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Pricing Risky Corporate Debt Using Default Probabilities

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by
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Abstract

We find that a momentum trading strategy can be improved by using the default probabilities that our model proposes. We determine default probabilities by modeling the total assets and liabilities by stochastic differential equations and we define the event of default to occur when total assets are lower than total liabilities. We observe that the bond prices of our model are rather sensitive to the 'flat' default rate assumption when compared to actual market prices. Our results provide an indication that the difference between 'risk-neutral' and 'real-world' default probabilities of a firm is based on the correlation of the asset ratio of that firm with the general market. We also find that the effect of a general market factor is not as influential on every firm as we initially expected.

KEYWORDS: STRUCTURAL MODEL, RISK-NEUTRAL DEFAULT PROBABILITIES, REAL-WORLD DEFAULT PROBABILITIES, RECOVERY RATE, RISKY CORPORATE BONDS, MOMENTUM INVESTING

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Contents

1 Introduction

We will focus on modeling risky corporate bonds. Corporate bonds are clearly affected by the default risks of the firms that underwrites the contracts. For the buying side of the bond market it is of course important to know these default probabilities. We will therefore propose a model to compute the default probabilities of a firm. The default probability of a firm may well be higher during a down-state of the economy respectively to an up-state, even if the fundamentals of the firm itself are not different. As one observes in any crisis, like for example the global financial crisis of 2008, defaults may trigger new defaults, which will cause default probabilities to rise even if the fundamentals of a firm are not affected. The default of a firm is very likely to introduce an anxious period on the market or on the sector of that firm. Such a default increases the doubts firms have over the creditworthiness of firms in general. For firms that already had a bad creditworthiness, this downstate of the market could cause them to not obtain any new loans, which causes their default probabilities to rise. Once a firm defaults, a bondholder is just another investor trying to get some of the promised payments back from the liquidator. The cash the investor eventually receives will be substantially less than the payment that was promised by the contract initially. It is therefore very important for an investor to be able to oversee the risks involving these financial contracts.

We already noted that the state of the economy and the state of other firms could influence the default probabilities of a firm. It is therefore not only relevant to consider the firm as just some individual entity, but one should also consider how other firms affect the firm and how it is affected by the general state of the economy. In this thesis we will therefore construct a model of the firm that takes into account the impact the state of the general economy and also the impact other firms in the market have on the firm.

For an investor operating in the bond market, default probabilities of corporations are the key elements in his portfolio or trading analysis. A firm that is defined by the market as risky will have to promise a higher return to persuade investors to buy his bonds despite the higher risk. When knowing the returns of various corporate bonds, it is of course important for an investor to know the risks that are involved with these bonds such that he is able to compare them and construct a portfolio. We will construct a model for the default probabilities which could be used in allocating a portfolio. Of course an investor should also consider the probability of multiple defaults or of defaults by contagion. These default probabilities are not part of this thesis, but they may be derived using the model and results of this thesis.

In this thesis we will compute the prices of corporate bonds as given by our model with parameters estimated on market data. Given our assumptions, we have that the default probability is the only risk involved in a financial contract with a firm. We are therefore able to compute the corporate bond prices. We will use our model by setting the parameters to empirically reasonable values and compare the results with actual market data. Even if the bond prices, as computed by our model, are not equal to the market prices, we will still use the default probabilities to show that they can be useful for improving a trading strategy.

In the theoretical section we are able to construct a rather simple formula for the price of a zero-coupon bond based on the underlying default probabilities. In the empirical section we observe the sensitivity of some of the bond prices to the parameters that we estimated on the actual market data. We find that for some firms the model computes default probabilities which are nearly negligible even for maturities far into the future. Many researchers actually argue that this is empirically not very sensible. We find that both extremes of the cumulative default distribution should be less extreme by comparing it with the market data. In our research we also observe the large influence of the 'flat' default-free interest rate assumption, which causes that one value might explain bonds with either long or short maturities, but it fails to perform reasonably well for both at the same time.

In section two we will first give a summary of research that is very relevant to the subject of this thesis. Section three lays the groundwork for the model which is specified in section four. Section five offers the details about the estimation procedure that we use to fit our model with actual market data and also gives two short examples. In section six we estimate the parameters we need and simulate our model using these estimates to compute the price of corporate bonds. Section seven shows some examples of improving a momentum strategy by using default probabilities, section eight summarizes our results and section nine concludes and provides some ideas for future research.

2 Overview of Literature

In the literature we find many papers that are relevant to our paper. Let us discuss some papers that modeled default probabilities in various different ways. From these papers we observe that default probabilities are useful for several practical issues as computing the prices of corporate bonds. In modeling default probabilities there are two common approaches in the literature. The structural approach, which has been done in many papers including Merton (1974), and Longstaff and Schwartz (1995), and the reduced form approach which has been used in for example the paper by Duffie and Singleton (1994).

Researchers using the structural approach compute default probabilities by defining some kind of firm value and define some threshold for which the firm defaults. Merton (1974) defines a stochastic differential equation to represent the firm value. Other papers define the value of the firm implicitly by defining the assets and liabilities. This is done by for example Valuzis (2008), and this will be done in our model as well. We consider this to be a more intuitive approach which will lead to a more intuitive definition of the default of a firm. Merton also defines both the value of the equity of a firm and the value of debt, where he notes that the value of equity is represented by a call option on the debt of the firm. In our paper we will not mind the value of equity and only focus on the value of a bond issued by the firm, which we link to the default probability of the firm. Merton uses a contingent claim analysis to derive the value of debt, just like for example Chance (1990), who uses a model similar to that of Merton in order to investigate the effects of default risk on the duration of zero coupon bonds. In his paper, Merton, also states clearly the assumptions he makes to construct his model. These assumptions include for example a flat interest rate curve for the default-free interest rate. He also assumes that the market in his model is free of arbitrage. By this assumption a portfolio with an initial investment of zero should also have a expected return of zero. Merton considers the default of a firm only at the maturity of the bond. At maturity the holder of the bond finds out whether the firm is still operating or if it has defaulted at some early point in time. This is a rather big simplification of reality. In our model we will simulate stochastic differential equations and allow firms to default at any time. In the simulations we will use Euler Discritizations and therefore firms may default within small time-steps and the effects will be known immediately after the small time-step has ended.

The paper by Merton is one of the building blocks for the structural approach of computing default probabilities and constructing models for the price of corporate bonds. Most of the assumptions he states in his paper are also used by other researchers to build their model upon. Longstaff and Schwartz (1995) also used a structural approach. They basically extended the model of Black and Cox (1976), by adding interest rate risk. In their approach they also allowed for deviation of the strict absolute priority of bonds. In our case we will only have one bond which is the zero-coupon bond. This representation will be used in an empirical section to price a coupon-paying bond, where we do not take into account the priority of bonds. Longstaff and Schwartz realized that the credit spreads for similar firms, with respect to default risk, can vary significantly due to the correlations of the assets of a firm with interest rates. Unlike Merton (1974), Longstaff and Schwartz model the interest rate by the Vasicek model. Longstaff and Schwartz did not agree with the default probabilities predicted by the model by Merton for short term maturities of firms. These probabilities where almost zero, which is empirically not very likely according to Longstaff and Schwartz. They state that their model resolves this issue and they eventually compute credit spreads that are more in line with the observed spreads.

There are of course also a lot of papers written that use the reduced form approach. In contrast to the structural model, the reduced form models are not based on any representation of the firm value. The researchers using this approach argue that the event of default is very complicated to define and is therefore easily misspecified. The reduced form approach therefore estimates some rate of default and then models the event of default by using an arrival process like a Poisson process. For example Fons (1994) used a marginal default rate, which he estimated based on the statistics of defaults by Moodys. He concludes that default rates are not historically stable, which is an important notion for the reduced form approach.

Another paper using the reduced form approach is the paper by Duffee (1999). Duffee actually models the instantaneous default rate by a square root diffusion process. He also allows for correlation between the default probability and the default-free interest rate. He then uses the extended Kalman filter to fit the yields of the bonds on the model that he proposes. The fitted prices of the model by Duffee are rather close to market prices, although the model implies that the volatility of the instantaneous default risk follows a square root process, which is not supported by the data.

One of the most important and often quoted papers of the reduced form approach is the one by Duffie and Singleton (1999). In their paper they use a default adjusted short rate. This rate is defined as $R = r + h_t L_t$, where h_t represents the hazard rate and L_t represents the fractional loss in case of default. The approach of using this default adjusted short rate was already used by Duffie and Singleton (1997) and also again by Dai and Singleton (2002), but those papers focused directly on R itself. In Duffie and Singleton (1999), the hazard rate, the fractional loss and also the default free interest rate are parameterized separately. In their paper they eventually define $s_t = h_t L_t$, because they did not manage to compute them separately, which they refer to as the risk-neutral mean loss rate. They eventually use their model to price credit derivatives.

Zhou (2001) wrote an interesting paper where he constructed a model by combining the two approaches, which had also been done by Madan and Unal (2000). The models in these paper use a structural approach, but also model some shocks by a Jump diffusion process. Zhou (2001) constructs a model which combines what he defines as expected defaults of the structural approach with the unexpected defaults of the reduced form approach. He uses these terms because he argues that the default caused by a Brownian Motion hitting some value can be expected to some extent by the slow decay of the process towards the default threshold. In his model he also allows for different recovery rates among the various bonds of a firm. Zhou (2001) concludes that adding the jump component makes the model much more flexible and enables the model to generate much more shapes for the term structure of credit spreads than other structural models. We should note that in the paper he did not test his model on empirical data.

In this thesis we will not use the reduced form approach but the structural approach. A disadvantage of the reduced form approach is the practical application. If one uses a 'hazard rate to model the defaults of a set of firms, one implicitly assumes that every company in that set is identical. Or, if one estimates a hazard rate on a data set, one assumes the default distribution of all those firms to be equal and also representative for the firms on which one uses the model. In practice one observes a lot of different companies with very different growth figures or asset to liability ratios. It is a rather big simplification to assume that many firms are equal with respect to the probability of default. In our paper we use the structural approach and model both the asset and the liabilities of a firm and define the event of default as the moment when the total amount of assets drops below the total amount of liabilities. This causes the default probability distribution to be very different among the firms. The moment of default is usually defined by some fixed default threshold K, as is also done by Black and Cox (1976), Longstaff and Schwartz (1995), and Zhou (2001). The firm is then assumed to default instantaneous when the firm value drops below K. Our default rate is based on the assets of the firm and not the firm value. If we define the firm value as the assets divided by the liabilities, or by some equivalent measure, we also have a fixed value for K. In our case the 'K' is just one. We will use a fixed recovery rate, which is a proportion of the promised payments an investors receives after the default of a firm.

The papers mentioned above considered some extension to model of the firm. Some extended the modeling of the firm value by allowing the default-free rate to be other than constant and the term-structure to be 'flat'. Longstaff and Schwartz (1995), for example, modeled the interest rate by a Vasicek model. Zhou (2001) extended the model by introducing shocks in the process of the firm value represented by a jump-diffusion process. We have not seen a paper where the authors extend the model by a general market term or any other common factor. It could be that a set of firms have an exposure to a more general external market which is very influential on all the firms combined. We will present a model of multiple firms and define a general market where we compute the effect of this general market on the assets and liabilities of the firms in the model. A bad performing market could explain relative high default probabilities and vice versa. Of course one could consider many other factors than just the market that we use, but we will leave this to future research.

3 General Setting of the Model

3.1 Model

In general there are many ways a firm may default, but in the end it is caused by the firm not being able to repay liabilities. We will make the assumption that a firm defaults in case the total assets are smaller than the total liabilities of the firm. The model of the firm will therefore just take into account the total assets and the total liabilities. In fact we will use the total assets and liabilities as stated on the annual report of a firm for our model and during the thesis we will just refer to them as the total assets and total liabilities of the firm. As we observe from annual reports the total assets and total liabilities of a firm fluctuate over time. Both the assets and the liabilities will therefore be modeled by means of some stochastic differential equation (SDE). With the SDEs we will also simulate monthly values for the assets and liabilities. In our model we assume that the assets as well as the liabilities are also affected by some common factor which we define as the general market factor. We assume the general market factor is characterized by a Geometric Brownian Motion. The general market factor will be just a market index. It is very common to model stock prices with a Geometric Brownian Motion, and because an index is some collection of stocks we model it also by a Geometric Brownian Motion. The risky component of this SDE is also present in the SDEs of the total assets and liabilities. The SDEs of both the total assets and the total liabilities consist of two risky components which we will refer to as 'individual' and 'general market' risk. Besides the risky components, both SDEs also have a drift term. Using the assets and the liabilities of the firm, we define the asset ratio as being the total assets divided by the total liabilities, which is in fact just the inverse of the well-known debt ratio. We assume the company will default when the asset ratio drops below one, which is equivalent to the total liabilities being larger than the total assets. Defining the default event to occur when the asset ratio drops below one is rather intuitive and could be considered to be naive. One is not able to determine the actual market value of the total assets and liabilities of a firm and this might cause the default event of the firm to not coincide with an asset ratio dropping below one. Our model will assume a default threshold of the asset ratio equal to one, but we will also investigate this assumption. By using the SDE of both the assets and the liabilities we derive the SDE of the asset ratio. With the SDE of the asset ratio, we are able to determine the probability that the firm defaults before some time in the future given the state of the funding rate at the current time zero. With the default probabilities we determine the price of a corporate bond of this specific firm.

We assume that all companies can be characterized by the same model, with the only difference being their specific parameter values. The full model will consist of multiple firms of which the individual risk components are likely to have a nonzero correlation with each-other, which makes it interesting to model the market as a whole and not just the firms individually.

3.2 Assumptions

In the analysis we will make use of the first fundamental theorem of asset pricing[1], which is stated as follows:

First Fundamental Theorem of Asset Pricing. [2]
The market as specified by an objective ("real-world") probability measure $\mathbb{P}$ and a collection of asset price processes $\{Y_i\}_t$ (i = 1,$\cdots$,m) is free of arbitrage if and only if, given any numéraire N, there is a measure $\mathbb{Q}_N$ (depending on N) which is equivalent to the objective measure $\mathbb{P}$, and which is such that all relative price processes $(Y_i)_t/N_t$ are $\mathbb{Q}_N$-martingales.

We also state an important theorem which we will use repeatedly and which is derived using both the Girsanov Theorem (Girsanov, 1960) and the First Fundamental Theorem of Asset Pricing.

Theorem 1. *Assume we have a financial market which is modeled by the following equations,*

$$\mathrm{d}X_t = \mu_X(t, X_t)\mathrm{d}t + \sigma_X(t, X_t)\mathrm{d}W_t$$
$$Y_t = \pi_Y(t, X_t).$$

where X_t represents the state variables and Y_t is any traded asset in the model. Then the model allows for no arbitrage if and only if there exists a function $\lambda = \lambda(t, x)$ and a scalar function $r = r(t, x)$ such that

$$\mu_Y - r\pi_Y = \sigma_Y \lambda$$

Proof. See Theorem 3.4.1 (Girsanov, 1960) and Theorem 4.2.3 of Financial Models by J.M. Schumacher, 2014. □

Recall that two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be equivalent if any event that has positive $\mathbb{P}$-probability also has positive $\mathbb{Q}$-probability, and vice versa. In our analysis we will use a default-free bond B_t as numéraire and we therefore will use the risk neutral measure[3] $\mathbb{Q}_B$. To simplify notation we will just write $\mathbb{Q}$ instead of $\mathbb{Q}_B$ below. We will also incorporate some rather general assumptions, which are in fact Assumptions A.1 through A.7 as stated in Merton (1974):

A.1 there are no transactions costs, taxes, or problems with indivisibilities of assets.
A.2 there are a sufficient number of investors with comparable wealth levels so that each investor believes that he can buy and sell as much of an asset as he wants at the market price.
A.3 there exists an exchange market for borrowing and lending at the same rate of interest.
A.4 short-sales of all assets, with full use of the proceeds, is allowed.
A.5 trading in assets takes place continuously in time.

[1]For details see Delbaen, F.Y., & Schachermayer, W. (1994).
[2]This formulation can be found in Financial Models, page 66. by J.M Schumacher, 2014
[3]For details see Delbaen, F. Y., & Schachermayer, W. (1995).

A.6 the Modigliani-Miller theorem that the value of the firm is invariant to its capital structure obtains.

A.7 the term-structure is “flat” and known with certainty. I.e., the price of a riskless (default-free) discount bond which promises a payment of one dollar at time τ in the future is $P(\tau) = e^{-r\tau}$, where r is the instantaneous riskless rate of interest, the same for all time.

4 Multiple firm model

In this section we will introduce a multiple firm model. We will start by defining an individual firm in the system. We already made the assumption that all firms are modeled with the same model. The total model will therefore just consist of a collection of firms modeled according to the model of one firm, but then we also allow for correlations between the individual risk components of any two firms.

4.1 One Firm

Let us first start by modeling any firm in the system and then extend it to a model of multiple firms. A firm generally has assets and liabilities which they state on the balance sheet of the annual report. We model both time series of the total assets and liabilities by a SDE. One could rewrite this SDE as a Geometric Brownian Motion and we therefore have that the assets nor the liabilities can become negative. Negative amounts of total assets and liabilities are not feasible so it should hold that this does not occur in the model as well. The instantaneous correlation between the assets and the liabilities is denoted by $\rho(A_i, L_i)$, which we assume to be constant over time. Let us now first state the SDEs of both the assets and liabilities in the equations below and explain the signs in these SDEs afterwards. The index i just states that the parameter is specific for each firm.

$$\frac{\mathrm{d}A_{i,t}}{A_{i,t}} = \{r + \lambda_{A,i}\sigma_{A,i} + \lambda_G\beta_i\sigma_G\}\mathrm{d}t + \sigma_{A,i}\mathrm{d}W_{A,i,t} + \beta_i\sigma_G\mathrm{d}W_{G,t} \tag{1}$$

$$\frac{\mathrm{d}L_{i,t}}{L_{i,t}} = \{r + \lambda_{L,i}\sigma_{L,i} + \lambda_G\alpha_i\sigma_G\}\mathrm{d}t + \sigma_{L,i}\mathrm{d}W_{L,i,t} + \alpha_i\sigma_G\mathrm{d}W_{G,t} \tag{2}$$

In the equations above we represent the continuous changes of the total assets and total liabilities proportional to the level of the assets or liabilities, as a SDE. We let r denote the default-free interest rate, which is constant over time and also for any maturity. First we will assume this rate to equal two percent. We choose two percent because it is a common number for an interest rate in academics, but due to the actual year for which we use it we might consider an other value. We should also mind that we are using the same rate for all maturities, which is also why we will consider other values in a later section. Both equations are very similar and have a drift term and two stochastic terms. Both the total assets and the liabilities have a risky component subject to the general market factor, which is denoted by $\mathrm{d}W_{G,t}$. The riskiness of the general market is expressed by σ_G, what could be considered as the general market volatility. For both SDEs we then estimate the influence of the general market factor on the process. For the total assets we denoted this by β_i and for the total liabilities by α_i. The other risk component, which was already defined as the 'individual risk', is denoted by $\mathrm{d}W_{A,i,t}$ for the assets of firms i and by $\mathrm{d}W_{L,i,t}$ for the liabilities of firm i. Also for these sources of risk the risks are defined by respectively $\sigma_{A,i}$ and $\sigma_{L,i}$. Note that to identify (1), we should impose the restriction that the correlation between $\mathrm{d}W_{A,i,t}$ and $\mathrm{d}W_{G,t}$ is zero. Of course this should also hold for (2), implying that the correlation between $\mathrm{d}W_{L,i,t}$ and $\mathrm{d}W_{G,t}$ is zero. Note that $\mathrm{d}W_{L,i,t}$, $\mathrm{d}W_{A,i,t}$ and $\mathrm{d}W_{G,t}$ are Brownian Motions under the 'real-world'-measure. We consider the 'real-world' measure more or less to be the initial measure and therefore do not provide an extra superscript to represent it. Both SDEs also have some drift. The drifts for both SDEs are represented by the default-free interest rate r and some positive risk premium represented by the price of risk times the sources of risk corresponding to both Brownian Motions. The prices of risks can simply be derived by applying Theorem 1. For the sake of completeness we stated them below.

$$\lambda_G = \frac{\mu_G - r}{\sigma_G}, \quad \lambda_{A,i} = \frac{\mu_{A,i} - r}{\sigma_{A,i}}, \quad \lambda_{L,i} = \frac{\mu_{L,i} - r}{\sigma_{L,i}} \tag{3}$$

Note that the μ_G is indeed the drift of the Geometric Brownian Motion of G_t, but $\mu_{A,i}$ and $\mu_{L,i}$ are not that easily interpreted. This is simply caused by the assets having multiple sources of risk. One should consider $\mu_{A,i}$ as the 'individual' drift which is the residual drift of A_t after subtracting the drift caused by the dependence on the general market factor. The same story of course holds for $\mu_{L,i}$.

Now that we have stated the model of the firm we want to use the model for pricing bonds. The model above clearly specifies the SDEs of total assets and liabilities of a firm under the 'real-world' measure, but for pricing financial products we should take expectations with respect to the 'risk-neutral' measure. To be able to compute such expectations we should first specify the SDEs under the 'risk-neutral' measure. How to obtain the correct specification of both the SDE of the total assets and the SDE of the total liabilities under the 'risk-neutral' measure is stated in the next theorem.

Theorem 2. *Let the assets and the liabilities be denoted by (1) and (2) respectively. We then have by Theorem 1 that the assets and liabilities are denoted by the stochastic differential equations under the risk neutral measure $\mathbb{Q}$ below. Note that r is the default-free interest rate and $W^{\mathbb{Q}}_{A,i,t}$, $W^{\mathbb{Q}}_{L,i,t}$ and $W^{\mathbb{Q}}_{G,t}$ are standard Brownian Motions under the 'risk-neutral' measure.*

$$\frac{\mathrm{d}A_{i,t}}{A_{i,t}} = r\mathrm{d}t + \sigma_{A,i}(\mathrm{d}W_{A,i,t} + \lambda_{A,i}\mathrm{d}t) + \beta_i\sigma_G(\mathrm{d}W_{G,t} + \lambda_G\mathrm{d}t) = r\mathrm{d}t + \sigma_{A,i}\mathrm{d}W^{\mathbb{Q}}_{A,i,t} + \beta_i\sigma_G\mathrm{d}W^{\mathbb{Q}}_{G,t} \tag{4}$$

$$\frac{\mathrm{d}L_{i,t}}{L_{i,t}} = r\mathrm{d}t + \sigma_{L,i}(\mathrm{d}W_{L,i,t} + \lambda_{L,i}\mathrm{d}t) + \alpha_i\sigma_G(\mathrm{d}W_{G,t} + \lambda_G\mathrm{d}t) = r\mathrm{d}t + \sigma_{L,i}\mathrm{d}W^{\mathbb{Q}}_{L,i,t} + \alpha_i\sigma_G\mathrm{d}W^{\mathbb{Q}}_{G,t} \tag{5}$$

Proof. See Appendix A □

By modeling both the assets and the liabilities of the firm we define the asset ratio of the firm as $f_{i,t} := \frac{A_{i,t}}{L_{i,t}}$. We then use the SDEs stated in (1) and (2) to derive the SDE of the funding ratio explicitly. This derivation consist of applying the "Ito-rule" on the product of $A_{i,t}$ and $\frac{1}{L_{i,t}}$. Let us define the event of default by setting some default threshold for the asset ratio for which a firm defaults instantaneous if the asset ratio drops below that value at any point in time. This way of defining the event of default is similar to Longstaff and Schwarz (1995) and Zhou (2001). We implicitly assume that the asset ratio of an operating firm has some value at least larger than one.[4] Every firm subsequently has a asset ratio bigger than one in the initial state of the model. To determine the probability of default before some time t for the firm, we should use the SDE of the funding rate and determine the probability that it hits a value less than or equal to one before some time t in the future. The explicit expression for the SDE of the asset ratio where the sources of risk are defined under the 'real-world' measure and for the 'risk-neutral' measure can be found in the theorems below.

Theorem 3. *Assume the stochastic differential equations of both $A_{i,t}$ and $L_{i,t}$ are denoted by Equations 1 and 2, where the sources of risk are Brownian Motions under the 'real-world' measure. The stochastic differential equation of $f_{i,t} := \frac{A_{i,t}}{L_{i,t}}$ under the 'real-world' measure $\mathbb{P}$ is then denoted by:*

$$\frac{\mathrm{d}f_{i,t}}{f_{i,t}} = \left\{\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 + \lambda_{A,i}\sigma_{A,i} - \lambda_{L,i}\sigma_{L,i} + \lambda_G\sigma_G(\beta_i - \alpha_i) - \sigma_{A,i}\sigma_{L,i}\rho(A_i, L_i) - \alpha_i\beta_i\sigma_G^2\right\}\mathrm{d}t + \sigma_{A,i}\mathrm{d}W_{A,i,t} - \sigma_{L,i}\mathrm{d}W_{L,i,t} + \sigma_G(\beta_i - \alpha_i)\mathrm{d}W_{G,t} \tag{6}$$

Proof. See Appendix A □

Theorem 4. *Assume the stochastic differential equations of both $A_{i,t}$ and $L_{i,t}$ are denoted by Equations 4 and 5 where the sources of risk are Brownian Motions under the 'risk-neutral' measure. The stochastic differential equation of $f_{i,t} := \frac{A_{i,t}}{L_{i,t}}$ under the 'risk-neutral' measure $\mathbb{Q}$ is then denoted by:*

$$\frac{\mathrm{d}f_{i,t}}{f_{i,t}} = \left\{\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 - \sigma_{L,i}\sigma_{A,i}\rho(A_i, L_i) - \alpha_i\beta_i\sigma_G^2\right\}\mathrm{d}t - \sigma_{L,i}\mathrm{d}W_{L,i,t}^{\mathbb{Q}} + \sigma_{A,i}\mathrm{d}W_{A,i,t}^{\mathbb{Q}} + \sigma_G(\beta_i - \alpha_i)\mathrm{d}W_{G,t}^{\mathbb{Q}} \tag{7}$$

Proof. See Appendix A □

The probability of default at time t given all information at time zero, is determined by the probability of the asset ratio hitting the lower boundary one, i.e. $\mathbb{P}(f_{t|0} \leq 1)$. We should therefore determine the probability that the asset ratio hits the boundary before some time t given all information at current time zero. There is an explicit formula for the current probability of a Geometric Brownian Motion hitting a lower bound before some future time t, given that the current state is above that lower bound (Borodin & Salminen, 2002, p.612). Using this formula we are able to derive the probability of default, which is stated in the next theorem.

Theorem 5. *Assume that the stochastic differential equation of the asset ratio of firm i is represented by the stochastic differential equation given by Equation 6, where the risk components are Brownian Motions under the 'real-world' measure. Let τ_i be the time of default of firm i. We*

[4] In a later section we will set this value equal to 0.95 and compare the differences.

then denote the probability of default at some time t under the 'real-world' measure, given all information at current time zero of firm i by:

$$p_{i,t} = \mathbb{P}(\inf_{0\le s\le t} f_{i,s} \le 1) = \mathbb{P}(\tau_i \le t) = 1 - \Phi\left(\frac{bt+a}{\sqrt{t}}\right) + e^{-2ab}\,\Phi\left(\frac{bt-a}{\sqrt{t}}\right) \tag{8}$$

$$\text{where } a = -\frac{1}{\sigma_{f,i}}\log(\frac{1}{f_0}), \quad b = \frac{\mu_{f,i}}{\sigma_{f,i}} - \frac{1}{2}\sigma_{f,i}$$

Where $f_{i,t}$ is defined as a Geometric Brownian Motion with

$$\mu_{f,i} = \sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 + \lambda_{A,i}\sigma_{A,i} - \lambda_{L,i}\sigma_{L,i} + \lambda_G\sigma_G(\beta_i - \alpha_i) - \sigma_{A,i}\sigma_{L,i}\rho(A_i, L_i) - \alpha_i\beta_i\sigma_G^2 \tag{9}$$

$$\sigma_{f,i} = \sqrt{\sigma_{L,i}^2 + \sigma_{A,i}^2 + \sigma_G^2(\beta_i - \alpha_i)^2 - 2\sigma_{L,i}\sigma_{A,i}\rho(A_i, L_i)} \tag{10}$$

Proof. See Appendix A. □

Theorem 6. *Assume that the stochastic differential equation of the asset ratio of firm i is represented by the stochastic differential equation given by Equation 7, where the risk components are Brownian Motions under the 'risk-neutral' measure. Again τ_i denotes the time of default of firm i. We then denote the probability of default at some time t under the 'risk-neutral' measure, given all information at current time zero of firm i by:*

$$q_{i,t} = \mathbb{Q}(\inf_{0\le s\le t} f_{i,s} \le 1) = \mathbb{Q}(\tau_i \le t) = 1 - \Phi\left(\frac{bt+a}{\sqrt{t}}\right) + e^{-2ab}\,\Phi\left(\frac{bt-a}{\sqrt{t}}\right) \tag{11}$$

$$\text{where } a = -\frac{1}{\sigma_{f,i}}\log(\frac{1}{f_0}), \quad b = \frac{\mu_{f,i}}{\sigma_{f,i}} - \frac{1}{2}\sigma_{f,i}$$

Where $f_{i,t}$ is defined as a Geometric Brownian Motion with

$$\mu_{f,i} = \sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 - \sigma_{L,i}\sigma_{A,i}\rho(A_i, L_i) - \alpha_i\beta_i\sigma_G^2 \tag{12}$$

$$\sigma_{f,i} = \sqrt{\sigma_{L,i}^2 + \sigma_{A,i}^2 + \sigma_G^2(\beta_i - \alpha_i)^2 - 2\sigma_{L,i}\sigma_{A,i}\rho(A_i, L_i)} \tag{13}$$

Proof. Equivalent to the proof of theorem 5. □

In both theorems above we have stated that we define $f_{i,t}$ as a Geometric Brownian Motion with some drift and some volatility. One might have already observed that the drift stated in Equation 12 is very similar to the drift stated in Equation 9. In fact the only difference is that the terms containing a price of risk λ have disappeared in Equation 12. The change of measure is in fact only a change of drift, because the new sources of risk are in fact again Brownian Motions under the 'risk-neutral' measure. If it then holds that $\lambda_{A,i}\sigma_{A,i} + \beta_i\lambda_G\sigma_G > \lambda_{L,i}\sigma_{L,i} + \alpha_i\lambda_G\sigma_G$, the canceling terms cause the drift of the asset ratio to decrease when shifting from the 'real-world' measure to the 'risk-neutral' measure. Note that these terms represent in fact the drift in excess of the default-free rate of both the total assets and the total liabilities. This inequality is therefore equivalent to the drift of the total assets being greater than the drift of the total liabilities. It is obvious that a higher drift of the asset ratio translates into a lower probability of default. We therefore observe that the default probabilities increase under the 'risk-neutral' measure with respect to the 'real-world' measure when the drift of the total assets is higher than the drift of the total liabilities and it decreases if the opposite holds.

The switch from the 'real-world' measure to the 'risk-neutral' measure is often intuitively interpreted as shifting more probability mass towards the bad states of the world. Following this line of reasoning, the default probabilities of firms should be higher under the 'risk-neutral' measure, because that are clearly bad states of the world. This result is also supported by papers as Berg (2009) and Hull et al. (2004). Hull et al. (2004) use a reduced form approach and compute default intensities. They empirically find a positive spread between the 'risk-neutral' and the 'real-world' default intensity. Hull et al. (2004) argue that the differences in default probabilities between the 'risk-neutral' and the 'real-world' measure represents a risk premium for the investors. In their paper they only observe default probabilities that are higher under the 'risk-neutral' measure than under the 'real-word' measure, and therefore only have positive risk premiums. We are curious about this matter and will therefore check our findings for such premiums in the empirical section. By basic microeconomics we know that investors are willing to pay a positive premium on top of the expected value of a financial product if that product is negatively correlated with the market. For financial products that are positively correlated with the market, investors are willing to pay less, which is equivalent to paying a negative premium or receiving a positive premium. According to Hull et al. (2004), when investors receive a positive premium, or pay a negative one, the 'risk-neutral' default probabilities are higher than the 'real-world' default probabilities. In our model we can translate the correlation of the bond price with the general market to the correlation of the asset ratio with the general market factor, because the bond price decreases when the asset ratio decreases. We therefore have that both correlations have the same sign. By checking Equation 6 we observe that the sign of the correlation of the asset ratio with the general market is determined by β_i - α_i. For our research to be consistent with the findings of Hull et al. (2004) we should find in the empirical section that for firms where the drift of the assets is bigger than the drift of the liabilities, it holds that $\beta_i - \alpha_i > 0$. Of course when the drift of the total assets is smaller than the drift of the liabilities the opposite should hold. When the drifts are equal and it holds that $\beta_i = \alpha_i$, both default distributions should be the same.

4.2 Multiple Firms

In the previous section we defined the model of just an individual firm. In the complete market model we have multiple firms who are all modeled individually by this one firm model. The firms in the market that we model are therefore all represented by their particular parameter values and their specific asset ratios. In such a market every firm has a SDE representing the total assets and a SDE representing the total liabilities. In the complete market model, the individual risk components of a firm i of both the assets and liabilities are allowed to have nonzero correlation to any other individual risk component. For example, the instantaneous correlation between the individual risk component of the total assets of firm i with the individual risk component of the total liabilities of firm j will be denoted by $\rho(A_i, L_j)$. Note that we still have that $\rho(A_i, G) = 0$ and $\rho(L_i, G) = 0$ for every firm i in the model by assumption.

4.3 Pricing Corporate Bonds

Now that we have derived the probability of default for any firm, we are able to derive the corresponding bond prices. In this section we compute the price of a corporate bond. The corporate bond which we will model is a so-called normalized zero-coupon bond. The name of the bond is rather intuitive, the bond will not pay any coupon before maturity and it pays off the face value, which is normalized to one, at maturity. The risk that is involved in such a financial contract is whether the issuer of the bond is still operating at the time of maturity and whether he is therefore able to make the payment as promised by the contract. The issuer will pay the face value, if he has not defaulted before the time of maturity. We assume that in case the issuer does default at some time before the maturity of the contract, the issuer will pay the holder of the bond some default value, which is substantially less due to bankruptcy costs. Unlike Merton(1974) we continuously model the defaults of firms and they are therefore able to default at any point in time. The payment in case of default will be paid at the same time the firm defaults. We will assume the bankruptcy costs to be proportional to the assets of the firm. We do not include any seniority of the bonds and assume the firm is able to repay the same proportion of the value of any contract a debt-holder has. We refer to this proportion by the so-called 'recovery rate' and define it to be R, which should not be confused with the lower case r which represents the default-free interest rate. One could also define different recovery rates for different kinds of institutions, but we just assume it to be constant for all firms and all bonds to keep our model rather simple. Other researchers, like for example Zhou (2001), assume the recovery rate to differ across various bonds of a firm and therefore define some function for the recovery rate. In the paper by Altman et al. (2005) it is derived that a recovery rate of 50% corresponds to a default rate of 2% per year, according to actual data. Just as Elsinger et al. (2006) we define the recovery rate to equal 50%.[5]

We define the price at current time zero of a normalized zero-coupon bond with maturity t for a firm i by $B_{i,t}$. In the assumptions we have stated that we assume that the conditions for the First Fundamental Theorem of Asset Pricing hold. The price of the bond is therefore equal to the expected discounted payoff. Note that this expectation is subject to the 'risk-neutral' measure. The term structure of the default-free interest rate is assumed to be 'flat', but the yield on this corporate bond can be interpreted as the default-free interest rate plus some extra return, which may be interpreted as a reward for bearing the risk of default. The price of the normalized zero coupon bond is given by the theorem below.

Theorem 7. *We specified the risk neutral measure $\mathbb{Q}$ as an abbreviation of the numéraire dependent measure $\mathbb{Q}_B$. Under this measure we may use that all present value compounding is equal to e^{-rt}, for any future time t to current time $zero$. Mind that due to the expected value with respect to the 'risk-neutral' measure, we use the 'risk-neutral' default probabilities $q_{i,t}$. As a result the current price of a normalized zero-coupon bond with maturity t, is given by:*

$$B_{i,t} = q_{i,t} \cdot R \cdot e^{-rt} + (1 - q_{i,t}) \cdot e^{-rt} \qquad \forall t > 0 \tag{14}$$

[5]For the simulations of the DJ15 market, which will be defined later, we observe an overall default percentage of 52.25 after 30 years, which is relatively close to a default rate of two percent per year.

Proof. See Appendix A □

Normalized zero-coupon bonds are often used to construct coupon paying bonds. This is done by separating all the payments of the coupon paying bonds based on the time of the payments. One is then able to use zero-coupon bonds to value all the separate payments. For us the problem is that to separate all payments, the values of these payments should be independent of each-other. In our case we have that the values are based on the default probabilities and are therefore not independent. We therefore approximate the bond price by a time discretization. Due to the maturity of the financial contracts and the computations that come along with that, we use monthly time-steps. When these time steps shrink towards zero, we expect a better fit of the model with respect to the actual prices. Due to computation limitations, we will leave this for future research.

Our approximation is a recursive method and we start at the first time-step. For the first time-step we compute the probability of default and multiply this with the payoff in case of default. For the next time-step we multiply the probability of not defaulting during the first time period with the probability of default during the second time-step and also with the payoff of defaulting at the second time-step. During this procedure we also mind actual payments of the coupons. We recursively use this algorithm until the maturity of the bond is reached. We then also compute the probability of no default during the maturity of the bond and multiply it with the face value and the last coupon payment at maturity. We add all these terms up and then get an approximation of the price of the coupon bond, which is in fact just the expected payoff under the 'risk-neutral' measure.

5 Estimation

In the previous section we stated the multiple firm model, the default probabilities and the price of a zero-coupon bond given the parameters of the model. To use our model we should know how to obtain the parameters given some data set. In this section we first consider how to estimate all the parameters in the model. We then use the estimation procedure to estimate the parameters of two firms in our data set. As an illustrative example we use the estimated parameters to compute default probabilities in a homogeneous market where we only model firms identical to one of the estimated firms. This is of course not an exact representation of the market, but we only use these computations to show how that the simulations yield the same default probabilities as computed by the formula of Theorems 5 and 6. We also show what happens to the default probabilities of a firm when we change the measure.

5.1 Theory

We have completed our multiple firm model which consists of a couple of SDEs for the total assets and liabilities of a firm and a general market factor. For the next sections we need to be able to estimate the parameters of the model. We first use the assumption that the general market G_t, is defined by a Geometric Brownian Motion. Then we derive the parameters of this SDE and note that the source of risk of this SDE is also present in all SDEs describing the assets and liabilities of the firms in the model. To derive the estimates for the parameters of a Geometric Brownian Motion we will make use of the Maximum Likelihood estimators (MLEs).

Theorem 8. *Let G be a Geometric Brownian Motion. The Maximum Likelihood estimators of this process G ($\hat{\mu}_{MLE}$, $\hat{\sigma}^2_{MLE}$) are defined as:*

$$\hat{\sigma}^2_{MLE} = \frac{1}{T}\sum_{t=1}^{T}\left(\log(\frac{G_t}{G_{t-1}}) - \frac{1}{T}\sum_{t=1}^{T}\log(\frac{G_t}{G_{t-1}})\right)^2, \qquad \hat{\mu}_{MLE} = \frac{1}{T}\sum_{t=1}^{T}\log(\frac{G_t}{G_{t-1}}) + \frac{1}{2}\hat{\sigma}^2_{MLE}$$

With the estimated SDE of the general market 'G_t', we have implicitly derived the estimators for the general market risk $\mathrm{d}\hat{W}_{G,t} = \frac{\frac{dG}{G} - \hat{\mu}_G \mathrm{d}t}{\hat{\sigma}_G}$ and also for $\hat{\lambda}_G = \frac{\mu_G - r}{\sigma_G}$. Both terms are also present in the SDEs of the total assets and total liabilities of a firm. These estimators may therefore be used in deriving the estimation for the SDE of the total assets and similarly for the SDE of the total liabilities. Recall that the SDE of total assets is $\frac{\mathrm{d}A_{i,t}}{A_{i,t}} = \{r + \lambda_{A,i}\sigma_{A,i} + \beta_i\lambda_{G,i}\sigma_{G,i}\}\mathrm{d}t + \sigma_{A,i}\mathrm{d}W_{A,i,t} + \beta_i\sigma_G\mathrm{d}W_{G,t}$. By estimating the SDE of the general market we also already estimated part of the SDE of both the assets and the liabilities. In the empirical section we actually observe the total assets and liabilities of firms. With the total assets known and also the general market parameters estimated, we estimate the SDE of the total assets by a simple Ordinary Least Squares (OLS) regression, namely;

$$y_{i,t} = \alpha_i + \beta_i x_t + \epsilon_{i,t}$$
$$\text{where } y_{i,t} = \frac{\mathrm{d}A_{i,t}}{A_{i,t}},\ \alpha_i = r + \lambda_{A,i}\sigma_{A,i} + \beta_i\lambda_G\sigma_G$$
$$x_t = \sigma_G\mathrm{d}W_{G,t}, \text{ and } \quad \epsilon_{i,t} = \sigma_{A,i}\mathrm{d}W_{A,i,t} \sim \mathrm{N}\left(0, \sigma^2_{A,i}\right).$$

One should observe that in the equation above the OLS regression for the total assets is stated. Therefore the α_i in that regression is just the ordinary constant and is not related to the α_i of the multiple firm model. The α_i of the SDE of the total liabilities is estimated by a similar equation where the β_i in the regression equation will represent the α_i of the model. For the total assets regression we have that the β_i of the model nicely coincides with the β_i of the regression.

For the regression we impose the zero conditional mean assumption: $\mathbb{E} = (\epsilon_t|X)$, where $X = x_1, x_2, .., x_t$, which implies that we should have that $\rho(A_i, G) = 0$ for every i. Note that the zero conditional mean assumption is therefore already implied by the model identifying restriction. We then have the following well-known OLS estimators for the OLS-regression defined above:

$$\hat{\alpha}_i = \frac{1}{T}\sum_{t=1}^{T}\frac{\mathrm{d}A_{i,t}}{A_{i,t}} - \hat{\beta}_i\frac{1}{T}\sum_{t=1}^{T}\sigma_G\mathrm{d}W_{G,t}, \quad \hat{\beta}_i = \frac{\mathrm{Cov}\left(\frac{\mathrm{d}A_{i,t}}{A_{i,t}}, \sigma_G\mathrm{d}W_{G,t}\right)}{\mathrm{Var}\left(\sigma_G\mathrm{d}W_{G,t}\right)}, \quad \hat{\sigma}^2_{A,i} = \mathrm{Var}\left(y_t - \hat{\alpha} - \hat{\beta}_i x_t\right)$$

Where the estimator of $\hat{\sigma}^2_{A,i}$ is just the residual variance of the observations minus the fitted values. Note that we already derived $\hat{\lambda}_G$ and $\hat{\sigma}_G$ above and are therefore now able to derive $\hat{\lambda}_{A,i}$ using the estimated constant α_i combined with the estimated value of $\sigma_{A,i}$. The $\hat{\beta}_i$ term is just the estimated value for the β_i of the SDE for the total assets and for the estimation procedure of the SDE of the total liabilities $\hat{\beta}_i$ is the estimated value for α_i. To complete our model we should still estimate the correlation between the risk factors. The fitted valued of the Brownian

Motions $dW_{A,i,t}, dW_{L,i,t}$ can be computed by using all the parameters estimated above. We already computed $d\hat{W}_{G,t}$ and we then compute the correlation coefficients by just computing the correlation between the time series of these estimated Brownian Motions.

5.2 Results

We have applied the above theory to two rather randomly chosen firms of the 30 currently in the Dow Jones, namely Du Pont and Walt Disney. The estimated values of the parameters of both companies can be found in Table 1. All estimates are based on the data of both firms annual balance sheets from 2004 up to 2014. For the general market G_t we use the annual values of the S&P500 index for the same time period and the estimated parameters obtained are $\mu_G = 0.0720$ and $\sigma_G = 0.1949$ and with our assumption of $r = 0,02$ we have that $\lambda_G = 0.2668$.

	α_i	β_i	$\rho(A_i, L_i)$	$\lambda_{A,i}$	$\lambda_{L,i}$	$\sigma_{A,i}$	$\sigma_{L,i}$	Drift d$A_{i,t}$	Drift d$L_{i,t}$
Du Pont	-0.5300	-0.0813	0.8088	0.2824	0.6611	0.0741	0.0851	0.0367	0.0487
Walt Disney	-0.0243	0.0757	0.2339	0.5413	0.1613	0.0415	0.0537	0.0464	0.0274

Table 1: Estimated parameters for the period 2004-2014

In Table 1 we observe the differences in α and β for both firms. Apparently the total liabilities of Du Pont are way more sensitive to the general market factor than those of Walt Disney. The next difference is the sign of β. For Du Pont we observe a negative sign, which implies that when the general market suffers a positive shock, it causes the total assets of Du Pont to suffer a negative shock and similarly for negative shocks in the general market. For Walt Disney we observe the opposite, the total assets get a positive shock if there is a positive shock in the general market. If we consider the magnitude of both betas we have approximately eight percent in absolute value for both. The dependency combined with the volatility of the general market result in a volatility of roughly 1.6% of the general market on the assets for both Du Pont and Walt Disney. This value is quite low compared to the 'individual' volatility of the assets of both firms, which are 7.41% and 4.15% respectively. We can therefore conclude that the assets of these companies bear little general market risk. For the alphas we already noted that the total liabilities of Du Pont are extremely more sensitive to the general market. If we multiply the alphas with the variance of the general market, we get a risk of roughly 10% for Du Pont and 0.4% for Walt Disney. The risk subject to the general market for the liabilities of Du Pont is therefore slightly bigger than the 'individual' risk component which equals 8.15%. This is quite a difference in comparison with the risk of the total assets of Du Pont. For Walt Disney the risk of the liabilities due to the general market is only 0.4%, which is very small compared to the individual risk which equals 5.37%.

Another very important contrast between both companies is the sign of the difference between the drift of $A_{i,t}$ and $L_{i,t}$. For Du Pont the estimated drift of the assets is lower than that of the liabilities, but for Walt Disney it is the opposite. Based on these estimates the difference between the total assets and total liabilities of Du Pont will decrease on average over time. For Walt Disney the difference will increase on average. So if one only takes the drift factors of the total assets and total liabilities into account the prospects of the future asset ratio of Walt Disney is more promising than that of Du Pont. Based on only these drift parameters one should expect a lower default probability for Walt Disney compared to Du Pont, ceteris paribus. Clearly this does not have to be the case, because the impact of all risk factors and the current asset ratio should also be taken into account when computing the probability of default.

Besides the parameters displayed in the table above, we also estimated correlations between the 'individual' risk components of the two firms. These correlations are displayed in the table below, for the sake of completeness.

	Assets (Du Pont)	Liabilities (Du Pont)	Assets (Walt Disney)	Liabilities (Walt Disney)
Assets (Du Pont)	1.0000	0.8088	0.0024	0.4594
Liabilities (Du Pont)	0.8088	1.0000	-0.2626	0.4571
Assets (Walt Disney)	0.0024	-0.2626	1.0000	0.2339
Liabilities (Walt Disney)	0.4594	0.4571	0.2339	1.0000

Table 2: Estimated instantaneous correlation coefficients of the individual risk components of Du Pont and Walt Disney over the period 2004-2014

From Table 2 we observe a strong correlation between the individual risk of the assets and the individual risk of the liabilities of Du Pont, which was already displayed in Table 1. This is in contrast to the correlation between the individual risks of the assets and liabilities of Walt Disney, which is a lot smaller. It is also rather remarkable that the cross correlation coefficient of the individual risk components of the assets of both firms is very close to zero, although the cross correlation of the liabilities is 0.4571. The high correlation between the individual risks of the liabilities indicates that there could be some other common factor on which the liabilities of both firms depend. For the assets the low correlation indicates that there is probably not another common factor.

5.2.1 Default Probabilities

For Du Pont we have that the drift of the assets is greater than the drift of the liabilities, and for Walt Disney we have the opposite. We already noted that in that case the default probabilities for Du Pont are lower under the 'risk-neutral' measure than under the 'real-world' measure and for Walt Disney this will be just the other way around. We have computed the default probabilities for both firms under both measures and the results are displayed in the graphs below. In the graphs not only the function values are plotted but also the default probabilities as obtained by the simulations.

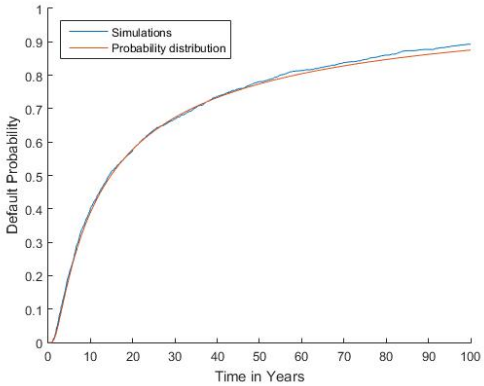

Figure 1: Cumulative default probabilities for Du Pont under the 'real-world' measure $\mathbb{P}$

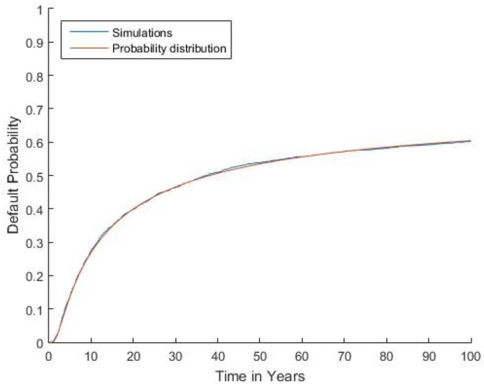

Figure 2: Cumulative default probabilities for Du Pont under the 'risk-neutral' measure $\mathbb{Q}$

The simulated default probabilities are obtained by simulating a homogeneous market consisting of multiple identical 'Du Pont' firms. These firms all have parameters equal to the estimated values for Du Pont. The defaults in the simulations are just counted and divided by the total number of firms, to approximate the probability of default. By simulating many markets and by letting the time-step be very small, we are able to approximate the cumulative default probability function. To obtain the values of both graphs above, we performed 1000 market simulations with a time-step of a 0.005th of a year. From the graphs above we observe that the simulations converge to the function values for a large amount of simulations and a small time-step. By this convergence we may conclude that the cumulative default probability function is correctly specified for our model. We also observe that we have lower default probabilities for the risk-neutral measure compared to the real-world measure for Du Pont, which is obvious from the estimated parameters, and may be explained by the risk-premium of such a contract. We will perform more estimations in later sections and dig deeper into whether the data support our theory of the risk premium.

We now show that for Walt Disney the difference in the probability of default caused by the change of measure, is just the opposite due to the difference in drift. We find that the cumulative default probabilities for Walt Disney are very low and we should therefore mind that these default probabilities are hard to approximate using simulations. We are in fact not able to get a fit of the simulated values which is just as smooth as for Du Pont, due to the high amount of simulations and small time steps which would be needed to do so.

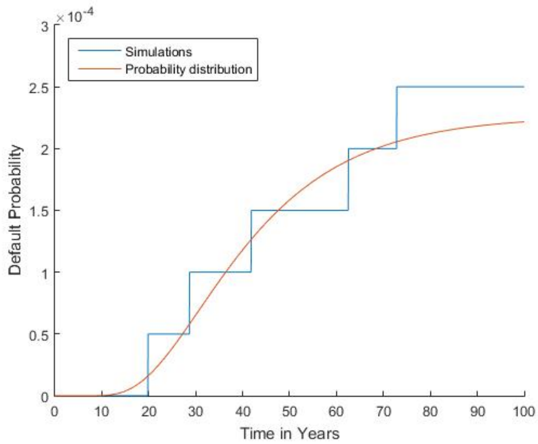

Figure 3: Cumulative default probabilities for Walt Disney under the 'real-world' measure $\mathbb{P}$

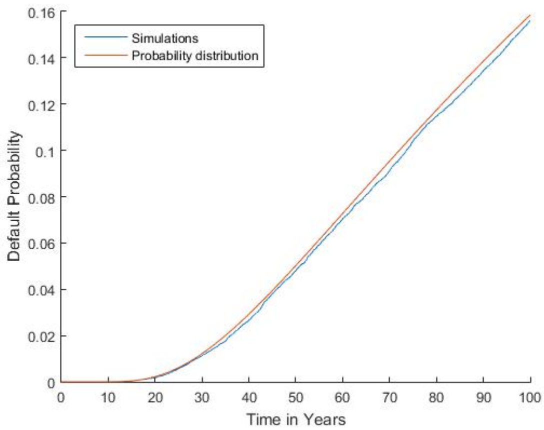

Figure 4: Cumulative default probabilities for Walt Disney under the 'risk-neutral' measure $\mathbb{Q}$

In the graphs above we clearly observe the difference in default probabilities for the two measures. Also for Walt Disney we observe that the function is indeed a correct specification, but due to the extremely small default probabilities under the 'real-world' measure, there are a lot more simulations needed with smaller time-step to let the simulation values converge to the function. For the simulated values displayed in the graphs above we used 10000 market simulations and a time-step of a 0.05th of a year.

In the graphs for both Du Pont and Walt Disney, we clearly observe the differences between the changes in the default probabilities of both firms when changing the measure. For Du Pont we observed in the estimation that the drift of the assets is lower than the drift of the total liabilities. We should mind that we estimated these parameters over only ten data points and that these parameters are therefore not statistically significant. On the one hand it is clear that the assets had a lower growth rate over the estimation period than the liabilities of Du Pont, but on the other hand it is rather difficult to assume that the drift of the total assets is lower than the drift of the total liabilities. At first this might seem unreasonable, because how can a firm be operating for so long if the liabilities grow on average more quickly than the assets. From the data we observe that many years a go the amounts of total assets and liabilities for each of these established firms were really small compared to the current amounts. The difference between the assets and liabilities at that time was relatively large compared to the current amounts, although the absolute difference was not that large at all. When the drift of the SDE of the liabilities exceeds the drift of the SDE of the assets, but the actual values are far from each-other, it could still take a long time before the liabilities exceed the assets based on only these drift parameters. This explains why a firm is also able to be operating for a long time although the drift of the SDE of the total liabilities exceeds the drift of the SDE of the total assets. It could of course also be that the assumption that the drift is a fixed value is to strict and it may be that the drift actually varies to some extent over time. Our model does not allow for such a drift and to account for a varying drift would be a rather fundamentally different model, which we leave for future research. We will proceed with our model but keep in mind that our assumption might be to strict.

5.2.2 Yield Curve

By computing bond prices it is not immediately clear what premium an investor receives for holding the bond until maturity. This is quite easy to see if we compute the yields for zero-coupon bonds of firms. In the yield curve we should have that the yields are higher than the default-free interest rate of two percent. If we assume a firm does not default during the maturity of the contract we are able to derive the yield-curve. In the case of no default, the contract only consists of an initial buyers fee and a payment from the seller at maturity and we compute the yield curve simply by $y_t = -\frac{1}{t}\log(B_{i,t})$, where $B_{i,t}$ is just the bond price for firm i given by (14). As an example the yield curves of both Du Pont and Walt Disney are displayed in the graphs below, which represent the annual yield of a zero-coupon bonds for any maturity.

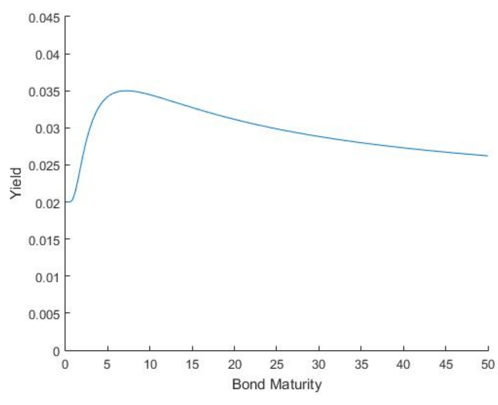

Figure 5: The Yield Curve of Du Pont

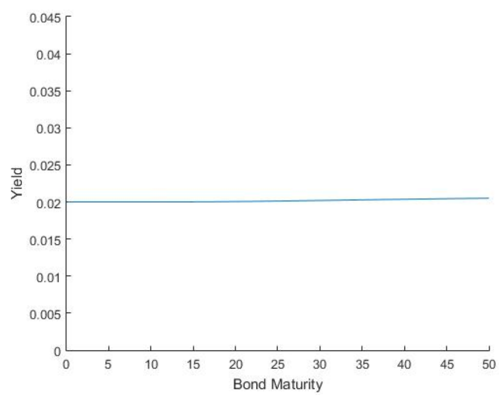

Figure 6: The Yield Curve of Walt Disney

The yield-curve of Du Pont clearly has a maximum on the above displayed interval. This maximum indicates that there is a zero-coupon bond for Du Pont with some maturity which gives the highest annual yield. This maximal annual yield is obtained at an maturity of 7.1883 years. In practice firms do not issue contracts with such maturities. The maximum yield for a maturity which is a natural number is seven with a yield of 3.5 percent. The cumulative default probability for this maturity is 28.84 percent. In contrast to the yield curve of Du Pont, the yield curve of Walt Disney is roughly equal to a constant value. Clearly the default probabilities of Walt Disney are so small that a bond of Walt Disney is very similar to a default-free bond and therefore only pays of a yield which is approximately equal to the default-free rate.

6 Simulating Markets

In this section we apply the model a couple of times with different estimated parameters based on some data set. Our data set consist of the annual reports of the firms who are currently in the Dow Jones. For the sake of completeness the list of all these firms can be found in Appendix C. The data set consists of the amount of total assets and total liabilities for each firm on an annual basis. Unfortunately, the amount of old annual reports available for each firm differs rather strongly among the firms of the Dow Jones. In the data set we only have a subset of fifteen firms of which we have obtained at least twenty years of annual report data. We therefore define a Dow Jones 15 Market (DJ15), which is just a collection of 15 of the 30 firms of which we could find balance sheet information starting at least at 1994. Also the names of these firms in the DJ15 market can be found in an overview in Appendix C. To be able to obtain more accurate parameter estimators we also take two firms of which we have found data starting at 1971. For both of these firms we compute the prices of several bonds with different maturities and coupon rates.

Besides the data from the annual report we also use data on corporate bond prices. For several firms we obtained bond prices by Datastream and we will compare these real bond prices with the bond prices computed by the model.

6.1 Estimation

For the DJ15 market we have the values of the total assets and total liabilities of all firms from 1994 up to 2014. We will estimate the parameters of the model for both the period 1994-2004 and also for 1994-2014. One should expect the estimates based on the period 1994-2014 to be closer to the ‘true values’, because these are based on more data. The period of 1994-2014 also incorporates the financial crisis. The financial crisis had an extraordinary effect on the growth of the total assets and total liabilities of many firms. When we now use the data of this time period, the estimated parameters incorporate this rather extreme period, which might cause noise in estimating the parameters. By extending the estimation period we therefore not only get more data points, but many of those data points are years where firms suffer from the financial crisis. We have assumed that the ‘true’ parameters are firms specific and constant over time, the ‘noise’ of the financial crisis may therefore cause the parameters to be estimated less accurately due to the big proportion of the sample being affected. We are not sure whether one of the estimated parameters sets will be better in terms of fitting on the actual bond prices. We therefore compare the parameter estimations for both periods and also compute bond prices with both parameter sets. Based on the bond prices we can also compare whether one estimation period outperforms the other. We do this because we want of course the best parameters to get a good idea how the model performs in computing bond prices. In the tables below the drift estimates for all firms for both estimation periods are displayed. We also choose to display the sign of the difference of the drifts and the sign of the difference of the influences of the general market factor, because this will be convenient for interpreting the results below. The other parameters, which we also estimated for both estimation periods, can be found in Appendix B.

	Apple Inc	Boeing	Walt Disney	Home Depot	IBM	Intel	JPMorgan	Coca Cola
Drift $\mathrm{d}A_{i,t}$	0,0552	0,0603	0,1974	0,2113	0,0327	0,1406	0,1701	0,0882
Drift $\mathrm{d}L_{i,t}$	0,0170	0,0959	0,1910	0,2212	0,0331	0,0940	0,1670	0,0702
Drift $\mathrm{d}A_{i,t}$ - Drift $\mathrm{d}L_{i,t}$	>0	<0	>0	<0	<0	>0	>0	>0
$\beta_i - \alpha_i$	>0	>0	>0	<0	<0	>0	>0	>0

Table 3: Drift parameters for the DJ15 market, estimated on the period 1994-2004

	McDonald's	3M	Microsoft	Nike	Pfizer	UnitedHealth	Wal-Mart
Drift $\mathrm{d}A_{i,t}$	0,0747	0,0499	0,3408	0,1346	0,2783	0,2487	0,1526
Drift $\mathrm{d}L_{i,t}$	0,0743	0,0527	0,3926	0,2056	0,2151	0,5433	0,1577
Drift $\mathrm{d}A_{i,t}$ - Drift $\mathrm{d}L_{i,t}$	>0	<0	<0	<0	>0	<0	<0
$\beta_i - \alpha_i$	<0	>0	>0	>0	<0	>0	<0

Table 4: Drift parameters for the DJ15 market, estimated on the period 1994-2004

	Apple Inc	Boeing	Walt Disney	Home Depot	IBM	Intel	JPMorgan	Coca Cola
Drift $\mathrm{d}A_{i,t}$	0.2335	0.0614	0,1226	0,1085	0,0202	0,1056	0,1308	0,1074
Drift $\mathrm{d}L_{i,t}$	0,2458	0,0855	0,1099	0,1493	0,0329	0,1323	0,1295	0,1237
Drift $\mathrm{d}A_{i,t}$ - Drift $\mathrm{d}L_{i,t}$	<0	<0	>0	<0	<0	<0	>0	<0
$\beta_i - \alpha_i$	>0	>0	>0	<0	>0	>0	>0	>0

Table 5: Drift parameters for the DJ15 market, estimated on the period 1994-2014

	McDonald's	3M	Microsoft	Nike	Pfizer	UnitedHealth	Wal-Mart
Drift $\mathrm{d}A_{i,t}$	0,0483	0,0468	0,2074	0,1125	0,1656	0,1875	0,1128
Drift $\mathrm{d}L_{i,t}$	0,0610	0,0576	0,2761	0,1530	0,1577	0,3340	0,1170
Drift $\mathrm{d}A_{i,t}$ - Drift $\mathrm{d}L_{i,t}$	<0	<0	<0	<0	>0	<0	<0
$\beta_i - \alpha_i$	>0	>0	>0	>0	<0	>0	<0

Table 6: Drift parameters for the DJ15 market, estimated on the period 1994-2014

In the tables above, in lines three and four we only mention the sign of the difference because the actual number is not very relevant. By this representation it is also easier to observe whether the estimated parameters satisfy the theory about a positive or negative premium. When the drift of the total assets is larger than the drift of the total liabilities we should have that the influence of the general market factor on the total assets (β_i) should be larger than the influence of the general market factor on the total liabilities (α_i) for the theory of the risk-premium to hold. In the tables above we should therefore have that the signs in the third and fourth line are the in the same direction for each firm.

For the estimation period 1994-2004 there are seven firms, out of the fifteen, that are in line with the theory of the risk-premium. By observing the actual numbers in Table 36 in Appendix B, we find that also another two firms have estimates with only minor deviations to both signs being the same. For the period of 1994-2014 there are only four firms left having both signs in the same direction. It is remarkable to see that these four firms are a subset of the seven of the other estimation period. Such a result is in favor of the theory and decreases the probability of the signs being randomly determined. The difference between the amount of firms satisfying or theory of the risk-premium of both periods also creates the impression that the 1994-2004 estimation period results in better estimates. Of course it is hard to draw any conclusions due to the small amount of firms. Most parameters are also not statistically significant due to the small sample size.

In the overview tables in Appendix B we also displayed the risks due to the general market factor on both the total assets and the total liabilities of the firm. From these results we clearly observe that there are some firms where the market risk factor represents an substantial amount of the total, risk but we also find some firms where the general market risk factor is not very influential. This indicates that some firms in the DJ15 are not very sensitive to a common factor as the general market factor.

We now estimate the parameters for Intel and Wal-Mart based on 43 years of data. This gives us from statistical point of view still a rather small sample and our theory therefore does not necessarily has to hold for these estimates. We compare the estimates with the previous estimates and we would like to observe some convergence to the values that are supporting our theory about the premiums. The estimated parameter values for Intel and Wal-Mart are stated in the table below. For the same period we again use the annual values of the S&P500 index to represent the general market factor. The estimated parameters are, $\mu_G = 0.0875$, $\sigma_G = 0.1748$ and $\lambda_G = 0.3862$.

	α_i	β_i	$\rho(A_i, L_i)$	$\lambda_{A,i}$	$\lambda_{L,i}$	$\sigma_{A,i}$	$\sigma_{L,i}$	Drift $\mathrm{d}A_{i,t}$	Drift $\mathrm{d}L_{i,t}$
Intel	0,2594	0,1226	0,8137	0,8638	0,4829	0,2522	0,6102	0,2462	0,3322
Wal-Mart	-0,1397	-0,0566	0,9285	1,3259	1,0803	0,1786	0,2306	0,2529	0,2597

Table 7: Estimated parameters for the period 1971-2014

Let us now first interpret the results in the table above to get a better understanding about the dynamics of the total assets and total liabilities for both firms. The risks involved for the assets due to the general market($\beta_i \sigma_G$) are 2.14 percent and 0.99 percent respectively. Compared to the individual risks($\sigma_{A,i}$) of 25.22 percent and 17.86 percent the risks due to the general market are rather low. For the liabilities we have that the risks due to the general market($\alpha_i \sigma_G$) are 10 percent and 3.22 percent and the individual risks($\sigma_{L,i}$) are 61 percent and 23 percent. For the liabilities we observe that the risks are a lot larger, and also as in the case for the assets the individual risks for the liabilities are much higher than the risks due to the general market. The coefficients of the individual risks actually the residual risks of the estimation. One might therefore conclude that the general market factor is not explaining a lot of the risk in the total assets and liabilities of a firm. If there are more risk factors that have significant effect on the assets and liabilities of the firm, these are now captured in the individual risk term. The proportion of individual risk to general market risk might therefore not be in line with the actual values. One could try to define some factor that represents the individual risk or define more factors to reduce the residual risk. We will leave such extension to our model for future research.

We relate the parameter estimates with the results of Hull et al. (2005). Intel clearly satisfies $\beta_i < \alpha_i$, and also the drift of the total liabilities is smaller than the drift of the assets. For the other estimators estimated on different time periods for Intel, these signs were not in line with the theory, which indicates that the parameters could indeed converge to the 'true' values. Both signs are in this case in line with the theory about the premiums a consumer is willing to pay. The asset ratio of Intel will on average react negatively at positive increments of the general market. This causes the default probabilities to raise and the bond price to decrease. If the general market decreases the bond price will on average increase. We therefore have that the bond of Intel yields a positive return in bad states of the economy and a consumer is therefore willing to pay a positive premium for such a financial product.

For Wal-Mart we almost have the opposite story of Intel. We observe that the parameters of Wal-Mart are such that $\beta_i > \alpha_i$, which indicates that the asset ratio of Wal-Mart is positively correlated with the general market factor. The other conditions which it should satisfy is that the drift of the assets exceeds the drift of the liabilities. This is unfortunately not the case, but the point estimates are very close to each-other. We should also note that even for the data period consisting of 43 years,we also do not have statistically significant estimators. Despite the significance, the point estimates give some indication that the theory of the premiums holds. To solve the problem of the small data set and to perform a better test on whether the theory of the premiums holds in actual markets, we encourage to use quarterly report in future research to obtain more data and probably better parameter estimates.

In the next section we will use the estimated parameter sets to simulate asset ratios and compute the default probabilities and use these to approximate the corporate bond prices. We compare the prices and relate this to the estimated parameters and also the bond specifics.

6.2 Corporate Bonds

In this section we use the estimated parameter values of the previous section and estimate the prices of corporate bonds implied by the model. In reality we do not find many firms that issue zero coupon paying bonds. Luckily we are able to approximate the price of any coupon paying bond by the recursive algorithm that we already noted in the theoretical part of this thesis. We simulate many DJ15 markets and compute the default probabilities for various maturities into the future for every firm at every monthly time instant. We then take the average of these default probabilities over all simulations. We then have cumulative default probabilities at every month for monthly time steps ranging far into the future. These will be needed in approximating the expected values of the payments of the coupon paying bonds. To be able to compute the bond prices we should of course know the future payments. We therefore only use fixed rate bonds of which the coupon payments are known with certainty. The bond data-set that we use only contains seven firms with bond prices starting at 2005 and we therefore only compute the prices of these seven bonds, although we simulate the entire market. The actual bonds all have different maturities and different coupon rates, which are displayed in Table 8 below.

	Boeing	IBM	MCD	3M	Nike	Pfizer	Wal-Mart
Coupon	8.75%	7%	6.375%	6.375%	5.15%	4.65%	6.75%
Maturity	09/31	10/25	01/28	02/28	10/15	02/18	10/23

Table 8: Overview of the specifics of the coupon paying bonds

We first perform simulations using the parameter estimates based on the period of 1994-2004. We use the parameter estimations for that period together with the asset ratio at the start of 2005 to compute the monthly bond prices of 2005. The asset ratios at the start of 2005 and the deviations of the prices as determined by our model from the actual market prices can be found in the tables below.

	Boeing	IBM	MCD	3M	Nike	Pfizer	Wal-Mart
Asset ratio	1.2511	1.3995	2.0414	2.0061	2.5239	2.1919	1.7415

Table 9: Asset Ratios at the start of 2005

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01/05	60,65%	55,15%	59,55%	60,91%	25,48%	10,00%	48,74%
02/05	54,20%	50,94%	54,85%	52,41%	24,34%	8,49%	44,88%
03/05	58,29%	58,16%	61,78%	60,24%	28,96%	10,05%	50,64%
04/05	55,65%	54,78%	58,45%	56,90%	25,03%	8,96%	45,40%
05/05	53,23%	53,12%	59,95%	54,40%	23,96%	7,68%	43,43%
06/05	51,24%	52,91%	56,06%	53,28%	23,70%	6,84%	44,69%
07/05	50,83%	52,51%	54,01%	53,15%	24,02%	7,87%	44,66%
08/05	52,08%	53,71%	55,68%	52,05%	25,07%	7,98%	46,07%
09/05	49,72%	54,38%	57,75%	53,09%	24,55%	4,54%	47,49%
10/05	53,86%	55,29%	64,04%	57,25%	24,69%	8,18%	48,43%
11/05	55,40%	56,49%	64,94%	61,26%	25,50%	9,70%	49,21%
12/05	54,22%	56,28%	61,13%	60,34%	25,68%	10,12%	49,51%
Average fit	54,11%	54,48%	59,02%	56,27%	25,08%	8,37%	46,93%

Table 10: Differences between the actual bond prices for 2005 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2004

From these results we quickly observe that the model prices are quite different from the actual market prices. Only for Pfizer we observe relatively small deviations of the actual bond prices. We also observe that all the bond prices estimated from our model are higher than the actual bond prices.

If we first assume that our parameter values for the recovery rate, the default-free rate and also our default threshold are correct[6], we observe in Equation 14 that the differences are caused by deviations of the 'risk-neutral' default probabilities in the model with the 'risk-neutral' default probabilities that are assumed by the actual market. Note that if one increases the default probabilities of a firm, the bond price decreases. The overpricing should therefore be caused by too low default probabilities for all maturities. The bond prices of Boeing, IBM, MCD, 3M and WMT all represent more or less the same extent of overpricing. It is therefore interesting to see on what grounds these firms separate themselves as a group from the other two. By observing the values of Table 9 and Table 10 we see multiple causes. Nike and Pfizer have the highest asset ratios, the lowest coupon rates and the shortest maturities. These values clearly affect the bond prices. For the estimated parameters we do not observe any value indicating this difference between the firms. Also the difference in drifts and the correlation with the general market are not consistent with the theory of the premium, which otherwise could have been some explanation for the difference between the bonds. When comparing the estimated parameter values we only observe that the signs of the market dependence of the asset ratio and the difference in drifts are only in line with our theory of the risk-premium for IBM and Wal-Mart. Unfortunately we do not observe a better fit for IBM and Wal-Mart compared to the bonds of other firms.

On the one hand it could be that the default probability distribution is not in line with what the markets belief it is, or it could be that the market value of the total assets and liabilities are rather different from the balance sheet amounts. This would change the asset ratio if the total assets and the total liabilities are not changed by the same proportion. We would therefore have to impose a different default threshold for the asset ratio. We will come back to a change in default threshold when we check all the general parameter values.

We now use the estimated parameters for the period 1994-2014 to compute the bond prices of the bonds stated in Table 8 for the year 2015. Our bond data does not contain all months of 2015, so we will only compute the prices for the first seven months, which should give us already a clear idea about the fit of the model. The results not only contain bond prices computed using the parameters estimated on the 1994-2014 period, but we also compute the bond prices using the parameters estimated on the 1994-2004 period, where we of course use the asset ratios at the start of 2015 as displayed in Table 11 below. Again we perform simulations of the DJ15 market and compare the prices given by the model with the real market prices. The differences between the real bond prices and the estimated values as well as the asset ratios at the end of 2014 can be found in the tables below.

	Boeing	IBM	MCD	3M	Nike	Pfizer	Wal-mart
Asset ratio	1.0970	1.1138	1.5996	1.7250	2.3925	1.7334	1.7239

Table 11: Asset Ratios at the start of 2015

[6]We will check the prices of the model for a different default-free rate, a different recovery rate and also for a different default value for the asset ratio.

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01/15	-15,16%	-10,99%	11,73%	8,08%	0,18%	-4,42%	4,49%
02/15	-12,32%	-9,35%	15,63%	9,37%	0,70%	-3,42%	6,94%
03/15	-15,85%	-9,01%	16,96%	10,52%	1,26%	-5,56%	7,64%
04/15	-16,56%	-13,29%	14,64%	8,23%	-0,72%	-5,64%	3,99%
05/15	-14,82%	-11,36%	18,02%	11,01%	-0,22%	-4,92%	6,68%
06/15	-13,88%	-9,77%	18,44%	13,54%	0,35%	-4,29%	8,30%
07/15	-15,59%	-9,11%	24,19%	14,78%	0,93%	-4,19%	10,55%
Average fit	-14,88%	-10,41%	17,09%	10,79%	0,35%	-4,64%	6,94%

Table 12: Difference between the actual bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2014

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01/15	-3,05%	2,54%	11,75%	10,20%	0,18%	-9,83%	4,40%
02/15	-0,16%	4,59%	15,65%	11,59%	0,70%	-8,88%	6,84%
03/15	-4,21%	5,28%	16,99%	12,78%	1,26%	-10,81%	7,53%
04/15	-5,28%	0,93%	14,67%	10,43%	-0,72%	-10,73%	3,89%
05/15	-2,62%	3,79%	18,06%	13,32%	-0,22%	-9,93%	6,57%
06/15	-0,66%	6,05%	18,49%	15,92%	0,35%	-9,01%	8,19%
07/15	-1,16%	6,95%	24,24%	17,19%	0,93%	-8,75%	10,44%
Average fit	-2,45%	4,30%	17,12%	13,06%	0,35%	-9,70%	6,84%

Table 13: Differences between the real bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2004

In the results above we observe different deviations than for our previous estimations of the bonds in the year 2005. There are two big differences with respect to that period. The asset ratios of all firms have dropped and of course the maturities of all bonds are ten years shorter than they were in 2005. This causes the bond prices of the model to drop and become even less than the actual market prices for some bonds. For Boeing, IBM and Pfizer we observe underpricing and for the other firms we again have overpricing. It is interesting to determine why the sign of the deviations is different for the bonds. The big drop for all bond prices will mostly be caused by the shorter maturity. We can therefore conclude that our model overvalues expected payments further into the future, which means that the default probabilities for longer maturities are in fact too low or that our default-free interest rate is not a good representation of the default-free interest rate for longer maturities in the market.

Let us now compare the companies to determine why the bond price of some dropped more than others. With regard to the asset ratios we observe that the asset ratio of Boeing, who experiences the biggest drop in the estimated bond prices, has dropped with 12%. Comparing this with IBM, MCD and Pfizer who all have an asset ratio in 2015 which is roughly 20% lower than in 2005, we find that the drop of the asset ratio can not be the main reason for the shift. Also for the parameter estimates we find no direct link. The only possible explanation left is the low value of the asset ratio of Boeing. The low bond values are a result of to high default probabilities for such a low asset ratio. We can therefore conclude that the default probabilities as computed are to high for relatively low asset ratios. This explanation also holds for the underpricing of the bond of IBM. The other firms have relative high asset ratios, which causes the overpricing and also the maturities affect the deviations.

In Table 13 we compute the bond prices based on parameter estimates of the period 1994-2004. Obviously we set the asset ratios at the start of the year 2015 again equal to the values given in Table 11. The changes we observe between these estimates and those in the preceding table are therefore only caused by the change of parameters. This gives us the opportunity to observe how these different parameter values affect the modeled bond prices. We roughly observe a closer fit for Boeing and IBM. Only for 3M and Pfizer the estimated bond prices have a relatively worse fit than for the other parameters and for MCD, Nike and Wal-Mart the fit is roughly the same. It therefore seems that, based on these numbers, the parameters estimated on the 1994-2004 period are closer to the 'true' values for all firms except 3M and Pfizer. Of course we should mind that it could also be just due to data mining, because the small data set on which these estimates are based. From these results we also observe that for some firms the change of parameter values does not change the bond price by much. The fit of these bond prices may be improved by changing our general parameter values. We will come back to this in the next section.

By these estimations and also the previous ones we have gained some understanding about the shortcomings of the model and the procedure of estimating parameters. Among other things we still do not have a clue about the difference in default probabilities, although we experienced that there are other factors which are generating much bigger fluctuations in the estimated bond prices. We are still very curious whether we are able to determine the parameters better with more data. To see whether more data results in better estimated bond prices we use the estimations of Intel and Wal-Mart for the 1971-2014 time period. With the estimated parameters we will simulate both firms individually and approximate the bond prices. For Intel we have data of the actual market prices of three different bonds and for Wal-Mart we have four different bonds. By having more bonds per firm we can observe how the bond prices of the model for the same firm differ if the maturities and coupon rates of the bonds are different. The maturities and coupon rates as well as the deviations of the modeled bond prices from the real bond prices can be found below, where we just define some simple names for notational convenience.

	Intel 1	Intel 2	Intel 3	Wal-Mart 1	Wal-Mart 2	Wal-Mart 3	Wal-Mart 4
Coupon	1.95%	3.3%	4.8%	6.75%	5.25%	5.375%	5.875%
Maturity	10/16	10/21	10/41	10/23	09/35	04/17	04/27

Table 14: Overview of the specifics of the coupon paying bonds

	Intel 1	Intel 2	Intel 3	Wal-Mart 1	Wal-Mart 2	Wal-Mart 3	Wal-Mart 4
01-2015	-3,90%	-5,06%	27,46%	3,72%	20,69%	-0,99%	7,89%
02-2015	-3,33%	-3,21%	34,01%	6,14%	25,74%	-0,49%	8,93%
03-2015	-3,08%	-2,92%	33,78%	6,84%	24,96%	0,00%	9,79%
04-2015	-4,03%	-5,63%	27,44%	3,20%	19,98%	-2,39%	5,93%
05-2015	-3,85%	-4,56%	37,10%	5,87%	27,64%	-1,82%	8,93%
06-2015	-3,56%	-3,18%	44,24%	7,48%	31,50%	-0,89%	11,75%
07-2015	-3,44%	-3,06%	45,63%	9,70%	32,13%	-0,61%	13,33%
Average fit	-3,60%	-3,94%	35,67%	6,14%	26,09%	-1,03%	9,51%

Table 15: Differences between the actual bond prices for 2015 of Intel and Wal-Mart and the bond prices as estimated by the model based on parameters estimated on data period 1971-2014

In the table above we clearly observe that the differences of the bond prices are not very firm specific. This means that the deviations are rather sensitive to the coupon rate and mostly the maturity. The second and third bond of Wal-Mart have coupon rates which are very close to each-other. The difference of these bonds is the maturity. We therefore observe the huge influence of the maturity, because for both bonds all other parameters affecting the bond price are equal. This result again strongly indicates that the default probabilities as stated by the model for long maturities are too low with respect to the probabilities incorporated in the market prices. We should mind that this only holds if we assume that the 'flat' interest rate r is not a large deviation of the actual term structure of interest rates. The actual term structure has of course a very different shape, and we will therefore investigate in the next section how different values for r influence the bond prices of the model. For Intel we have that the asset ratio at the start of 2015 is 2,6140 and for Wal-Mart we already stated that it equals 1.7239. These asset ratios are relatively high, which actually causes the default probabilities to be very low. If the cumulative default probability is going to zero, we observe from (14) that the price of a bond is only influenced by the discounting. The mispricing in Table 15 is therefore largely caused by our assumption of a 'flat' default-free interest rate. By the results we can conclude that the interest rate of two percent is a good representation of the default-free rate for payments in the neighborhood of the year 2018, but it fails to approximate the default-free interest rates for a lot longer or shorter maturities. We will therefore change the interest rate to different values in the next section.

In the estimations in this section we have used everywhere that we assume a default-free interest rate of two percent and a recovery rate of fifty percent. We also assumed that a firm defaults instantaneously at the moment the value of the assets drops below that of the liabilities, or stated differently, if the asset ratio drops below one. At least for the 'flat' interest rate we obviously saw that the assumption is to strict and this might also be the case for the other general parameters. In the next section we therefore redo the simulations for different settings for these parameters.

6.3 Parameter Checks

In the computations above we used our model and we assumed a default-free interest rate of two percent, a recovery rate of 50 percent and a default threshold for the asset ratio of one. It could be that the difference between the modeled bond prices and the actual market prices are caused by any of these general parameter values. We will therefore question these values and redo the computations for both the DJ15 market and Intel and Wal-Mart for the year 2015.

In the year of 2015 the actual market has experienced very low interest rates. Note that in our model we made a simplification by assuming a 'flat' interest rate curve. We could assume the German government bonds to pay a default-free interest rate. In 2015 these rates where very low, even negative for short-term maturities. Due to the 'flat' interest rate curve we should set r somewhere between the short en long term interest rates such that it is a relatively good approximation of reality. We have chosen it to be two percent in our general computations, but this might not be a good approximation for the term structure in 2015. Due to the negative interest rates for German government bonds we will redo calculations with an default-free interest rate equal to one percent. As we already saw in the previous section the interest rates should be rather different for short and long maturities. To get a better fit on the long maturities we will also redo the calculations with a default-free interest rate of four percent. Using both of these interest rates and comparing them with the general setting might give us interesting results. We keep in mind that a lower default-free interest rate would increase the current value of the bond payments due to a lower discount rate, and of course a higher default-free interest rate causes the opposite.

The recovery rate has been set to fifty percent, which corresponds to the default rate of roughly two percent according to Altman et al. (2005). A lower recovery rate would lead to lower bond prices, according to (14). A recovery rate of forty percent is still realistic according to Altman et al. (2005), so we will redo calculations for this value. We could of course also redo calculations with a higher recovery rate of for example 60 percent, which would surely increase the bond prices, but this is not really realistic according to the paper by Altman et al. (2005) with the rate of default observed in our model. We will therefore only redo calculations for a recovery rate of forty percent, because we do not want to change the parameters to unrealistic values just such that the model would produce bond prices that are closer to the actual prices.

In our model we assumed that a firm defaults instantaneously if the asset ratio drops below one. This is quite an intuitive but strong assumption, and some researchers like Jarrow and Turnbull (1995), argue that one should not impose such a strict assumption. The total asset and liabilities of a firm are non-traded assets and their market value is therefore hard to determine. This might cause the actual value of the total assets and total liabilities as believed by the market to deviate from the book value as given on the balance sheet. Due to this noise in the valuation of the total liabilities it therefore might be the case that the market assumes the default threshold of the total assets divided by the total liabilities as stated on the balance sheet to be lower than one. We will check the change of the bond prices when we set the default threshold equal to 0.95.

6.3.1 Default-free interest rate

Let us now first redo the calculations for an interest rate equal to one percent. Of course nothing changes with the estimation procedure, so the parameters are just the same as those that we used earlier. The results of the computations are displayed below.

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01-2015	-6,95%	-4,58%	23,24%	18,73%	0,92%	-1,65%	12,11%
02-2015	-3,87%	-2,92%	27,43%	20,31%	1,36%	-0,67%	14,64%
03-2015	-7,95%	-2,67%	28,79%	21,46%	1,84%	-2,78%	15,29%
04-2015	-9,42%	-7,31%	26,13%	18,78%	-0,22%	-3,01%	11,47%
05-2015	-7,19%	-4,95%	29,73%	21,77%	0,20%	-2,30%	14,25%
06-2015	-5,86%	-3,27%	30,35%	24,43%	0,68%	-1,75%	15,90%
07-2015	-7,22%	-2,61%	36,56%	25,70%	1,18%	-1,79%	18,20%
Average fit	-6,92%	-4,04%	28,89%	21,60%	0,85%	-1,99%	14,55%

Table 16: Differences between the actual bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2014 and r=0.01

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01-2015	6,66%	10,64%	23,27%	21,39%	0,92%	-7,34%	12,00%
02-2015	10,18%	12,81%	27,47%	23,08%	1,36%	-6,16%	14,53%
03-2015	5,93%	13,59%	28,83%	24,30%	1,84%	-8,14%	15,17%
04-2015	4,66%	9,06%	26,18%	21,61%	-0,22%	-8,08%	11,35%
05-2015	7,67%	12,18%	29,80%	24,70%	0,20%	-7,05%	14,13%
06-2015	9,62%	14,47%	30,42%	27,45%	0,68%	-6,33%	15,78%
07-2015	9,20%	15,49%	36,64%	28,75%	1,18%	-5,99%	18,08%
Average fit	7,70%	12,61%	28,94%	24,47%	0,85%	-7,01%	14,43%

Table 17: Differences between the actual bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2004 and r=0.01

Let us first compare the results in both tables above with the results for the same parameters but with a default-free interest rate of two percent. We observe an upward shift for all the bond prices, which is of course expected for a lower discount rate. The overall fitting of the bond prices gets slightly worse with respect to the two percent case. Of course the bonds with a longer maturity get a larger shift upwards. We expected the bond prices to be better fit for short maturities and worse for long maturities. We indeed observe roughly a better fit for the bond of Pfizer. On the other hand the bond of Wal-Mart deviates more and indicates that bonds of maturing in 2023 and later should obtain a better fit of a higher default-free interest rate. For Boeing and IBM we observe that the default probabilities are rather influential and are also very sensitive to the estimated parameters. Due to their maturities we should expect rather heavily overpricing for these bonds compared to the other bonds. We therefore again conclude that the default probabilities of firms with relatively low asset ratios are too high with respect to the 'risk-neutral' default probabilities that the market prices incorporate.

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01-2015	-28,76%	-22,24%	-7,67%	-9,89%	-1,29%	-9,72%	-9,03%
02-2015	-26,24%	-20,68%	-4,29%	-9,05%	-0,61%	-8,62%	-6,74%
03-2015	-29,89%	-20,42%	-3,03%	-7,96%	0,11%	-10,51%	-5,97%
04-2015	-30,83%	-24,36%	-4,79%	-9,75%	-1,71%	-10,50%	-9,33%
05-2015	-29,19%	-22,42%	-1,82%	-7,27%	-1,05%	-9,72%	-6,83%
06-2015	-28,60%	-20,93%	-1,73%	-5,02%	-0,32%	-9,07%	-5,26%
07-2015	-29,78%	-20,53%	3,21%	-3,84%	0,43%	-8,87%	-3,13%
Average fit	-29,04%	-21,66%	-2,87%	-7,54%	-0,63%	-9,57%	-6,61%

Table 18: Differences between the actual bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2014 and r=0.04

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01-2015	-19,10%	-11,56%	-7,66%	-8,63%	-1,29%	-14,59%	-9,09%
02-2015	-16,36%	-9,63%	-4,27%	-7,73%	-0,61%	-13,40%	-6,80%
03-2015	-20,11%	-8,87%	-3,01%	-6,59%	0,11%	-15,32%	-6,04%
04-2015	-20,96%	-12,90%	-4,77%	-8,39%	-1,71%	-15,03%	-9,40%
05-2015	-18,54%	-10,32%	-1,80%	-5,84%	-1,05%	-14,14%	-6,91%
06-2015	-16,86%	-8,23%	-1,70%	-3,52%	-0,32%	-13,14%	-5,33%
07-2015	-17,62%	-7,24%	3,25%	-2,31%	0,43%	-12,86%	-3,21%
Average fit	-18,51%	-9,82%	-2,85%	-6,14%	-0,63%	-14,07%	-6,68%

Table 19: Differences between the actual bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2004 and r=0.04

In both tables above we present the results of the computations with a default-free interest rate equal to four percent. We expect that such a default-free interest rate will cause the bonds with longer maturities to be priced better in absolute terms. The higher interest rate causes higher discount rates and therefore reduces all bond prices. An interesting result of these computations is the relative small deviation of the bond price for McDonald's, which has a very big deviation in the general results. In the preceding sections we already observed that the default probabilities for McDonald's are not at all high. We therefore find that the interest rate of four percent is better representing bonds with longer maturities than the two percent rate. By comparing McDonald's with 3M we observe that the default probabilities do matter, because for these two bonds the maturity and coupon are almost exactly equal. We therefore conclude that even for rather safe firms the bond prices are sensitive to the specification of the default probabilities.

Based on these results we consider a 'flat' interest rate slightly below four percent to be a good representation of the term structure of interest rates for long maturities in 2015. The deviations depending on the maturities are then caused by deviations of the default probabilities with respect to the default probabilities as believed by the market. Due to the different maturities, coupon rates and also initial asset ratios, it is hard to determine from these estimates what is mostly influencing the bond prices and in what direction. To be able to hold the asset ratio constant and investigate the effects of the maturity and the coupon rate we also redo the computations for the multiple bond contracts of Intel and Wal-Mart. The results of these computations can be found in both tables below.

	Intel 1	Intel 2	Intel 3	Wal-Mart 1	Wal-Mart 2	Wal-Mart 3	Wal-Mart 4
01-2015	-2,26%	0,67%	51,65%	11,20%	38,63%	1,14%	18,41%
02-2015	-1,73%	2,63%	59,44%	13,69%	44,27%	1,57%	19,43%
03-2015	-1,61%	2,74%	58,86%	14,32%	43,21%	1,98%	20,24%
04-2015	-2,61%	-0,10%	51,63%	10,51%	37,70%	-0,49%	16,13%
05-2015	-2,57%	0,85%	62,83%	13,27%	46,38%	0,00%	19,31%
06-2015	-2,29%	2,29%	71,27%	14,88%	50,63%	0,87%	22,28%
07-2015	-2,24%	2,36%	72,81%	17,17%	51,25%	1,06%	23,91%
Average fit	-2,19%	1,63%	61,21%	13,58%	44,58%	0,88%	19,96%

Table 20: Differences between the actual bond prices for 2015 of Intel and Wal-Mart and the bond prices as estimated by the model based on parameters estimated on data period 1971-2014 and r=0.01

	Intel 1	Intel 2	Intel 3	Wal-Mart 1	Wal-Mart 2	Wal-Mart 3	Wal-Mart 4
01-2015	-7,12%	-15,49%	-7,54%	-9,57%	-7,16%	-5,12%	-9,99%
02-2015	-6,43%	-13,72%	-2,65%	-7,30%	-3,11%	-4,47%	-8,97%
03-2015	-6,09%	-13,46%	-2,83%	-6,54%	-3,57%	-3,84%	-8,11%
04-2015	-6,89%	-15,93%	-7,88%	-9,89%	-7,73%	-6,08%	-11,55%
05-2015	-6,53%	-14,84%	-0,74%	-7,41%	-1,70%	-5,37%	-8,91%
06-2015	-5,94%	-13,31%	4,77%	-5,85%	1,44%	-4,32%	-6,39%
07-2015	-5,66%	-13,07%	5,95%	-3,73%	2,10%	-3,89%	-4,90%
Average fit	-6,38%	-14,26%	-1,56%	-7,18%	-2,82%	-4,73%	-8,40%

Table 21: Differences between the actual bond prices for 2015 of Intel and Wal-Mart and the bond prices as estimated by the model based on parameters estimated on data period 1971-2014 and r=0.04

Let us first compare both estimated bond prices with the results of the computations with the two percent ‘flat’-rate. For Intel 1, Intel 2 and Wal-Mart 3 we observe smaller deviations from the actual bond prices than for the two percent case. These three bonds are also the bonds with the shortest maturity, so it is in line with our expectations. Again the large upward shift of the bonds with longer maturities clearly indicates that a ‘flat’ default-free interest rate is a too strict assumption to capture all bond prices. It could be very interesting to allow for a non-flat term structure. This could of course be simply approximated by setting a couple of interest rates for different maturities or one could use a more extensive approach and model the interest rate by a Vasicek-model just like Longstaff and Schwartz (1995).

In Table 21 we observe that the four percent flat rate is a bit to high for the maturities of the bonds that we consider. For both firms one might consider the results rather unexpected to some extent. The mispricing is not linear with respect to the maturity in the sense that when reducing the maturity one expects lower bond prices. When starting with the prices of Wal-Mart 2 we expect bonds with shorter maturities to be underpriced more heavily with respect to the actual bond prices. This indeed holds for Wal-Mart 4, but not for Wal-mart 1 and Wal-Mart 3. Here we actually see that the underpricing becomes smaller. Because we are considering the same firm this change cannot be due to other parameters. The price is actually higher than we expected and is than caused by the default probabilities being higher than expected. We then may conclude that for a firm with a relative high asset ratio the short-term default probabilities are relatively low.

Of course we could also reverse this argument. We should in that case start with the price of the bond with the shortest maturity and look at the changes when increasing the maturity. We will not do this because we assume the price of the bonds with relatively long maturities to be better priced when using the default-free interest rate of four percent relatively to the bonds with shorter maturities. The results of Intel are also in line with our conclusion.

6.3.2 Default Asset Ratio

We now change the default threshold of the asset ratio and keep all other parameters equal to the general setting. Note that initially we have set it equal to one, but we now redo the calculations for a lower default threshold, which is just a rather arbitrary lower number, namely 0.95. We do not argue that this is empirically a better or optimal threshold, but we just want to observe the results of such a change in the general parameter value.

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01-2015	-5,81%	-2,37%	11,85%	8,97%	0,18%	-3,28%	4,51%
02-2015	-2,59%	-0,54%	15,75%	10,28%	0,70%	-2,32%	6,96%
03-2015	-6,10%	-0,01%	17,08%	11,46%	1,26%	-4,47%	7,65%
04-2015	-7,48%	-4,47%	14,76%	9,10%	-0,72%	-4,58%	4,00%
05-2015	-5,63%	-1,86%	18,15%	11,96%	-0,22%	-3,87%	6,69%
06-2015	-4,95%	-0,11%	18,58%	14,48%	0,35%	-3,24%	8,31%
07-2015	-6,37%	0,52%	24,34%	15,71%	0,93%	-3,17%	10,56%
Average fit	-5,56%	-1,26%	17,22%	11,71%	0,35%	-3,56%	6,95%

Table 22: Differences between the actual bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2014 and a default asset ratio of 0.95

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01-2015	6,95%	7,97%	11,86%	10,57%	0,18%	-7,73%	4,46%
02-2015	10,76%	10,17%	15,77%	11,97%	0,70%	-6,63%	6,91%
03-2015	6,99%	10,97%	17,10%	13,18%	1,26%	-8,59%	7,60%
04-2015	6,06%	6,49%	14,78%	10,81%	-0,72%	-8,59%	3,95%
05-2015	9,76%	9,53%	18,17%	13,71%	-0,22%	-7,68%	6,64%
06-2015	12,47%	11,90%	18,61%	16,31%	0,35%	-6,96%	8,26%
07-2015	12,21%	12,92%	24,37%	17,59%	0,93%	-6,69%	10,51%
Average fit	9,31%	9,99%	17,24%	13,45%	0,35%	-7,55%	6,91%

Table 23: Differences between the actual bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2004 and a default asset ratio of 0.95

Both tables above should be compared to the general setting with a default asset ratio threshold of one and a 'flat'-rate of two percent. For the first set of estimators we observe smaller deviations of the bond prices in absolute value than for the initial setting. This is unfortunately not the case for the second set of estimators. The change in the default threshold causes upward shifts for all bonds prices, because the default probabilities have all decreased. The second set of parameters initially had a better fit, but due to the upward shift the first set now has a better fit. This overall fit of the bond prices due to the estimated parameters seems to be rather arbitrary and we are not able to draw any conclusion from it. Due to the dependence of the default threshold on the estimated parameters we can not draw a general conclusion for this value of the default threshold with respect to the initial value. Again we note that the bonds of Boeing and IBM react most heavily because of their relatively high default probabilities in the initial setting. 3M and Pfizer did change slightly but the other bond prices remained rather equal to the previous estimates.

Let us now redo the computations for the various bond for Intel and Wal-Mart with a default asset ratio of 0.95. The results are displayed in the table below.

	Intel 1	Intel 2	Intel 3	Wal-Mart 1	Wal-Mart 2	Wal-Mart 3	Wal-Mart 4
01-2015	-3,50%	-4,35%	28,46%	4,04%	21,46%	-0,99%	8,40%
02-2015	-2,96%	-2,54%	34,98%	6,48%	26,54%	-0,49%	9,45%
03-2015	-2,75%	-2,33%	34,62%	7,16%	25,73%	0,01%	10,29%
04-2015	-3,62%	-4,92%	28,44%	3,52%	20,73%	-2,39%	6,43%
05-2015	-3,40%	-3,84%	38,18%	6,20%	28,44%	-1,82%	9,43%
06-2015	-3,05%	-2,43%	45,39%	7,81%	32,32%	-0,89%	12,27%
07-2015	-2,98%	-2,39%	46,65%	10,04%	32,94%	-0,61%	13,84%
Average fit	-3,18%	-3,26%	36,67%	6,46%	26,88%	-1,03%	10,01%

Table 24: Differences between the actual bond prices for 2015 of Intel and Wal-Mart and the bond prices as estimated by the model based on parameters estimated on data period 1971-2014 and a default asset ratio of 0.95

The results of the bond of Intel and Wal-Mart are not that different from the results in the initial setting. The change of the default asset ratio threshold does not affect the default probabilities of both firms by much, because these default probabilities are already very small. The change of the asset ratio is therefore not providing any big observable difference with the initial case. A conclusion that we can draw from these results is that for some firms the default probabilities, as computed by the model, are that small that small changes in the default threshold are not very relevant.

6.3.3 Recovery Rate

In our model we also use the recovery rate which we assumed to be the same for all firms and all bonds. We will still assume this, but we redo the computations with a different value. In the initial setting we used a recovery rate equal to 50 percent, which is supported by Altman et al. (2005). We set the recovery rate now equal to forty percent, which is still a realistic value according to Altman et al. (2005). The results of these computations are given by the tables below.

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01-2015	-23,21%	-15,93%	11,67%	7,23%	0,18%	-5,15%	4,49%
02-2015	-20,70%	-14,54%	15,57%	8,49%	0,70%	-4,12%	6,94%
03-2015	-24,43%	-14,10%	16,89%	9,64%	1,26%	-6,21%	7,63%
04-2015	-25,71%	-18,52%	14,58%	7,32%	-0,72%	-6,30%	3,98%
05-2015	-23,90%	-16,46%	17,96%	10,10%	-0,22%	-5,55%	6,67%
06-2015	-23,38%	-14,85%	18,38%	12,61%	0,35%	-4,95%	8,29%
07-2015	-24,86%	-14,41%	24,13%	13,82%	0,93%	-4,95%	10,54%
Average fit	-23,74%	-15,54%	17,02%	9,89%	0,35%	-5,32%	6,93%

Table 25: Differences between the actual bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2014 and a Recovery Rate of 40 percent

	Boeing	IBM	MCD	3M	Nike	Pfizer	WMT
01-2015	-8,66%	0,60%	11,70%	9,95%	0,18%	-11,64%	4,37%
02-2015	-5,99%	2,74%	15,61%	11,34%	0,70%	-10,37%	6,81%
03-2015	-10,31%	3,38%	16,94%	12,53%	1,26%	-12,12%	7,50%
04-2015	-11,66%	-0,90%	14,62%	10,17%	-0,72%	-11,91%	3,85%
05-2015	-9,36%	1,97%	18,01%	13,06%	-0,22%	-10,91%	6,54%
06-2015	-8,13%	4,11%	18,44%	15,65%	0,35%	-10,18%	8,16%
07-2015	-8,86%	5,15%	24,19%	16,92%	0,93%	-9,74%	10,39%
Average fit	-9,00%	2,43%	17,07%	12,80%	0,35%	-10,98%	6,80%

Table 26: Differences between the actual bond prices for 2015 of the DJ15 market and the bond prices as estimated by the model based on parameters estimated on data period 1994-2004 and a Recovery Rate of 40 percent

Comparing Table 25 with the results of the general setting we observe that the bonds of Boeing and IBM are estimated much worse and the others only have minor differences. This seperation between the bonds is of course caused by the low asset ratios of Boeing and IBM, which cause the default probabilities to be relatively high. High default probabilities cause the bond price to be more dependent on the recovery rate and we therefore see a big drop for these two bonds. Clearly the change of the recovery rate has a negative impact on the estimated bonds for Boeing and IBM and roughly no effect on the estimated bond prices of the other firms. For the results in Table 26 we observe roughly the same changes, which also do not improve the overall fit. We may therefore conclude that the general setting where the recovery rate equals fifty percent is reasonable. For the sake of completeness we now also redo the computations with a recovery rate of forty percent for the 43-years Intel and Wal-Mart case. The results are stated in the table below.

	Intel 1	Intel 2	Intel 3	Wal-Mart 1	Wal-Mart 2	Wal-Mart 3	Wal-Mart 4
01-2015	-4,25%	-6,46%	24,92%	3,52%	19,77%	-1,00%	7,48%
02-2015	-3,71%	-4,68%	31,26%	5,94%	24,78%	-0,49%	8,52%
03-2015	-3,53%	-4,58%	30,76%	6,63%	24,01%	0,00%	9,37%
04-2015	-4,49%	-7,31%	24,46%	3,00%	19,07%	-2,39%	5,53%
05-2015	-4,29%	-6,30%	33,84%	5,66%	26,66%	-1,82%	8,51%
06-2015	-3,84%	-4,78%	41,03%	7,27%	30,50%	-0,89%	11,33%
07-2015	-3,72%	-4,68%	42,35%	9,51%	31,15%	-0,61%	12,92%
Average fit	-3,98%	-5,54%	32,66%	5,93%	25,14%	-1,03%	9,09%

Table 27: Differences between the actual bond prices for 2015 of Intel and Wal-Mart and the bond prices as estimated by the model based on parameters estimated on data period 1971-2014 and a Recovery Rate of 40 percent

We already knew that both firms have a relatively high asset ratio and therefore relatively low default probabilities. The dependence of the bond price on the recovery rate is therefore not very large and the change of the recovery rate is therefore also not very apparent in these results.

7 Momentum Strategy

In the previous sections we have estimated parameters and used the model to compute default probabilities and bond prices. Due to the data set being small, the parameters where not statistically significant and also the bond prices of the model where overall not extremely close to the actual bond prices. Now we want to determine whether the default probabilities can still be useful for an investor. In this section we will use the default probabilities to construct several momentum strategies and we want to determine whether we are able to improve the returns by setting additional constraints on the default probabilities as computed by our model.

The momentum strategy is a well-known investment strategy.[7] The strategy is basically based on exploiting the anomaly that when asset prices rise they have a larger probability of rising again in the next period than falling. Most momentum strategies are based on monthly returns of asset prices, which are rather short time periods. For longer time periods the literature states that past winners turn out to be losers in the next years and vice versa. DeBondt and Thaler

[7]For an overview of momentum strategies we refer to Jegadeesh, N., & Titman, S. (1993).

(1985) find this effect. When we define our strategies later on, we will therefore consider both strategies investing in ‘winners’ and also strategies investing in ‘losers’.

Momentum strategies can be defined in many different ways, but the basic idea of momentum strategy is to buy the ‘winners’ and to short sell the ‘losers’. One of course could also choose to only distribute wealth over all ‘winners’, which is less risky than if one also short sells the ‘losers’. Momentum strategies may also differ due to how they define the ‘winners’ and ‘losers’. These definitions can namely be influenced by factors like for example the period that is taken into account to determine the ‘winners’ and the ‘losers’. We will refer to the amount of periods the strategy takes into account for determining the ‘winners’ as the momentum.

Momentum strategies for stocks are basically trying to determine which stocks are likely to become more expensive in the next period and which are likely to become less expensive based on passed price changes. Increasing stock prices of long positions and decreasing stock prices of short positions yield positive returns of the momentum strategy. The momentum strategies stated below will redistribute the wealth every period to the bonds that satisfy the constraint of the strategy. The strategies distribute the total wealth evenly over all bonds satisfying the momentum constraint and then sell the whole portfolio after the next period ends. It is therefore also for these bond momentum strategies important to buy bonds that will be worth more in the next period. The wealth obtained from selling the portfolio is instantaneously redistributed over those bonds that satisfy the constraints in the next period. Note that the strategies have equal wealth in all bonds satisfying the constraint. If there is no bond satisfying the constraint we just assume the investor invests his wealth completely in the default-free asset until the end of the period. In the computations below we will assume the general parameters to equal the values in the general setting of our model and we re-balance the portfolio annually.

Let us now define our momentum trading strategies. We define four different non-short selling momentum strategies, where we let the momentum vary from one up to three periods for each strategy. We therefore get in fact twelve trading strategies. In contrast to other momentum strategies we use the ‘risk-neutral’ default probabilities of the firm as a base for determining the ‘winners’. This causes the strategies to be related to the behavior of the asset ratios of the firms, which should behave in the same way as the bond prices according to our model.

The first strategy buys bonds of firms of which the default probabilities have decreased during all the periods considered by the momentum of the strategy. The second strategy is quite similar, but it buys bonds of firms of which the default probabilities have decreased in the previous period but have increased in the periods before. Note that these strategies yield the same strategy for a momentum of only one period, but they are obviously different for a momentum of two and three. The third strategy is just the opposite of the first strategy. The third strategy buys bonds bonds of firms of which the default probabilities have increased during all the periods of the momentum. Again the fourth strategy buys bonds of firms of which the default probabilities have increased in the previous period but decreased in the periods which were before that period, but were still within the momentum.

7.1 Computations

We will use data for the years 1994-2014. The first ten years are used to estimate the parameters and for the other years we only use the actual asset ratios to determine the one-year default probabilities. We choose to use the risk-neutral default probabilities and we only consider one-year default probabilities, because the strategy re-balances after every year. The only risk of buying some bond is therefore a default of the firm in the next year. As stated earlier our data set only contains bond data starting at 2005 for seven bonds of the firms. We will therefore only consider these seven bonds in the analysis in this section. Using the estimated parameters we compute the one-year default probabilities for every firm using the asset ratios as stated by their annual report. We use the 'risk-neutral' default probabilities, which we also used to define the momentum strategies. So in the computations we use data of actual bond prices and one-year default probabilities of the firms computed be the model. We should also carefully note that the results will be subject to the survival bias.

For every strategy we will compute the return for each year. The return value stated in the results is just the average over all returns and the risk of the returns is the standard deviation over the returns. After computing the results for the strategies as stated above we impose another restriction of the strategy by means of the one-year default probability. First we impose the restriction of an upper bound on the one-year default probabilities and redo the calculations and compare these with the returns of the strategy without this extra restriction. We also redo calculations with a lower bound. Note that we consider either a lower or an upper bound, but not both at the same time. This would lead to very large tables and it does not lead to more interesting results. The results of all these strategies are stated in the tables below.

		normal	UB= 10ˆ(-50)	10ˆ(-20)	10ˆ(-7)	10ˆ(-5)	0.001	0.01	0.1
momentum 1	Return	0,0716	0,0172	0,0632	0,0761	0,0745	0,0731	0,0716	0,0716
	Risk	0,0590	0,0211	0,0671	0,0690	0,0638	0,0596	0,0590	0,0590
momentum 2	Return	0,0734	0,0172	0,0597	0,0710	0,0725	0,0734	0,0734	0,0734
	Risk	0,0684	0,0211	0,0690	0,0714	0,0686	0,0684	0,0684	0,0684
momentum 3	Return	0,0526	0,0172	0,0542	0,0510	0,0526	0,0526	0,0526	0,0526
	Risk	0,0620	0,0211	0,0670	0,0642	0,0620	0,0620	0,0620	0,0620

Table 28: Results of the first momentum trading strategy with an upper bound on the default probabilities

		LB= 10ˆ(-50)	10ˆ(-20)	10ˆ(-7)	10ˆ(-5)	0.001	0.01	0.1
momentum 1	Return	0,0718	0,0703	0,0385	0,0508	0,0195	0,0200	0,0200
	Risk	0,0588	0,0494	0,0401	0,0611	0,0034	0,0000	0,0000
momentum 2	Return	0,0743	0,0512	0,0222	0,0247	0,0200	0,0200	0,0200
	Risk	0,0681	0,0524	0,0176	0,0148	0,0000	0,0000	0,0000
momentum 3	Return	0,0513	0,0212	0,0175	0,0200	0,0200	0,0200	0,0200
	Risk	0,0610	0,0150	0,0080	0,0000	0,0000	0,0000	0,0000

Table 29: Results of the first momentum trading strategy with a lower bound on the default probabilities

In the two tables above we have displayed all the results for our first momentum strategy. The returns of the unrestricted strategies are very high compared to the two percent default-free interest rate, but the corresponding risks are also high. By imposing an upper-bound on the default probabilities we manage to improve the returns of the momentum strategy for the momentum one and three. Our main interest is whether we are able to obtain a higher return for the restricted case compared to the unrestricted case. To compare the highest return of the restricted strategies with the return of the unrestricted strategy we use the Sharpe Ratio. For the unrestricted momentum one and three we obtain a Sharpe Ratio of 0.8746 and 0.5258 respectively. For the corresponding restricted strategies with the highest return we obtain Sharpe Ratios of 0.8130 and 0.5104 respectively. The higher returns therefore do not lead to improvements when compared by the Sharpe Ratio. For the lower-bound restrictions in Table 29 we observe higher returns of the restricted strategies for the momentum one and two. The Sharpe Ratios of the restricted strategies are 0.8810 and 0.7974 respectively, and for the unrestricted momentum one and two, the Sharpe Ratios are 0.8746 and 0.7807 respectively. The Sharpe Ratios are higher for both restricted strategies. We therefore may conclude that we have improved the performance of the first strategy by setting a lower bound.

		normal	UB= 10ˆ(-50)	10ˆ(-20)	10ˆ(-7)	10ˆ(-5)	0.001	0.01	0.1
momentum 2	Return	0,0525	0,0200	0,0417	0,0471	0,0431	0,0544	0,0525	0,0525
	Risk	0,0392	0,0000	0,0392	0,0398	0,0313	0,0406	0,0392	0,0392
momentum 3	Return	0,0244	0,0200	0,0249	0,0260	0,0260	0,0261	0,0244	0,0244
	Risk	0,0094	0,0000	0,0140	0,0134	0,0134	0,0135	0,0094	0,0094

Table 30: Results of the second momentum trading strategy with an upper bound on the default probabilities

		LB= 10ˆ(-50)	10ˆ(-20)	10ˆ(-7)	10ˆ(-5)	0.001	0.01	0.1
momentum 2	Return	0,0525	0,0531	0,0392	0,0487	0,0191	0,0200	0,0200
	Risk	0,0392	0,0405	0,0378	0,0616	0,0030	0,0000	0,0000
momentum 3	Return	0,0244	0,0242	0,0214	0,0214	0,0191	0,0200	0,0200
	Risk	0,0094	0,0104	0,0082	0,0082	0,0030	0,0000	0,0000

Table 31: Results of the second momentum trading strategy with a lower bound on the default probabilities

In the tables above there are no results for the momentum one of the second strategy, because it would results in just the same returns as for the first strategy. For both momentum two and three we observe returns exceeding the unrestricted strategy, using an upper-bound. The unrestricted strategies have Sharpe Ratios of 0.8291 and 0.4681 respectively. The restricted strategies with the highest return have Sharpe Ratios of 0.8473 and 0.4519 respectively. We therefore have improved the second strategy with a momentum equal to two using an upper-bound. For the lower-bounds we only observe a return exceeding the unrestricted return for momentum two. The Sharpe Ratio for this specific strategy is 0.8173. This is not higher than that of the unrestricted strategy. We therefore conclude that we are able to improve the second momentum strategy by setting some upper-bound.

		normal	UB= 10^(-50)	10^(-20)	10^(-7)	10^(-5)	0.001	0.01	0.1
momentum 1	Return	0,0603	0,0200	0,0551	0,0648	0,0612	0,0525	0,0554	0,0551
	Risk	0,0553	0,0000	0,0602	0,0835	0,0718	0,0552	0,0558	0,0562
momentum 2	Return	0,0553	0,0200	0,0267	0,0534	0,0447	0,0447	0,0453	0,0453
	Risk	0,0451	0,0000	0,0213	0,0710	0,0465	0,0465	0,0473	0,0473
momentum 3	Return	0,0414	0,0200	0,0200	0,0289	0,0289	0,0289	0,0289	0,0289
	Risk	0,0353	0,0000	0,0000	0,0282	0,0282	0,0282	0,0282	0,0282

Table 32: Results of the third momentum trading strategy with an upper bound on the default probabilities

		LB= 10^(-50)	10^(-20)	10^(-7)	10^(-5)	0.001	0.01	0.1
momentum 1	Return	0,0603	0,0514	0,0513	0,0398	0,0383	0,0300	0,0352
	Risk	0,0553	0,0511	0,0397	0,0384	0,0389	0,0357	0,0287
momentum 2	Return	0,0553	0,0550	0,0414	0,0365	0,0365	0,0310	0,0310
	Risk	0,0451	0,0449	0,0283	0,0273	0,0273	0,0275	0,0275
momentum 3	Return	0,0414	0,0414	0,0325	0,0325	0,0325	0,0325	0,0325
	Risk	0,0353	0,0353	0,0263	0,0263	0,0263	0,0263	0,0263

Table 33: Results of the third momentum trading strategy with a lower bound on the default probabilities

We will now compare the results of the unrestricted third strategy for momentum one up to three with the restricted strategies. In Table 32 we only observe a higher restricted return for the momentum one strategy. The Sharpe Ratio of the restricted strategy is 0.5366 which is clearly lower than the Sharpe Ratio of the unrestricted strategy which yields a Sharpe Ratio of 0.7288. For the lower-bound restricted strategies we do not observe any return which is higher than the unrestricted case. We conclude that we are not able to improve the third momentum strategy by using one-year default probability restrictions.

		normal	UB= 10^(-50)	10^(-20)	10^(-7)	10^(-5)	0.001	0.01	0.1
momentum 2	Return	0,0496	0,0200	0,0484	0,0570	0,0569	0,0482	0,0501	0,0498
	Risk	0,0605	0,0000	0,0600	0,0808	0,0758	0,0596	0,0602	0,0606
momentum 3	Return	0,0437	0,0200	0,0389	0,0542	0,0524	0,0437	0,0437	0,0437
	Risk	0,0568	0,0000	0,0599	0,0774	0,0742	0,0568	0,0568	0,0568

Table 34: Results of the fourth momentum trading strategy with an upper bound on the default probabilities

		LB= 10^(-50)	10^(-20)	10^(-7)	10^(-5)	0.001	0.01	0.1
momentum 2	Return	0,0496	0,0415	0,0453	0,0293	0,0278	0,0189	0,0242
	Risk	0,0605	0,0553	0,0495	0,0342	0,0343	0,0222	0,0131
momentum 3	Return	0,0437	0,0350	0,0352	0,0215	0,0200	0,0200	0,0200
	Risk	0,0568	0,0496	0,0431	0,0048	0,0000	0,0000	0,0000

Table 35: Results of the fourth momentum trading strategy with a lower bound on the default probabilities

In the tables above one finds the results of the fourth momentum strategy. We first compare the unrestricted strategies with the upper-bound restricted strategies. For both the momentum two and three we observe returns of a restricted strategy exceeding the return of the unrestricted strategy. The Sharpe Ratios for the unrestricted strategies are 0.4893 and 0.4173 for respectively the momentum two and three strategy. The Sharpe Ratios of the restricted strategies are 0.4579 and 0.4419. We therefore have improved the momentum three strategy but not the momentum two strategy. For the lower-bound restrictions we do not observe returns exceeding the unrestricted returns. Overall we conclude that we are able to improve also the fourth momentum strategy.

8 Summary of Results

In this thesis we started by building a theoretical framework and in the later sections we exposed the model to actual market data. Under our assumptions we developed a rather intuitive event of default. From the model we also derived a rather simple equation for the price of a zero-coupon bond. To determine the price of such a bond we defined the asset ratio under the 'real-world' measure and the 'risk-neutral' measure. In the theoretical part we already noticed that the change of measure is just a change of drift and that the drift of the asset ratio under the 'risk-neutral' measure is lower if the drift of the total assets is higher than the drift of the total liabilities under the 'real-world' measure. And it is lower when the opposite holds. Of course when only considering the drift, a higher drift of the asset ratio results in a lower default probability of the firm when everything else is kept equal.

In the empirical section we find firms where the 'risk-neutral' default probabilities are higher than the 'real-world' default probabilities and also firms where it is the opposite way around. We graphically displayed this in section 5.2.1. In that section we also observed that for some firms the default probabilities are extremely low, such that the bond of such a firm can be interpreted more or less as risk-less with respect to the default of the firm. We have estimated the model on a subset of the Dow Jones which we have referred to as the DJ15. We again find multiple firms for both changes in default probabilities when changing the measure. For the 1994-2004 estimation period we found seven firms that were in line with the theory of the premium and for the 1994-2014 estimation period we only found four. We observed that these four were actually a subset of the seven firms of the other estimation period. We also found that the general market risk factor is for some firms only capturing a small part of the total risk influencing the total assets and total liabilities and for other firms a more substantial amount. This result indicates that, contrary to our initial beliefs, there are firms for which the risk in the total assets and liabilities is only slightly influenced by a common factor. On the other hand we also observe firms for which the general market factor indeed explains a substantial amount of the risk of the total assets and total liabilities.

For the case where we used 43 years of data we found that the estimates of Intel are in line with the theory of the risk-premium and those of Wal-Mart are only slightly deviating. Of course we can not conclude anything from only these two firms, but it seems to indicate that when we use a lot of data and our estimates are closer to the 'true' values, the theory of the risk-premium holds.

In the bond pricing section we find that bond prices of some firms are more sensitive to the estimated parameters than other firms due to the default probabilities the model implies. We observe that the bond prices of some firms are almost insensitive to the parameters estimators due to the low default probabilities of these firms. From the computations we actually also observe that the default probabilities as computed by the model, are rather high for firms with respectively low asset ratios compared to the default probabilities as incorporated by the market prices. We also find that the default probabilities of firms with respectively high asset ratios are too low for short-term maturities with respect to the default probabilities as incorporated by the market prices. In computing the bond prices we used an algorithm to approximate the bond prices. There we used monthly time intervals, which can of course be reduced to smaller periods to obtain possibly bond prices closer to the actual prices. One should mind that when decreasing the time-steps, more computation power is needed. From the results in section 6.3 we clearly observe dependence of the differences in the average fit of a bond with respect to the default-free interest rate. We find that the prices of bonds with short-term and long-term maturities cannot be fitted well by using one 'flat' default-free interest rate. Changing the default threshold does not yield results from which we could conclude that the market default threshold is different from our theoretical threshold value. By redoing our computations with another value for the recovery rate, we observed that there is no strong reason to deviate from the initial value of fifty percent.

In the trading strategy section we did not perform a complete investigation of our momentum strategies. This could have been done more extensively to be able to get very nice results, but our main goal was to show that we are able to improve these strategies by imposing extra conditions on the one-year default probabilities of the firms. We clearly obtained some returns that exceeded the 'unrestricted' strategies and which were also performing better by comparing them with the Sharpe Ratio. We should mind that we only used some arbitrary boundaries. One could also determine the boundary that optimizes the return and gives a higher Sharpe Ratio than the 'unrestricted' strategy. We also did not specify any boundary for which every strategy will be improved and we should also mind that these results do not imply same results for other data sets. We want to stress that the only purpose of section seven is to show that we are able to improve our proposed momentum strategies by imposing extra constraints on the 'risk-neutral' default probabilities. It is of course interesting to develop the idea of that section more extensively, which we leave to future research.

9 Conclusions and Recommendations

This thesis is mainly about pricing corporate risky debt and we want to compute prices as close as possible to actual market prices. We have constructed a model that we test empirically. The model that we propose is able to compute bond prices which can become close to actual bond prices for some parameter values, but the prices may also deviate a lot for other parameter values. We therefore conclude that we are able to compute bond prices close to actual prices, but we are not able to do so for all bonds with the same general parameter values.

We are convinced that the performance of the model can be improved such that it will fit the actual bond prices even better. This could be done by using more data, extending the model by allowing for different interest rates for different maturities and by reducing the time-steps in the approximation algorithm. The biggest issue that we find, is that the 'flat' interest rate is a bad approximation of the interest rate of all maturities to such an extent that the model is not able to fit the prices of bonds with different maturities well for just one value of the default-free interest rate.

In the estimation section we also found relatively large estimates for the individual risk with respect to the influence of the general market risk on the assets and the liabilities for some firms. This indicates that not all the firms in the DJ15 market are substantially influenced by the common factor. We should note that there are also firms for which we do observe substantial values for the general market factor. From this result we may conclude that the common factor is not very influential for all firms.

We find that the default probabilities in our model for firms with respectively low asset ratios are relatively high with respect to the default probabilities incorporated in the market prices. We also find that the default probabilities for firms with relatively high asset ratios are too low for short-term maturities with respect to the default probabilities as incorporated in the market prices. For some firms we also observed that the default probabilities are so low that the return on a zero-coupon bond for such a firm is approximately equal to the default-free interest rate. This again indicates that our model provides too low default probabilities for firms with respectively high asset ratios. Overall we conclude that our model overestimates the high default probabilities and underestimates the low default probabilities.

In our computations we are also cautious with the parameters which we estimate on market data. An important conclusion that we may draw from our estimations is that we need more data to obtain statistically significant parameter estimates. This should be achieved by using the balance sheets of quarterly reports. Based on our results we expect that when using more data, the parameter estimates will be closer to the 'true' values and be in line with the theory of the risk-premium. This theory uses the basic property of risk premiums from microeconomics to explain why 'risk-neutral' default probabilities can be either greater or smaller than 'real-world' default probabilities based on the risk-premium investors demand depending on the correlation of the asset with the general market.

The data set that we used in this thesis is based on the firms in the Dow Jones index. The Dow Jones includes lots of companies from very different industries. It might be a better idea to collect data from firms within one industry, because one can then use a general market factor which is more industry specific and will therefore affect the firms assets and liabilities probably more which might result in better estimated parameters. A possible extension that we are interested in is how the model would perform in estimating a financial market consisting of multiple banks, insurers and asset managers, because of the inter-linkages of these firms. The assets and liabilities of those firms will clearly be affected heavily by the state of the stock market or other common financial factors.

In the application of the momentum strategy we indeed achieved to improve the momentum strategy by using the default probabilities of the model. Although this is a nice result, it is of course just a beginning. It is very interesting to investigate how investors could use default probabilities of firms to optimize their trading strategies. It may cause them to do a better job at portfolio optimization by having a better understanding of the risk involving different companies.

The theoretical framework which we used might also be useful for institutions or regulators on financial markets. Although they might want to extent the model to be able to compute contagion probabilities, they can use the default probabilities in their assessment of various possible regulatory impacts and on how financial injections like during the financial crisis would save welfare.

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10 Appendices

10.1 A - Proof of theorems

Proof of theorem 3.
Let $f_{i,t} = \frac{A_{i,t}}{L_{i,t}}$, with $A_{i,t}$ and $L_{i,t}$ as defined by their stochastic differential equations given by (1) and (2) respectively.
Using the product rule of Ito-calculus: $\mathrm{d}f_{i,t} = A_{i,t}\mathrm{d}\frac{1}{L_{i,t}} + \frac{1}{L_{i,t}}\mathrm{d}A_{i,t} + \mathrm{d}\langle A_{i,t}, \frac{1}{L_{i,t}}\rangle$
Again by Ito we get: $\mathrm{d}\frac{1}{L_{i,t}} = -\frac{1}{L_{i,t}^2}\mathrm{d}L_{i,t} + \frac{1}{L_{i,t}^3}\mathrm{d}\langle L_{i,t}, L_{i,t}\rangle$
$= -\frac{1}{L_{i,t}^2}\mathrm{d}L_{i,t} + \frac{1}{L_{i,t}^3}\mathrm{d}\langle L_{i,t}\left(\sigma_{L,i}\mathrm{d}W_{L,i,t} + \alpha_i\sigma_G\mathrm{d}W_{G,t}\right), L_{i,t}\left(\sigma_{L,i}\mathrm{d}W_{L,i,t} + \alpha_i\sigma_G\mathrm{d}W_{G,t}\right)\rangle$
$= -\frac{1}{L_{i,t}^2}\mathrm{d}L_{i,t} + \frac{1}{L_{i,t}}\mathrm{d}\langle \sigma_{L,i}\mathrm{d}W_{L,i,t} + \alpha_i\sigma_G\mathrm{d}W_{G,t}, \sigma_{L,i}\mathrm{d}W_{L,i,t} + \alpha_i\sigma_G\mathrm{d}W_{G,t}\rangle$
$= -\frac{1}{L_{i,t}^2}\mathrm{d}L_{i,t} + \frac{1}{L_{i,t}}\left(\sigma_{L,i}^2 + 2\alpha_i\sigma_G\rho(L_i, G) + \alpha_i^2\sigma_G^2\right)\mathrm{d}t$
$= \frac{1}{L_{i,t}}\{(\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2)\mathrm{d}t - (r + \lambda_{L,i}\sigma_{L,i} + \lambda_G\alpha_i\sigma_G)\mathrm{d}t - \sigma_{L,i}\mathrm{d}W_{L,i,t} - \alpha_i\sigma_G\mathrm{d}W_{G,t}\}$
$= \frac{1}{L_{i,t}}\{(\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 - r - \lambda_{L,i}\sigma_{L,i} - \lambda_G\alpha_i\sigma_G)\mathrm{d}t - \sigma_{L,i}\mathrm{d}W_{L,i,t} - \alpha_i\sigma_G\mathrm{d}W_{G,t}\}$
Plugging this result into the first equation:
$\mathrm{d}f_{i,t} = \frac{A_{i,t}}{L_{i,t}}\Big((\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 - r - \lambda_{L,i}\sigma_{L,i} - \lambda_G\alpha_i\sigma_G)\mathrm{d}t - \sigma_{L,i}\mathrm{d}W_{L,i,t} - \alpha_i\sigma_G\mathrm{d}W_{G,t}\Big) +$
$\frac{A_{i,t}}{L_{i,t}}\left(\{r + \lambda_{A,i}\sigma_{A,i} + \lambda_G\beta_i\sigma_G\}\mathrm{d}t + \sigma_{A,i}\mathrm{d}W_{A,i,t} + \beta_i\sigma_G\mathrm{d}W_{G,t}\right) -$
$\frac{A_{i,t}}{L_{i,t}}\mathrm{d}\langle \sigma_{A,i}\mathrm{d}W_{A,i,t} + \beta_i\sigma_G\mathrm{d}W_{G,t}, -\sigma_{L,i}\mathrm{d}W_{L,i,t} - \alpha_i\sigma_G\mathrm{d}W_{G,t}\rangle \Leftrightarrow$
$\frac{\mathrm{d}f_{i,t}}{f_{i,t}} = \Big\{\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 + \lambda_{A,i}\sigma_{A,i} - \lambda_{L,i}\sigma_{L,i} + \lambda_G\sigma_G(\beta_i - \alpha_i)\Big\} + \sigma_{A,i}\mathrm{d}W_{A,i,t} - \sigma_{L,i}\mathrm{d}W_{L,i,t} +$
$\sigma_G(\beta_i - \alpha_i)\mathrm{d}W_{G,t} + \frac{A_{i,t}}{L_{i,t}}\left(-\sigma_{A,i}\sigma_{L,i}\rho(A_i, L_i) - \sigma_{A,i}\alpha_i\sigma_G\rho(A_i, G) - \beta_i\sigma_G\sigma_{L,i}\rho(G, L_i) - \alpha_i\beta_i\sigma_G^2\right)\mathrm{d}t \Leftrightarrow$

$\frac{\mathrm{d}f_{i,t}}{f_{i,t}} = \Big\{\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 + \lambda_{A,i}\sigma_{A,i} - \lambda_{L,i}\sigma_{L,i} + \lambda_G\sigma_G(\beta_i - \alpha_i) - \sigma_{A,i}\sigma_{L,i}\rho(A_i, L_i) - \alpha_i\beta_i\sigma_G^2\Big\}\mathrm{d}t +$
$\sigma_{A,i}\mathrm{d}W_{A,i,t} - \sigma_{L,i}\mathrm{d}W_{L,i,t} + \sigma_G(\beta_i - \alpha_i)\mathrm{d}W_{G,t}$

□

Proof of theorem 4.
We use the result of Theorem 3

$$\begin{aligned}
\frac{\mathrm{d}f_{i,t}}{f_{i,t}} =& \left\{\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 + \lambda_{A,i}\sigma_{A,i} - \lambda_{L,i}\sigma_{L,i} + \lambda_G\sigma_G(\beta_i - \alpha_i) - \sigma_{A,i}\sigma_{L,i}\rho(A_i, L_i) - \alpha_i\beta_i\sigma_G^2\right\}\mathrm{d}t \\
&+ \sigma_{A,i}\mathrm{d}W_{A,i,t} - \sigma_{L,i}\mathrm{d}W_{L,i,t} + \sigma_G(\beta_i - \alpha_i)\mathrm{d}W_{G,t} \\
=& \{\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 - \sigma_{A,i}\sigma_{L,i}\rho(A_i, L_i) - \alpha_i\beta_i\sigma_G^2\}\mathrm{d}t + \sigma_{A,i}\left(\mathrm{d}W_{A,i,t} + \lambda_{A,i}\sigma_{A,i}\mathrm{d}t\right) \\
&- \sigma_{L,i}\left(\mathrm{d}W_{L,i,t} + \lambda_{L,i}\mathrm{d}t\right) + \sigma_G(\beta_i - \alpha_i)\left(\mathrm{d}W_{G,t} + \lambda_G\mathrm{d}t\right) \\
=& \{\sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 - \sigma_{A,i}\sigma_{L,i}\rho(A_i, L_i) - \alpha_i\beta_i\sigma_G^2\}\mathrm{d}t + \sigma_{A,i}\mathrm{d}W_{A,i,t}^{\mathbb{Q}} - \sigma_{L,i}\mathrm{d}W_{L,i,t}^{\mathbb{Q}} \\
&+ \sigma_G(\beta_i - \alpha_i)\mathrm{d}W_{G,t}^{\mathbb{Q}}
\end{aligned}$$

□

Proof of theorem 5.
For this proof we need that $f_{i,t}$ is represented by a Geometric Brownian Motion. To transform Equation 6 to a Geometric Brownian Motion we should combine the three Brownian Motions to one. This can easily be done, because the sum of random normal variables is also a random normal variable. We now derive the combined variance
$Var\left(\sigma_{A,i}\mathrm{d}W_{A,i,t} - \sigma_{L,i}\mathrm{d}W_{L,i,t} + \sigma_G(\beta_i - \alpha_i)\mathrm{d}W_{G,t}\right) = \sigma_{A,i}^2 + \sigma_{L,i}^2 + \sigma_G^2(\beta_i - \alpha_i)^2$
$- 2\sigma_{L,i}\sigma_{A,i}\rho(L_i, A_i) - 2\sigma_{L,i}\sigma_G(\beta_i - \alpha_i)\rho(L_i, G) + 2\sigma_{A,i}\sigma_G(\beta_i - \alpha_i)\rho(A_i, G) =$
$\sigma_{A,i}^2 + \sigma_{L,i}^2 + \sigma_G^2(\beta_i - \alpha_i)^2 - 2\sigma_{L,i}\sigma_{A,i}\rho(A_i, L_i)$
Where the last equality holds by our identifying restriction of $\rho(A_i, G) = 0$ and $\rho(L_i, G) = 0$.
We therefore have a Geometric Brownian Motion with
$\mu_{f,i} = \sigma_{L,i}^2 + \alpha_i^2\sigma_G^2 + \mu_{A,i} - \mu_{L,i} - \sigma_{L,i}\sigma_{A,i}\rho(A_i, L_i) - \alpha_i\beta_i\sigma_G^2$
$\sigma_{f,i} = \sqrt{\sigma_{L,i}^2 + \sigma_{A,i}^2 + \sigma_G^2(\beta_i - \alpha_i)^2 - 2\sigma_{L,i}\sigma_{A,i}}$
This proof uses a general Geometric Brownian Motion representation and the specific parameters may be plugged into the result for both measures. Let $T_{a,b}$ be the time a Brownian Motion first hits the line $a + bt$. We will rewrite the Geometric Brownian Motion and define some a and b. The probability of $T_{a,b} \leq t$ has a general formula, for which we refer to the Handbook of Brownian Motion by Borodin & Salminen.
For a Geometric Brownian Motion f_t we have that $f_t = f_0\, e^{(\mu_f - \frac{1}{2}\sigma_f^2)t + \sigma_f W_t}$.

$$f_t \leq 1 \Leftrightarrow f_0\, e^{(\mu_f - \frac{1}{2}\sigma_f^2)t + \sigma_f W_t} \leq 1 \Leftrightarrow \left(\mu_f - \frac{1}{2}\sigma_f^2\right) t + \sigma_f W_t \leq \log(\frac{1}{f_0})$$
$$\Leftrightarrow W_t \leq \frac{\log(\frac{1}{f_0}) - (\mu_f - \frac{1}{2}\sigma_f^2)t}{\sigma_f} = \frac{1}{\sigma_f}\log(\frac{1}{f_0}) - \frac{\mu_f - \frac{1}{2}\sigma_f^2}{\sigma_f}\, t$$

We therefore have a linear boundary $a + bt$. The general formula which we will apply should have $a > 0$ and $-\infty \leq b \leq \infty$. In our case a is negative, but because the Brownian Motion is normally distributed with a zero mean, it is symmetric and we multiply the boundary by -1. Due to the symmetric property of the Brownian Motion the probability of hitting $-a - bt$ is the same as for hitting $a + bt$. We therefore have that $a = -\frac{1}{\sigma_f}\log(\frac{1}{f_0})$ and $b = \frac{\mu_f}{\sigma_f} - \frac{1}{2}\sigma_f$. We now use that $P(T_{a,b} \leq t) = 1 - \Phi\left(\frac{bt-a}{\sqrt{t}}\right) + e^{-2ab}\,\Phi\left(\frac{bt-a}{\sqrt{t}}\right)$ to conclude with the probability of default:

$$\mathrm{P}(\inf_{0 \leq s \leq t} f_{i,s} \leq 1) = P(T_{a,b} \leq t) = 1 - \Phi\left(\frac{bt + a}{\sqrt{t}}\right) + e^{-2ab}\,\Phi\left(\frac{bt - a}{\sqrt{t}}\right)$$
$$\text{where } a = -\frac{1}{\sigma_{f,i}}\log(\frac{1}{f_0}), \quad b = \frac{\mu_{f,i}}{\sigma_{f,i}} - \frac{1}{2}\sigma_{f,i}$$

□

Proof of theorem 7.
By the first fundamental theorem of asset pricing we have that the price of a contract equals the present value of the expected payoff, i.e.: $price = \mathbb{E}_t^{\mathbb{Q}_B}\left[value(m) \cdot e^{-r \cdot m}\right]$, where $\mathbb{E}_t^{\mathbb{Q}_B}$ denotes the expectation operator at time t under the risk-neutral measure with the default-free bond as

numéraire and where the payoff is some nominal value at time m. Note that due to the measure $\mathbb{Q}_B$ and our 'flat' term structure assumption, we can just take the discounting term out of the expectation operator and that the probabilities of default are under the 'risk-neutral' measure, i.e. $q_{i,t}$. The 'normalized zero-coupon bond' pays off 1 at time m if the company is not bankrupt and the present value of one times the recovery rate R at the moment the company defaults. In the proof below we will denote the time of default by T_d:

$$\begin{aligned} B_{i,t} = \mathbb{E}^{\mathbb{Q}_B}\left[\int_0^t \text{payoff}(s) \cdot e^{-r \cdot s} \mathrm{d}s\right] &= \mathbb{E}^{\mathbb{Q}_B}\left[\text{payoff if default}(T_d) + \text{payoff if no default}(t)\right] \\ &= q_{i,t} \cdot e^{-r \cdot T_d} \cdot R \cdot e^{-r(t-T_d)} + (1-q_{i,t}) \cdot 1 \cdot e^{-rt} \\ &= q_{i,t} \cdot R \cdot e^{-rt} + (1-q_{i,t}) \cdot e^{-rt} \qquad \forall t > 0 \end{aligned}$$

□

10.2 B - Results of Computations

For the estimation period of 1994-2004 we have that $\mu_G = 0,1152$, $\sigma_G = 0,1986$ and $\lambda_G = 0,4794$ and for the estimation period of 1994-2014 we have that $\mu_G = 0,0938$, $\sigma_G = 0,1979$ and $\lambda_G = 0,3730$, where we assume $r = 2$ for both prices of risk.

	α_i	β_i	$\rho(A_i, L_i)$	$\lambda_{A,i}$	$\lambda_{L,i}$	$\sigma_{A,i}$	$\sigma_{L,i}$	Drift $dA_{i,t}$	Drift $dL_{i,t}$	$\beta_i * \sigma_G$	$\alpha_i * \sigma_G$
Apple Inc	-0,1111	0,0347	0,5734	0,1874	0,0424	0,1702	0,1779	0,0552	0,0170	0,0069	-0,0221
Boeing	-0,0622	-0,0090	0,9573	0,4736	0,6634	0,0869	0,1234	0,0603	0,0959	-0,0018	-0,0124
Walt Disney	0,6506	0,7395	0,9943	0,2514	0,2363	0,4257	0,4615	0,1974	0,1910	0,1469	0,1292
Home Depot	0,1520	0,0660	-0,2588	3,3935	1,3470	0,0545	0,1386	0,2113	0,2212	0,0131	0,0302
IBM	0,0268	-0,0362	0,8521	0,4380	0,2291	0,0369	0,0462	0,0327	0,0331	-0,0072	0,0053
Intel	0,2638	0,3178	0,8339	0,6407	0,2395	0,1411	0,2040	0,1406	0,0940	0,0631	0,0524
JPMorgan	0,5538	0,5734	0,9973	0,3970	0,3870	0,2406	0,2436	0,1701	0,1670	0,1139	0,1100
Coca-Cola	-0,1855	-0,1281	0,7933	1,2664	0,4954	0,0635	0,1370	0,0882	0,0702	-0,0254	-0,0368
McDonald's	0,0751	0,0216	0,4075	1,5696	0,9382	0,0335	0,0502	0,0747	0,0743	0,0043	0,0149
3M	-0,1032	-0,0741	0,4844	0,5139	0,6734	0,0720	0,0632	0,0499	0,0527	-0,0147	-0,0205
Microsoft	0,3101	0,4797	0,5300	1,7494	1,0096	0,1573	0,3398	0,3408	0,3926	0,0953	0,0616
Nike	-0,0197	0,0946	0,8630	0,7772	0,6099	0,1359	0,3074	0,1346	0,2056	0,0188	-0,0039
Pfizer	-0,7461	-1,1196	0,9554	1,0058	1,2098	0,3628	0,2200	0,2783	0,2151	-0,2223	-0,1482
UHG	-0,7354	-0,0101	0,8359	0,9692	0,6057	0,2369	0,9796	0,2487	0,5433	-0,0020	-0,1460
Wal-Mart	0,0988	0,0388	0,9897	1,1537	0,7107	0,1117	0,1806	0,1526	0,1577	0,0077	0,0196

Table 36: Estimated parameter values of the DJ15 market, based on the period 1994-2004

	α_i	β_i	$\rho(A_i, L_i)$	$\lambda_{A,i}$	$\lambda_{L,i}$	$\sigma_{A,i}$	$\sigma_{L,i}$	Drift $dA_{i,t}$	Drift $dL_{i,t}$	$\beta_i * \sigma_G$	$\alpha_i * \sigma_G$
Apple Inc	-0,4513	-0,3047	0,7899	1,0102	0,8783	0,2401	0,3060	0,2335	0,2458	-0,0605	-0,0896
Boeing	-0,0947	0,0750	0,7720	0,3782	0,7571	0,0907	0,0984	0,0614	0,0855	0,0149	-0,0188
Walt Disney	0,4219	0,5099	0,9868	0,1751	0,1484	0,3089	0,3352	0,1226	0,1099	0,1013	0,0838
Home Depot	0,2930	0,1544	0,5248	0,5866	0,5985	0,1258	0,1694	0,1085	0,1493	0,0307	0,0582
IBM	-0,0262	0,0489	0,7119	-0,0779	0,2221	0,0577	0,0694	0,0202	0,0329	0,0097	-0,0052
Intel	0,1256	0,3085	0,6488	0,4852	0,3913	0,1160	0,2565	0,1056	0,1323	0,0613	0,0250
JPMorgan	0,1042	0,1212	0,9981	0,4765	0,4751	0,2083	0,2095	0,1308	0,1295	0,0241	0,0207
Coca-Cola	-0,0139	0,0255	0,9518	0,5851	0,4433	0,1452	0,2368	0,1074	0,1237	0,0051	-0,0028
McDonald's	0,0270	0,0679	0,5405	0,4587	0,7304	0,0475	0,0527	0,0483	0,0610	0,0135	0,0054
3M	-0,2036	-0,0575	0,1555	0,4875	0,8109	0,0662	0,0703	0,0468	0,0576	-0,0114	-0,0404
Microsoft	0,2380	0,3610	0,5693	0,7434	0,8560	0,2058	0,2727	0,2074	0,2761	0,0717	0,0473
Nike	-0,0479	0,0333	0,8366	0,8788	0,5946	0,1016	0,2314	0,1125	0,1530	0,0066	-0,0095
Pfizer	-0,2101	-0,3875	0,8995	0,4859	0,4495	0,3755	0,3508	0,1656	0,1577	-0,0770	-0,0417
UHG	-0,3639	-0,0075	0,7904	0,8480	0,4868	0,1984	0,7162	0,1875	0,3340	-0,0015	-0,0723
Wal-Mart	0,0406	0,0222	0,9634	0,9499	0,6421	0,0955	0,1450	0,1128	0,1170	0,0044	0,0081

Table 37: Estimated parameter values of the DJ15 market, based on the period 1994-2014

10.3 C - Additional Details

	Dow Jones 30	Dow Jones 15
firms	Apple Inc.	Apple Inc.
	American Express Company	The Boeing Company
	The Boeing Company	Walt Disney Co
	Caterpillar Inc.	Home Depot Inc.
	Cisco System, Inc.	International Business Machines Corp.
	Chevron Corporation	Intel Corporation
	E I Du Pont De Nemours and Co	JPMorgan Chase & Co.
	Walt Disney Co	The Coca-Cola Co.
	General Electric Company	McDonald's Corporation
	Goldman Sachs Group Inc.	3M Co.
	Home Depot Inc.	Microsoft Corporation
	International Business Machines Corp.	Nike Inc.
	Intel Corporation	Pfizer Inc.
	Johnson & Johnson	UnitedHealth Group Inc.
	JPMorgan Chase & Co.	Wal-Mart Stores, Inc.
	The Coca-Cola Co.	
	McDonald's Corporation	
	3M Co.	
	Merck & Co., Inc.	
	Microsoft Corporation	
	Nike Inc.	
	Pfizer Inc.	
	Procter & Gamble Co.	
	Travelers Companies Inc.	
	UnitedHealth Group Inc.	
	United Technologies Corporation	
	Visa Inc.	
	Verizon Communications Inc.	
	Wal-Mart Stores, Inc.	
	Exxon Mobil Corporation	

Table 38: Companies represented in the two characteristic markets