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Pricing and Hedging in Incomplete Markets with Model Ambiguity

Anne Balter and Antoon Pelsser

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Anne G. Balter
Department of Econometrics and Operations Research, Tilburg University, NWO, Netspar
5000 LE Tilburg, The Netherlands, a.g.balter@uvt.nl

Antoon Pelsser
Department of Quantitative Economics, Maastricht University, Netspar, Kleynen Consultants,
6200 MD Maastricht, The Netherlands, a.pelsser@maastrichtuniversity.nl

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We search for a trading strategy and the associated robust price of unhedgeable assets in incomplete markets under the acknowledgement of model uncertainty. Our set-up is that we postulate an agent who wants to maximise the expected surplus by choosing an optimal investment strategy. Furthermore, we assume that the agent is concerned about model misspecification. This robust optimal control problem under model uncertainty leads to (i) risk-neutral pricing for the traded risky assets, and (ii) adjusting the drift of the nontraded risk drivers in a conservative direction. The direction depends on the agent's long or short position, and the adjustment that ensures a robust strategy leads to what is known as "actuarial" or "prudential" pricing. Our results extend to a multivariate setting. We prove existence and uniqueness of the robust price in an incomplete market via the link between the semilinear partial differential equation and backward stochastic differential equations.

Key words: model uncertainty; indifference pricing; hedging; incomplete markets; robustness; optimisation
MSC2000 subject classification: Primary: 91B06; 91B30; 49L99; 49J35; secondary: 62P05; 35A01; 35A02

1. Introduction. When markets are complete and arbitrage-free then there exists a unique pricing measure such that all contracts have a unique price. However real world processes and situations are not perfectly hedgeable. Therefore there is the risk of a mismatch between optimal policies based on a complete market assumption and the consequence the theoretical strategy has in practice. In this paper, we want to minimise this mismatch by a robust strategy. Pension funds, insurance companies and many other parties need to price liabilities that are not or only partially hedgeable. Multiple equivalent martingale measures exist in an incomplete market. Moreover the underlying model of both hedgeable and unhedgeable risk factors can be wrong. In financial and economic decision making allowing for model uncertainty has an impact on the optimal strategies. Our goal is to find a pricing method for assets in incomplete markets under the acknowledgement of model uncertainty.

The standard optimal strategy is obtained when the agent maximises his surplus based on the assumption that his model description is correct. However, if the agent is uncertain about the model he anticipates to a worst-case happening. To ensure a robust strategy he pretends that mother nature picks the worst-case model among a set of plausible model, whereafter he maximises the surplus given this worst-case model. Without uncertainty, the standard optimal strategy can harm if mother nature selects the worst-case model afterwards, while the robust strategy is less vulnerable to perturbations of the underlying model. Hence mother nature plays the minimising role and the agent the maximising role. Gilboa and Schmeidler [15] introduced the maxmin method to deal with multiple priors in a Bayesian setting. Note the difference between risk and uncertainty. Risk describes the situation that a specific event might happen with a known probability distribution whereas for uncertainty[1] the probability distribution is not precisely known. Hence, by considering a profit-maximising agent who is ambiguous about the model we focus on the latter, namely model uncertainty.

Maximising profit is a frequently observed objective of doing business. The objective coincides with a linear expectation. Profit maximisation in an incomplete market without trading constraints leads to an unbounded problem, which is economically unlikely. However, we show in this paper that profit maximisation with model ambiguity results in a bounded optimisation problem. Since insurance companies and pension funds want to offset the risk of their liabilities that are unhedgeable, profit maximisation in an incomplete market is motivated. Ambiguity about the underlying model leads to a change in the company's standard delta-hedge position. Hence, the mere objective of maximising the surplus, which is defined as the difference between the investment position and the liabilities, implies an adjusted delta-hedging strategy. Examples of pricing in incomplete markets are; pricing and hedging extremely long dated obligations such as pensions that have to be paid on a horizon up to a century (since life expectancy increases the horizon grows even further) or pricing insurance contracts linked to instruments that are not or barely traded, such as the number of survivors (this example is shown in § 6.2), temperature or natural disasters.

The specification of a concave utility function can also overcome the unboundedness of the optimal investment strategies which are implied by the profit maximisation problem. Merton [39] solves the investment problem in a complete market setting for power utility. The large equity premium implies more risky investments than data shows us. The difference between the optimal and the observed investment strategy arises from the equity premium puzzle. The puzzle might be explained by the aversion of model uncertainty (Biagini and Pınar [6]). In Bansal and Yaron [2]'s long-run risk model, time-varying economic uncertainty justifies the puzzle. Maximising utility in incomplete markets under model uncertainty has been considered in different settings. Bordigoni et al. [8] introduce uncertainty as an additional penalty term measured by its entropy. Matoussi et al. [38] deal with uncertainty of the volatility. Hernández-Hernández and Schied [27, 28] consider logarithmic and power utility and Mania and Tevzadze [37] solve the utility maximisation via duality. The utility specification is rather a feature that is related to individual investors, while the long-term goal of companies is maximising profit. Thus, unlike most other literature, we do not specify a utility function but we set profit maximisation as objective motivated by a business viewpoint. Moreover, this objective ensures a focus on model uncertainty rather than on risk-aversion preferences.

To characterise model uncertainty, the agent specifies plausible alternative models by a set surrounding the baseline model. The baseline model is his best guess on which the agent would rely if no uncertainty existed. Our theorem relies on the presence of model uncertainty and is described by an ellipsoid bounded by the amount of uncertainty. In Balter and Pelsser [1] we focus on the characterisation of the set of plausible alternative models. Ben-Tal et al. [5] focus on ϕ-divergences in robust optimisation. The uncertainty set is identified by the confidence set using a specific ϕ-divergence function, which measures the distance between two probability measures according to a specific weight function. Cochrane and Saa-Requejo [11] allow trades in only those assets that have plausible mean-variances. The Good-Deal-Bounds method states that the existence of assets with extreme market prices of risk are unlikely to be true ("too good to be true"). Amongst others, Björk and Slinko [7], Černỳ and Hodges [10], Becherer [4] and Klöppel and Schweizer [32] extended this concept. Hansen et al. [23] introduce the concept of Model Confidence Set (MCS). This algorithm is a sequential method that starts with a collection of possible models and eliminates undesired models based on a test statistic, a level of confidence and the historic data. Barrieu and El Karoui [3] consider a buyer and seller perspective in the two player game, where the pricing measure is known as the solution of an inf-convolution problem. Gundel and Weber [21] and Goldfarb and Iyengar [19] deal with robustness in portfolio optimisation. A literature review on

[1] We use "ambiguity" and "model uncertainty" interchangeably.

ambiguity in asset pricing in specific is given by Guidolin and Rinaldi [20]. Contrary to the bounded sets around the baseline model, Hansen and Sargent [22] define a penalty for *all* alternative models. They add the relative entropy as a penalty term, which is a weighted divergence between the baseline model and alternative models, to the original objective.

In this paper the price of nontraded liabilities that takes model uncertainty into account is implied by the robust trading strategies. The robust price, given by a semilinear partial difference equation, is derived via the concept of utility indifference pricing. Pricing and hedging in incomplete markets is of great interest in financial and economic modelling. In the pricing literature, both in complete and incomplete markets, the focus lies on pricing (complex) contracts where model ambiguity is often ignored. We show in this paper that a profit-maximising objective for a company with an ambiguity averse attitude, leads to a unique time-consistent robust price and to a unique time-consistent robust hedging strategy for both a long and a short position in an incomplete market. We illustrate our approach in the context of pensions and insurances, though it can be applied more generally.

The remainder of this paper is organised as follows. At first, we introduce the uncertain financial market in § 2. In § 3 we solve the robust optimisation problem and in § 4 we derive the optimal strategies and indifference price, summarised by our main contribution Theorem 1 and Corollary 1. In § 5 we show the robust price of the illiquid liability in the extreme setting of a pure incomplete market. And we check the disappearance of the effect of uncertainty on pricing in a pure complete market setting. The economic interpretations of the optimal solutions are explained in detail by the two examples in § 6. While our approach can be applied more generally, as shown in the first example, the second example is illustrated in the context of asset-liability management which is related to insurance and actuarial problems. This life insurance contract application showcases the approach naturally. Finally, § 7 concludes.

2. Financial market with model ambiguity. We consider an economic agent who has a liability $L(\cdot)$ that must be paid at time T. The agent wants to hedge his liability that depends on both hedgeable and unhedgeable risk factors, by finding the optimal hedging portfolio. Note that the agent is only interested in the maximum profit, which causes a hedging type interpretation. The hedging portfolio $A(\cdot)$ is represented by the positions in the hedgeable risk factors, the traded assets. This set-up is similar as in an asset-liability management (ALM) problem. The presence of nonhedgeable risk leads to market incompleteness. In an incomplete market multiple equivalent martingale measures exist, each corresponding to a different no-arbitrage price.

Moreover, the agent is uncertain about the models underlying the hedgeable and unhedgeable risk factors. Therefore the objective reduces to a robust portfolio optimisation problem in which the agent targets to hedge the liability subject to his uncertainty. Gilboa and Schmeidler [15] introduced the $\max\min$ concept that is utilised to obtain a robust solution. In this setting, the agent robustifies his decision by introducing a counterplayer who minimises his objective by selecting the worst-case model from the uncertainty set. Without uncertainty, the objective of the agent is to match the liability as good as possible by the hedging portfolio, which is summarised by maximising the surplus of his position of assets minus liabilities. After introducing the minimising counterplayer, by which he anticipates on the effect of uncertainty, the worst-case probability measure is revealed and the agent derives his robust investment strategy. By utility indifference pricing, we derive the unique robust price of the liability that is driven by the worst-case probability measure. We derive both the upper and lower bound on the indifference pricing. These can be interpreted as the bid and ask price.

We transform the stochastic optimisation into an optimal control problem by the Hamilton-Jacobi-Bellman equation (HJB). The nonlinear partial differential equation (PDE) can be represented as a forward backward stochastic differential equation (FBSDE). This connection is used to prove the existence and uniqueness of the optimal strategy.

The liabilities may depend on both hedgeable and unhedgeable risk. Hedgeable risk represents the risk that underlies the liquid and traded assets, i.e. the agent can have short and long positions in these assets. If only this type of risk is present, this market is called a complete market. Whereas in an incomplete market, additional risk factors that are not traded are present which are therefore unhedgeable. We consider the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$. Throughout, the explicit stochasticity dependence on ω will be suppressed whenever possible. Let there be n tradeable assets $X = (X_1, ..., X_n)$ and l untradeable risk factors $Y = (Y_1, ..., Y_l)$ and let there be a bank account X_0 on which one can go short or long for the interest rate $r(t, X_0(t), X(t))$. These three processes follow the stochastic differential equation (SDE)

$$d\begin{pmatrix} X_0(t) \\ X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} r(t, X_0(t), X(t)) \\ \mu^X(t, X(t), Y(t)) \\ \mu^Y(t, X(t), Y(t)) \end{pmatrix} dt + \begin{pmatrix} 0_n & 0_l \\ \Sigma^{XX}(t, X(t), Y(t)) & \Sigma^{XY}(t, X(t), Y(t)) \\ \Sigma^{YX}(t, X(t), Y(t)) & \Sigma^{YY}(t, X(t), Y(t)) \end{pmatrix}^{1/2} d\begin{pmatrix} W^X(t) \\ W^Y(t) \end{pmatrix}, \tag{1}$$

where $\mu^X(t, X(t), Y(t))$ and $\mu^Y(t, X(t), Y(t))$ are $\mathbb{R}^n$ and $\mathbb{R}^l$-valued. The dimension of covariance matrices are $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times l}$, $\mathbb{R}^{l\times n}$ and $\mathbb{R}^{l\times l}$ for $\Sigma^{XX}(t, X(t), Y(t))$, $\Sigma^{YX}(t, X(t), Y(t))$, $\Sigma^{XY}(t, X(t), Y(t))$ and $\Sigma^{YY}(t, X(t), Y(t))$ respectively. Both drift vectors are allowed to depend on the tradeable and untradeable assets. And W^X and W^Y are the n-dimensional and l-dimensional Brownian motion under $\mathbb{P}$ respectively. In (1) we display each block matrix of the $[2n \times 2l]$ covariance matrix $\Sigma(t, X(t), Y(t))$ separately. The superscript XX refers to the covariance matrix of the hedgeable process, superscript $XY = (YX)'$ refers to the correlation between the hedgeable and unhedgeable factors and superscript YY refers to the covariance matrix of the unhedgeable process. We assume that the covariance matrix $\Sigma(t, X(t), Y(t))$ is positive definite and thus invertible. The subscript attached to 0_n and 0_l indicate the dimension of the zero vectors. The interest rate that drives the bank account can either be constant or follow a stochastic process. The interest rate obtained on the bank account is the integral over the short rate $r(t, X_0(t), X(t))$ that we assume to be progressively measurable and which maps to $\mathbb{R}$, in other words we do not exclude negative interest rates. Without the bank account, the above SDE reduces to the matrix-vector form

$$dZ(t) = \mu(t, Z(t))dt + \Sigma(t, Z(t))^{1/2} dW^Z(t), \tag{2}$$

where $Z = (X, Y)'$.

In order to have a classical solution of the partial differential equation (PDE) that we derive, we need the following assumption.

ASSUMPTION 1. *We assume that*

(i) the drift and covariance matrix related to both processes X and Y are twice differentiable and have uniformly bounded partial derivatives for all orders up to two,

$$\mu(t, Z(t)), \Sigma(t, Z(t)), r(t, X_0(t), X(t)) \in C_b^2, \tag{A1}$$

(ii) and that the terminal condition of the liability is Lipschitz

$$L(T, Z(T)) \in C_L,$$

where C_b^ℓ is the space of those functions that are ℓ-times continuously differentiable and all derivatives up to order ℓ are uniformly bounded and C_L represents Lipschitz continuity.

The goal of maximising the profit (equivalently payoff or surplus) has a widespread applicability and intuition described in the introduction. The agent's choice variable at every point in time is $\theta^X(t)$ the allocation of his wealth over the different financial assets for the next time step. The investment strategy θ^X is the n-dimensional progressively measurable process and we assume that $\int_0^\cdot (\theta^X)'(s)\Sigma^{XX}(s, Z(s))\theta^X(s)ds$ is finite $\mathbb{P}$ - a.s. The amount of wealth held in the self-financing portfolio at time t is denoted by $A(t, X(t))$, which is the integral of the self-financing SDE. The agent chooses the hedge position $\theta^X(t)$ in the financial assets that maximises his expected profit at terminal time T. The strategy $\theta^X(t)$ is admissible for the initial wealth $A(0)$ if the solution to the self-financing condition, which is defined $\mathbb{P}$-a.s., remains $\mathbb{P}$-a.s. nonnegative at all times. Let Θ be the set of all admissible θ^X from the initial wealth $A(0)$.

The (nonrobust) objective is to maximise the expected surplus at time T of the investment position minus the liabilities. In other words, the objective is to replicate the contract that depends on both hedgeable and unhedgeable risk by a portfolio of traded assets.

$$\max_{\theta^X(t)\in\Theta} \mathbb{E}^{\mathbb{P}}[A(T,X(T)) - L(T,X(T),Y(T))], \tag{3}$$

where Θ are all admissible trading strategies and $\mathbb{P}$ denotes the baseline model. The surplus at terminal time T is a known function. Profit maximisation without trading constraints leads in most cases to an unbounded problem. By specifying a utility function this problem can be overcome. However this implies that the (risk-neutral) objective of profit maximisation is lost and that a utility function has to be specified. To focus on model uncertainty rather than the interplay with risk-aversion, we assume neutrality with respect to risk.

We specify model uncertainty by a set of different probability measures, where each measure belongs to an alternative model. By Girsanov theorem the set $\mathscr{L}$ can be identified by

$$\begin{aligned}\mathscr{L} = \Big\{&\mathbb{L} \sim \mathbb{P} : R(t) = \frac{d\mathbb{L}}{d\mathbb{P}}\Big|_t, dR(t) = \lambda(t,X,Y)R(t)dW^Z(t) \\ &\text{with } |\lambda(t,X,Y)| \le k \text{ where } k \text{ satisfies Assumption 2}\Big\},\end{aligned} \tag{4}$$

where $R(t)$ is the Radon-Nikodym derivative (i.e. likelihood ratio) that describes the change from probability measure $\mathbb{P}$ to $\mathbb{L}$, $\mathbb{P}$ represents the agent's baseline model, $\mathbb{L}$ the alternative model and the probability measures $\mathbb{P}$ and $\mathbb{L}$ are equivalent which implies that they have the same null sets. The change of measure is driven by the parameter $\lambda(t,X,Y)$ and is allowed to depend on the stochastic processes X and Y. The uncertain drift adjustment is assumed to be progressively measurable and to satisfy Novikov's condition

$$\mathbb{E}^{\mathbb{P}}[e^{\frac{1}{2}\int_0^T \lambda^2(t,X(t),Y(t))dt}] < \infty.$$

We allow for stochastic $\lambda(t,X,Y)$ which implies that the alternative model under $\mathbb{L}$ can have a different distribution than the baseline model. Contrary to constant or deterministic $\lambda(t)$'s that take only parameter uncertainty into account. In the next subsection we further elaborate on λ. The width of this set of alternatives indicates the amount of ambiguity and is represented by the scalar $k > 0$. The larger k, the larger becomes the set of alternative models and the more uncertain the agent is[2].

In order to have a bounded problem, we assume that the uncertainty around the drift $\mu(t,Z(t))$ includes the interest rate $r(t,X_0(t),X(t))$. If this assumption is violated, implying that even the worst-case drift of the risky assets is larger than the interest rate, then it is optimal to always invest an infinite amount in the risky asset. This leads to an unbounded problem. Hence Assumption 2, which incorporates the interest rate in the uncertainty set such that the worst-case drift on the risky assets is possibly worse than the investing on the bank account, bounds the problem of the risk-neutral agent.

ASSUMPTION 2. *We assume that the radius on the uncertainty set $\mathscr{L}$ is large enough such that it includes interest rate*

$$k^2 - (\mu^X(t,Z) - r(t,X_0,X))'(\Sigma^{XX}(t,Z))^{-1}(\mu^X(t,Z) - r(t,X_0,X)) \ge 0. \tag{A2}$$

Note that every agent can have a different degree of uncertainty, i.e. a different value k, as long as it fulfills Assumption 2. Since most variables are a function of t and $\{X_0(t), X(t), Y(t)\}$ we suppress expressing the time dependence of the function twice.

[2] See Balter and Pelsser [1] for more about the size and characterisation of the set of alternative models.

The robust equivalent of objective (3) is

$$\max_{\theta^X(t)\in\Theta}\min_{\mathbb{L}\in\mathscr{L}} \mathbb{E}^{\mathbb{L}}[A(T,X(T))-L(T,X(T),Y(T))]. \tag{5}$$

The acknowledgment of uncertainty makes the agent looking for a strategy that is least affected under scenarios that emerge from different models than the baseline model. In anticipation of the worst-case model, he solves a maxmin problem. To robustify the investment strategy, the inner part of the optimisation is played by a so called “mother nature” who acts as a malevolent factor that minimises the surplus by choosing the worst-case measure $\mathbb{L}^*$. Whereas the agent searches for the best hedge strategy that is least affected by mother nature’s choice.

Given the initial wealth $A(0) > 0$, the investment strategy θ^X is admissible with respect to the robust problem if the measurability, integrability and nonnegativity assumptions made at the beginning of this section hold $\mathbb{L}$-a.s. for all $\mathbb{L} \in \mathscr{L}$.

The agent can trade in the hedgeable risk, but not in the unhedgeable risk. Therefore the amount he invests in the first n assets needs to be chosen such that he maximises the surplus while the other l hedge positions with respect to the unhedgeable risk are restricted to zero. Thus the investment strategy equals $\theta(t) := (\overbrace{\theta_1(t),\dots,\theta_n(t)}^{\text{n}}\overbrace{0,\dots,0}^{\text{l}}) = (\theta^X(t),\theta^Y(t))$. The restriction on the unhedgeable part can be summarised by the constraint $\Xi\theta(t) = 0_l$, where Ξ is an $[l \times (n+l)]$ matrix equal to $\left[0_{[l\times n]}|I_{[l\times l]}\right]$, with on the left an $[l \times n]$ zero-matrix and next to it an $[l \times l]$ identity matrix.

The amount invested on the bank account is denoted by $\theta_0(t)$ and to later simplify notation we introduce the combing vector $\bar{\theta}(t) = (\theta_0(t),\theta^X(t))'$ and $\bar{X}(t) = (X_0(t), X(t))'$. How much the agent goes long or short on the bank account is captured by the position $\theta_0(t)$ on which we do not assume any short selling constraints. Each $\theta_i(t)$ with $i > 0$ represents the number of assets the agents buys of asset $X_i(t)$. Consequently the amount invested is $\theta_i(t)\cdot X_i(t)$ at time t. The total value of the assets $A(t)$ includes the bank account, therefore $A(t) = \bar{\theta}(t)'\bar{X}(t)$. Note that the change in the value of the investment position is denoted by $dA(t)$, while we highlight the explicit dependence on the hedgeable risk factors by $A(t,X(t))$ contrary to the liabilities which might depend on nonhedgeable risk as well. Hence, we use both the notation $A(t)$ and $A(t,X(t))$ depending on the context, where the latter reminds us explicitly of its dependence structure. At every point in time a change in the value of the assets can only occur due to a change in the underlying values, stated differently gains or losses are not possible by a re-allocation over the different assets. This is incorporated by the self-financing condition.

PROPOSITION 1. *By the self-financing condition, the total value of wealth evolves by*

$$d\tilde{A}(t) = \tilde{\theta}^X(t)'dX(t), \tag{6}$$

where $\tilde{\theta}^X$ is the $[n\times 1]$ vector consisting of the investment strategies in the traded assets $X_1,\dots,X_n$ in terms of units of the bank account which is the numéraire. And $dX(t)$ are the stochastic differential equations of the hedgeable risk, as described by equation (1).

Proof of Proposition 1 The self-financing condition is

$$dA(t) = \bar{\theta}(t)'d\bar{X}(t), \tag{7}$$

where the first term of $\bar{X}(t)$ is represented by the bank account $dX_0(t) = r(t,X_0(t),X(t))X_0(t)dt$, accordingly $\bar{X}(t)$ is a vector of size $[(n+1)\times 1]$. By Itō’s lemma one can see that a change in asset value may not be caused by a re-allocation of the value over the available assets, this indirectly leads to the second term of $dA(t) = \bar{\theta}(t)'d\bar{X}(t) + \bar{X}(t)'d\bar{\theta}(t)$ to be zero.

The definition of $A(t)$ is

$$A(t) = \bar{\theta}(t)'\bar{X}(t). \tag{8}$$

Rewriting the amount invested in the bank account in terms of the self-financing condition yields

$$A(t) = \theta_0(t)X_0(t) + \theta^X(t)'X(t) \tag{9}$$
$$dA(t) = (\theta_0(t)r(t, X_0(t), X(t))X_0(t) + \theta^X(t)'\mu^X(t, Z(t)))dt + \theta^X(t)'\left(\Sigma^{XX}\right)^{1/2}(t, Z(t)dW^X(t). \tag{10}$$

Let the bank account $X_0(t)$ be the numéraire, then by Itō's lemma the following holds

$$d\tilde{A}(t) = d\left(\frac{A}{X_0}\right)(t) \tag{11}$$
$$= \frac{1}{X_0(t)}\Big\{\big(\theta_0(t)r(t, X_0(t), X(t))X_0(t) + \theta^X(t)'\mu^X(t, Z(t)) - A(t)r(t, X_0(t), X(t))\big)dt + \theta^X(t)'\left(\Sigma^{XX}\right)^{1/2}(t, Z(t))dW^X(t)\Big\}. \tag{12}$$

The definition (9) as function for the amount invested in the bank account $\theta_0(t) = \frac{1}{X_0(t)}(A(t) - \theta^X(t)'X(t))$ can be substituted in (12)

$$d\tilde{A}(t) = \tilde{\theta}^X(t)'dX(t), \tag{13}$$

where

$$\tilde{\theta}^X(t) = \frac{\theta^X(t)}{X_0(t)}. \tag{14}$$

□

3. Optimisation. First we transform the stochastic optimisation into an optimal control problem by the HJB-equation. Then we optimise the objective subject to the constraints such that we obtain the robust strategies. By the link between nonlinear PDEs and FBSDEs with Lipschitz drivers we prove the uniqueness and existence of the viscosity and classical strategy under different assumptions on the drift and diffusion parameters of the price processes.

The uncertainty set is described by alternative probability measures that are linked to the baseline model. All alternatives that are within a radius of k are assumed to be plausible models and form the basis of the robustification of the investment strategy. By Girsanov's theorem we can express the characteristics of the uncertainty set explicitly by equation (4). The change in measure yields the transition in equation (1) from the process $\{W^Z(t)\}_{0\le t\le T}$ which is a standard Brownian motion (i.e. $\mathcal{N}(0, t)$) under probability measure $\mathbb{P}$ to $\{W^Z(t) + \int_0^t \lambda(s, Z)ds\}_{0\le t\le T}$ which is a standard Brownian motion under the alternative probability measure $\mathbb{L}$. We set $\epsilon(t, X(t), Y(t)) := \Sigma^{1/2}(t, X(t), Y(t))\lambda(t, X(t), Y(t))$ for ease of analytical exposition. And we suppress the explicit dependence on t and $Z(t)$ as much as possible to enhance readability, i.e. $\mu^X(t, X(t), Y(t)) \equiv \mu^X(t, X, Y) \equiv \mu^X(t, Z(t)) \equiv \mu^X(t, Z) \equiv \mu^X$. The characterisation of the uncertainty set leads to

$$\mathcal{L}^\epsilon = \left\{\mu(t, Z) + \epsilon(t, Z) | \epsilon(t, Z)'\Sigma^{-1}(t, Z)\epsilon(t, Z) \le k^2 \text{ where } k \text{ satisfies Assumption 2}\right\}. \tag{15}$$

In one dimension the uncertainty set reduces to an interval, in two dimensions to an ellipse and in multidimensions to an ellipsoid with the baseline model as middle point.

As an intermediate step we consider the Feynman-Kaç equation (FK) whereafter we proceed with the optimisation of the associated partial differential equation. First we also divide the liability by the numéraire, which gives $\tilde{L}(t, Z(t)) = \frac{L(t,Z(t))}{X_0(t)}$. The Feynman-Kaç theorem states that the conditional expectation

$$\mathbb{E}^{\mathbb{L}}[\tilde{A}(T, X(T)) - \tilde{L}(T, Z(T)) | \tilde{A}(t) = \tilde{A}, X(t) = X, Y(t) = Y] = \bar{v}(t, \tilde{A}, X, Y), \tag{16}$$

where the measure $\mathbb{L}$ represents the measure $\left(dW^X(t) + \epsilon^X(t, Z)dt, dW^Y(t) + \epsilon^Y(t, Z)dt\right)$ and $\bar{v}(t, \tilde{A}, X, Y)$ satisfies the PDE

$$\partial_t \bar{v} + \tilde{\theta}^X(t)'(q^X(t, Z) + \epsilon^X(t, Z))\partial_{\tilde{A}}\bar{v} + \left(\mu^X(t, Z) + \epsilon^X(t, Z)\right)'\partial_X \bar{v} + \left(\mu^Y(t, Z) + \epsilon^Y(t, Z)\right)'\partial_Y \bar{v} + \tfrac{1}{2}\text{tr}(\Delta(\tilde{A}, Z) \cdot \bar{v}(t, \tilde{A}, Z) \cdot \Phi\Sigma(t, Z)\Phi') = 0, \tag{17}$$

where $q = (q^X, q^Y)' = \mu(t, Z(t)) - r(t, X_0(t), X(t))Z(t)$. The operator Δ gives the partial derivatives, and the operator Φ enlarges the covariance matrix $\Sigma(t, Z)$ with the covariance matrices with respect to $\tilde{A}(t, X)$,

$$\Delta(\tilde{A}, Z) = \begin{bmatrix} \partial_{\tilde{A}\tilde{A}} & \partial_{\tilde{A}X} & \partial_{\tilde{A}Y} \\ \partial_{X\tilde{A}} & \partial_{XX} & \partial_{XY} \\ \partial_{Y\tilde{A}} & \partial_{YX} & \partial_{YY} \end{bmatrix}, \Phi = \begin{bmatrix} [\tilde{\theta}^X \ 0_l]' \\ I_{[n+l]} \end{bmatrix}, \tag{18}$$

where we use the operator ∂ for the partial derivatives. The PDE remains linear in $\tilde{A}$, because for every $\tilde{\theta}^X(t, Z(t))$ and every $\epsilon^X(t, Z(t))$ as fixed stochastics the PDE is linear in all directions of $\bar{v}(\cdot)$ and its derivatives.

When the agent is uncertain and anticipates to this, he optimises the value function

$$\max_{\theta(t)\in\Theta} \min_{\epsilon(t)\in\mathscr{L}^\epsilon} \bar{v}(t, \tilde{A}, X, Y) = v(t, \tilde{A}, X, Y). \tag{19}$$

The robust optimisation problem can be interpreted as a two player game where the agent wants to maximise the surplus whereas the robustness role is played by the counter player “mother nature” who minimises the surplus. The robust optimised value $v(t, \tilde{A}, X, Y)$ is given by the HJB-equation,

$$\begin{aligned} &\partial_t v + \max_{\tilde{\theta}(t)} \min_{\epsilon(t,Z)} \Big\{ \tilde{\theta}^X(t)'(q^X(t,Z) + \epsilon^X(t,Z))\partial_{\tilde{A}} v + \partial'_Z v(\mu(t,Z) + \epsilon(t,Z)) + \\ &\tfrac{1}{2}\partial_{\tilde{A}\tilde{A}} v \tilde{\theta}^X(t)'\Sigma^{XX}(t,Z)\tilde{\theta}^X(t) + \partial'_{\tilde{A}X} v \Sigma^{XX}(t,Z)\tilde{\theta}^X(t) + \partial'_{\tilde{A}Y} v \Sigma^{YX}(t,Z)\tilde{\theta}^X(t) \Big\} + \\ &\tfrac{1}{2}\mathrm{tr}(\Delta(Z)\cdot v(t,\tilde{A},Z)\Sigma(t,Z) = 0 && (20) \\ \text{s.t.}\quad &\epsilon(t,Z)'\Sigma(t,Z)^{-1}\epsilon(t,Z) \le k^2 \\ &\Xi\tilde{\theta} = 0_l \\ \text{with}\quad &v(T,\tilde{A},X,Y) = \tilde{A}(T,X) - \tilde{L}(T,Z), && (21) \end{aligned}$$

where (21) is the boundary condition at time T and $\Delta(Z)$ is the lower right second order derivative matrix with respect to the variables Z only. Note that the other terms of $\Delta(\tilde{A}, Z)$ include the control variable θ^X and are therefore explicitly written out. The HJB is intuitively equal to “the FK with optimal $\tilde{\theta}^X(t)$ and $\epsilon(t)$ per dt”. The difference between FK and HJB is that now $\tilde{\theta}(t)$ and $\epsilon(t)$ may depend on the value function and its derivatives which makes the PDE nonlinear. Therefore the value function $v(t, \tilde{A}, X, Y)$ can no longer be represented by a linear expectation. We show that the value for $v(t, \tilde{A}, X, Y)$ can be represented by a nonlinear expectation $\mathscr{E}[\cdot]$. Since the terminal condition is linear in $\tilde{A}$, we propose the Ansatz that $v(\cdot)$ is linear in $\tilde{A}$

$$v(t, \tilde{A}, X, Y) = \tilde{A}(t, X) - \tilde{w}(t, X, Y). \tag{22}$$

Despite the nonlinearity caused by the max min the linearity in $\tilde{A}$ remains due to the specific boundary condition of the surplus. The function $\tilde{w}(\cdot)$ satisfies

$$\begin{aligned} &-\partial_t \tilde{w} + \max_{\tilde{\theta}(t)} \min_{\epsilon(t,Z)} \Big\{ \tilde{\theta}^X(t)'(q^X(t,Z) + \epsilon^X(t,Z)) \\ &-\partial'_Z \tilde{w}(\mu(t,Z) + \epsilon(t,Z)) \Big\} - \tfrac{1}{2}\mathrm{tr}(\Delta(Z)\cdot \tilde{w}(t,Z)\Sigma(t,Z)) = 0 && (23) \\ \text{s.t.}\quad &\epsilon(t,Z)'\Sigma^{-1}(t,Z)\epsilon(t,Z) \le k^2 \\ &\Xi\tilde{\theta} = 0_l \\ \text{with}\quad &\tilde{w}(T,X,Y) = \tilde{L}(T,Z). \end{aligned}$$

The agent maximises $-\tilde{w}(t, X, Y)$ which is the difference between the nonlinear expectation of the surplus and the value of the assets. At time T, $\tilde{w}(T, X, Y)$ is exactly the liability value divided by the numéraire. Note that the optimised $\tilde{w}$ may depend on θ and ϵ. Supporting the mindset of the agent he maximises minus $\tilde{w}$ which can be interpreted as an equivalent for the discounted value of the liabilities. Since the only variables that depend on the max min problem are $\theta(t)$ and $\mu(t, Z)$, where $\mu(t, Z)$ can be rewritten as $\mu(t,Z)+\epsilon(t,Z)$, and ϵ is the dependent variable, the above objective function can be simplified by eliminating the terms that are independent on the decision variables. First we concentrate on the max min, which we define by

$$m(t,X,Y)=\max_{\tilde{\theta}^X(t)}\min_{\epsilon(t,Z)}\left\{\tilde{\theta}^X(t)'(q^X(t,Z)+\epsilon^X(t,Z))-\partial_Z'\tilde{w}(\mu(t,Z)+\epsilon(t,Z))\right\}. \tag{24}$$

We optimise $m(t,X,Y)$ for every time step dt and plug in the optimal $m^*(t,X,Y)$ in PDE (23) to obtain the optimal nonlinear PDE. For both control variables we obtain a candidate solution that satisfies the first order conditions. In the proof of Corollary 1 together with Verification Lemma 2 we show that the minimisation and maximisation both yield a unique solution. The optimal hedging strategy and drift distortions in terms of the numéraire, are

$$\begin{aligned}\tilde{\theta}^* &= \begin{bmatrix}\partial_X\tilde{w}+(\Sigma^{XX})^{-1}\Sigma^{XY}\partial_Y\tilde{w}+\tilde{h}(\Sigma^{XX})^{-1}q^X\\ 0_l\end{bmatrix} \\ \epsilon^* &= \begin{bmatrix}-q^X\\ \left(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)\tilde{h}^{-1}\partial_Y\tilde{w}-\Sigma^{YX}(\Sigma^{XX})^{-1}q^X\end{bmatrix},\end{aligned} \tag{25}$$

where $\tilde{h}=\sqrt{\frac{\partial_Y'\tilde{w}\left(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)\partial_Y\tilde{w}}{k^2-(q^X)'(\Sigma^{XX})^{-1}q^X}}$. Plugging the optimal $m^*(t,X,Y)$ into PDE (23) results in the nonlinear PDE

$$\begin{aligned}&-\partial_t\tilde{w}-\partial_X'\tilde{w}\cdot r\cdot X-\partial_Y'\tilde{w}\cdot\left(\mu^Y-\Sigma^{YX}(\Sigma^{XX})^{-1}q^X\right)-\tfrac{1}{2}\mathrm{tr}\left(\Delta(Z)\cdot\tilde{w}\cdot\Sigma\right)\\ &-c\sqrt{\partial_Y'\tilde{w}\cdot(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY})\cdot\partial_Y\tilde{w}}=0,\end{aligned} \tag{26}$$

where $c=\sqrt{k^2-(q^X)'(\Sigma^{XX})^{-1}q^X}$.

The fact that the square root has to be positive leads to the requirement that the risky asset should not be strictly better than or equal to the bank account. The admissibility of the square root leads to Assumption 2, which implies that the interest rate should be inside the uncertainty ellipse. This yields that the uncertain return on risky assets can possibly be lower than the interest rate. If the agent is confident that even in the worst case a positive excess return can be made by investing in the financial market, then the agent will try to invest a massive amount in the financial market and has a confident expectation to get very rich. However, the assumption ensures that $\max_\theta\min_\epsilon\mathbb{E}[A(T,X(T))-L(T,X(T),Y(T))]<\infty$. Note that the risk-neutral agent is indifferent with respect to risk since he is maximising the surplus. We concentrate on model misspecification purely. If there is only one hedgeable risk factor in the economy then (A2) states that the market price of risk should be within the agent's set of alternatives and cannot be "too good". This corresponds with the idea of the Good-Deal-Bounds methodology of Cochrane and Saa-Requejo [11].

Because the financial components X of the risk vector are perfectly replicated by the delta-hedge $\partial_X\tilde{w}$, the ambiguity regarding the drift of the hedgeable risk is eliminated and replaced by the interest rate r. The ambiguity regarding the drift of the unhedgeable processes Y is decomposed into the part consisting of the variance of each component of Y plus terms consisting of the covariances with all risk factors. This is known as the Schur complement $S=\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}$ which is the conditional variance of the unhedgeable risk given the hedgeable risk. We now derive the optimal pricing rule of the liability by its indifference value, after which we explain the implication and interpretation of the optimal strategies in more detail.

4. Utility indifference pricing and optimal pricing result. In complete markets, any derivative or claim can be priced uniquely by replicating the payoff with available products. The weighted average of the market prices of these available products determines the fair value of the derivative. However, in reality markets are incomplete, implying that there is risk that cannot be replicated. Not only nontraded assets make the market incomplete, but also the presence of transaction costs or portfolio constraints. The law of one price is violated since many equivalent martingale measures exist and each determines a different no-arbitrage price. Hodges and Neuberger [29] price options under transaction costs in the Black-Scholes model by the concept of utility indifference pricing. The concept of utility indifference pricing we employ is based on the price of a claim such that an agent who is maximising his utility is indifferent between having the claim or not. The agent is willing to accept the obligation to fulfill the liability if he receives a certain amount of money today, such that he is not worse off in terms of expected utility than he would be without having the obligation to pay the liability. Hence, the utility indifference price π is the price at which the investor is indifferent (in the sense that his expected utility under optimal trading is unchanged) between receiving nothing in combination with not having the liability $L(T,Z(T))$ and receiving π now in combination with having the obligation to pay $L(T,Z(T))$ at time T. The utility indifference price is closely related to the certainty equivalent amount, which is the certain amount of money that makes the agent indifferent between the return from a gamble and receiving a certain cash value.

Other techniques to price in incomplete markets are super- and subreplication, or to specify a penalty function among the equivalent martingale measures. The most famous example of the latter method is to add the relative entropy to the objective, which weights distributions that are further away from the baseline more heavily.

For an overview on utility indifference pricing see Henderson and Hobson [26]. For recent work on utility indifference pricing, and in specific for exponential utility which leads to analytical solutions see Henderson [24, 25], Miao and Wang [40], Hu et al. [30], Musiela and Zariphopoulou [41], Young and Zariphopoulou [47] and Zariphopoulou [48] among others.

Remember the Ansatz of (22) to be $v(t,\tilde{A},X,Y)=\tilde{A}(t,X)-\tilde{w}(t,X,Y)$. The indifference price $\pi(t,X,Y)$ is the extra cash needed to make the agent indifferent between being and not being liable. From equation (22) the growth of the bank account is known. Without liability contract the initial value of assets generates an expected surplus of $v(t,\tilde{A},X,Y)=\tilde{A}(t,X)$ at time t. Note that without contract the nonlinear expectation becomes linear, while the expected surplus at $0<t<T$ based on the initial assets plus the liability contract is implied by the nonlinear PDE. Together with the extra cash, the expected surplus is $v(t,\tilde{A},X,Y)=\tilde{A}(t,X)+\tilde{\pi}(t,X,Y)-\tilde{w}(t,X,Y)$. Being indifferent between these two situations leads to

$$\begin{aligned}\tilde{A}(t,X) &= \tilde{A}(t,X)+\tilde{\pi}(t,X,Y)-\tilde{w}(t,X,Y) && (27)\\ \tilde{\pi}(t,X,Y) &= \tilde{w}(t,X,Y), && (28)\end{aligned}$$

where $\tilde{w}(t,X,Y)=w(t,X,Y)/X_0(t)=e^{-\int_0^t r(s,X_0(s),X(s))ds}w(t,X,Y)$ and $\pi(t,X,Y)=w(t,X,Y)$. Rephrasing the PDE in terms of $\pi(t,X,Y)$ gives

$$\begin{aligned}&\partial_t\pi+\partial_X'\pi\cdot r\cdot X+\partial_Y'\pi\left(\mu^Y-\Sigma^{YX}(\Sigma^{XX})^{-1}q^X\right)+\tfrac{1}{2}\mathrm{tr}\left(\partial_{XX}\pi\Sigma^{XX}+2\partial_{XY}\pi\Sigma^{XY}+\partial_{YY}\pi\Sigma^{YY}\right)\\&-r\cdot\pi+c\sqrt{\partial_Y'\pi\left(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)\partial_Y\pi}=0.\end{aligned} \tag{29}$$

Note, that the equation is a semilinear PDE that describes the behaviour of the liability $\pi(t,Z)$ as a function of t and Z. The PDE is subject to the boundary condition $\pi(T,Z)=L(T,X,Y)$ which is the value of the insurance contract at time T. Thus we derived the PDE for which we can prove existence and uniqueness of the solution. The theorem is proven based on the auxiliary representation of the semilinear PDE as a FBSDE problem.

4.1. PDEs and FBSDEs. The general relation between partial differential equations (PDEs) and forward backward stochastic differential equations (FBSDEs) connects the semilinear PDE (30)

$$\begin{aligned}&\partial_t\pi(t,\hat{X})+\tfrac{1}{2}\mathrm{tr}\left(\Sigma(t,\hat{X})\partial_{\hat{X}\hat{X}}\pi(t,\hat{X})\right)+\mu(t,\hat{X})\partial_{\hat{X}}\pi(t,\hat{X})+g(t,\hat{X},\pi,\partial_{\hat{X}}\pi\cdot\Sigma(t,\hat{X}))=0\\&\pi(T,\hat{X})=f(T,\hat{X}),\end{aligned} \tag{30}$$

with the decoupled FBSDE (31)

$$\begin{aligned} d\hat{X}(t) &= \mu(t,\hat{X})dt + \Sigma(t,\hat{X})^{1/2}dW(t) \\ d\hat{Y}(t) &= -g(t,\hat{X},\hat{Y},\hat{Z})dt + \hat{Z}dW(t). \end{aligned} \tag{31}$$

If the FBSDE has a classical solution then $\hat{Z} = \partial_{\hat{X}}\hat{Y}\cdot\Sigma(t,\hat{X})$. Our notation yields the connection of the systems by $\hat{X} = Z = (X,Y)'$, $\hat{Y} = \pi$ and the PDE is semilinear due to $\sqrt{\partial_Y\pi\cdot S\cdot\partial_Y\pi}$ which is captured by the driver g.

A viscosity solution, which implies that π need not be everywhere differentiable, of the FBSDE, is also a viscosity solution of the PDE. The relation between the FBSDE and PDE is weak in the sense that several other viscosity solutions to the PDE might exist as well. On the other hand, a classical solution of the FBSDE which is unique, is also the unique classical solution of the PDE. Hence, the relation between FBSDEs and PDEs is stronger under assumptions by which classical solutions are implied. Existence and uniqueness of viscosity and classical solutions from the FBSDE literature, is translated to the PDE setting. We summarise the most important theorems

LEMMA 1 **(PDE and FBSDE)**. *The following results based on the PDE (30) and FBSDE (31) relation are known;*

If	then the	solution π	by	
$\mu,\Sigma\in C_L$ and $f,g\in C$	viscosity	exists	Evans [13].	(L1)
$\mu,\Sigma,f,g\in C_L$	viscosity	is unique	Pardoux and Peng [42], Ma and Yong [35], Evans [13].	(L2)
$\mu,\Sigma\in C_b^1$ and $f,g\in C_L$ and (A3)	viscosity	is C_b^1	Zhang [49] .	(L3)
$\mu,\Sigma,f,g\in C_b^3$	classical	is unique and C^2	Pardoux and Peng [42], Ma et al. [34], El Karoui et al. [12] .	(L4)
$\mu,\Sigma\in C_b^2$ and $f,g\in C_L$ and (A3)	classical	is unique and C^2	Zhang [49].	(L5)

where C stands for continuity, C_b^ℓ for functions that are ℓ-times continuously differentiable and all derivatives up to order ℓ are uniformly bounded and C_L represents Lipschitz continuity[3].

Since the optimal strategies θ^* and ϵ^* depend on $\partial_Z\pi$, and a viscosity solution does not imply differentiability of the solution π we need at least (L3) which ensures differentiability of order one of π. Moreover, the semilinearity of the optimal PDE relates to a nondifferentiable driver of the associated FBSDE. Therefore result (L4), which is standard in the FBSDE literature and leads to a classical solution is not applicable since the driver is not differentiable.

Lemma (L3) and (L5) rely on the additional assumptions [49]

ASSUMPTION 3. *Let $K>0$ be the Lipschitz constant*

(i) The functions $\mu,\Sigma\in C_b^1$ and

$$\sup_{0\le t\le T}\left\{|\mu(t,0)|+|\Sigma(t,0)|\right\}\le K.$$

(ii) We assume that Σ satisfies

$$\Sigma(t,Z)\ge\frac{1}{K}I_{(n+l)}, \tag{A3}$$

where $I_{[n+l]}$ is the $[n+l]\times[n+l]$ identity matrix.

[3] Note that we express the continuity and differentiability explicitly with respect to $Z(t)$. Differentiability of π with respect to t is needed for the verification lemma which is satisfied by the definition of the classical solution. Therefore we do not explicitly refer to differentiability of a function with respect to the first dependent variable t but focus on Z.

(iii) The driver $f, g \in C_L$ and

$$\sup_{0\le t\le T} |g(t,0,0,0) + f(t,0)| \le K.$$

Assumption (A3i) states that the drift and diffusion of the forward process should be bounded if $Z = 0$. Assumption (A3ii) states that the diagonal entries of the covariance matrix, which are the variances of the components of the random vector $\hat{X}$, are all positive. This is implied by the positive definite assumption on Σ. Additionally, the variances should also be bounded away from zero. And assumption (A3iii) is the Lipschitz condition on the nonlinear driver and terminal condition, as well as the assumption that the driver and terminal condition are bounded in the initial zero states.

Since we need the existence of $\partial_Z \pi$, the viscosity solution as described by (L3) would be sufficient to prove existence of the investment strategy. Moreover, uniqueness of the viscosity solution is implied by (L3) as well. A stronger relation with the FBSDEs is obtained for the classical solution. This additional assumption in (L5) of twice differentiability is a condition on which the verification lemma relies. Hence, we use the result of Zhang [49] for which we need to prove that the forward diffusion process fulfills the requirement that the first two derivatives with respect to Z, i.e. both the hedgeable and unhedgeable risk factors, are bounded. The forward process under the optimal measure has become

$$d\begin{pmatrix} X(t) \\ Y(t)\end{pmatrix} = \begin{pmatrix} r(t,X_0(t),X(t))\cdot X(t) \\ \mu^Y(t,Z) - \Sigma^{YX}(t,Z)\cdot\left(\Sigma^{XX}(t,Z)\right)^{-1} q^X(t,Z)\end{pmatrix} dt + \Sigma(t,Z(t))^{1/2} W^Z(t). \tag{32}$$

We introduce the assumption

ASSUMPTION 4. *For all $(t,X,Y) \in \{[0,T]\times\mathbb{R}\times\mathbb{R}\}$*

$$\Sigma^{YX}(t,Z)\cdot\left(\Sigma^{XX}(t,Z)\right)^{-1} q^X(t,Z) \text{ and } r(t,X_0,X)\cdot X \text{ are } C_b^2. \tag{A4}$$

Based on assumption (A4) together with (A3) for the optimal forward process, we can apply (L5). Brigo et al. [9] also make use of not needing the driver to be differentiable, to price nonlinear valuation equations under credit and funding effects. Brigo et al. [9] initially have a driver that does not satisfy the Lipschitz condition, therefore they move this component from the backward driver to the forward drift to have applicability of (L5).

Pham [45] extends the Merton investment problem for stochastic volatility and portfolio constraints. He proves existence and uniqueness of the classical solution of the semilinear PDE based on hypothesis (3a) in Pham [45], which is in the same spirit as assumption (A4) here.

In our context, the connection between nonlinear PDEs and FBSDEs can be summarised by

$$\left.\begin{array}{l} \text{PDE} \\ \mathscr{L}\cdot\pi - \\ g(t,Z,\pi,\partial_Z\pi\cdot\Sigma) = 0 \\ \pi(T,Z) = L(T,Z) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \text{BSDE} \\ d\hat{X}(t) = \mu^*(t,\hat{X})dt + \Sigma^*(t,\hat{X})^{1/2} dW(t) \\ d\hat{Y}(t) = -g(t,\hat{X},\hat{Y},\hat{Z})dt + \hat{Z}dW(t) \\ \hat{Y}(T) = f(T,\hat{X}). \end{array}\right. \tag{33}$$

On the left hand side, the PDE is expressed in term of the linear Feynman-Kaç part, denoted by the operator

$$\begin{aligned} \mathscr{L}\cdot\pi \;=\; & \Big(\partial_t + rX'\partial_X + (\mu^Y - \Sigma^{YX}(\Sigma^{XX})^{-1}q^X)'\partial_Y + \\ & \tfrac{1}{2}\operatorname{tr}\left(\partial'_{XX}\Sigma^{XX} + 2\partial'_{XY}\Sigma^{XY} + \partial'_{YY}\Sigma^{YY}\right)\Big)\cdot\pi, \end{aligned} \tag{34}$$

and the nonlinear part by

$$g(t,Z,\pi,\partial_Z\pi\cdot\Sigma) \;=\; r\pi - c\sqrt{\partial'_Y\pi\,(\Sigma^{YY} - \Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY})\,\partial_Y\pi}. \tag{35}$$

The first equation on the right hand side of (33) is the forward process where μ^* are the drifts of X and Y under the optimal measure, hence

$$\mu^* = \begin{bmatrix} r(t, X_0, X) \cdot X \\ \mu^Y(t, Z) - \Sigma^{YX}(t, Z) \cdot (\Sigma^{XX}(t, Z))^{-1} q^X(t, Z) \end{bmatrix}. \tag{36}$$

The covariance matrix Σ^* belongs to the trace terms of the PDE, hence $\Sigma^* = \Sigma(t, Z)$. The second equation is the backward process with the driver $g(\cdot)$ and $\hat{Z} = \partial_{\hat{X}} \hat{Y} \cdot \Sigma(t, \hat{X})$. Recall that $\hat{X} = Z = (X, Y)'$ and $\hat{Y} = \pi$.

Hence, we proceed by assuming that (A1), (A2), $\{\mu^*, \Sigma^*\} \in \{$(A3i) and (A3ii)$\}$, (A3iii) and (A4) hold, in order to apply Theorem 2.4.1 of Zhang [49] from which it follows that π is the classical solution whose derivatives exist. The terminal condition $L(\cdot) = f(\cdot)$.

ASSUMPTION 5. *Let μ^* be the optimal drift of the forward process and let Σ^* be the optimal covariance matrix of the forward process, then we assume that*

$$\mu^*, \Sigma^* \in \{\textit{(A3i), (A3ii)}\} \textit{ and that } f, g \in \textit{ (A3iii).} \tag{A5}$$

The solution of the robust objective (19) characterised by the optimisation problem (20) is proven to be optimal based on the following verification lemma.

LEMMA 2 **(Verification)**. *If*

(i) There exists a function $v(t, \tilde{A}, Z) : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^{n+l} \to \mathbb{R}$ which is C on its domain and C^2 with respect to Z, verifying $v(T, \tilde{A}, Z) = \tilde{A}(T, X(T)) - \tilde{L}(T, X(T), Y(T))$;

(ii) For any $\theta \in \Theta$ there exists an optimal solution ϵ^* with associated measure $\mathbb{L}^{(\theta)}$ of the inner minimisation in (20), such that

$$\Psi(t) = \Psi(t)^{(\theta)} = v(t, \tilde{A}(t), X(t), Y(t)), \tag{37}$$

is a $\mathbb{L}^{(\theta)}$-supermartingale;

(iii) There exist some $\theta^* \in \Theta$ such that the corresponding Ψ^* is a $\mathbb{L}^{(\theta^*)}$-martingale.

Then θ^* is optimal for the problem (20) and $v(0, \tilde{A}, Z)$ coincides with $\upsilon(0, \tilde{A}, X, Y)$.

Proof of Lemma 2 The lemma is similar as the Davis-Varaiya Martingale Principle of Optimal Control (Rogers [46]), which Biagini and Pınar [6] modified to the robust max min setting. We slightly adopt this to the pure terminal wealth case, including the liability in terms of linear utility.

By the supermartingale property of Ψ under $\mathbb{L}^{(\theta)}$, and by $v(T, \cdot) = \tilde{A}(T, \cdot) - \tilde{L}(T, \cdot)$ we have

$$\begin{aligned} \mathbb{E}^{\mathbb{L}^{(\theta)}}[\Psi(T)] &= \mathbb{E}^{\mathbb{L}^{(\theta)}}[v(T, \tilde{A}(T), Z(T))] \\ &= \mathbb{E}^{\mathbb{L}^{(\theta)}}[\tilde{A}(T, X(T)) - \tilde{L}(T, X(T), Y(T))] \\ &\leq \Psi(0) = v(0, \tilde{A}(0), Z(0)). \end{aligned} \tag{38}$$

Taking the supremum over Θ gives $\tilde{v}(0, \tilde{A}(0), Z(0)) = \sup_A \mathbb{E}^{\mathbb{L}^{(\theta)}}[\tilde{A}(T, X(T)) - \tilde{L}(T, X(T), Y(T))] \leq v(0, \tilde{A}(0), Z(0))$. This is the static equivalent of the dynamic problem where there is a one-to-one link between the path dependent investment strategy $\{\theta(t)\}_{0 \leq t}$ and the asset value $A(t, X(t))$. Since by assumption for some θ^* the process Ψ^* is a martingale under $\mathbb{L}^{(\theta^*)}$, then $\mathbb{E}^{\mathbb{L}^{(\theta^*)}}[\Psi(T)] = \Psi(0) = v(0, \tilde{A}(0), Z(0))$. Hence θ^* is optimal. This coincides with the intuition that optimality is obtained whenever the complete budget is utilised. □

If we find a function v that satisfies the condition of the verification lemma, then we know that the corresponding solutions are optimal. The optimal investment and worst-case drift adjustment depend on the first derivative of π. In order to guarantee that υ, which is one-to-one linked with $\tilde{w}$ and the indifference price π, is twice differentiable, the classical solution of the partial differential equation is required. The differentiability is ensured by (A1), (A2), (A4) and (A5). Hence, the first premise of Lemma 2 is fulfilled.

Using Itō's formula under $\mathbb{L}^{(\theta)}$ to the process Ψ verifies a stochastic differential equation. Moreover, the supermartingale property under every $\mathbb{L}^{(\theta)}$ and the martingale property for some $\mathbb{L}^{(\theta^*)}$ are the other two premises of Lemma 2. These coincide with the drift of Ψ to be zero for the supremum over $\theta \in \Theta$ of the infimum over $\epsilon \in \mathscr{L}^{\epsilon}$. Hence, the Hamilton-Jacobi-Bellman equation is derived from the verification requirements.

4.2. Optimal pricing result. Combining the PDE of the utility indifference price (29) with Verification Lemma 2 we derive here the main result of the paper.

THEOREM 1 **(Existence and uniqueness of classical solution).** *Let $\pi(t,X,Y)$ be the indifference price of an agent who maximises the expected[4] surplus $\mathbb{E}[A(T,X)-L(T,X,Y)|t,Z]$ based on the SDE*

$$d\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} \mu^X(t,X,Y) \\ \mu^Y(t,X,Y) \end{pmatrix} dt + \Sigma^{1/2}(t,X,Y)\, d\begin{pmatrix} W^X(t) \\ W^Y(t) \end{pmatrix},$$

where X is traded, Y is nontraded and the stacked vector is denoted by $Z=(X,Y)'$. Under model ambiguity, which is characterised by the change of measure $dW^Z(t)+\lambda(t,X,Y)dt$ where $\lambda(t,X,Y) = \Sigma(t,X,Y)^{-1/2}\epsilon(t,X,Y)$ with $\lambda(t,X,Y)'\lambda(t,X,Y) \le k^2$, the indifference price $\pi(t,X,Y)$ in an incomplete market is given by the PDE

$$\begin{aligned} &\partial_t\pi + \partial'_X\pi\cdot r\cdot X + \partial'_Y\pi\cdot\left(\mu^Y - \Sigma^{YX}(\Sigma^{XX})^{-1}q^X\right) + \\ &c\sqrt{\partial'_Y\pi\cdot(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY})\cdot\partial_Y\pi} + \\ &\tfrac{1}{2}tr\left(\partial_{XX}\pi\Sigma^{XX} + 2\partial_{XY}\pi\Sigma^{XY} + \partial_{YY}\pi\Sigma^{YY}\right) - r\cdot\pi = 0, \end{aligned}$$

with terminal value $\pi(T,X,Y) = L(T,X,Y)$, and where $c=\sqrt{k^2 - q_X{}'(\Sigma^{XX})^{-1}q_X}$. Under Assumptions 1, 2, 4, 5 the classical solution exists and by the connection with the FBSDE-framework, the classical solution is unique, which implies time differentiability and twice differentiability of π.

REMARK 1. A semilinear multivariate PDE can be interpreted as a FBSDE. The link between semilinear PDEs and FBSDEs has been shown by Pardoux and Peng [42] and is only an if-and-only-if relation for classical solutions which require the driver to be C^3. Without the existence of the first three derivatives, at least the viscosity solution of the PDE exists. In order to have a unique viscosity solution the driver has to be Lipschitz [13]. We also need the first derivatives of π, since θ^* and ϵ^* are functions of $\partial_Z\pi$. A slightly stronger version of the viscosity solution in the sense of (L3) guarantees this. By the verification lemma we need also the second derivatives of π. Following Zhang [49] the classical solution satisfies these requirements based on the assumption that $g \in C_L$ and $\mu, \Sigma \in C_b^2$.

Proof of Theorem 1 By applying Itō's lemma to π under the new measure $\mathbb{L}^*$ we get

$$\begin{aligned} d\pi(t) \;=\; &\partial_t\pi dt + \partial'_X\pi\left(rXdt + (\Sigma^{XX})^{1/2}dW^X\right) + \partial'_Y\pi\left((\mu^Y - \Sigma^{YX}(\Sigma^{XX})^{-1}q^X)dt + (\Sigma^{YY})^{1/2}dW^Y\right) + \\ &\tfrac{1}{2}\mathrm{tr}\left(\partial_{XX}\pi\Sigma^{XX} + 2\partial_{XY}\pi\Sigma^{XY} + \partial_{YY}\pi\Sigma^{YY}\right). \end{aligned} \tag{39}$$

Furthermore the FBSDE representation gives the relation

$$d\hat{Y}(t) = -g(t,\hat{X},\hat{Y},\hat{Z})dt + \hat{Z}dW(t). \tag{40}$$

The terms related to the Brownian motions are combined into the $\hat{Z}$ term. Thus $\hat{Z} = (\partial'_X\pi(\Sigma^{XX})^{1/2}, \partial'_Y\pi(\Sigma^{YY})^{1/2})'$. Since $\pi=\hat{Y}$, the definition of $\hat{Z}$ as the derivatives of $\hat{Y}$ is fulfilled. Based on (39) and (40) the driver $g(\cdot)$ equals

$$-g(t,\hat{X},\hat{Y},\hat{Z}) \;=\; \partial_t\hat{Y} + \partial'_X\hat{Y}rX + \partial'_Y\hat{Y}(\mu^Y - \Sigma^{YX}(\Sigma^{XX})^{-1}q^X) + \tfrac{1}{2}\mathrm{tr}\left(\partial_{XX}\hat{Y}\Sigma^{XX} + 2\partial_{XY}\hat{Y}\Sigma^{XY} + \partial_{YY}\hat{Y}\Sigma^{YY}\right).$$

[4] The conditional expression $|t,Z$ is a shorthand notation for $|A(t)=A, X(t)=X, Y(t)=Y$.

Recall from equation (35) that the driver implied by the forward relation equals

$$g(t,Z,\pi,\partial_Z\pi\cdot\Sigma)=g(t,\hat{X},\hat{Y},\hat{Z}) \;=\; r(t,X_0,X)\pi-c\sqrt{\partial_Y'\pi\,(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY})\,\partial_Y\pi}.$$

Together with Assumptions 2, it follows that the driver g is Lipschitz continuous. In conjunction with Assumptions 1, 4 and 5 we can apply Lemma (L5) [49]. Hence, the PDE has a classical solution which is unique. And from the definition of a classical solution, if follows that π is differentiable.

□

REMARK 2. The max min problem leads to an upper bound on the set of no-arbitrage prices of the liability for $c\sqrt{\partial_Y'\pi S\partial_Y\pi}>0$ and to a lower bound for $c\sqrt{\partial_Y'\pi S\partial_Y\pi}<0$. Hence, these could be interpreted as the bid and ask price. Note that the problem reduces to the linear expectation if $c\sqrt{\partial_Y'\pi S\partial_Y\pi}=0$. The max max problem would lead to an unbounded problem.

REMARK 3. A shorthand notation for the FBSDE solution and the indifference price is

$$\pi(t,X,Y)=e^{-\int_t^T r(s,X_0(s),X(s))ds}\mathscr{E}^{g^*}\left[L(T,X,Y)|t,Z\right],$$

where $\mathscr{E}^{g^*}$ denotes the *nonlinear* expectation with respect to the optimal driver $g^*(t,Z,\pi,\partial_Z\pi\Sigma)=r(t,X_0,X)\pi-c\sqrt{\partial_Y'\pi\,(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY})\,\partial_Y\pi}$ that is implied by the optimal strategies (θ^*,ϵ^*).

The solution of the agent's problem in which he maximises profit, while being concerned about the underlying model, leads to the explicit hedging demand and worst-case model as stated in the following corollary.

COROLLARY 1 **(Optimal hedging portfolio and robustness adjustments by Theorem 1)**. *Let $\pi(t,X,Y)$ be the indifference price of an agent who maximises the expected surplus $\mathbb{E}[A(T,X)-L(T,X,Y)|t,Z]$ under model ambiguity and in an incomplete market. Under the assumptions of Theorem 1 the agent's robust optimal dynamic hedging portfolio consists of the investment strategy*

$$\theta^* \;=\; \begin{bmatrix}\partial_X\pi+(\Sigma^{XX})^{-1}\Sigma^{XY}\partial_Y\pi+h(\Sigma^{XX})^{-1}q^X\\ 0_l\end{bmatrix},$$

and the ambiguity ensures the optimal solution to be robust for drift adjustments

$$\epsilon^* \;=\; \begin{bmatrix}-q^X\\ \left(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)h^{-1}\partial_Y\pi-\Sigma^{YX}(\Sigma^{XX})^{-1}q^X\end{bmatrix},$$

where $h=\sqrt{\frac{\partial_Y'\pi\left(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)\partial_Y\pi}{k^2-(q^X)'(\Sigma^{XX})^{-1}q^X}}$.

Proof of Corollary 1 Expression (25) is obtained by solving the first order conditions implied by the Lagrangian of the optimisation problem. Subsequently, w and its derivatives are expressed in term of π by relation (28).

Biagini and Pınar [6] consider a robust max min setting for which they prove optimality of the strategies by considering subcases that follow from the power utility. A similar second order argument is what we employ too. To prove optimality, first consider the inner minimisation

$$\begin{aligned}\min_{\epsilon}\quad & \tilde{\theta}'\cdot q+\epsilon'(\tilde{\theta}-\partial_Z\tilde{w})\\ \text{s.t.}\quad & \epsilon'\Sigma^{-1}\epsilon\le k^2.\end{aligned}\tag{41}$$

This is a linear objective with a quadratic constraint, from which we know that it has a unique minimum.

After plugging in the optimal value for ϵ^*, the objective for the agent becomes

$$\begin{aligned}\max_{\tilde{\theta}}\quad & \tilde{\theta}'q-k\sqrt{(\tilde{\theta}-\partial_Z\tilde{w})'\Sigma(\tilde{\theta}-\partial_Z\tilde{w})}\\ \text{s.t.}\quad & \Xi\tilde{\theta}=0.\end{aligned}\tag{42}$$

This is a quadric objective with linear constraints. The covariance matrix Σ is positive definite and thus the square root is a convex function. Since the optimal ϵ is the negative root, the agent maximises a concave function which results in a unique maximum (to see this, let $\hat{\theta} = \tilde{\theta} - \partial_Z \tilde{w}$, then $\max_{\hat{\theta}} (\hat{\theta} + \partial_Z \tilde{w})' q - k\sqrt{\hat{\theta}'\Sigma\hat{\theta}}$ s.t. $\Xi\hat{\theta} = \Xi\partial_Z \tilde{w}$). The optimal strategy of mother nature is $\epsilon^* = -k \frac{\Sigma(\tilde{\theta} - \partial_Z \tilde{w})}{\sqrt{(\tilde{\theta} - \partial_Z \tilde{w})'\Sigma(\tilde{\theta} - \partial_Z \tilde{w})}}$.

The optimisation problem from (41) yields the Lagrangian

$$L(\tilde{\theta}, \epsilon, \lambda_0, \lambda) = \tilde{\theta}' q + \epsilon'(\tilde{\theta} - \partial_Z \tilde{w}) - \lambda_0 \tfrac{1}{2}(\epsilon'\Sigma^{-1}\epsilon - k^2) - \lambda'(\Xi\tilde{\theta} - 0_l),$$

where λ has dimension $[l \times 1]$ and λ_0 is a scalar.
The FOC are

$$\begin{aligned} \frac{\partial L}{\partial \tilde{\theta}} &= q + \epsilon - \Xi'\lambda = 0_{n+l} \\ \frac{\partial L}{\partial \epsilon} &= -\partial_Z \tilde{w} + \tilde{\theta} - \lambda_0 \Sigma^{-1}\epsilon = 0_{n+l} \\ \frac{\partial L}{\partial \lambda} &= -\Xi\tilde{\theta} = 0_l \\ \frac{\partial L}{\partial \lambda_0} &= -\tfrac{1}{2}\left(\epsilon'\Sigma^{-1}\epsilon - k^2\right) = 0. \end{aligned} \tag{43}$$

Note that although Ξ is rank deficient, $\Xi\Sigma^{-1}\Xi'$ has full rank. The first three blocks of equations are linear and yield

$$\begin{aligned} \lambda^* &= (\Xi\Sigma^{-1}\Xi')^{-1}\Xi(\Sigma^{-1}q - \lambda_0^{-1}\partial_Z \tilde{w}) \\ \tilde{\theta}^* &= \partial_Z \tilde{w} + \lambda_0 \Sigma^{-1}(\Xi'\lambda^* - q) \\ \epsilon^* &= \Xi'\lambda^* - q. \end{aligned} \tag{44}$$

Now we can plug in λ in ϵ^* and solve the constraint $\epsilon'\Sigma^{-1}\epsilon = k^2$ for λ_0,

$$\lambda_0^{-1} = \pm\sqrt{\frac{q'\Sigma^{-1}\Xi'(\Xi\Sigma^{-1}\Xi')^{-1}\Xi\Sigma^{-1}q - q'\Sigma^{-1}q + k^2}{\partial_Z' \tilde{w}\Xi'(\Xi\Sigma^{-1}\Xi')^{-1}\Xi\partial_Z \tilde{w}}}. \tag{45}$$

Since mother nature is minimising, the second order derivative with respect to ϵ has to be positive to ensure a minimum. Therefore λ_0^{-1} is minus the square root term in equation (45). Let $\Psi = \Xi'(\Xi\Sigma^{-1}\Xi')^{-1}\Xi$, which implies Ψ to be the $[(n+l) \times (n+l)]$ null matrix with the Schur complement on the lower right $[l \times l]$ block, hence $\Psi = \begin{bmatrix} 0_{[n\times n]} & 0_{[n\times l]} \\ 0_{[l\times n]} & \Sigma^{YY} - \Sigma^{YX}\left(\Sigma^{XX}\right)^{-1}\Sigma^{XY} \end{bmatrix}$. This results in

$$\begin{aligned} \epsilon^* &= \Psi\left(-\sqrt{\frac{q'\Sigma^{-1}X\Sigma^{-1}q - q'\Sigma^{-1}q + k^2}{\partial_Z' \tilde{w} X \partial_Z \tilde{w}}}\partial_Z \tilde{w} + \Sigma^{-1}q\right) - q \\ &= \begin{bmatrix} -q_x \\ S\sqrt{\frac{k^2 - (q^X)'(\Sigma^{XX})^{-1}q^X}{\partial_Z' \tilde{w}\left(\Sigma^{YY} - \Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)\partial_Z \tilde{w}}}\partial_Z \tilde{w} - \Sigma^{YX}\left(\Sigma^{XX}\right)^{-1}q^X \end{bmatrix}, \end{aligned}$$

since $\Psi\Sigma^{-1} = \begin{bmatrix} 0_{[n\times n]} & 0_{[n\times l]} \\ -\Sigma^{YX}\left(\Sigma^{XX}\right)^{-1} & I_{[l\times l]} \end{bmatrix}$ and $q'\Sigma^{-1}\Psi\Sigma^{-1}q - q'\Sigma^{-1}q = \left(-\left(q^X\right)'\left(\Sigma^{XX}\right)^{-1}q^X, 0\right)'$.
□

The connection between PDEs and FBSDEs proves the uniqueness and solvability of the theorem. Algorithms to find numerical solutions of FBSDEs are well-known [16, 17, 18]. See for applications of FBSDEs El Karoui et al. [12]. Time-consistent ambiguity averse preferences including jump processes are priced by Laeven and Stadje [33]. Pelsser and Stadje [44] arrive at the same class of pricing operators by imposing time- and market-consistency. The time-consistency is in our case directly fulfilled by the initial HJB formulation of the optimisation problem. Time-consistent coherent risk measures have a Lipschitz driver which coincides with the ellipsoid uncertainty constraint.

The agent who wants to maximise his surplus and acknowledges the ambiguity of the underlying model, acts by the "robust method of pricing" that we derived. A practitioner's methodology to price in incomplete markets is the industry standardised Cost-of-Capital method [31]. This method that insurance companies use quantifies the market value of the replicating portfolio plus a mark-up for the unhedgeable risks, which relies mostly on the subjective quantification of risk. See Filipović and Vogelpoth [14] for a critical discussion of the Swiss Solvency Test on which the Cost-of-Capital method (CoC) is based. This method leads to a pricing operator that has similar characteristics as our result. The indifference pricing operator from Theorem 1 can be interpreted as a best estimate, which is the conditional expectation (if $g(\cdot) = 0$ then we have a linear PDE), plus a constant (c) times the standard deviation of the unhedgeable component. The standard deviation per dt is a penalty that is added. If $dt \to 0$ this is normally distributed and hence the penalty consists of the quantiles from the normal distribution. The normality assumption is fulfilled since we are in a diffusion setting. The decomposition of the result of Theorem 1 corresponds with the interpretation of the Cost-of-Capital method by $c\sqrt{\partial_Y' \pi S \partial_Y \pi}$ = "CoC" per dt. Pelsser and Ghalehjooghi [43] show that in a diffusion setting the CoC method and the standard deviation pricing principle have the same limit.

The interpretation of Corollary 1 is discussed in detail in the next two sections. Each case or example is dedicated to a specific part of the optimal robust strategy.

5. Uncertainty in complete and incomplete markets. In this section we show the application of Theorem 1 and Corollary 1 for the boundary cases. On the one hand we consider a market with only hedgeable risk and on the other hand we consider a market with purely unhedgeable risk. The first case yields the complete market setting which is an assumption that is often made in the literature. The characteristics that go along with completeness enlarge the analytically and numerically solvability of many problems, such as option pricing. However, this assumption lacks features that are present in practice. The imperfection of economic and financial markets, due to transaction costs or illiquidity, makes the market incomplete. Financial instruments that are linked to such nontraded underlyings still need to be quantified in order to calculate the present value. This is of high importance to insurance companies and pension funds, whose contracts are based on extremely long-dated interest rates, on mortality rates and on probabilities of natural hazards among others. Moreover, additional to incompleteness, the specification and assumptions of the model that describe the process of the assets contain uncertainty. The two most extreme cases we can encounter is either uncertainty with respect to purely hedgeable risk or uncertainty with respect to purely unhedgeable risk. In § 6 we consider two applications that incorporate uncertainty in a mixed setting.

5.1. Pure hedgeable risk. Financial theory tells us that pricing in a complete market setting should reduce to pricing under the risk-neutral measure. Hence, uncertainty should vanish under the completeness assumption. This is confirmed for our setting.

In a market where only hedgeable risk is present, all risk is traded. This is what we call a complete market setting. For illustrative purpose of Theorem 1, we assume the bank account to be driven by $dX_0 = rX_0 dt$, where r is constant. If we assume that there is no unhedgeable risk in the market, then $l = 0$. The indifference pricing operator is given by the linear PDE

$$\partial_t \pi + \partial_X' \pi r X + \tfrac{1}{2} \mathrm{tr}\left(\partial_{XX} \pi \Sigma^{XX}\right) - r\pi = 0. \tag{46}$$

Note that the drift of the process X is r, the constant interest rate, and not μ^X. Therefore the price of the replicating portfolio is known as the "risk-neutral price". By Corollary 1 the optimal hedging strategy is $\theta_X^* = \partial_X \pi$, which is the delta-hedge that perfectly replicates the derivative contract. The optimal robustness factors are $\epsilon_X^* = -q^X$ for the adjustments on all n traded assets. Based on merely two assumptions,

(i) an agent who wants to maximise his profit while (ii) he is uncertain about the underlying model; it follows that the optimal strategies lead to delta-hedging and risk-neutral pricing. Hence, the complete market setting leads to the elimination of ambiguity and the robust hedge replicates the derivative contract perfectly without specifying any probability measure upfront.

In the Black-Scholes economy, the vector of hedgeable processes is reduced to the one-dimensional traded stock $S(t)$. Examples of liabilities written on this stock are for instance vanilla options such as a call option, $L(T,S(T)) = \max(S(T)-K,0)$. The familiar Black-Scholes equation is equivalent to equation (46) and often expressed in the following notation

$$\frac{\partial L}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 L}{\partial S^2} + rS\frac{\partial L}{\partial S} - rL = 0, \tag{47}$$

where $\sigma^2 = \Sigma^{XX}$.

5.2. Pure unhedgeable risk. The other extreme is the case when there are no tradeable assets at the agent's disposal, then $n = 0$. This implies that there are no assets available to hedge the risk. Therefore the asset side will grow according to the rate r. The corresponding indifference price is given by the PDE

$$\partial_t \pi + \partial_Y' \pi \mu^Y + k\sqrt{\partial_Y' \pi \Sigma^{YY} \partial_Y \pi} + \tfrac{1}{2}\mathrm{tr}\left(\partial_{YY}\Sigma^{YY}\right) - r\pi = 0. \tag{48}$$

By definition one cannot trade unhedgeable risk. Together with the absence of traded assets, except the presence of the bank account, the only admissible strategy is $\theta_X = 0$ and $\theta_0 = 1$. The response of mother nature is $\epsilon_Y^* = \frac{k}{\sqrt{\partial_Y' \pi' \Sigma^{YY} \partial_Y \pi}} \Sigma^{YY} \partial_Y \pi$. The drift adjustment ϵ_Y^* is the adjustment of the drift of the unhedgeable process (μ^Y) in the "prudent" direction. Mother nature adds a penalty term to the drift proportional to the standard deviation of the unhedgeable risk multiplied by the ambiguity specification k. The adjusted drift is equivalent to the robust pricing method that is known as "actuarial pricing". In a one-dimensional setting the drift under the robust measure is $\mu^Y \pm k\sqrt{\Sigma^{YY}}$ where the plus and minus depend on the direction of the agent's preference. In a long position the worst case is accomplished by a negative relation and in a short position robustness is obtained by an increase in the drift. The agent typically has a short position, since π plays the role of a liability contract. In other words, he has the obligation to pay at time T and thus a positive relation determines here the robust price of the liability.

6. Examples. If there is only one risk factor of each type and the partial derivative $\partial_y \pi$ is either monotonically increasing or decreasing, then the nonlinear driver becomes linear because the absolute signs of $\sqrt{\partial_y' \pi S \partial_y \pi}$ can be replaced by $\pm\partial_y \pi$ depending on the sign of $\partial_y \pi$ and the position of the agent. The PDE becomes linear and we can express the solution, using the Feynman-Kaç formula

$$\pi(t,x,y) = e^{-r(T-t)}\mathbb{E}^{\mathbb{L}^*}\left[L(T,x,y)|t,x,y\right], \tag{49}$$

where $\mathbb{L}^*$ is the measure with the adjusted means for both risk factors, i.e. $r \cdot x$ for the x process and $\mu_y + \epsilon_y$ for the y process. Since $n = 1$ and $l = 1$ we denote the vector X and Y which are one-dimensional random variables by x and y. The terminal condition is $\pi(T,x,y) = L(T,x,y)$. In the general multidimensional case, the optimal measure $\mathbb{L}^*$ can be interpreted as the intersection of the ellipsoid and the risk-neutral measures $\mathscr{L} \cap \mathscr{Q}$, corresponding with the inf-convolution of Barrieu and El Karoui [3]. By specifying the ellipsoid this intersection determines the optimal solution.

The first example is based on an underlying asset that can be quite general, from real estate to weather indicators. This generality goes beyond the asset-liability management of insurance companies or pension funds which is illustrated by the second example as an insurance contract.

6.1. Correlated risk. Consider a nontraded asset that is correlated with a traded asset. Assume there is a risky asset x, the bank account x_0 and a nontraded asset y with the following dynamics

$$d\begin{bmatrix} x(t) \\ y(t) \\ x_0(t) \end{bmatrix} = \begin{bmatrix} \mu_x x(t) \\ \mu_y y(t) \\ r x_0(t) \end{bmatrix} dt + \begin{bmatrix} \begin{bmatrix} \sigma_x^2 x(t)^2 & \rho\sigma_x x(t)\sigma_y y(t) \\ \rho\sigma_x x(t)\sigma_y y(t) & \sigma_y^2 y(t)^2 \end{bmatrix}^{1/2} \\ 0_{[1\times 2]} \end{bmatrix} d\begin{bmatrix} W^x \\ W^y \end{bmatrix}. \tag{50}$$

Both the traded and nontraded asset are assumed to follow a geometric Brownian motion and these processes are correlated by ρ. In many financial optimisation problems, such as the Merton portfolio problem, an agent is postulated who maximises expected utility from terminal wealth and/or intertemporal consumption. His control variables are how much of his wealth he allocates to risky assets, and depending on the objective, the remainder is invested on the bank account in return for the riskfree rate or a part of his wealth is consumed. In the classical Merton set-up the agent is assumed to have power utility and he is certain about the geometric Brownian motion model specification for the stock price process. The results have been extended in many directions, including alternative preference specifications, more general asset dynamics, the inclusion of parameter or model uncertainty, incorporation of mortality and default risk, and allowing for incomplete markets.

With this financial example we illustrate the effect of uncertainty in the drifts of both the traded and nontraded asset. In a complete market the fair valuation of products is calculated by the risk-neutral probabilities deduced from the available market prices. Contrary to this, in an incomplete market there are products which rely on risk factors that are not liquidly available in the market. Moreover, in an incomplete market the risk-neutral measure is no longer unique. Therefore uncertainty arises naturally among the risk-neutral measures. We have shown that uncertainty about purely hedgeable risk does not survive since the worst-case measure picked by mother nature leads to the risk-neutral pricing measure. Note that the admissibility Assumption 2 needs to be fulfilled to generate this result. If the assumption that the interest rate is inside the uncertainty set is met, then it is upfront never optimal to go extremely long or short in the risky asset since the uncertain return can be lower than the interest rate. Hence, the assumption on the amount and direction of uncertainty confirms a natural reasoning pattern. This shows the well-known pricing risk-neutral pricing rule, in which the drift in the Black-Scholes formula is replaced the riskfree rate. More formally, by Theorem 1 we know that the process of the risky asset will be driven by the interest rate, i.e. the drift term changes from μ_x to r due to $\epsilon_x = -(\mu_x x - r x)$. However the ambiguity remains for the uncertain drift of the unhedgeable risk process and is implied by the two dimensional ellipse $\epsilon'\Sigma^{-1}\epsilon \le k^2$. There is no uncertainty left in the x process, the optimal drift adjustment ϵ_x corresponds to the market price of risk. While the unhedgeable process depends on the uncertainty factor k. The adjusted drift term of the unhedgeable risk can be expressed in term of the constant k and the market price of risk ϵ_x.

Due to the uncertainty the pricing operator is nonlinear, which is captured by the driver

$$g(t,z,\pi,\partial_z\pi\Sigma) \;=\; r(t,x_0)\pi(t) \pm c|\partial_y\pi(t)|\sqrt{y^2(t)\sigma_y^2(1-\rho^2)}. \tag{51}$$

Therefore the drift adjustment ϵ_y determines two prices that can be interpreted as either going long or short or as a bid-ask price. Madan and Cherny [36] introduced the field of "conic finance", in which they extend the law of one price for transaction costs. The bid and ask price that is present in an incomplete market is modelled by a convex cone that contains all nonnegative variables. Depending on the position of the agent, i.e. a short or long position, the worst-case driver has either a positive or negative dependence on $\partial_y\pi(t)$. The driver determines the robust price of the nontraded or illiquid asset and the $\pm$-sign is determined by the prudential direction. Or stated differently, we obtain both a robust ask and a robust bid price. By Corollary 1 the optimal drift adjustments are

$$\begin{bmatrix} \epsilon_x^* \\ \epsilon_y^* \end{bmatrix} = \begin{bmatrix} -\mu_x x(t) + r x(t) \\ -\rho\sigma_y y(t)\frac{\mu_x - r}{\sigma_x} \pm \sigma_y y(t)\sqrt{1-\rho^2}\sqrt{k^2 - \left(\frac{\mu_x - r}{\sigma_x}\right)^2} \end{bmatrix}. \tag{52}$$

The optimal hedge position is

$$\theta_x^* = \partial_x \pi + \partial_y \pi \frac{\sigma_y y(t)}{\sigma_x x(t)} \rho \pm \partial_y \pi \frac{\sigma_y y(t)}{\sigma_x x(t)} \sqrt{\frac{1-\rho^2}{k^2 - \left(\frac{\mu_x - r}{\sigma_x}\right)^2}} \frac{\mu_x - r}{\sigma_x}, \tag{53}$$

where the first term $\partial_x \pi$ is the delta-hedging part linked to purely hedgeable risk. The second term $\partial_y \pi \frac{\sigma_y y(t)}{\sigma_x x(t)} \rho$ implies delta-hedging for the unhedgeable risk weighted by the relative standard deviations. And the last term is the product of the residual of the second term, the unhedgeable risk, $\partial_y \pi \frac{\sigma_y y(t)}{\sigma_x x(t)} \sqrt{1-\rho^2}$ and $\left((\mu_x - r)/\sigma_x\right) / \sqrt{k^2 - \left((\mu_x - r)/\sigma_x\right)^2}$. Note that latter term goes to infinity when the market price of risk approaches k, while for small market prices of risk it is approximately the market price of risk scaled down by a factor k. Thus for market prices of risk that are approaching the ones that are "too good to be true" the agent invests huge amounts in the underlying asset.

The robust price of the nontraded asset can be characterised by the stochastic differential equation

$$dy(t) = \left(\mu_y - \rho \sigma_y \frac{\mu_x - r}{\sigma_x} \pm \sigma_y \sqrt{1-\rho^2} \sqrt{k^2 - \left(\frac{\mu_x - r}{\sigma_x}\right)^2}\right) y(t) dt + \tag{54}$$

$$\sigma_y y(t) \left(\rho dW^x + \sqrt{1-\rho^2} dW^y\right). \tag{55}$$

6.2. Life insurance contract. Consider a life insurance contract with one traded and one nontraded asset. In this unit linked contract, the survivors receive the value of the stock at time T bounded by a minimum guarantee g. In this case $n = 1$, $l = 1$ and S is the stock price that follows a lognormal distribution. For ease of exposition, N, the number of survivors in the policy, also follows a lognormal distribution[5]. The stochastic processes are

$$d \begin{bmatrix} S(t) \\ N(t) \end{bmatrix} = \begin{bmatrix} \mu S(t) \\ \nu N(t) \end{bmatrix} dt + \begin{bmatrix} \sigma S(t) & 0 \\ 0 & \beta N(t) \end{bmatrix} \begin{bmatrix} dW^S \\ dW^N \end{bmatrix}.$$

We assume no correlation between the two processes, implying that we assume that there is no causal relation between the price of the stock and the number of survivors. Since the liability the insurer faces is

$$L(T, S(T), N(T)) = \max(S(T), g) N(T),$$

it follows that $\partial_N \pi > 0$ at time T. Therefore π is monotone increasing in N and consequently the PDE is

$$\partial_t \pi + \partial_S \pi r S(t) + \partial_N \pi \left(\nu N(t) + \sqrt{k^2 - \frac{\left(\mu S(t) - r S(t)\right)^2}{\sigma^2 S^2(t)}} \sigma_N N(t)\right) + \tfrac{1}{2}\left(\partial_{SS} \pi \sigma^2 S^2(t) + \partial_{NN} \pi \beta^2 N^2(t)\right) - r\pi = 0.$$

By Theorem 1, the optimal hedging portfolio is

$$\theta_x^* = \partial_S \pi + \partial_N \pi \frac{\beta N(t)}{\sigma S(t)} \frac{(\mu - r)/\sigma}{\sqrt{k^2 - \left(\frac{\mu - r}{\sigma}\right)^2}}. \tag{56}$$

Similar as in the previous example the optimal hedging portfolio consists of a delta-hedging part. And now that there is no correlation between the hedgeable and unhedgeable risk, the investment in the traded asset proportional to the correlation is zero. The second term of the hedge position is the market

[5] Assumption of lognormality for ease of exposition. Can be unrealistic since non-zero probability of $N(t) < N(t+dt)$. However, if the drift is negative (enough) this probability becomes small. The realistic counterpart with decreasing number of survivors also leads to Lipschitz continuity of the terminal condition. Which can be obtained by a change of variables to $\max(\tilde{S}(T), g \cdot N(T))$ where $\tilde{S}(t) = S(t) \cdot N(t)$.

confidence term, proportional to the relative standard deviations. The larger the uncertainty set, the less aggressive the insurance company hedges the unhedgeable risk by investing in the tradeable assets. And vice versa, if the market price of risk becomes large, investing in the traded asset is more and more profitable.

The robust price of the life insurance contract is $\pi(t, S(t), N(t))$. By solving the PDE we can find the function of $\pi(T-dt, S(t), N(t))$ and its derivatives recursively until time t. The solution can be written as a conditional expectation by the Feynman-Kaç formula

$$\pi(t, S(t), N(t)) = e^{-r(T-t)}\mathbb{E}^{\mathbb{L}^*}\left[\pi(T, S(T), N(T)) | S(t) = s, N(t) = n\right]. \tag{57}$$

Note that the expectation is taken under the probability measure $\mathbb{L}$ that belongs to the adjusted risk-neutral drift of the hedgeable component and the additional prudence factor of the unhedgeable part. Measure $\mathbb{L}^*$ belongs to the processes having drifts:

$$\begin{cases} rS(t) \\ \nu N(t) + \beta N(t)\sqrt{k^2 - \left(\frac{\mu - r}{\sigma}\right)^2} \end{cases}. \tag{58}$$

The interpretation of risk measure $\mathbb{L}^*$ is that the financial risk is perfectly replicated such that the ambiguity is eliminated and the drift is replaced by the riskfree rate. The ambiguity of the unhedgeable process is now the intersection of the ambiguity set $\mathscr{L}$ and the line r. The intersection of this line and the ellipsoid has two solutions, corresponding with the sign of the liability function. In this case the zero correlation between the two processes causes the ellipsoid to be an exact circle.

For this example, we can solve the conditional expectation analytically

$$\begin{aligned} \pi(t, S(t), N(t)) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{L}^*}\left[\max(S(T), g)N(T)|\mathscr{F}_t\right] \\ &= \left(S(t)\Phi(d_1) - e^{-r(T-t)}g\Phi(d_2) + e^{-r(T-t)}g\right)\cdot N(t)e^{\left(\nu+\beta\sqrt{k^2-\left((\mu-r)/\sigma\right)^2}\right)(T-t)}, \end{aligned} \tag{59}$$

where $d_{1,2} = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{S(t)}{g}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right)$.

7. Conclusion. For an agent (i) who wants to maximise his expected surplus, and (ii) who is uncertain about the modelled economy, we obtain the pricing rule described by the semilinear PDE for the pricing operator.

Firstly, model ambiguity in a complete market leads to risk-neutral pricing. Mother nature minimises the objective of the agent to ensure robustness. When all asset are hedgeable, mother nature's choice eliminates model ambiguity. The agent replicates the derivative contract perfectly by the delta-hedge.

Secondly, when there are only unhedgeable risk factors and no traded assets in an economy, then model ambiguity results in the action of "actuarial pricing". This is a conservative way of pricing where the uncertain drift is adjusted in the "prudent" direction. The penalty term that mother nature adds is proportional to the standard deviation of the unhedgeable risk multiplied by the amount of ambiguity (the size of the ellipsoid). Thus the larger the volatility or the larger the initial uncertainty, the larger becomes the drift adjustment that ensures a robust price of the contract the agent is liable to.

Thirdly, in a model with both hedgeable and unhedgeable risk the agent will price market-consistently and actuarial prudentially. The traded risky assets are priced by the interest rate, whereas the drift adjustments of the nontraded assets are twofold. The drift adjustment can be either negative or positive. The prudent direction depends on the payoff structure of the agent and can be interpreted as the bid or the ask price. Interpreted differently, the worst-case adjustment has either a positive or a negative impact on the price of the liability. The robust relation depends on whether the agent has a short or long position.

The optimal investment strategies that drive the hedging portfolio consists of (i) the delta-hedge linked to hedgeable risk, (ii) the delta-hedge linked to unhedgeable risk proportional to the correlation between the traded and nontraded assets and their standard deviations, plus (iii) the product of the residual of the correlated delta-hedge of the unhedgeable risk and a market confidence term. The market confidence term has as effect that the investment in risky assets goes to infinity when the market prices of risk become "too good to be true", while low market prices of risk lead to an additional term of the market price of risk itself scaled down by the size of the uncertainty set.

For some special cases we can solve the pricing semilinear PDE explicitly, as shown by the given examples. Moreover, we prove existence and uniqueness of the robust price of the liabilities by connecting the optimal semilinear PDE to a FBSDE with a Lipschitz driver. The classical solution of the PDE also proves optimality of the robust investment strategies.

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