomogeneous. In Theorem 2.3 we show that, under mild conditions, the presence of a stochastic long rate in a factor model with two state variables implies that the term-structure is unbounded. Finally, in Proposition 3.1 we construct a factor model with a stochastic long rate which is finite almost surely.

2 THE LONG RATE IN FACTOR MODELS

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a d-dimensional P-Brownian motion W. Consider a time-inhomogeneous affine term structure model in which the forward rates $r\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfy

$$r(t,x) \;:=\; f(t,t+x) \;=\; h_0(x) + \sum_{i=1}^{d} \lambda_i(x) Y_i(t)\,, \tag{2.1}$$

where $h_0\colon \mathbb{R}_+ \to \mathbb{R}$ and $\lambda_i\colon \mathbb{R}_+ \to \mathbb{R}$, $i \in \{1,\dots,d\}$, are continuously differentiable functions. The stochastic factor Y is defined by [3]

$$Y(t) \;=\; \int_0^t D(s,Y(s))\,ds + \int_0^t V(s,Y(s))dW(s)\,, \tag{2.2}$$

where D and V are functions $\mathbb{R}_+ \times \mathbb{R}^{d\times 1} \to \mathbb{R}^{d\times 1}$ and $\mathbb{R}_+ \times \mathbb{R}^{d\times 1} \to \mathbb{R}^{d\times d}$ respectively such that, for fixed $t \geq 0$, $D(t,\cdot)$ and $V(t,\cdot)$ are continuous and (2.2) admits a strong solution. We will refer to the components of Y as the state variables. Notice that the number of driving factors may be smaller than the number of state variables, for instance if V has rows consisting only of zeroes. The zero-coupon rates satisfy

$$z(t,x) \;=\; H(x) + \sum_{i=1}^{d} \Lambda_i(x) Y_i(t)\;, \qquad H(x) = \frac{1}{x}\int_0^x h_0(s)\,ds\;, \qquad \Lambda(x) = \frac{1}{x}\int_0^x \lambda(s)\,ds\;. \tag{2.3}$$

For $x=0$ let $\Lambda(x)$ and $H(x)$ be defined by their righthand side limits $h_0(0)$ and $\lambda(0)$. Notice that this setup includes[4] the model of El Karoui et al. (1997) but we do not require D and V to be affine or time-homogeneous

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[2] In models where the short rate $r(t)$ has a factor structure, the forward rate $f(t,t+x) = -\partial_x \log P(t,t+x)$ with $P(t,t+x) = \mathbb{E}_t\left[e^{-\int_t^{t+x} r(s)\,ds}\right]$ inherits this factor structure only if the bond price $P(t,t+x)$ has an (exponential) affine representation.

[3] Notice that the assumption $Y(0)=0$ is not restrictive. If $Y(0)=y_0 \neq 0$ then we may apply our analysis to $Z(t) := Y(t) - y_0$ satisfying $dZ(t) \;=\; D(t,Z(t)+y_0)\,dt + V(t,Z(t)+y_0)dW(t)$ and $Z(0)=0$.

[4] To obtain the setup in El Karoui et al. (1997) set $A(T-t) = \int_0^{T-t} \lambda(s)\,ds$, $\;b(T-t) = \int_0^{T-t} h_0(s)\,ds$ and $T = t+x$, then $-\log P(t,T) = A(T-t)Y(t) + b(T-t)$.

functions of the factor process $Y(t)$. The long forward rate f_∞ and the long zero-coupon rate z_∞ are defined by

$$f_\infty(t) = \lim_{x\to\infty} f(t, t+x) \qquad \text{and} \qquad z_\infty(t) = \lim_{x\to\infty} z(t,x)\ ,$$

provided that the corresponding limits exist. In the special case where $\lim_{x\to\infty} h_0(x)$ and $\lim_{x\to\infty} \lambda(x)$ exist it follows by l'Hôpital's rule that

$$z_\infty(t) = \lim_{x\to\infty}\left[\frac{\int_0^x h_0(s)\,ds}{x} + \sum_{i=1}^d \frac{\int_0^x \lambda_i(s)\,ds}{x} Y_i(t)\right] = \lim_{x\to\infty} h_0(x) + \sum_{i=1}^d \Big[\lim_{x\to\infty} \lambda_i(x)\Big] Y_i(t) = f_\infty(t). \tag{2.4}$$

While the Dybvig-Ingersoll-Ross theorem states that the long forward rate is nondecreasing — provided it exists and bonds of all maturities are traded — the following result shows that in factor models with a non-deterministic long rate a linear combination of the state variables driving the forward rate is itself nondecreasing. The proof (and the proofs of subsequent results) can be found in Section 4.

Lemma 2.1. *If there is no arbitrage and*

(i). The long forward rate exists almost surely, for every $t \geq 0$, and is non-deterministic for $t > 0$;

(ii). The elements of $\{1, Y_1(t_0), \ldots, Y_d(t_0)\}$ are almost surely linearly independent for at least one $t_0 \in [0,\infty[$;

then

1. *$\lim_{x\to\infty} \lambda(x) = \zeta \neq 0$ exists and $\zeta V = 0$;*
2. *The term-structure model in Eqs. (2.1)–(2.3) has an equivalent representation in which the non-decreasing long forward rate coincides with one of the state variables.*

If the conditions of Lemma 2.1 are satisfied then the long forward rate and the long zero-coupon rate coincide. Indeed, Eq. (2.4) holds due to part 1 of Lemma 2.1 and condition (i) with $t = 0$. It follows that the results in Lemma 2.1 not only apply for the long forward rate but also for the long zero-coupon rate.

We can also formulate the following variation of Lemma 2.1 in which the conditions have been slightly weakened.

Lemma 2.2. *If 'long zero-coupon rate' is substituted instead of 'long forward rate' in condition (i) and in part 2 of Lemma 2.1, then the result remains true with '$\lambda(x)$' in part 1 replaced by '$\Lambda(x)$'.*

In the following theorem we establish that in any two-dimensional factor model that admits a non-deterministic long rate the term structure is eventually unbounded. We show that this result holds if the factor process is asymptotically non-deterministic and the forward curve flattens for large expiration dates. The first requirement excludes models in which forward rates are eventually deterministic, while the second condition reflects the fact that little market information is available about the far future, hence the instantaneous rate for lending or borrowing should not differ much between long-dated maturities.

Theorem 2.3. *Consider the factor model in Eq. (2.1) with $d = 2$. Suppose that the conditions (i) and (ii) of Lemma 2.1 hold, no arbitrage possibilities exist and*

(iii). The factor process is asymptotically non-deterministic almost surely, that is, $P(\lim_{t\to\infty} V(t, Y(t)) = 0) = 0$;

(iv). For long maturities the forward rate becomes flat, that is, $\lim_{x\to\infty} \partial_x h_0(x) = 0$ and $\lim_{x\to\infty} \partial_x \lambda_i(x) = 0$ for $i \in \{1,2\}$;

then $\lim_{t\to\infty} f_\infty(t) = \infty$ almost surely and the term-structure $x \mapsto r(t,x)$ is unbounded over time, i.e. there exists a random variable $X\colon \Omega \to [0,\infty[$ such that $\lim_{t\to\infty} r(t,X) = \infty$ almost surely.

We thus find that in all factor models with two state variables satisfying the conditions of Theorem 2.3 not only the long forward rate tends to infinity over time, but also that there exists at least one finite expiration date for which the corresponding forward rate is unbounded. Notice that the finite expiration date for which this happens may depend on the stochastic scenario.

In the proof of Theorem 2.3 a path-by-path argument is used to establish the almost sure unboundedness of the term-structure; hence if the factor process is asymptotically deterministic on some set with strictly positive probability, then the result holds everywhere except on this set.

3 AN EXAMPLE

In this Section we show that the result of Theorem 2.3 does not necessarily hold for $d > 2$ by constructing an explicit example of an affine factor model (with $d = 3$) in which the long rate is finite and non-deterministic, yet the factor process is not asymptotically deterministic. Let $\mu_1, \kappa_1, \kappa_2, \sigma_1$ and σ_2 be constants and consider the process Y defined by

$$\begin{aligned} Y_1(t) &= \int_0^t [\mu_1 - \kappa_1 Y_1(s)]\, ds + \int_0^t \sigma_1 \sqrt{Y_1(s)} dW_1(s)\,, \\ Y_2(t) &= 1 - \int_0^t \kappa_2 Y_2(s)\, ds + \int_0^t \sigma_2 \sqrt{Y_2(s)} dW_2(s)\,, \\ Y_3(t) &= \int_0^t Y_2(s)\, ds\ . \end{aligned} \tag{3.1}$$

The second component of the factor process is positive, continuous and mean reverting to zero in finite time with probability one, hence the third state variable is nondecreasing and finite almost surely. By the Yamada-Watanabe theorem, see Karatzas and Shreve (1991, Thm. 5.2.13), a strong solution exists for the square-root processes $Y_1(t)$ and $Y_2(t)$. The next Proposition shows that the long rate in the factor model defined by Eq. (3.1) can be made finite and non-deterministic.

Proposition 3.1. *Let $2\mu_1 > \sigma_1^2$, $\sigma_2 > 0$, $\lambda_1(0) > 0$, $\lambda_2(0) > 0$ and set $\lambda_3(0) = 1$. The factor model*

$$r(t,x) = h_0(x) + \sum_{i=1}^{3} \lambda_i(x) Y_i(t)\ , \tag{3.2}$$

with $Y(t)$ as in Eq. (3.1) is arbitrage-free if we choose

$$\begin{aligned} \lambda_1(x) &= \lambda_1(0) \left(\cosh\left(\frac{1}{2}\gamma x\right) + \frac{\kappa_1}{\gamma} \sinh\left(\frac{1}{2}\gamma x\right) \right)^{-2}, \\ \lambda_2(x) &= \frac{2}{\psi} \left(\phi + \frac{1}{2}\psi x - \left(\frac{C_1 \partial_1 \operatorname{Ai}(\phi + \frac{1}{2}\psi x) - C_2 \partial_1 \operatorname{Bi}(\phi + \frac{1}{2}\psi x)}{C_1 \operatorname{Ai}(\phi + \frac{1}{2}\psi x) - C_2 \operatorname{Bi}(\phi + \frac{1}{2}\psi x)} \right)^2 \right), \\ \lambda_3(x) &= 1, \end{aligned} \tag{3.3}$$

and

$$h_0(x) = h_0(0) + \mu_1 \int_0^x \lambda_1(s)\, ds\ . \tag{3.4}$$

in which $\gamma = (\kappa_1^2 + 2\lambda_1(0)\sigma_1^2)^{\frac{1}{2}}$, $\phi = (\kappa_2^2 + 2\lambda_2(0)\sigma_2^2)/\psi^2$, $\psi = (2\sigma_2)^{2/3}$ while

$$C_1 = \kappa \operatorname{Bi}(\phi) - \psi \partial_1 \operatorname{Bi}(\phi) \qquad \text{and} \qquad C_2 = \kappa \operatorname{Ai}(\phi) - \psi \partial_1 \operatorname{Ai}(\phi)\ , \tag{3.5}$$

and where Ai *and* Bi *denote the Airy functions of the first and second kind respectively.*

The long forward rate in this model is finite almost surely and satisfies

$$f_\infty(t) = h_0(0) + \frac{2\mu_1}{\gamma + \kappa_1} + \int_0^t Y_2(s)\, ds\ . \tag{3.6}$$

The requirement in Theorem 2.3 which states that the factor process must be asymptotically non-deterministic cannot be omitted. Indeed, observe from Proposition 3.1 that the factor model defined by Y_2 and Y_3 has a bounded and non-deterministic long rate but the volatility of the factor process vanishes over time.

4 PROOFS

Proof of Lemma 2.1:

We subsequently prove the two parts in the statement of the Lemma.

1. Take t_0 as in condition (ii). By assumption (i) there is a set $U \subset \Omega$ with $P(U) = 0$ such that the long forward rate $f_\infty(t_0)$ exists for every $\omega \in U^c$. Consider the set $V = \{(Y_1(t_0, \omega), \ldots, Y_d(t_0, \omega)) : \omega \in U^c\}$ and suppose that $\dim(V) < d$, that is, there exists some $\beta \in \mathbb{R}^{d+1}$, $\beta \neq 0$ such that $\sum_{j=1}^d \beta_j v_j = \beta_0$ whenever $v \in V$. Then

$$\beta_0 \;=\; \sum_{j=1}^d \beta_j Y_j(t_0, \omega)\;, \qquad \text{for all } \omega \in U^c,$$

which contradicts the assumption that the elements of $\{1, Y_1(t_0), \ldots, Y_d(t_0)\}$ are (almost surely) linearly independent. Hence there must be $\omega_1, \ldots, \omega_d \in U^c$ such that $y_i := (Y_1(t_0, \omega_i), \ldots, Y_d(t_0, \omega_i))$, $i \in \{1, \ldots, d\}$, span $\mathbb{R}^d$. Write $\Upsilon = (y_1, \ldots, y_d)^T$, then Υ is an $d \times d$ matrix of full rank, therefore Υ^{-1} exists. By assumption (i) and Eq. (2.1) we have that $\lim_{x\to\infty} h_0(x)$ exists (take $t = 0$) and $\lim_{x\to\infty} \Upsilon\lambda(x)$ exists almost surely (take $t = t_0$). It follows that

$$\lim_{x\to\infty} \lambda(x) \;=\; \lim_{x\to\infty} \Upsilon^{-1}(\Upsilon\lambda(x)) \;=\; \Upsilon^{-1} \lim_{x\to\infty} \Upsilon\lambda(x) \qquad \text{exists}\;.$$

From Eqs. (2.1) and (2.2) we obtain

$$f(t, t+x) - f(0, x) \;=\; \sum_{i=1}^d \lambda_i(x) \left[\int_0^t D_i(s, Y(s))\, ds + \int_0^t \sum_{j=1}^d V_{ij}(s, Y(s)) dW_j(s)\right]. \tag{4.1}$$

We thus find that, for all $t \geq 0$, the long rate satsifies

$$f_\infty(t) - f_\infty(0) \;=\; \sum_{i=1}^d \zeta_i \left[\int_0^t D_i(s, Y(s))\, ds + \int_0^t \sum_{j=1}^d V_{ij}(s, Y(s)) dW_j(s)\right]. \tag{4.2}$$

The Dybvig-Ingersoll-Ross theorem[5] implies that if the long rate exists as an almost sure limit, then it is a nondecreasing process; therefore it has finite first-order variation and thus cannot have a nonzero diffusion term. It follows that $\zeta V = 0$ almost surely for every $t \geq 0$ and

$$f_\infty(t) \;=\; f_\infty(0) + \int_0^t \left[\sum_{i=1}^d \zeta_i D_i(s, Y(s))\right] ds\,. \tag{4.3}$$

If $\zeta = 0$ then by Eq. (4.2) the long rate is constant which contradicts requirement (i). Therefore $\lambda_i(x)$ does not converge to 0, for at least one $i \in \{1, \ldots, d\}$. It follows that V is singular. This proves part 1.

2. If $d = 1$, that is, if the factor process is one-dimensional, then part 2 clearly holds since $f_\infty(t)$ and $Y_1(t)$ only differ by a constant shift $\lim_{x\to\infty} h_0(x)$ and a factor ζ_1. Note however that the case $d = 1$ is not of interest to us since, due to part 1, either $V = 0$ or $\zeta = 0$ must hold. In both cases the long rate is deterministic which contradicts requirement (i).

   Assume that $d \geq 2$. In order for the long rate to be stochastic, there must be an $i^* \in \{1, \ldots, d\}$ such that $\zeta_{i^*} \neq 0$. Define the (invertible) $d \times d$ matrix M by

$$M \;=\; \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ \zeta_1 & \zeta_2 & \cdots & \zeta_{i^*} & \cdots & \cdots & \zeta_d \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}. \tag{4.4}$$

   The matrix M has zero entries outside the main diagonal and the i^*-th row. Define

$$\widetilde{Y}(t) := MY(t) \qquad \text{and} \qquad \widetilde{\lambda}(x) := \lambda(x)M^{-1}\;. \tag{4.5}$$

[5] See for example Hubalek et al. (2002) or Theorem 2.17 in Goldammer and Schmock (2012).

The process $\widetilde{Y}(t)$ satisfies $\widetilde{Y}(0) = 0$ and

$$d\widetilde{Y}(t) \;=\; \widetilde{D}(t, \widetilde{Y}(t))\, dt + \widetilde{V}(t, \widetilde{Y}(t)) dW(t) \;, \tag{4.6}$$

in which $\widetilde{D}(t, y) = MD(t, M^{-1}y)$ and $\widetilde{V}(t, y) = MV(t, M^{-1}y)$. The factor model defined through h_0, $\widetilde{\lambda}$ and $\widetilde{Y}$ is equivalent to the one defined by h_0, λ and Y. Indeed,

$$r(t, x) \;=\; h_0(x) + \sum_{i=1}^{d} \lambda_i(x) Y_i(t) \;=\; h_0(x) + \sum_{i=1}^{d} \lambda_i(x) M^{-1} M Y_i(t) \;=\; h_0(x) + \sum_{i=1}^{d} \widetilde{\lambda}_i(x) \widetilde{Y}_i(t) \;.$$

The dynamics of the i^*-th state variable satisfy

$$d\widetilde{Y}_{i^*}(t) = \sum_{i=1}^{d} \zeta_i dY_i(t) = \sum_{i=1}^{d} \zeta_i \left[D_i(t, Y(t))\, dt + \sum_{j=1}^{d} V_{ij}(t, Y(t)) dW_j(t) \right] \;=\; df_\infty(t) \;.$$

We have thus established that the model (2.1) has a representation in which the long rate occurs as one of the state variables. Due to the Dybvig-Ingersoll-Ross theorem the long rate is non-decreasing almost surely. This completes the proof.

□

Proof of Lemma 2.2

As in the proof of Lemma 2.1 one can show that the almost sure existence of a non-deterministic long zero-coupon rate together with the independence of the elements of $\{1, Y_1(t_0), \ldots, Y_d(t_0)\}$ for some $t_0 \in [0, \infty[$ implies that $\lim_{x\to\infty} \Lambda(x) = \zeta$ exists and $\zeta \neq 0$ while $\zeta V = 0$. Using the transformation $\widetilde{Y}(t) = MY(t)$ and $\widetilde{\Lambda}(t) = \Lambda(t) M^{-1}$, where M is defined as in Eq. (4.4), a representation can be established in which the long zero-coupon rate coincides with one of the state variables. The rest of the proof is similar to the proof for Lemma 2.1.

□

Proof of Theorem 2.3:

From the proof of Lemma 2.1 we have that assumptions (i) and (ii) imply that $\zeta \neq 0$. Assume, without loss of generality, that $\zeta_2 \neq 0$ and apply the coordinate transformation defined in Eq. (4.5), i.e. $\widetilde{Y}_1(t) = Y_1(t)$, $\widetilde{Y}_2(t) = \zeta_1 \widetilde{Y}_1(t) + \zeta_2 \widetilde{Y}_2(t)$ and $\widetilde{\lambda}_1(x) = \lambda_1(x) - \frac{\zeta_1}{\zeta_2} \lambda_2(x)$ while $\widetilde{\lambda}_2(x) = \frac{1}{\zeta_2} \lambda_2(x)$. The factor process (2.2) can thus be written as

$$\begin{aligned} \widetilde{Y}_1(t) &= \int_0^t \widetilde{D}_1(s, \widetilde{Y}(s))\, ds + \int_0^t \widetilde{V}_{11}(s, \widetilde{Y}(s)) dW_1(s) + \int_0^t \widetilde{V}_{12}(s, \widetilde{Y}(s)) dW_2(s) \;, \\ \widetilde{Y}_2(t) &= \int_0^t \widetilde{D}_2(s, \widetilde{Y}(s))\, ds \;. \end{aligned} \tag{4.7}$$

Due to the Dybvig-Ingersoll-Ross theorem, the second row of the volatility matrix must be zero, i.e. $\widetilde{V}_{21} = \widetilde{V}_{22} = 0$.

Property (iv) is preserved under the transformation in Eq. (4.5), that is $\lim_{x\to\infty} \partial_x \widetilde{\lambda}(x) = 0$. Similarly, since $\widetilde{V}(t, \widetilde{Y}(t)) = MV(t, M^{-1}\widetilde{Y}(t)) = MV(t, Y(t))$ and since M is invertible we have that $\widetilde{V}(t, \widetilde{Y}(t)) = 0$ if and only if $V(t, Y(t)) = 0$; hence property (iii) is also preserved under the transformation.

The HJM drift condition in Musiela form[6] requires that for all $\widetilde{Y}(t)$

$$\sum_{i=1}^{2} \widetilde{\lambda}_i(x) \widetilde{D}_i(t, \widetilde{Y}(t)) \;=\; \partial_x r(t, x) \;+\; \sum_{j=1}^{2} \widetilde{\sigma}_j(t, x, \widetilde{Y}(t)) \int_0^x \widetilde{\sigma}_j(t, u, \widetilde{Y}(t))\, du - \sum_{j=1}^{2} \varphi_j(t) \widetilde{\sigma}_j(t, x, \widetilde{Y}(t)) \;, \tag{4.8}$$

in which $\widetilde{\sigma}_j(t, x, y) = \sum_{i=1}^{2} \widetilde{\lambda}_i(x) \widetilde{V}_{ij}(t, y)$ for $y \in \mathbb{R}^{d\times 1}$ and $j \in \{1, 2\}$, while $\varphi(t)$ denotes the market price of risk. The market price of risk is adapted to the filtration generated by W and appears since we have defined the

[6] See for example Björk and Svensson (2001, Definition 4.2).

factor process Y under the probability measure P which is not necessarily a martingale measure for the model (2.1). The second term on the right-hand side can be written as

$$\begin{aligned}\sum_{j=1}^{2} \widetilde{\sigma}_j(t,x,\widetilde{Y}(t)) \int_0^x \widetilde{\sigma}_j(t,u,\widetilde{Y}(t))\, du &= \sum_{j=1}^{2} \left(\sum_{i=1}^{2} \widetilde{\lambda}_i(x) \widetilde{V}_{ij}(t,\widetilde{Y}(t)) \right) \int_0^x \left(\sum_{k=1}^{2} \widetilde{\lambda}_k(u) \widetilde{V}_{kj}(t,\widetilde{Y}(t)) \right) du \\ &= \sum_{i=1}^{2}\sum_{k=1}^{2} \left(\widetilde{H}_{ik}(t,\widetilde{Y}(t))\, \widetilde{\lambda}_i(x) \int_0^x \widetilde{\lambda}_k(u)\, du \right) , \end{aligned} \tag{4.9}$$

where $\widetilde{H}_{ik}(t,\widetilde{Y}(t)) := \sum_{j=1}^{2} \widetilde{V}_{ij}(t,\widetilde{Y}(t)) \widetilde{V}_{kj}(t,\widetilde{Y}(t))$ is the (i,k)-th element of the symmetric matrix $\widetilde{H} = \widetilde{V}\widetilde{V}'$, which equals $\frac{d}{dt}\langle \widetilde{Y} \rangle_t$.

Integrating the drift condition Eq. (4.8) on the interval $[0,t]$ and using $\widetilde{V}_{21} = \widetilde{V}_{22} = 0$ (hence $\widetilde{H}_{12} = \widetilde{H}_{21} = \widetilde{H}_{22} = 0$) we find that

$$\begin{aligned}\sum_{i=1}^{2} \widetilde{\lambda}_i(x) \int_0^t \widetilde{D}_i(s,\widetilde{Y}(s))\, ds &= \partial_x h_0(x) t + \sum_{i=1}^{2} \partial_x \widetilde{\lambda}_i(x) \int_0^t \widetilde{Y}_i(s)\, ds + L(x) \int_0^t \widetilde{H}_{11}(s,\widetilde{Y}(s))\, ds \\ &\quad - \widetilde{\lambda}_1(x) \int_0^t \sum_{j=1}^{2} \widetilde{V}_{1j}(s,\widetilde{Y}(s)) \varphi_j(s)\, ds , \end{aligned} \tag{4.10}$$

in which $L(x) = \widetilde{\lambda}_1(x) \int_0^x \widetilde{\lambda}_1(u)\, du$. Due to the coordinate transformation in Eq. (4.5) we have $\lim_{x\to\infty} \widetilde{\lambda}_1(x) = 0$ and $\lim_{x\to\infty} \widetilde{\lambda}_2(x) = 1$. By taking the limit $x \to \infty$ in Eq. (4.10) we obtain, using assumption (iv), that

$$f_\infty(t) - f_\infty(0) = \int_0^t \widetilde{D}_2(s,\widetilde{Y}(s))\, ds \;=\; \lim_{x\to\infty} L(x) \int_0^t \widetilde{H}_{11}(s,\widetilde{Y}(s))\, ds . \tag{4.11}$$

By condition (i) of Lemma 2.1, the left-hand side exists almost surely for every $t \geq 0$, hence the right-hand side — and thus $\lim_{x\to\infty} L(x)$ — exists almost surely. Notice that $\int_0^t \widetilde{H}_{11}(s,\widetilde{Y}(s))\, ds = 0$ would imply that $\widetilde{H}_{11}(t,\widetilde{Y}(t)) = 0$ for any $t > 0$ and this contradicts (i).

By Eq. (4.11) and since the long rate is not constant (assumption (i) of Lemma 2.1) we have $\lim_{x\to\infty} L(x) \neq 0$. Moreover, from Eq. (4.7) it follows that $\lim_{x\to\infty} L(x) < \infty$. Indeed, the Itô integral in Eq. (4.7) — and hence $\widetilde{Y}(t)$ — has a version with continuous sample paths. It follows that $\widetilde{Y}(t) < \infty$ for finite $t \geq 0$. By continuity also $\widetilde{D}(t,\widetilde{Y}(t))$, $\widetilde{V}(t,\widetilde{Y}(t))$ and $\widetilde{H}_{11}(t,\widetilde{Y}(t))$ are finite. The functions $\partial_x h_0(x)$, $\partial_x \lambda(x)$ and $\lambda(x)$ are continuous with finite limits for $x \to \infty$; hence they are bounded functions of x. From these estimates we have that all terms in Eq. (4.10) — except possibly the term involving $L(x)$ — are bounded for finite $t \geq 0$. We thus conclude that $\lim_{x\to\infty} L(x) < \infty$.

The Dybvig-Ingersoll-Ross theorem implies that the drift $\widetilde{D}_2(t,\widetilde{Y}(t))$ is nonnegative, therefore $\lim_{t\to\infty} \widetilde{Y}_2(t) = \lim_{t\to\infty} \int_0^t \widetilde{D}_2(s,\widetilde{Y}(s))\, ds$ either equals a finite positive value or is infinite. To ensure that $\widetilde{Y}_2(t)$ stays bounded we must have that $\lim_{t\to\infty} \widetilde{D}_2(t,\widetilde{Y}(t)) = 0$. By Eq. (4.11) this implies

$$0 \;=\; \lim_{t\to\infty} \widetilde{H}_{11}(t,\widetilde{Y}(t)) \;=\; \lim_{t\to\infty} \left[\widetilde{V}_{11}^2(t,\widetilde{Y}(t)) + \widetilde{V}_{12}^2(t,\widetilde{Y}(t)) \right] .$$

Recall that $\widetilde{V}_{21} = \widetilde{V}_{22} = 0$ holds due to the coordinate transformation in Eq. (4.5). Hence $\widetilde{V} \equiv 0$ and this contradicts (iii). Therefore $\lim_{t\to\infty} \widetilde{Y}_2(t) = \infty$ almost surely and, consequently, $\lim_{t\to\infty} f_\infty(t) = \infty$ almost surely.

It remains to prove that the term structure $x \mapsto r(t,x)$ is unbounded in time. Take any $0 \leq x_0 < \infty$ such that $\widetilde{\lambda}_1(x_0) \neq 0$ and consider the term structure at $x = x_0$, i.e.

$$r(t,x_0) \;=\; h_0(x_0) + Z(t) , \qquad Z(t) \;:=\; \widetilde{\lambda}_1(x_0)\widetilde{Y}_1(t) + \widetilde{\lambda}_2(x_0)\widetilde{Y}_2(t) .$$

Let $U = \{\omega \in \Omega : t \mapsto Z(\omega,t)$ is bounded on $[0,\infty[\}$. The sample paths of $Z(t)$, and hence $r(\cdot,x_0)$, are unbounded on $\Omega \setminus U$.

Since $\lim_{x\to\infty} \widetilde{\lambda}_1(x) = 0$ and $\lim_{x\to\infty} \widetilde{\lambda}_2(x) = 1$ it follows that $\widetilde{\lambda}_1$ is not proportional to $\widetilde{\lambda}_2$ and hence there exists an $0 \le x_1 < \infty$ such that $c := \widetilde{\lambda}_1(x_0)\widetilde{\lambda}_2(x_1) - \widetilde{\lambda}_2(x_0)\widetilde{\lambda}_1(x_1) \neq 0$. The term structure in $x = x_1$ satisfies

$$r(t, x_1) \;=\; h_0(x_1) + a_1 Z(t) + a_2 \widetilde{Y}_2(t)\,,$$

in which $a_1 = \widetilde{\lambda}_1(x_1)/\widetilde{\lambda}_1(x_0)$ and $a_2 = c/\widetilde{\lambda}_1(x_0) \neq 0$. It follows that $r(\cdot, x_1)$ is unbounded on $[0,\infty[$ for $\omega \in U$. Set $X := x_0 + (x_1 - x_0)1_U$ then $\lim_{t\to\infty} r(t, X) = \infty$ almost surely.

□

Proof of Proposition 3.1:

By integrating the drift condition Eq. (4.8) on the interval $[0, x]$ we find that to exclude arbitrage the functions $A(x) = \int_0^x \lambda(s)\,ds$ and $b(x) = \int_0^x h_0(s)\,ds$ must satisfy

$$\begin{pmatrix}\partial_x A_1(x)\\ \partial_x A_2(x)\\ \partial_x A_3(x)\end{pmatrix} \;=\; \begin{pmatrix}\partial_x A_1(0)\\ \partial_x A_2(0)\\ \partial_x A_3(0)\end{pmatrix} + \begin{pmatrix}-\kappa_1 & 0 & 0\\ 0 & -\kappa_2 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}A_1(x)\\ A_2(x)\\ A_3(x)\end{pmatrix} - \frac{1}{2}\begin{pmatrix}\sigma_1^2 A_1^2(x)\\ \sigma_2^2 A_2^2(x)\\ 0\end{pmatrix}, \tag{4.12}$$

and

$$\partial_x b(x) \;=\; \partial_x b(0) + \mu_1 A_1(x)\;. \tag{4.13}$$

The first and third ODE admit closed-form solutions:

$$A_1(x) \;=\; 2\lambda_1(0)\left(\kappa_1 + \gamma \coth\left(\frac{1}{2}\gamma x\right)\right)^{-1} \qquad \text{and} \qquad A_3(x) \;=\; x\;.$$

The solution to the second ODE can be expressed in terms of solutions of the Stokes equation. Indeed, if we apply the transformations

$$A_2(x) = \frac{2\partial_x w(x)}{\sigma_2^2 w(x)}\;, \qquad w(x) = e^{-\frac{1}{2}\kappa_2 x} z(x)\;, \qquad z(x) = v(\phi + \frac{1}{2}\psi x)\;, \tag{4.14}$$

then $v(x)$ satisfies the Stokes equation $0 = x v(x) - \partial_{xx} v(x)$. The solution to this equation is $v(x) = C_3 \operatorname{Ai}(x) + C_4 \operatorname{Bi}(x)$ in which Ai and Bi denote the Airy functions of the first and second kind respectively, and the values of C_3 and C_4 are determined by the boundary condition $A_2(0) = 0$. It follows that

$$A_2(x) \;=\; -\frac{\kappa_2}{\sigma_2^2} + \frac{4}{\psi^2}\,\frac{C_1\partial_1 \operatorname{Ai}(\phi + \frac{1}{2}\psi x) - C_2 \partial_1 \operatorname{Bi}(\phi + \frac{1}{2}\psi x)}{C_1 \operatorname{Ai}(\phi + \frac{1}{2}\psi x) - C_2 \operatorname{Bi}(\phi + \frac{1}{2}\psi x)}\;.$$

Differentiation yields Eq. (3.3).

By repeatedly applying the identities $\partial_{xx} \operatorname{Ai}(x) = x \operatorname{Ai}(x)$ and $\partial_{xx} \operatorname{Bi}(x) = x \operatorname{Bi}(x)$ together with l'Hopital's rule one can show that

$$\lim_{x\to\infty} \lambda(x) \;=\; (0,0,1) \qquad \text{and} \qquad \lim_{x\to\infty} h_0(x) \;=\; h_0(0) + \frac{2\mu_1}{\gamma + \kappa_1}\;. \tag{4.15}$$

Hence the third state variable $Y_3(t)$ coincides with the long rate and Eq. (3.6) follows. The long rate is finite since $Y_2(t)$ hits zero in finite time with probability one.

□

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