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Optimal prepayment of Dutch mortgages*

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Valuation of the prepayment option in Dutch mortgages is complicated. In the Netherlands, mortgagors are not allowed to prepay the full mortgage loan without a compensating penalty. Only a limited amount of the initial mortgage loan can be prepaid penalty-free. We introduce a general model formulation for the valuation of limited callable mortgages, based on binomial trees. This model can be used for determining both the optimal prepayment strategy and the value of embedded prepayment options. For some mortgage types the prepayment option can be valued exactly, whereas other types require approximative methods for efficient valuation. The heuristic we propose here determines the prepayment option value efficiently and accurately for general mortgage types.

Keywords and Phrases: mortgage valuation, partial prepayments, binomial trees.

1 Introduction

A mortgage loan is a long-term loan secured by a collateral, usually real estate. The mortgagor borrows money from the mortgagee and pays back the loan according to an agreed upon amortization schedule. Typical contracts have a maturity of 30 years. Mortgage contracts are written with various embedded options. For example, at periodic interest rate adjustment dates, borrowers can be guaranteed the lowest interest rate over the last 2 years, or borrowers can set the period over which the rate remains constant. The most important option is the right to prepay the loan before maturity. Rational borrowers will prepay, or refinance, their mortgage if interest rates are sufficiently low.

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Prepayments are an important phenomenon in the Dutch mortgage market. In a series of empirical studies of prepayments in the Dutch market, Alink (2002), Charlier and Van Bussel (2003) and Hayre (2003) report that the proportion of newly issued and refinanced mortgages in the mortgage pool has more than quadrupled in the last 15 years. This increase is completely due to the refinance of existing loans, driven by the significant mortgage rate decrease in this period. Consequently, the importance of optimal, interest rate driven, prepayment and refinancing has increased. But although various articles analyze observed prepayment behavior, less is known about the optimal prepayment policies of borrowers.

Optimal strategies have been derived for mortgage prepayment behavior in the US (see Kau *et al.*, 1990, 1993; McConnell and Singh, 1994). These studies develop techniques that are similar to valuation methods for American options, based on binomial lattices. These techniques are not applicable for Dutch contracts, because US and Dutch contracts differ in one important aspect. In the US, mortgage loans can be prepaid fully and penalty-free. In the Netherlands, mortgagors are only allowed to prepay a fixed percentage (usually 10% or 20%) of the initial mortgage loan each calendar year. If a larger prepayment is made, a penalty equal to the present value of the expected profit of the excess prepayment has to be paid. Therefore, prepaying more than allowed is never optimal.

The partial prepayment option leads to much more complicated optimal prepayment strategies. At any point in time, the option value of future prepayments depends on the history of previous prepayments. This path dependence precludes an efficient solution using a backward recursion in a binomial lattice, as in Kau *et al.* (1990, 1993). Instead, we must apply a non-recombining tree approach, because the history of prepayment decisions is relevant for the current mortgage value, thereby introducing path dependencies.

In this paper, we present two solution methods. The first is a general linear programming (LP) formulation of the problem. This provides an exact optimal prepayment solution within a full binomial tree. As the number of nodes in a full binomial tree grows exponentially, the exact LP solution will not be efficiently computable for large models with many time periods. For an efficient approximate solution, we reduce the full tree by considering a predefined prepayment strategy. This is our second approach, based on a combination of a non-recombining tree method when required and an efficient lattice method when possible. A conceptually similar approach is used by Nielsen and Poulsen (2004), to price mortgage contracts with delivery options (introducing path dependencies), where optimality decisions are only taken in a small subset of time periods. Between decision dates, the unique scenario path of the mortgage value is given by a lattice. At decision dates, the state–space behaves as a tree, branching out and not recombining. The number of states increases exponentially with the number of decision dates.

In the following sections we formulate an LP model for the valuation of partially callable annuity mortgages. The LP formulation can also capture other amortization schemes, such as linear and interest-only mortgages. All time periods in our model allow for prepayment of a part of the mortgage loan, involving the use of a complete non-recombining tree. LP is applied both to obtain an exact mortgage value and prepayment strategy and, using duality theory, to derive bounds on the optimal mortgage value whenever the optimal prepayment strategy cannot be determined efficiently. Section 2 introduces the mathematical framework and builds the LP model. The dual problem is formulated in section 3. The implications of the LP formulation for fully callable mortgages are provided in section 4, based on duality theory. Section 5 solves an accurate heuristic for the original LP model, obtaining bounds on the mortgage value. We also narrow the gap between upper and lower bound on the mortgage value, in order to derive an accurate approximation. Results are provided in section 6.

2 Mathematical framework

The problem is formulated on a non-recombining tree. The states in a non-recombining tree are labelled as in Figure 1. The root node at $t=0$ has label 1, and the two nodes at time $t=1$ are labelled 2 and 3. Generally, the transitions are given by

$$i \begin{array}{l} \nearrow \quad 2i \\ \searrow \quad 2i+1. \end{array}$$

All transitions occur with probability 1/2. The time period $t(i)$ corresponding to state i is

$$t(i) = \lfloor {}^2\log(i) \rfloor.$$

The final period T has nodes $2^T, \ldots, 2^{T+1}-1$. The unique predecessor of state i, if not the root node, is $\lfloor i/2 \rfloor$. A state i, for which $t(i)=T$, is called a leaf node. All nodes that are neither the root node nor a leaf node are called intermediate nodes.

Nodes are associated with one-period interest rates r_i. An interest rate scenario is represented by a path from the root node to a leaf. These interest rates determine the value of cash flows at each node of the tree. Formally, the state price λ_i is defined as the root price of a security that pays out 1 in state i and 0 in all other states. The state price is recursively defined by

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_i &= \frac{1}{2} \frac{\lambda_{\lfloor i/2 \rfloor}}{1+r_{\lfloor i/2 \rfloor}}, \quad i>1 \end{aligned} \tag{1}$$

(see Duffie, 2001). State prices are used for discounting cash flows along a scenario path. The present value of an asset, paying a cash flow of c_i in state i and 0 in

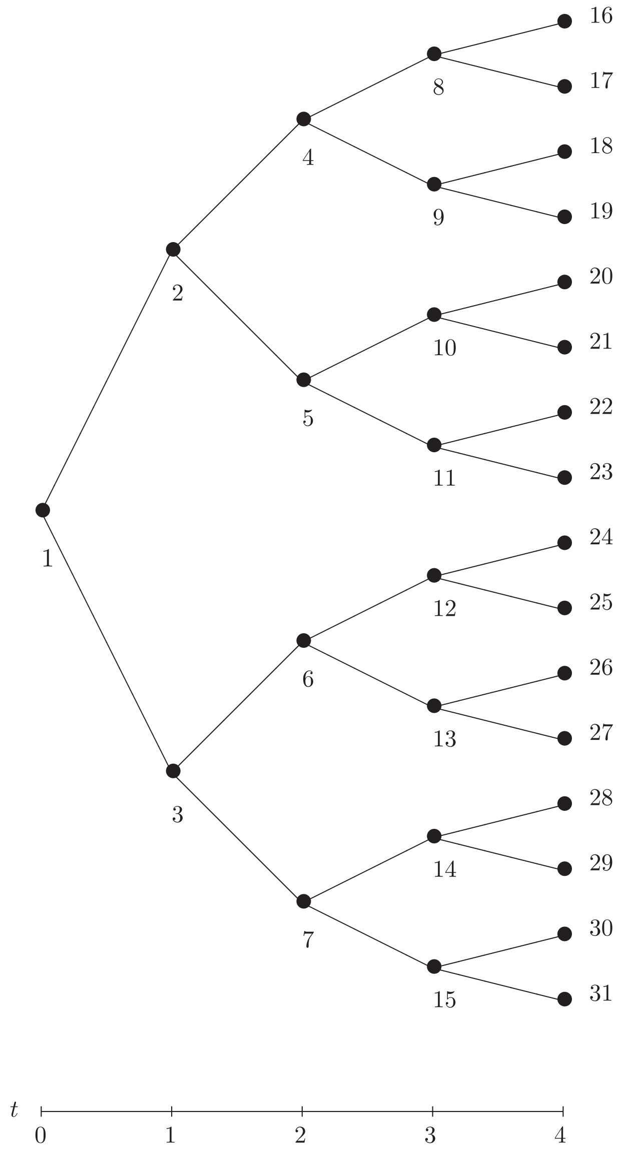

Fig. 1. Non-recombining binomial tree: The figure shows the first four time steps of a non-recombining binomial tree and the applied node labelling. Transitions have probability 1/2.

all other states, is equal to $\lambda_i c_i$. As a mortgage is a portfolio of such assets, the mortgage value

$$V = \sum_i \lambda_i c_i. \tag{2}$$

We next develop a model for the cash flows. The mortgage has a maturity of T periods and starts with a principal U_1 at $t=0$. At every node i the borrower must pay interest at a rate y on the remaining unpaid balance $U_{\lfloor i/2 \rfloor}$ of the previous period. In addition, the borrower makes two additional cash flows: scheduled redemption of the principal and prepayments.

We focus on mortgage contracts with a regular redemption schedule, of which an annuity mortgage is most popular and well known. An annuity contract is characterized by constant scheduled cash flows over the remaining lifetime of the contract. Let $L \geq T$ be the initial maturity of the contract, and $n_i = L - t(i)$ the remaining lifetime at node i. Moreover define the unpaid balance U_i in state i as the amount of money still owed to the mortgagee at this state. The annuity M_i in state i is equal to the unpaid balance in the previous period times the annuity factor

$$f(y, n_i) = \frac{y}{1-(1+y)^{-n_i}},$$

leading to

$$M_i = U_{\lfloor i/2 \rfloor} \cdot f(y, n_i + 1) \tag{3}$$

Without prepayments the cash flow is constant over all states, that is $M_i = M$. In this case, the annuity M implies that the principal is repaid exactly at maturity. Linear and interest-only mortgages are easily modelled by only adapting $f(y, n_i)$ to $y + n_i^{-1}$ and y, respectively.

An annuity mortgage *with partial prepayments* is defined by the principal amount U_1, a periodic contract rate y, a maturity L and K consecutive subintervals I_k of $[0, \ldots, T]$, containing all distinct $t(i)$ in increasing order. The endpoints of all subintervals belong to the set $\{0, \ldots, T\}$ such that each time period is in exactly one interval. The first set starts at $t = 0$, and the last set ends with $t = T$. We consider a single fixed rate period, ending at T. In each interval I_k, the total amount that can be repaid in this interval is restricted to be less than or equal to X_k. In most cases, all intervals have equal length (usually a calendar year) and the prepayment is restricted to at most a fixed percentage α of the principal amount: $X_k = X = \alpha \cdot U_1$ for all K.

The actual prepayment in state i is denoted by x_i. Total cash flow at state i is thus

$$c_i = M_i + x_i. \tag{4}$$

At the end of each fixed rate period, the remaining loan balance can be fully repaid without penalty. The mortgage price in leaf nodes is therefore equal to the remaining unpaid balance. Consequently, in the optimization model the prepayment amount x_i can be set to zero in all leaf nodes. In fact, in leaf nodes the mortgagor is indifferent to prepay. Furthermore, we assume that no prepayment takes place at $t = 0$ (this could be accounted for in the initial loan amount).

The original unpaid balance U_1 can be scaled to 1. The first class of constraints models the unpaid balance in an intermediate state i:

$$\begin{aligned} U_i &= U_{\lfloor i/2 \rfloor} \cdot (1+y) - c_i \\ &= U_{\lfloor i/2 \rfloor} \cdot \big(1 + y - f(y, n_i + 1)\big) - x_i, \quad 1 < i < 2^T. \end{aligned} \tag{5}$$

Every period, the unpaid balance increases at rate y. A regular amount M_i is paid in state i according to (3). Additionally, the mortgagor must decide whether to prepay an amount up to the allowed $X = \alpha \cdot U_1$, with α the maximally allowed prepayment percentage. Because no prepayment occurs in a final state, the unpaid balance in such state equals

$$U_i = U_{\lfloor i/2 \rfloor} \cdot (1 + y - f(y, n_i + 1)), \quad i \geq 2^T.$$

The next class of constraints models the upper bound on the total prepayments within a given time interval. Let us denote by Q_k the set of all paths for which the first state on the path belongs to the layer corresponding to the begin point of interval I_k and the last state on the path belongs to the layer corresponding to the endpoint of the interval I_k, $k = 1, \ldots, K$. Then the additional prepayment amount x_i is restricted by

$$\sum_{i \in Q} x_i \leq X, \quad Q \in Q_k, \ k = 1, \ldots, K. \tag{6}$$

We consider a constant prepayment amount $X = X_k$ and all subintervals make up exactly one calendar year.

An optimal prepayment strategy for the mortgagor is the strategy that minimizes the present value of all payments. These payments include regular payments M_i, additional payments x_i and, if any, redemption of the remaining contract value at the leaf nodes. Payments are discounted by means of the state prices λ_i. The mortgage value is now represented by

$$V = \sum_{i=1}^{2^T - 1} \lambda_i [U_{\lfloor i/2 \rfloor} \cdot f(y, n_i + 1) + x_i] + \sum_{i=2^T}^{2^{T+1} - 1} \lambda_i U_{\lfloor i/2 \rfloor}(1 + y). \tag{7}$$

Now, the LP objective for pricing annuity mortgages with partial prepayments is to

$$\text{minimize} \sum_{i=1}^{2^T - 1} \lambda_i [U_{\lfloor i/2 \rfloor} \cdot f(y, n_i + 1) + x_i] + \sum_{i=2^T}^{2^{T+1} - 1} \lambda_i U_{\lfloor i/2 \rfloor}(1 + y)$$

subject to

$$\begin{aligned}
U_1 &= 1 \\
U_i &= U_{\lfloor i/2 \rfloor} \cdot (1 + y - f(y, n_i + 1)) - x_i, \quad i = 1, \ldots, 2^T - 1 \\
\sum_{i \in Q} x_i &\leq X, \quad Q \in Q_k, \ k = 1, \ldots, K \\
U_i &\geq 0, \quad \forall i \\
x_i &\geq 0, \quad \forall i
\end{aligned}$$

The last two restrictions state that the borrower can never prepay more than the unpaid balance and that prepaid amounts cannot be taken out again. In practice, some mortgage contracts allow taking out earlier prepaid loan amounts. In that case, x_i

can be restricted to be larger than minus the sum of all previous prepayments, or larger than some contract specification restricting the maximal amount to take back.

An upper bound on the mortgage value can be obtained by constructing a feasible solution to the general (primal) problem. No prepayment, equating all x variables to zero, is a trivial feasible solution for which the objective boils down to discounting future regular periodical payments. Consequently, the value of a non-callable mortgage is a trivial upper bound on the value of a partial prepayment mortgage with the same contract rate and time to maturity. In order to find the mortgage value with partial prepayments, the variables x_i (and the resulting U_i) of this LP model must be optimized. Together, the variables x_i constitute a prepayment strategy.

As an example of the model formulation, consider a problem instance defined on the state–space given in Figure 1. We assume that we have two time intervals, $I_1=[0, 1]$ and $I_2=[2, 3]$. [Time intervals with the year split through nodes, such that one time period belongs to both the previous and the upcoming year, requires two prepayment variables for each end-of-calendar year node. This can be achieved by assigning one of the prepayments to each of the edges incident to the end-of-calendar year node. For the purpose of the example, this would complicate the formulations and increase the number of variables unnecessarily.] Furthermore, we face a constant maximum prepayment percentage X and a contract lifetime of four periods, that is, the final tree period marks the end of the contract. The model is given below in standard format. For this small-scale example, the contract is fully amortized at the end of the fixed rate period (i.e. at $t=4$). In case the mortgage lifetime is longer than the fixed rate period, an analogous formulation can be applied, only changing n_i. The objective of the example is to

$$\begin{aligned}\text{minimize } & \lambda_2[U_1\cdot f(y,4)+x_2]+\cdots+\lambda_{15}[U_7\cdot f(y,2)+x_{15}]\\ & +\lambda_{16}U_8(1+y)+\cdots+\lambda_{31}U_{15}(1+y)\end{aligned}$$

subject to

$$\begin{aligned}U_1&=1\\ U_2-U_1(1+y)+U_1f(y,4)+x_2&=0\\ &\vdots\\ U_{15}-U_7(1+y)+U_7f(y,2)+x_{15}&=0\\ -x_2&\geq -X\\ -x_3&\geq -X\\ -x_4-x_8&\geq -X\\ &\vdots\\ -x_7-x_{15}&\geq -X\\ U_i&\geq 0,\quad \forall i\\ x_i&\geq 0,\quad \forall i\end{aligned}$$

3 Dual formulation

In order to make statements about the optimality of a solution, we apply duality theory. Before deriving the general dual problem formulation, we provide the dual of the example at the end of the previous section. Dual variables v_i and z_i are introduced, the first corresponding to the constraint set (5), and the second to the restrictions (6). For each state of the tree there exists one v_i, while a z_i is required only for periods concluding a calendar year as these determine the number of calendar year restrictions (6). Denote the set of nodes concluding the calendar years by C. Both v_i and z_i have labels equal to the corresponding state, such that the z_i labels are not continuous. For instance, in the state–space example z_4 does not exist because state 4 does not mark a calendar year end. The dual formulation of the four-period example is given by

$$\text{maximize} - X\sum_{i\in C} z_i + v_1$$

subject to

$$\begin{aligned}
&v_1 - v_2(1+y) + v_2 f(y,4) - v_3(1+y) + v_3 f(y,4) \le \lambda_2 f(y,4) + \lambda_3 f(y,4)\\
&v_2 - v_4(1+y) + v_4 f(y,3) - v_5(1+y) + v_5 f(y,3) \le \lambda_4 f(y,3) + \lambda_5 f(y,3)\\
&v_3 - v_6(1+y) + v_6 f(y,3) - v_7(1+y) + v_7 f(y,3) \le \lambda_6 f(y,3) + \lambda_7 f(y,3)\\
&\qquad\vdots\\
&v_7 - v_{14}(1+y) + v_{14} f(y,2) - v_{15}(1+y) + v_{15} f(y,2) \le \lambda_{14} f(y,2) + \lambda_{15} f(y,2)\\
&v_8 \le (\lambda_{16} + \lambda_{17})(1+y)\\
&\qquad\vdots\\
&v_{15} \le (\lambda_{30} + \lambda_{31})(1+y)\\
&-z_2 + v_2 \le \lambda_2\\
&-z_3 + v_3 \le \lambda_3\\
&-z_8 - z_9 + v_4 \le \lambda_4\\
&-z_{10} - z_{11} + v_5 \le \lambda_5\\
&-z_{12} - z_{13} + v_6 \le \lambda_6\\
&-z_{14} - z_{15} + v_7 \le \lambda_7\\
&-z_8 + v_8 \le \lambda_8\\
&\qquad\vdots\\
&-z_{15} + v_{15} \le \lambda_{15}\\
&z_i \ge 0, \quad \forall i \in C
\end{aligned}$$

Let $C_i \subset C$ denote the set containing all states, marking the end of the calendar year to which state i belongs, that are attainable from state i. For instance, considering intermediate state 4, $C_4 = \{8, 9\}$. Moreover, for notational convenience, define the function $g(i)$ to be

$$g(i) = (v_{2i} + v_{2i+1})(1 + y - f(y, n_i)) + (\lambda_{2i} + \lambda_{2i+1}) f(y, n_i).$$

Final period states $i = 2^T, \ldots, 2^{T+1} - 1$ have $v_i = \lambda_i$, which can be observed when including the balance constraints $U_i \geq 0$ for these states explicitly in the problem formulation and rewriting the objective to include the remaining unpaid balance U_i for leaf nodes separately, discounted by λ_i. For penultimate states, $g(i)$ can therefore be simplified to

$$g(i) = (\lambda_{2i} + \lambda_{2i+1})(1 + y) = \lambda_i \frac{1 + y}{1 + r_i}.$$

The complete definition of the function $g(i)$ is then

$$g(i) = \begin{cases} (v_{2i} + v_{2i+1})(1 + y - f(y, n_i)) & \\ \quad + (\lambda_{2i} + \lambda_{2i+1}) f(y, n_i), & i = 1, \ldots, 2^{T-1} - 1 \\ (\lambda_{2i} + \lambda_{2i+1})(1 + y), & i = 2^{T-1}, \ldots, 2^T - 1. \end{cases} \tag{8}$$

Now, the general formulation of the dual problem to value annuity mortgages with partial prepayments is the following:

$$\text{maximize} - X \sum_{i \in C} z_i + v_1$$

subject to

$$v_i \leq g(i), \quad i = 1, \ldots, 2^T - 1$$

$$-\sum_{\ell \in C_i} z_\ell + v_i \leq \lambda_i, \quad i = 2, \ldots, 2^T - 1$$

$$z_i \geq 0, \quad \forall i \in C$$

Complementary slackness conditions can be used to find dual variables based on the primal solution. If a primal inequality contains slack, then the corresponding dual variable equals zero. For the restrictions in our mortgage valuation problem, this implies:

$$\sum_{i \in Q} x_i < X \Rightarrow z_\ell = 0, \tag{9}$$

where ℓ is the last node, at the time interval end, of path Q. Typically, prepayment is restricted per calendar year, such that path Q covers 1 year. Node ℓ is then the last node of the year. Condition (9) states that if prepayment during scenario path interval Q is less than the maximally allowed amount, the dual variable z_ℓ can be fixed to 0.

When the dual solution is known, complementary slackness can be used to obtain a partial solution to the primal LP problem:

$$z_\ell > 0 \Rightarrow \sum_{i \in Q} x_i = X. \tag{10}$$

This complementary slackness condition states that if the dual variable z_ℓ, belonging to state ℓ, is positive, then a maximal prepayment is optimal along path Q, which ends in node ℓ and covers exactly 1 year.

Complementary slackness conditions with respect to the inequalities of the dual formulation can be derived similarly. These conditions read, $\forall i = 2, \ldots, 2^T - 1$,

$$-\sum_{\ell \in C_i} z_\ell + v_i < \lambda_i \Rightarrow x_i = 0 \tag{11}$$

$$x_i > 0 \Rightarrow -\sum_{\ell \in C_i} z_\ell + v_i = \lambda_i \tag{12}$$

and

$$v_i < g(i) \Rightarrow U_i = 0 \tag{13}$$

$$U_i > 0 \Rightarrow v_i = g(i) \tag{14}$$

From the dual problem formulation it follows that the dual variables v_i must be less than or equal to both $g(i)$ and $\lambda_i + \sum_{\ell \in C_i} z_\ell$. As v_1 (the dual variable to be maximized) is determined by a backward recursion approach depending on all future v_i, we may state that

$$v_i = \min(g(i), \lambda_i + \sum_{\ell \in C_i} z_\ell), \quad \forall i = 2, \ldots, 2^T - 1.$$

Hence for given z, the complete dual solution and the corresponding mortgage value can be obtained by backward recursion. The optimal prepayment strategy in state i can be partly derived from this minimum evaluation to obtain v_i, as will be shown by Theorems 1 and 2.

Theorem 1. *If*

$$\lambda_i + \sum_{\ell \in C_i} z_\ell < g(i),$$

then a final prepayment of the remaining loan is optimal in state i.

Proof. Suppose that

$$\lambda_i + \sum_{\ell \in C_i} z_\ell < g(i).$$

Then

$$v_i = \lambda_i + \sum_{\ell \in C_i} z_\ell < g(i),$$

and $U_i = 0$ because of complementary slackness condition (13). A full prepayment of the remaining loan is optimal. Similarly, if full prepayment is not optimal in state

i, then $U_i > 0$. By complementary slackness condition (14), $v_i = g(i)$, which can only be true if

$$g(i) \le \lambda_i + \sum_{\ell \in C_i} z_\ell. \qquad \square$$

THEOREM 2. *If*

$$g(i) < \lambda_i + \sum_{\ell \in C_i} z_\ell,$$

then no positive prepayment of a (partially) callable mortgage is optimal in state i.

PROOF. Suppose that

$$g(i) < \lambda_i + \sum_{\ell \in C_i} z_\ell.$$

Then

$$v_i = g(i) < \lambda_i + \sum_{\ell \in C_i} z_\ell,$$

and $x_i = 0$ because of complementary slackness condition (11). No prepayment is optimal. Similarly, if a positive prepayment is optimal in state i, then $x_i > 0$. By complementary slackness condition (12),

$$v_i = \lambda_i + \sum_{\ell \in C_i} z_\ell,$$

which can only be true if

$$\lambda_i + \sum_{\ell \in C_i} z_\ell \le g(i). \qquad \square$$

As a direct result from complementary slackness, the theorems imply that for a non-final partial prepayment,

$$\lambda_i + \sum_{\ell \in C_i} z_\ell = g(i)$$

must hold. The theorems on optimal prepayment are difficult to use for partially callable mortgages, because all non-final partial prepayment decisions cannot be determined by either

$$\lambda_i + \sum_{\ell \in C_i} z_\ell < g(i) \quad \text{or} \quad \lambda_i + \sum_{\ell \in C_i} z_\ell > g(i).$$

For fully callable mortgages however, the optimal prepayment strategy follows easily, as will be discussed in section 4.

4 Implications for fully callable mortgages

Mortgage valuation including full prepayment is a relaxation of the original problem formulated in section 2, omitting the limited prepayment restriction (6). Stated differently, the *maximum* prepayment amount X is infinite for fully callable mortgages. *Actual* prepayments must still satisfy the conditions

$$x_i \geq 0, \quad \forall i$$
$$U_i \geq 0, \quad \forall i.$$

As a result,

$$\sum_{i \in Q} x_i < X, \quad \forall Q$$

is a valid constraint for fully callable mortgages as well, assuming X to be infinitely large. By complementary slackness condition (9),

$$z_\ell = 0, \quad \forall \ell \in C_i, \ \forall i.$$

The equations with respect to the dual variables v_i follow directly from the dual programming formulation and the fact that $z_i = 0, \forall i \in C$. Therefore, the value of a fully callable mortgage is equal to the dual objective v_1, where v_1 is given by

$$\begin{aligned} v_1 &= g(1), \\ v_i &= \min\left(g(i), \lambda_i\right), \quad i = 2, \ldots, 2^T - 1. \end{aligned} \tag{15}$$

Terminal values to the backward recursion of v_i are either provided at the final states, for which $v_i = \lambda_i$, or at the penultimate states, at which v_i only depends on state prices and the contract rate, according to the definition of $g(i)$ in (8). This approach is comparable with the standard backward recursion applied for the valuation of American options.

Optimal prepayment conditions for a fully callable mortgage are based on complementary slackness and can be easily derived from the theorems on optimal prepayment in section 3. The optimal prepayment strategy of a fully callable mortgage depends solely on $g(i)$ and the state prices λ_i, according to (15). Theorem 1 implies that full prepayment of a fully callable mortgage is optimal in state i if $\lambda_i < g(i)$. [From the definition of $g(i)$, the recursive defining of the state prices and the restrictions of the dual problem it is easily shown that if $\lambda_i < g(i)$, then $r_i < y$. This implies that interest rates are lower than the contract rate whenever full prepayment is optimal. The converse however, is not necessarily true.] No positive prepayment of a fully callable mortgage is optimal in state i if $\lambda_i > g(i)$, according to Theorem 2. If $\lambda_i = g(i)$, a mortgagor is indifferent to prepay or not. In that case, 'no prepayment' or 'full prepayment' is not dominated by any strategy involving partial prepayments.

Any dual feasible solution provides a lower bound on the mortgage value. Consequently, the value of a partially callable mortgage is bounded from below by the value of a fully callable mortgage with the same contract rate and time to maturity. The lower bound can be improved by increasing z_i for some i. Although, according to the dual problem formulation, this decreases the lower bound directly, v_i [and v_1 by backward recursion condition (15)] can increase because of constraint relaxation. If the increase in v_1 is larger than the rise of $X\Sigma_{i \in C} z_i$, raising some z_i can improve the dual solution and hence the lower bound on the mortgage price.

As the problem formulation is based on a non-recombining tree, only small problem instances can be solved to optimality. For long-term, partially callable mortgage contracts the optimal prepayment strategy cannot be determined efficiently. Section 5 introduces a heuristic to derive the optimal prepayment strategy based on a lattice approach. This approximative strategy is used to improve the bounds on the mortgage price.

5 Improving the partial prepayment strategy

Small problem instances can be solved exactly by either primal or dual formulation, based on a non-recombining tree approach. For large instances (a common fixed rate period is 10 years; with monthly periods our problem size equals 120 periods, resulting in 2^{120} final states), such formulation is not efficiently solvable. Therefore, we must focus on obtaining upper and lower bounds on the mortgage value by constructing primal and dual feasible solutions, respectively. Any primal feasible solution (i.e. an allowed prepayment strategy) implies an *upper* bound on the optimal value of a partially callable mortgage. This section constructs a primal feasible solution, based on a lattice approach to retain computational efficiency.

The size of the original lattice equals the length of the first fixed rate period. During this period a large number of prepayment decisions must be taken. According to the proposed heuristic each prepayment originates a new mortgage loan with a smaller unpaid balance, periodical payment and time to maturity. These new mortgage loans are valued by a sublattice of the original lattice, using the corresponding interest rates.

Figure 2 shows the decomposition process based on the full prepayment boundary. This boundary is derived according to the optimal prepayment strategy of a fully callable mortgage. Optimal valuation of fully callable mortgages and the derivation of the full prepayment boundary can be performed efficiently. All nodes below the full prepayment boundary are considered as states in which full prepayment (if allowed) is optimal. Full prepayment is not optimal in nodes above the full prepayment boundary.

Optimal prepayment of a partially callable mortgage can be both earlier and later than an optimal full prepayment. It might be optimal to postpone a partial prepayment if only limited prepayment is allowed. The reason is that higher interest payments are compensated by a lower future unpaid balance. A later prepayment reduces this unpaid balance more than an early prepayment, as regular redemption reduces the unpaid balance more before than after an additional prepayment. A lower unpaid balance leads to lower future periodical payments. If these lower payments (more than) offset the disadvantageous higher interest payments due to postponing prepayment, a later prepayment might be optimal. Consequently, for a partially

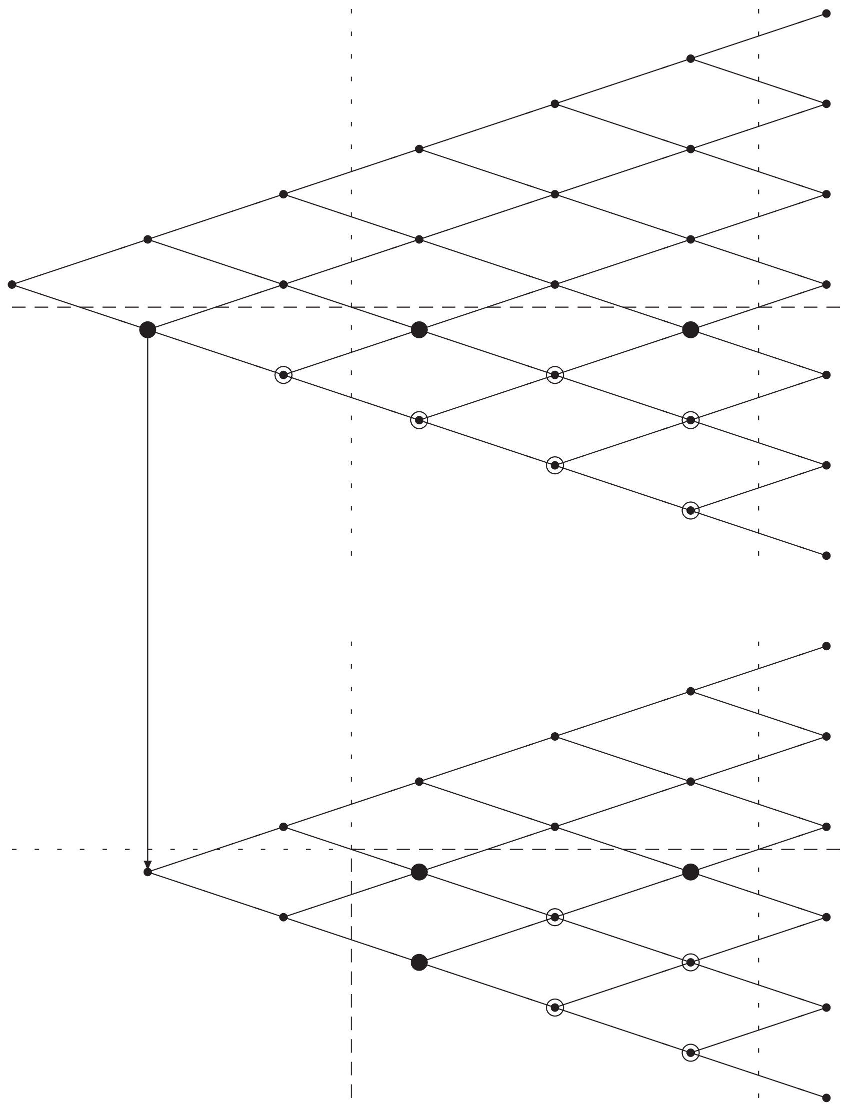

Fig. 2. Decomposition based on full prepayment boundary: The figure shows the main lattice and one of the first level sublattices after a decomposition based on the full prepayment boundary (the horizontal dashed line). All encircled nodes are candidate prepayment nodes. All solid encircled nodes are nodes in which a first prepayment is considered and from which a new sublattice is constructed. Vertical dashed lines represent calendar years. The effective prepayment boundary (longer dashes) is a combination of the full prepayment boundary and one of the calendar-year restrictions. Prepayment in the first candidate prepayment node of the main lattice (i.e. the root node of the sublattice) implies that the next prepayment cannot be in the same calendar year.

callable mortgage 'no prepayment' can be the optimal decision in a node below the full prepayment boundary. Notice that postponing prepayments can only be profitable for mortgages with a regular amortization schedule. Moreover, for fully callable mortgages there is no gain of postponing a prepayment. The unpaid balance after full prepayment is zero, the resulting periodical payments are zero as well, and these payments can therefore not be used as compensation for higher interest payments.

An optimal prepayment strategy might also involve a partial prepayment in a node above the full prepayment boundary. Such an early prepayment can be optimal in December to exercise a prepayment option just before the end of a calendar year, the option expiration date. An extra prepayment reduces the future unpaid balance and periodical payments. If the resulting lower payments more than offset the disadvantageous prepayment in December, an early partial prepayment can be optimal. This effect holds for all partially callable mortgages, independent of the amortization scheme.

As an optimal partial prepayment can be both earlier and later than an optimal full prepayment, the full prepayment boundary provides a feasible prepayment strategy, but not necessarily the optimal strategy. To construct a primal feasible solution we assume that no prepayment occurs in nodes above the full prepayment boundary and a partial prepayment occurs in nodes below the full prepayment boundary immediately after this boundary is crossed. Additionally, we assume that a partial prepayment amount is always equal to the maximally allowed amount, unless the remaining loan is smaller than the maximal prepayment. In the latter case we assume a final prepayment of the remaining loan.

Our approximation to the optimal prepayment strategy involves no prepayment in nodes above the full prepayment boundary. This part of the valuation process can be performed by a single lattice approach. Furthermore, a maximally allowed prepayment ($x_i = X$) is included whenever the full prepayment boundary is crossed downwards. After each prepayment a new sublattice is constructed based on the remaining mortgage lifetime and unpaid balance. The prepayment boundary in each sublattice is similar to the boundary in the original lattice, except for prepayment to start at the first month of a new calendar year. The prepayment node in the parent lattice is the root node of the sublattice.

One of the first-level sublattices (after the first partial prepayment), including full prepayment boundaries adapted for calendar-year restrictions, is depicted in Figure 2. The number of levels of sublattices is equal to the maximum number of prepayments. In case of prepayments limited to 20%, the number of levels is bounded by five. The number of sublattices increases with rate T per level. Denote the number of levels by K. A recursion through each sublattice to determine the mortgage price requires a computation time of $O(T^2)$, implying a total computational effort of $O(T^{K+2})$.

Although computation time is of a polynomial order (compared with exponential for a non-recombining tree), the polynomial degree is still large. Efficiency can be improved by performing a recursion only once for all sublattices rooted in the same node. Suppose node i can be reached by two different paths. For the first path a recursion is required to determine the price P_1 corresponding to unpaid balance U_1 in node i. The unpaid balance according to the second path reaching node i

equals U_2. Now the price can be scaled to be $P_2 = U_2 \cdot P_1/U_1$. However, additional prepayments are not scalable because these depend on the initial loan and not on the remaining loan. These cash flows are excluded from the traditional valuation procedure, but added separately and discounted at the appropriate discount factors. The scaling approach is more efficient than the standard approach as long as the decrease in the number of recursions is not outweighed by the preprocessing phase of calculating discount factors. This is typically the case for large instances with many prepayment opportunities. Computation time for the scalable decomposition method is of $O(T^4)$, as at most one recursion of $O(T^2)$ is required for each node in the original lattice.

Mortgage values following from the scalable decomposition method slightly differ from values according to the standard decomposition method. Prepayment in node i according to the standard decomposition method is based on the unpaid balance and price in node i of the parent lattice. The scalable method, having no recursion in most (sub)lattices and therefore no truly optimal strategy of consecutive prepayments, can only compare unpaid balance and price at the root of the child lattice. The standard decomposition method is more accurate, although differences in mortgage values are negligible.

Partially callable mortgages with a fixed rate period of 5 years can be valued by the (scalable) decomposition method based on the full prepayment boundary, providing an upper bound on the mortgage value. As many lattices must be stored in memory simultaneously for large instances, loans with 10-year fixed rate periods can only be valued when the number of sublattices is limited. To obtain accurate approximations of the optimal prepayment strategy, we have restricted prepayment opportunities in various ways to improve efficiency. One could choose for allowing prepayment only once or twice per calendar year. However, shifting the prepayment boundary downwards provided the best upper bound on the mortgage value.

A *lower* bound on the optimal value of a partially callable mortgage is provided by any dual feasible solution. The value of a fully callable mortgage, having all z-variables equal to 0, is a straightforward lower bound. Lower bound improvements are obtained by increasing the z-variables that correspond to low interest rate states. Although we can achieve an improved lower bound, this bound is worse than the upper bound derived previously. For this reason, we will rely on a practical lower bound on the mortgage value in section 6.

6 Results

Results are provided in terms of fair rates. We define the fair rate as the contract rate that makes the present value of the sum of all cash flows equal to the principal value. If the contract rate is higher (lower) than the fair rate, implying a mortgage value higher (lower) than the principal value due to high (low) interest payments, the bank makes a profit (loss) on the contract. Choosing an initial contract rate, the fair rate is determined iteratively by increasing (decreasing) the contract rate when the mortgage value appears to be lower (higher) than the principal value.

An upper (lower) bound on the mortgage value corresponds to a lower (upper) bound on the fair rate. Consequently, a primal feasible solution to the model formulated in section 2, obtained exactly or heuristically by applying the approximative algorithm proposed in the previous section, provides a lower bound on the fair rate. A dual feasible solution, providing a lower bound on the optimal mortgage value, gives an upper bound on the fair rate.

The fair rate of a partially callable interest-only mortgage is a practical upper bound on the fair rate of a partially callable annuity and can be determined efficiently (see KUIJPERS, 2004). As all term structures are upward-sloping, an interest-only mortgage faces an unattractive redemption schedule. The fair rate of an interest-only mortgage is therefore higher than the fair rate of an annuity or linear mortgage with similar characteristics.

Bounds on the fair rates are calculated for both 5- and 10-year fixed rate periods. We consider partially callable mortgages excluding commission and including a 1% commission on four dates. Annuity and linear mortgages are included. The bounds define a range for the optimal fair rate of partially callable annuity and linear mortgages. A narrow range between lower and upper bound indicates that the proposed heuristic is accurate.

For 5-year fixed rate periods no computational problems arise. When prepaying the maximally allowed amount in any node below the full prepayment boundary and not prepaying anything in any node above, a tight lower bound on the fair rate is obtained. As can be concluded from Table 1, the lower bound differs between 3 and 8 basis points from the upper bound, defined by the fair rate of an interest-only mortgage with similar conditions. Therefore, the lower bound is a very accurate approximation of the optimal fair contract rate. Moreover, the optimal prepayment strategy will not differ largely from the full prepayment boundary.

Table 1. Fair rates for a 5-year fixed rate period.

		No commission		1% commission	
Type	Date	LB	UB	LB	UB
Annuity	1 January 2002	4.72	4.75	4.41	4.46
	1 January 2003	3.73	3.76	3.43	3.48
	1 January 2004	3.74	3.77	3.44	3.49
	1 January 2005	3.26	3.29	2.96	3.00
Linear	1 January 2002	4.69	4.75	4.38	4.46
	1 January 2003	3.71	3.76	3.41	3.48
	1 January 2004	3.72	3.77	3.41	3.49
	1 January 2005	3.25	3.29	2.94	3.00

Notes: This table provides lower bounds on fair rates of partially callable annuity and linear mortgages. Upper bounds correspond to fair rates of partially callable interest-only mortgages. The underlying interest rate lattice consists of monthly periods. Mortgage contracts have a 5-year fixed rate period and exclude commission, respectively include a 1% commission.

Table 2. Fair rates for a 10-year fixed rate period.

		No commission		1% commission	
Type	Date	LB	UB	LB	UB
Annuity	1 January 2002	5.32	5.44	5.11	5.24
	1 January 2003	4.45	4.56	4.25	4.38
	1 January 2004	4.46	4.58	4.26	4.39
	1 January 2005	3.87	4.01	3.67	3.81
Linear	1 January 2002	5.27	5.44	5.05	5.24
	1 January 2003	4.40	4.56	4.19	4.38
	1 January 2004	4.40	4.58	4.19	4.39
	1 January 2005	3.84	4.01	3.62	3.81

Notes: This table provides lower bounds on fair rates of partially callable annuity and linear mortgages. Upper bounds correspond to fair rates of partially callable interest-only mortgages. The underlying interest rate lattice consists of monthly periods. Mortgage contracts have a 10-year fixed rate period and exclude commission, respectively include a 1% commission.

Table 2 provides fair rate results for 10-year fixed rate periods. Prepayment is restricted to the bottom 22 nodes (per period) of the original lattice and the corresponding nodes in all sublattices, as long as these are located below the full prepayment boundary. This prepayment strategy restricts the number of sublattices, although still capturing prepayment gains from large interest rate declines. The difference between lower and upper bound can rise up to 20 basis points, although the lower bound is considerably improved compared with the initial lower bound, that is, the fair rate of a non-callable mortgage.

7 Concluding remarks

A LP formulation has been introduced for the valuation and optimal prepayment of (partially) callable mortgages. We have also derived optimal prepayment conditions for fully callable mortgage contracts based on state prices and following from duality theory.

A fully callable mortgage can be modelled by a lattice approach. Partially callable annuity and linear mortgages can only be priced to optimality by an inefficient non-recombining tree approach. To enhance efficiency, we propose a lattice-based method to obtain an accurate lower bound on the fair rate for these mortgage types.

As, for upward sloping term structures, the fair rate of a partially callable interest-only mortgage, which can be efficiently priced to optimality, provides a practical upper bound on the fair rate of a partially callable annuity, a narrow range for the optimal fair rate is derived. This indicates that the proposed heuristic is accurate.

Related to the LP formulation, we propose two directions for future research on the optimal valuation of partially callable annuities. First, a theoretical upper bound on the fair rate can be derived by improving the basic dual feasible solution, represented by the full prepayment strategy. The upper bound can be improved by increasing z-variables corresponding to low interest rate states. Then, by backward recursion, the dual objective (that is, the mortgage value) increases and the fair rate decreases. The number of z-variables is exponential and therefore many z-variables must be increased from zero to an (*a priori* unknown) positive value to achieve a significant improvement.

A second direction for further research is based on approximating the fair rate of a partially callable annuity. As not all states in a non-recombining tree can be included, we might consider a tree defined on a subset of scenario paths. Valuation based on this subtree generates approximative mortgage prices and fair rates. Approximations are more accurate for finer subtrees. However, approximations can lead to both higher and lower fair rates than optimal. As a consequence, measuring the accuracy of the approximative fair rate is not possible without the use of fair rate bounds derived in this paper.

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