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An Artificial Financial Market with Fundamentalists and Trend Followers

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An Artificial Financial Market with Fundamentalists and Trend Followers

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Abstract

In this master thesis we will investigate an agent-based artificial financial market designed by Hendri Adriaens (2008). We will use his model as a benchmark and investigate how well different specifications of an economy consisting of fundamentalists and trend followers performs in terms of reproducing the stylized facts of asset returns of Cont (2001). We compare multiple simulations of each specification with each other and with the benchmark model. We find that an artificial financial market consisting of fundamentalists and trend followers, with heterogeneous adjustment speeds and rates of discounting old asset prices, can reproduce some additional stylized facts that are not reproduced by the benchmark model, most notably no autocorrelation in returns and gain/loss asymmetry.
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1 Introduction

In the light of the recent financial crisis, which is to a large extent caused by the systematic mispricings of securities based on US sub-prime mortgages, it becomes once again clear how hard it is for traditional financial models to describe financial markets and the behavior of their participants. Traditional financial models, which are based on the efficient market hypothesis and homogeneous rational expectations of economic agents, fail to capture the full dynamics of financial markets. There is an extensive amount of financial literature aiming at understanding the formation of asset prices. Researchers have found numerous anomalies and puzzles in financial market data, such as excess volatility (Shiller, 2003); and multiple properties of financial markets, like the clustering of volatility and heavy tails in the distribution of asset returns. Those properties were denoted stylized facts by Cont (2001).

Over the last decades the research field of Agent-Based Computational Economics (ACE) has made major developments. ACE models view financial markets as an interacting group of learning, rationally bounded agents and use computer simulations to provide numerical solutions for complex problems in situations where analytical solutions are impossible to find\(^1\). As a result, ACE can be used to model more complex financial markets with more heterogeneity among beliefs of agents about the movements of asset prices. These beliefs might be varying over time and do not have to be purely rational. The ability to investigate more complex models about financial markets can provide insights in the puzzles and anomalies found in financial literature, and in the formation of asset prices in financial markets.

Numerous artificial financial markets have been constructed based on ACE that try to model a complex financial market capable of reproducing stylized facts of actual financial markets\(^2\). Adriaens (2008) has designed an artificial market in

\(^1\)see LeBaron (2007) for an overview
\(^2\)see f.i. LeBaron (2007), Hi and Li (2007), Gaunersdorfer and Hommes (2007) and Adriaens (2008)
which agents, with heterogeneous beliefs, trade multiple assets based on econometric predictions of future asset prices or returns. In his artificial financial market there are two types of agents, one type using the fundamental value of assets to determine his future positions and another type holding a mean-variance efficient portfolio.

This master thesis contributed to the literature by introducing trend followers to the model of Adriaens and investigating the influence of this type of agents on the artificial financial market. Trend followers, who for instance also appear in the models of Hi and Li (2007) and Gaunersdorfer and Hommes (2007), are defined as agents who assume that the market is not efficient with respect to information in such a way that all information available to participants is captured in the asset price. As a result they assume that prices of assets that went up in the past, continue to go up in the future. Trend followers take positions in the market based on a trend in past asset prices.

We will investigate different specifications of moving average processes used by trend followers to determine their future demand for an asset in an artificial financial market with fundamentalists and trend followers. Subsequently, we investigate two specifications of financial markets with trend followers, using an exponential moving average, to determine their future demand. First, we increase the heterogeneity among agents. Second, will allow part of the agents to adapt their beliefs and switch between behaving as a fundamentalist and a trend follower. Last, we will introduce trend followers to the benchmark model to investigate the influence of trend followers on the performance of this model.

We will test the performance of all specifications of financial markets mentioned above in reproducing the stylized facts (Cont, 2001). That way testing their performance to mimic a actual financial market. Here we make another extension to the research of Adriaens (2008) by investigating multiple replications of the simulation to reduce simulation inaccuracy. We find that the artificial financial market models with an exponential moving average are capable of replicating some of the stylized facts that are not replicated by the models of Adriaens, for instance: absence of autocorrelation and gain/loss asymmetry. On the other hand there are still stylized facts that are not replicated by these models or that are replicated only in a very small number of simulations.

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3 This is often called herding behavior in the literature (See for instance (Barbaris and Thaler, 2003))
This thesis is organized as follows: Chapter 2 discusses the background in financial literature on beliefs of economic agents. We will discuss the traditional hypothesis of rational expectations and no arbitrage opportunities on which the efficient market hypothesis is based. We will also discuss how the fundamental value on which the trading strategy of fundamentalists is based follows from the efficient market hypothesis, and discuss the interpretation of the efficient market hypothesis by trend followers. Next, we discuss contributions to the literature that question the rational expectations and no arbitrage hypothesis, in particular behavioral finance. We will see that there is little consensus in the literature about the validity of both hypotheses and see how ACE can be an interesting tool to model behavior of agents.

Chapter 3 introduces the artificial financial market designed by Adriaens (2008). In this market 200 economic agents trade multiple risky assets and one riskless asset. The economic agents have a position in all of the risky assets and adjust their demand using forecasts of economic variables based on their beliefs. A temporary Walrasian equilibrium is used to determine the market price and clear the market. There is also a regulator active in the market, regulating trade in order to try to prevent the economy from crashing. The model of Adriaens is used as a benchmark for our research.

Chapter 4 provides backgrounds on the demand calculations of trend followers as well as different specifications of economies in which trend followers are active. Economy 1 is an economy with fundamentalists and trend followers, where trend followers use a simple moving average function to determine their demand for the risky assets. Economy 2 is an economy with fundamentalists and trend followers, where trend followers use a weighted moving average instead of the simple moving average. Economy 3 is an economy with fundamentalists and trend followers, where trend followers use an exponential moving average. This specification is in line with the specification trend followers of Hi & Li in their model (Hi and Li, 2007). Because the summary statistics and autocorrelation plots of this specification are most promising in replicating the stylized facts (Cont, 2001) we use trend followers with an exponential moving average for the specifications in the remainder of our research project. Economy 4 is a extension of economy 3 where some parameter values are different for each agent to create more heterogeneity. Economy 5 is an extension
to economy 4 where part of the agents is not consistent in their beliefs and changes between fundamentalist and trend follower based on the past performance of both strategies. Last, in economy 6 we add trend followers, with the specification used in economy 4, to the benchmark model to investigate the influence of trend followers on the performance of this model to replicate the stylized facts.

Chapter 5 provides some theoretical background on the econometric analysis of simulation models, based on Li et al. (2009), and discusses the performance of the discussed economies in replicating the stylized facts. We compare the last 1000 of 5000 periods simulation of 25 replications. We use the fraction of replications in which the individual stylized facts are significant according to econometric tests to compare their performance. We will see that economy 3 and 4 perform best and are capable of reproducing most stylized facts. Furthermore, we see that multiple replications of the benchmark model reveal slightly different results for some stylized facts as the single replication analysis provided by Adriaens (2008). “Slow decay of autocorrelation in absolute returns” and “Volume/volatility clustering” that were significant in Adriaens analysis of the benchmark model turn out to be insignificant in a substantial part of the simulation replications used in our analysis. On the other hand, “Agregational Gausianity” that was insignificant in Adriaens analysis turns out to be significant in a substantial number of replications.

Chapter 6 concludes and provides a discussion on the performance of the artificial financial market with fundamentalists and trend followers; and the performance of agent-based computational economics in general. We will also give suggestions for further research based on our results.
2 Background on the Efficient Market Hypothesis

This chapter provides a background on the traditional asset pricing approach, assuming rational expectations and the absence of arbitrage opportunities. Those two simple assumptions provide powerful models, for instance, the implication that asset prices reflect a fundamental value. However, empirical tests reveal numerous anomalies indicating violations of this approach.

The first section of this chapter explains the efficient market hypothesis with its underlying beliefs, which has been the dominant view on financial markets from the beginning of the field of empirical finance. The second section will elaborate on critique on the efficient market hypothesis that has emerged from the 1990s onwards. This critique is based on the difficulties the efficient market hypothesis has in explaining realized asset returns in financial markets and anomalies found in data series of financial markets. The third and last section concludes.

2.1 Rational Expectations, No Arbitrage, and the Efficient Market Hypothesis

This section focuses on the traditional view of financial markets that has been the dominant theory in empirical finance from the 60s onwards that was widely accepted until the end of the last century. The first subsection describes the beliefs of economic agents and traditional hypotheses that follow from assumptions of those beliefs. The second section describes the efficient market hypothesis.
2.1.1 Beliefs of Economic Agents and Traditional Hypotheses

Economic agents in a financial market use econometric models to make predictions of asset returns based on information available to them at this point in time, and based on their beliefs about the development of the financial market in the future. An econometric model of a financial market at time \( t \) can be described as \( E_t = (\omega_t, f_t, g_t, z_t) \), where \( \omega_t \) represents the state of the economy at time \( t \); \( f_t \) represents a vector of endogenous variables at time \( t \) that are to be explained by the model; \( g_t \) represents a vector of exogenous variables at time \( t \); and \( z_t \) represents a vector of all publicly available information at time \( t \). This and the following notations are roughly based on Melenberg (2008).

The variables in the model can be determined as follows: The exogenous variables \( g_t \) conditional on the previous state of the economy follow from some probability distribution \( P_{g_t} (g_t | \omega_{t-1}) \sim P_{\omega_t} \). The vector of endogenous variables \( f_t \) is determined as a transformation of the exogenous variables \( g_t \) and the previous state of the economy \( \omega_{t-1} \) by means of some econometric model: \( f_t = f_t (\omega_{t-1}, g_t) \). The state of the economy at time \( t \), \( \omega_t \), is determined by the endogenous variables \( f_t \), the exogenous variables \( g_t \) and the previous state of the economy \( \omega_{t-1}, \omega_t = (\omega_{t-1}, f_t, g_t) \).

Last, the vector containing all publicly available information \( z_t \) consists of \( (z_{t-1}, \nu_t) \); where \( \nu_t \) represents a vector of the components of \( f_t \) and \( g_t \) that are publicly available.

The time \( t + \tau \) payoff of the \( J \) assets that are traded at time \( t \), denoted by: \( X_{t,t+\tau} \in \mathbb{R}^J \), is defined as: \( X_{t,t+\tau} = S_{t,t+\tau} + D_{t,t+\tau} \). The vector \( X_{t,t+\tau} \) is a subvector of \( X_{t+\tau} \) that only includes the \( t + \tau \) period payoffs of assets that are available in the market at time \( t \).

In every time period \( t \) agent \( i \) forms a belief over the future asset payoffs for his entire planning horizon \( T_i \) \( (X_{t,t+1}, \ldots, X_{t,t+T_i}) \). In the case that the agent only uses one probability distribution to construct his set of beliefs \( B_{it} \) the agent is said to have unambiguous beliefs and quantifies the uncertainty of his beliefs as risk. If on the other hand \( B_{it} \) contains more than one probability distribution, the agents beliefs are said to be ambiguous.

Traditional econometric models used in finance usually use two assumptions...
about the beliefs of an agent underlying the econometric model. They assume rational expectations and the exclusion of arbitrage opportunities. The next two subsections explain each of the two assumptions in more detail.

Rational Expectations

Agent \( i \in \mathcal{I} \) is said to have rational expectations if in period \( t \) that agent employs the correct probability distribution over the payoffs of all \( J \) assets in future periods \( t + 1, \ldots, t + \tau (X_{t,t+1}, \ldots, X_{t,t+\tau}) \), given all information available to that investor in period \( t \). In the context of the rational expectations hypothesis, the actual probability distribution of the exogenous variables given the previous state of the world, \( g_t|\omega_{t-1} \sim P^g_t \), is assumed to be the probability distribution of \( g_t \) conditional on \((g_1, \ldots, g_{t-1})\).

Because \( f_t \) can be rewritten as \( f_t = f_t(\omega_0, g_1, \ldots, g_t) \), the probability distribution of \( f_t \) is induced by the distribution of \( g_t \) conditional on \( g_1, \ldots, g_{t-1} \). The rational expectations hypothesis can then be formulated in the following way:

\[
H_{RE} : \mathcal{B}_it = \{P_{it}\} = \{P_{X_{t,t+1}, \ldots, X_{t,t+\tau}|g_1, \ldots, g_{t-1}}\}
\]

Where \( \mathcal{B}_it \) is the set of beliefs of agent \( i \) in period \( t \) and \( P_{it} \) is the probability distribution used by agent \( i \) at time \( t \).

No Arbitrage Opportunities

A single period arbitrage opportunity arises with the possibility to construct a portfolio of assets at time \( t \) that has a price of zero; and that has a non negative payoff with probability one and a strictly positive payoff with a probability larger than zero.

If more than one period is concerned, a time \( t \) arbitrage opportunity occurs if it is possible for an agent \( i \in \mathcal{I} \) to construct a self financing portfolio that has a price at time \( t \) which is equal to 0; and that at time \( t + T \) with \( T \geq 1 \) has a nonnegative payoff with probability one, and a strictly positive payoff with a probability larger than zero.

A self financing portfolio strategy is defined as a portfolio \((h_t, h_{t+1}, \ldots, h_{t+T-1})\)
satisfying
\[ h_{t+\tau-1} \cdot (S_{t,t+\tau} + D_{t,t+\tau}) = h_{t+\tau} \cdot S_{t,t+\tau} \]
where \( \tau = 1, \ldots, T - 1 \). For such a portfolio an arbitrage opportunity arises when:

\[
\text{Arbitrage Opportunity : } \begin{cases} 
  h_t \cdot S_t = 0 \\
  P_{it}(h_{t+T-1} \cdot X_{t,t+T} \geq 0) = 1 \\
  P_{it}(h_{t+T-1} \cdot X_{t,t+T} > 0) > 0 
\end{cases} \tag{2.1}
\]

Given the set of beliefs \( B_{it} \) a self financing portfolio strategy is a time \( t \) arbitrage opportunity if it is a time \( t \) arbitrage opportunity according to at least one \( P_{it} \in B_{it} \). The no arbitrage hypothesis states that the vector of asset prices \( S_t \) is set such that there are no time \( t \) arbitrage opportunities according to belief set \( B_{it} \). In other words:

\( H_{NA} \): The price \( S_t \) is such that it is impossible to construct a portfolio \( h_t \) that satisfies equation 2.1, for all \( i \in I \).

### 2.1.2 Efficient Market Hypothesis

The efficient market hypothesis assumes that prices of traded assets reflect all available information and instantly change to reflect new information when that information becomes available to economic agents. A formal definition is for instance provided by Jensen (1968). He states: “A market is efficient with respect to information set \( \Omega_t \) if it is impossible to make economic profits by trading on the basis of information set \( \Omega_t \)”.

There are three forms of efficiency, based of the amount of information represented in the information set \( \Omega_t \)

- **Weak form efficiency** implies that only past and current prices are incorporated in \( \Omega_t \). So, the market is efficient with respect to the set of endogenous variables \( f_t \). As a result, it is not possible for agents to earn excess return by trading on historical prices. There is no serial dependence in asset price series and hence asset prices follow a random walk. This formulations implies that technical analyses (trading of financial assets based on past returns) will not result into better performance. In other
words, excess returns based on technical analysis are assumed to be the result of 'luck' under this hypothesis

Semi strong form efficiency implies that the market is efficient with respect to all publicly available information \( z_t = (z_{t-1}, \nu_t) \). Asset prices adjust to publicly available information very rapidly. Hence it is not possible for economic agents to earn excess return by trading on publicly available information. Under this hypothesis technical nor fundamental analysis can provide an agent with information for predicting excess returns.

Strong form efficiency implies that markets are efficient with respect to all publicly and private available information. So the information set \( \Omega_t \) includes \( (\Omega_{t-1}, f_t, g_t) \). This definition implies that agents cannot consistency beat the market by using any of the information present in \( f_t, g_t \).

In this section one possible definition of the efficient market hypothesis is quantified as the fundamental value. This formulation states the correct price of an asset is given by the discounted stream of dividend payments and the value of the asset at the end of the agents planning horizon. Hence this formulation implies the semi strong form of the efficient market hypothesis assuming the market is efficient with respect to prices and dividends. In the next paragraph the Fundamental Theorem of Asset Pricing is defined. in the subsequent paragraph the Fundamental value is defined based on this Fundamental Theorem.

**First Fundamental Theorem of Asset Pricing**

We start by assuming that \( H_{RE} \) holds at time \( t \) for at least one agent \( i \in I \). In a one period setting we may simplify the no arbitrage hypothesis to a single period no arbitrage hypothesis \( H_{N.A1} \), saying exploiting arbitrage is impossible in the next time period \( t \). The First Fundamental Theorem of Asset Pricing links the no arbitrage hypothesis to the existence of a non negative stochastic discount factor: \( m_{i,t+1} = m_{i,t+1}(\omega_t, g_{t+1}) \). We define \( E_t(\cdot) \) as the expectation of the probability distribution of the future values of \( f_t \) and \( g_t \) conditional on the entire history of \( g_1, \ldots, g_{t+1} \), which is a generalization of the probability distribution \( \mathbb{P}_{X_t+1, \ldots, X_{t+r+1}|g_1, \ldots, g_{t-1}} \) that is assumed to hold according to \( H_{RE} \).
The FTAP states that:

\[ H_{NA1}(i, t) \iff S_t = \mathbb{E}_{\mathbb{P}_t}(m_{i,t+1} X_{t,t+1}) \]

with

\[ \mathbb{P}_t(m_{i,t+1} > 0) = 1 \]

### Fundamental Value

One very simple model to determine asset prices based on exogenous information is the efficient market model. This model characterizes the price of an asset as a fundamental value based on the Efficient Market Hypothesis (EMH). The EMH assumes that \( H_{RE}(i, t + \tau) \) and \( H_{NA1}(i, t + \tau) \) hold for at least one agent \( i \in \mathcal{I}_{t+\tau} \) for each \( \tau \geq 0 \). The FTAP states that:

\[ h_{t,t+\tau} \cdot S_{t,t+\tau} = \mathbb{E}_{t+\tau}(m_{i,t+\tau+1}(h_{t,t+\tau} \cdot X_{t,t+\tau+1})) \]

We rewrite the stochastic discount factor as \( m_{i,t+\tau+1} = 1/(1 + \delta_{t,t+\tau}(h_{t,t} + t + \tau - 1)) \), where \( \delta \) is a discount rate. Using this, we get equation 2.2:

\[ h_t \cdot S_t = \mathbb{E}_t \left( \sum_{\tau=1}^{\infty} \frac{h_{t,t+\tau-1} \cdot D_{t,t+\tau}}{\Pi_{j=1}^{\tau-1} 1 + \delta_{t,t+\tau}(h_{t,t+\tau-1})} \right) + \lim_{T \to \infty} \mathbb{E}_t \left( \frac{h_{t,t+T-1} \cdot S_{t,t+T}}{\Pi_{j=1}^{T-1} 1 + \delta_{t,t+j}(h_{t,t+j-1})} \right) \]

For the derivation of this equation see the appendix.

The right hand side of equation 2.2 divides the price of a portfolio of assets at time \( t \) into two parts. The first part of the equation consists of the discounted stream of dividend payments, and is called the fundamental value. The second part the equation consists of the final price of the portfolio of assets if \( T \to \infty \). This part is called the bubble term. The bubble term can be assumed zero according to Santos and Woodford (1997), they argue that the denominator of the equation \( \left( \Pi_{j=1}^{T-1} 1 + \delta_{t,t+j}(h_{t,t+j-1}) \right) \) grows faster over time as the numerator \( (h_{t,t+T-1} \cdot S_{t,t+T}) \). This way they assume the correct price of an asset or portfolio of assets is the discounted stream of dividend payments or the fundamental value.

Agents believing in the EMH and the fundamental value that follows from this hypothesis are called fundamentalists. The model is very compact and only based on two assumptions (often combined with the assumption that the bubble term for
2 Background on the Efficient Market Hypothesis

\( T \to \infty \) is zero). But the validity of the model is more and more questioned in the literature as will be discussed in the next section.

2.2 Limits to Arbitrage and Rational Expectations

The last decades there have been numerous studies in the field of empirical finance questioning the EMH and the joint validity of the two hypotheses mentioned in section 2.1.1 for at least some investor \( i \in I \).

Empirical evidence of the difference between realized asset returns in financial markets and predicted returns by traditional financial models are, for instance, mentioned by Shiller (2003). He points out that stocks show excess volatility relative to what would be predicted by the EMH. He argues that the excess volatility implies that changes in prices occur for no fundamental reason at all, but are caused by mass psychology. Another example is the Equity Premium Puzzle described by Mehra and E. (2003). These authors observe that there much higher return on equity stock compared to government bonds in the United States. In order to reconcile this difference, agents must have an implausibly high risk aversion according to efficient market models.

The apparent difference between market returns of asset and predicted returns by traditional models suggests the EMH might be (partially) violated. In other words \( H_{NA1} \) and \( H_{RE} \) might not hold at the same time. Most research that tries to bridge the gap between the EMH and real asset returns assumes one of the two hypotheses does not hold (exactly).

In particular the field of Behavioral finance assumes limits to arbitrage resulting in a violation of \( H_{NA1} \). Following the definition of Barbaris and Thaler (2003): “In an efficient market no investment strategy can earn excess risk-adjusted average returns, or average returns greater than that is warranted for its risk”. However empirical studies of financial markets, as the ones described above, show large deviations of asset prices from their fundamental value. Behavioral finance argues that those deviations are caused by agents who are not fully rational. Advocates of
the EMH would argue that rational agents would take advantage of the “mispricing” that is caused by those irrational investors and that the irrational investors will lose money, would eventually learn from their mistakes, and become fundamentalists or leave the market. Behavioral finance literature, for instance (Barbaris and Thaler, 2003), argues that it will not always be possible for fundamentalists, and in particular arbitrageurs, to exploit those mispricings because strategies to correct this mispricing might by both too costly and risky, making them unattractive for agents. As a result they conclude that there are limits to arbitrage allowing irrational noise traders to effect market prices. Behavioral finance continues by using psychological insights to explain anomalies caused by those noise traders.

By assuming limits to arbitrage, behavioral finance implicitly maintains the rational expectations hypothesis. This choice is debatable. Adriaens (2008) for instance argues: “... violations of the rational expectations hypothesis might also be a potential channel through which to explain the anomalous findings. But without the rational expectations hypothesis the concept of fundamental value no longer exist, so that price deviations from fundamental values cannot really be studied, and noise traders can no longer be 'blamed' for mispricings”. One could also use existing empirical evidence for maintaining the no arbitrage hypothesis. This way implying that the anomalous findings are caused by limits to rational expectations.

Adriaens also points out that it is difficult to objectively test the hypothesis of rational expectations given limits to arbitrage. It is unclear what the restrictions on the actual probability distribution $P_t$ are in this situation, due to the influence that noise traders have on the outcomes of the financial market.

When testing the no arbitrage hypothesis under the assumption of rational expectations. A rejection does not necessary mean limits to arbitrage, as is for instance discussed by Adriaens (2008). This is only true if limits to rational expectations can be excluded. Limits to rational expectations become an interesting alternative when there cannot be found empirical evidence for limits to arbitrage, which is indeed the case. There is little evidence in literature for a large scale limit to arbitrage. Examples as (Shiller, 2003), and (Barbaris and Thaler, 2003) can be explained also as examples of limits to rational expectations instead of limits to arbitrage is done for instance by Adriaens (2008).
2.3 Conclusions

There is an extensive amount of empirical studies pointing at anomalous developments of realized asset returns in financial markets compared to predictions by traditional models based on the efficient market hypothesis. However, causes of these anomalous findings are much debated in literature. Behavioral finance points at limits to arbitrage, where other researchers point at the lag of large scale empirical evidence for this conclusion and the impossibility to test the validity of the rational expectations hypothesis given limits to arbitrage, implying limits to rational expectations might also be a possible cause for the anomalous findings. Again other researchers, for instance, (Malkiel, 2003) point out that given the anomalous findings, economic agents do not succeed in profiting from deviations of the market prices from their fundamental values, concluding that hence the market is efficient despite the anomalies.

It turns out there is much discussion in scientific literature about the nature of anomalous findings in realized asset returns, and the way that those findings should be explained. An interesting way of modeling would be to find a way to impose as little assumptions on agents behavior in the modeled financial market as possible. We will continue this research project by investigating a simulations model in the field of agent based computational finance with heterogeneous agents. This way trying to investigate the influence of agents with different beliefs on the evolution of the financial market. By investigating till what extend different specifications of agents in the market influences the presence the stylized facts in financial markets, it might be possible to create a artificial market that is more realistic.

2.A Appendix

Equation 2.2:
Note that we use $Y_{t, t+\tau}$ to denote the subvector of $Y_{t+\tau}$ that only included values of assets that are available in the market in period $t$.

If we consider $\tau = 0$. Then for each portfolio $h_t \in \mathbb{R}^J$ the FTAP states that:

$$h_{t,t} \cdot S_{t,t} = \mathbb{E}_t(m_{i,t+1}(h_{t,t} \cdot X_{t,t+1}))$$
$$= \mathbb{E}_t(m_{i,t+1})\mathbb{E}_t((h_{t,t} \cdot X_{t,t+1})) + \mathbb{C}ov_t(m_{i,t+1}(h_{t,t} \cdot X_{t,t+1}))$$
Where $E_t(\cdot)$ is an expectation using all information up to time $t$, and $m_{i,t+1}$ is a stochastic discount factor for agent $i$ in period $t+1$. This can be rewritten as:

$$E_t(h_{t,t} \cdot X_{t,t+1}) = \frac{1}{E_t(m_{i,t+1})} (h_{t,t} \cdot S_{t,t} - \text{Cov}_t(m_{i,t+1}(h_{t,t} \cdot X_{t,t+1})))$$

$$= \frac{h_{t,t} \cdot S_{t,t}}{E_t(m_{i,t+1})} - \frac{h_{t,t} \cdot S_{t,t} \text{Cov}_t(m_{i,t+1}(h_{t,t} \cdot X_{t,t+1}))}{E_t(m_{i,t+1})}$$

$$= h_{t,t} \cdot S_{t,t} \left( \frac{1}{E_t(m_{i,t+1})} - \text{Cov}_t(\tilde{m}_{i,t+1}(h_{t,t} \cdot R_{t,t+1})) \right)$$

Where $R_{t,t+1} = X_{t,t+1}/S_{t,t+1}$ is the gross return of portfolio $h_{t,t}$, and $\tilde{m}_{i,t+1} = \frac{m_{i,t+1}}{E_t(m_{i,t+1})}$ is the normalization of the stochastic discount factor.

Note that when considering the payoff of a riskfree asset between period $t$ and $t+1$ the payoff $X_{t,t+1} = 1$. Hence, the FTAP yields for this riskfree asset: $S_{t,t+1} = E_t(m_{i,t+1}X_{t,t+1}) = E_t(m_{i,t+1} \cdot 1)$. So, if we define the gross return on the riskfree asset in period $t$ and $t+1$ for assets available in period $t$ as $R^f_{t,t+1}$, we can write $R^f_{t,t+1} = X_{t,t+1}/S_{t,t+1} = 1/E_t(m_{i,t+1})$, and it follows that the net return on the riskfree asset $r^f_{t+1} = 1/E_t(m_{i,t+1}) - 1$.

To continue we introduce:

$$\delta_{t,t+1}(h_{t,t}) = r^f_{t+1} - \text{Cov}_t(\tilde{m}_{i,t+1}(h_{t,t} \cdot R_{t,t+1}))$$

$\delta_{t,t+1}(h_{t,t})$ can be interpreted as the sum of the return on the riskfree asset $r^f_{t,t+1}$ and a risk premium $-\text{Cov}_t(\tilde{m}_{i,t+1}(h_{t,t} \cdot R_{t,t+1}))$ of holding a portfolio of $h_{t,t}$ assets. We can simplify the the FTAP for $\tau = 0$ further to:

$$h_{t,t} \cdot S_{t,t} = E_t \left( \frac{h_{t,t} \cdot X_{t,t+1}}{1 + \delta_{t,t+1}(h_{t,t})} \right) = \left( \frac{h_{t,t} \cdot S_{t,t+1} + h_{t,t} \cdot D_{t,t+1}}{1 + \delta_{t,t+1}(h_{t,t})} \right)$$

(2.3)

Next, for $\tau = 1$ the FTAP states:

$$h_{t,t+1} \cdot S_{t,t+1} = E_{t+1}(m_{i,t+2}h_{t,t+1} \cdot X_{t,t+2})$$

Where $E_{t+1}(\cdot)$ is an expectation using all information up to time $t+1$, and $m_{i,t+2}$ is a stochastic discount factor for agent $i$ in period $t+2$.

This statement can be rewritten in the same way as the FTAP for $\tau = 0$ above as:

$$h_{t,t+1} \cdot S_{t,t+1} = E_{t+1} \left( \frac{h_{t,t+1} \cdot X_{t,t+2}}{1 + \delta_{t,t+2}(h_{t,t+1})} \right) = E_{t+1} \left( \frac{h_{t,t+1} \cdot S_{t,t+2} + h_{t,t+1} \cdot D_{t,t+2}}{1 + \delta_{t,t+2}(h_{t,t+1})} \right)$$

(2.4)

with

$$\delta_{t,t+2}(h_{t,t+1}) = r^f_{t+2} - \text{Cov}_{t+1}(\tilde{m}_{i,t+2}, h_{t,t+1} \cdot R_{t,t+2})$$
Using the law of iterated expectations \( \mathbb{E}_t(Y) = \mathbb{E}_t(\mathbb{E}_{t+1}(Y)) \) we can substitute equation 2.4 into equation 2.3 to get:

\[
\begin{align*}
ht \cdot St &= \mathbb{E}_t \left( \frac{ht \cdot St+1+ht \cdot Dt+1}{1+\delta_{t+1}(ht)} \right) \\
&= \mathbb{E}_t \left( \frac{ht \cdot Dt+1}{1+\delta_{t+1}(ht)} + \frac{1}{1+\delta_{t+1}(ht)} \right) \mathbb{E}_{t+1} \left( \frac{ht+1 \cdot St+1+ht+1 \cdot Dt+1}{1+\delta_{t+1+1}(ht+1)} \right) \\
&= \mathbb{E}_t \left( \sum_{\tau=1}^{\tau-1} \frac{ht+\tau \cdot Dt+\tau}{1+\delta_{t+\tau+j}(ht+\nu-1)} \right) + \mathbb{E}_t \left( \frac{ht+T \cdot St+T}{1+\delta_{t+T+j}(ht+\nu-1)} \right)
\end{align*}
\]

Continuing the iteration with \( \tau = 2, \ldots, T \) gives:

\[
\begin{align*}
ht \cdot St &= \mathbb{E}_t \left( \sum_{\tau=1}^{T} \frac{ht+\tau \cdot Dt+\tau}{1+\delta_{t+\tau+j}(ht+\nu-1)} \right) + \mathbb{E}_t \left( \frac{ht+T \cdot St+T}{1+\delta_{t+T+j}(ht+\nu-1)} \right)
\end{align*}
\]

Last, taking the limit of \( T \to \infty \) we end up with equation 2.2

\[
\begin{align*}
ht \cdot St &= \mathbb{E}_t \left( \sum_{\tau=1}^{\infty} \frac{ht+\tau \cdot Dt+\tau}{1+\delta_{t+\tau+j}(ht+\nu-1)} \right) + \lim_{T \to \infty} \mathbb{E}_t \left( \frac{ht+T \cdot St+T}{1+\delta_{t+T+j}(ht+\nu-1)} \right)
\end{align*}
\]
3 Benchmark Artificial Financial Market

In this chapter we will describe the benchmark model used for this research project. This model was designed by Hendri Adriaens as part of his PhD research (Adriaens, 2008). This model belongs to the field of Agent-based Computational Economics (ACE). According to LeBaron (2007) those models view financial markets as interacting groups of learning, rationally bounded agents. They concentrate on situations where analytical solutions are hard or impossible to find, and computer simulations might provide researchers with the necessary computational tools to provide a good solution. This way, traditional financial models can also be extended by investigating interactions of agents and heterogeneity among agents.

The possibility to provide a detailed analysis of agents’ behavior in artificial financial markets can be an useful tool to investigate much debated characteristics of the financial markets such as the efficient market hypothesis and stylized facts of asset returns. Traditional financial models do not account for many of these stylized facts\(^1\). The aim of the artificial financial market designed by Adriaens is to investigate whether such a model with realistic settings for the economic agents is capable of generating a number of stylized facts simultaneously, and hence a better description of a real financial market than traditional financial models.

The model consists of an artificial financial market with several different types of agents who each have different beliefs of future returns and different views of the world. The economy is characterized as a pure exchange economy where firms pay dividends, agents determine their demand for available assets, the market maker sets temporary equilibrium prices, and a regulator guarantees ordinary trading between

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\(^1\)an extensive summary of the stylized facts of asset returns is provided by Cont (2001)
agents. In the benchmark setting there are two types of agents in the economy. The first type uses the fundamental value of assets to determine their demand for the risky assets. The second type uses mean-variance optimization to determine their demands. All agents use econometric models to forecast returns and dividends on which they base their investment decision for the next period. The economy contains multiple risky assets as well as one riskfree asset.

In the first section of this chapter the characteristics of the benchmark agent based financial model are described in detail. In the second section parameter selection and simulation runs of the benchmark model are discussed.

3.1 Characteristics of the Benchmark Model

There are four types of participants in the artificial financial model of Adriaens (2008). First of all, there are firms that have risky assets on the market and pay dividends. Second, there are economic agents who determine their demand of the risky assets every period based on their believes and forecasts. Third, a market maker ensures an equilibrium of the market every period. Last, a regulator prevents crashes of the artificial financial market. The following sections will describe all participants in detail.

3.1.1 Assets

At time $t$ the economy contains $J + 1$ assets, which can be subdivided into $J$ ownerships of firms (risky assets) and one numéraire (riskfree asset). The price of risky asset $j$ in period $t$ is characterized by the variable $S_{jt}$ and the dividend payoff of asset $j$ in period $t$ by $D_{jt}$. $S_{jt}$ is set by the market, while $D_{jt}$ is determined by firm $j$. The riskfree asset, asset 0, has a price ($S_{0t}$) that is equal to 1 in all periods and a dividend payment ($D_{0t}$) that is equal to 0 in all periods.

The dividend process is assumed to be exogenously determined and given by the following dividend payment rule:

$$\log(D_{j,t+1}) = \epsilon_{Dj} + \log(D_{jt}) + \epsilon_{D,j,t+1}$$
with \( j = 1, \ldots, J \). In this equation \( c_{Dj} \) is the growth rate of the log dividends of asset \( j \) and \( \epsilon_{D,j,t+1} \) is a noise term. The dividend process is independent across firms.

The gross return of asset \( j \) in period \( t \) is defined by:

\[
r_{jt} = \frac{S_{jt} + D_{jt}}{S_{jt-1}}
\]

and the net return by: \( R_{jt} = r_{jt} - 1 \). It is assumed that \( D_{jt} \) is only observed after trading took place in period \( t \) to avoid informational differences between agents.

### 3.1.2 Agents

There are \( I \) agents participating in the economy, who hold and trade all available assets. At each period \( t \) each agent \( i \) decides on his portfolio holdings:

\[
h_{it} = (h_{i,0,t}, h_{i,1,t}, \ldots, h_{i,J,t})'
\]

where \( h_{i,j,t} \) are the portfolio holdings measured in the number of assets of asset type \( j \) agent \( i \) holds at the end of period \( t \). The initial endowments for agent \( i \) at the beginning of period \( t \) are hence given by \( h_{i,t-1} \).

\( u_{it} \) is defined as a vector of the value of each assets holdings of agent \( i \) in period \( t \) given by:

\[
u_{it} = h_{it} \odot S_t = \begin{pmatrix}
  h_{i,0,t} S_{0t} \\
  h_{i,1,t} S_{1t} \\
  \vdots \\
  h_{i,J,t} S_{Jt}
\end{pmatrix}
\]

\( W_{it} \) is the wealth of agent \( i \) at the beginning of period \( t \), which is given by:

\[
W_{it} = S_t \cdot h_{i,t-1} = S_{0t} h_{i,0,t-1} + S_{1t} h_{i,1,t-1} + \cdots + S_{Jt} h_{i,J,t-1}
\]

There are two types of agents in the benchmark model. The first type believes that a process underlying the asset price contains information about the fundamental value of the asset. This type is called *fundamentalist*. The second type of agent invests in a mean-variance efficient portfolio. This type is called *MV agent*. Both type of agents will be described in detail below.
Fundamentalists

The fundamentalist agents trade based on the price difference between the previous period prices and the perceived fundamental price during that period. Fundamentalists increase their demand for an asset when the price of the asset in the previous period is lower than the perceived fundamental price, and decrease their demand when the price is higher.

Adriaens (2008) assumes that fundamentalists in the model use a discounted dividend model to model the fundamental price of an asset. Agents hence discount expected dividends of each asset \( j \) independently as follows:

\[
S_{f,i,j,t} = \mathbb{E}_{it} \left( \sum_{\tau=1}^{\infty} \frac{D_{j,t+\tau}}{(1 + r_{D_j})^\tau} \right)
\]

In this equation \( \mathbb{E}_{it} \) is the expectation used by agent \( i \) to predict the return of asset \( j \) incorporating all information available to him up to time \( t \), and \( r_{D_j}^* \) is the discount rate of the dividend of asset \( j \). Under appropriate regularity conditions, the expression above can be simplified using the Gordon growth model (Gordon, 1962):

\[
S_{f,i,j,t} = \mathbb{E}_{it} (D_{j,t+1}) \frac{1 + g_j}{r_{D_j}^* - g_j} = \mathbb{E}_{it} (D_{j,t+1}) r_{D_j}^* \\
\]

where \( g_j \) is the expected growth rate of the dividend of asset \( j \), \( r_{D_j}^* \) is the expected return on the asset, and \( r_{D_j} \) is defined as \( r_{D_j} = \frac{r_{D_j}^* - g_j}{1 + g_j} \).

The fundamentalist needs to estimate \( \mathbb{E}_{it} (D_{j,t+1}) \). In the benchmark model it is assumed that the fundamentalist uses the last available dividend realization as an estimate. So, \( \mathbb{E}_{it} (D_{j,t+1}) = D_{j,t-1} \).

The fundamentalists will adjust their portfolio according to the difference of the fundamental price and price in the previous period. This is modeled in the following way:

\[
u_{i,j,t} = u_{i,j,t-1} + \alpha_i \text{sgn}(S_{f,i,j,t} - S_{j,t-1}) \left( \frac{|S_{f,i,j,t} - S_{j,t-1}|}{S_{f,i,j,t}} \right)^{\theta_i} S_{f,i,j,t}
\]

where \( \alpha_i > 0 \) is the adjustment speed of agent \( i \), \( \theta_i > 0 \) is the responsiveness to differences between the perceived fundamental value of asset \( j \) and the price of that asset the previous period by agent \( i \), and \( \text{sgn}(\cdot) \) is a sign function.
In words the above equation states that the demand of agent \( i \) for asset \( j \) differs from the demand in the previous period \( t - 1 \) based on a function of the perceived fundamental value of asset \( j \) by agent \( i \) and the price of asset \( j \) in the previous period. If the perceived fundamental value is larger than the price of the asset in the previous period, \( \text{sgn}(S_{f,i,j,t} - S_{j,t-1}) \) is positive and \( |S_{f,i,j,t} - S_{j,t-1}|/S_{f,i,j,t} \) is between 0 and 1. As a result, the agent will increase his demand for asset \( j \) proportional to the perceived fundamental value. If the perceived fundamental value is smaller than the price of the asset in the previous period the sign function is negative and \( |S_{f,i,j,t} - S_{j,t-1}|/S_{f,i,j,t} \) is larger than 0. So, the agent will decrease its demand proportional to the perceived fundamental value. If both values are exactly the same the agents demand will not change.

Furthermore, the larger the speed adjustment parameter \( \alpha_i \), the larger the demand adjustment. The larger the responsiveness parameter \( \theta_i \), the the lower the adjustment in demand. Provided that \( |S_{f,i,j,t} - S_{j,t-1}|/S_{f,i,j,t} < 1; \) which is the case while \( S_{j,t-1}, \leq 2S_{f,i,j,t} \) as will be in almost every situation.

**MV agents**

The MV-agents in the model always buy a mean-variance efficient portfolio by solving the following optimal investment problem.

\[
\max_u \quad E_t(W_{i,t+T_i}) - \gamma_{it} \text{Var}_t(W_{i,t+T_i}) \\
\text{s.t.} \quad W_{i,t+	au+1} = r'_{i,t+	au} + u_{i,t+	au} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad i', u_{i,t+	au} = W_{i,t+	au} \quad \tau = 0, 1, \ldots, T_i - 1
\]

where \( T_i \) is the investment horizon of agent \( i \) and \( \text{Var}_t(\cdot) \) is the variance operator for agent \( i \) using all information available to him until period \( t \). \( \gamma_{it} \) is the risk aversion parameter of agent \( i \), which can be time varying.

This problem can be solved for the entire planning horizon under the assumption of (perceived) iid returns\(^2\). Because the agents in the benchmark model are allowed to re-evaluate their portfolio every time period and implement their strategy only one period ahead, only the demands, for the different assets, that optimize the MV-portfolio in the next period (\( \tau = 0 \)) are relevant. Those asset demands are given

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\(^2\)for general result of this optimization see (Adriaens, 2008)
by:

$$\tilde{u}_{it} = \frac{r_{0,t+1} \left( \mathbb{E}_it \left( R_{t+1} R'_{t+1} \right) \right)^{-1} \mathbb{E}_it \left( R_{t+1} \right)}{2r_{it} \prod_{k=0}^{t} r_{t+k+1} \left( 1 - B_{i,t+k} \right)}$$

$$u_{i,0,t} = W_{it} - i' \tilde{u}_{it}$$

with

$$B_{i,t+k} = \mathbb{E}_{it} (R'_{t+k+1}) \left( \mathbb{E}_{it} \left( R_{t+k+1} R'_{t+k+1} \right) \right)^{-1} \mathbb{E}_{it} \left( R_{t+k+1} \right)$$

In these equations $\tilde{u}_{it} \in \mathbb{R}^J$ is a vector of agent $i$’s demand for all $J$ risky assets in period $t$ and $u_{i,0,t}$ is the demand for the riskfree asset. $i$ is a vector of ones with length $J$. The total demand $u_{ij} = (\tilde{u}_{it} \ u_{i,0,t})'$

For a planning horizon of $T_i = 1$ the demands can be simplified further to the standard Markowitz solution of the allocation problem.

The MV-agent needs to estimate the expected returns and variance of the returns of all assets. In the benchmark model, MV-agents will use the sample analogues over their entire memory as estimates for the expected return and covariance matrix. So,

$$\mathbb{E}_{it} (R_{t+1}) = \frac{1}{M_i} \sum_{k=1}^{M_i} R_{t-k}$$

and

$$\text{Cov}_{it} (R_{t+1}) = \frac{1}{M_i} \sum_{k=1}^{M_i} \left( R_{t-k} - \mathbb{E}_{it} (R_{t+1}) \right) \left( R_{t-k} - \mathbb{E}_{it} (R_{t+1}) \right)'$$

### 3.1.3 Market Maker

The role of the market maker in the model is to equate demand and supply in the artificial financial market. In Adriaens benchmark model, a temporary Walrasian equilibrium is used by the market maker in every time period. According to this theory the equilibrium prices $S_{jt}$ are given by:

$$S_{jt} = \frac{\sum_{i=1}^{I} \tilde{u}_{i,j,t-1}}{\sum_{i=1}^{I} \tilde{h}_{i,j,t-1}}$$

where the total supply of risky assets is given by $\sum_{i=1}^{I} \tilde{h}_{i,j,t-1}$, and the total demand of risky assets in terms of the numéraire is given by $\sum_{i=1}^{I} \tilde{u}_{i,j,t-1}$. 
3.1.4 Regulator

The regulator checks the asset prices that are proposed by the market maker to prevent a situation where the market maker suggests prices that cause the market to explode or to crash. The regulator limits the price difference in such a way that the realized return is within four standard deviations computed over the returns of the 2000 most recent periods. When the regulator intervenes and the price difference is limited; the demand of agents is adjusted accordingly but the supply does not change. The level of four times the standard deviation is chosen by Adriaens (2008) to allow for a relatively big shock to asset prices, but at the same time to avoid crashes.

3.2 Parameter Selection for the Benchmark Model

For the benchmark model the same parameter values as used in Adriaens (2008). This means that there are $I = 200$ agents in the market, who trade $J = 3$ assets, and there is one riskfree asset available. Furthermore, the dividend process is modeled on a weekly basis. The parameters of this process are $c_{D,j} = \frac{0.02}{52}$ for all $j$ assets; and $\epsilon_{D,j,t+1}$ are i.i.d. $N(0, \frac{0.06^2}{52})$ for all $j$ risky assets. The discount factor for fundamentalists is the same as the growth rate for the log dividends, so $r_{D,j} = \frac{0.02}{52}$ for all assets.

MV-agents adjust their risk aversion parameter $\delta_{it}$ in the following way.

$$\delta_{it} = \frac{\delta_{i,t-1}}{1 + c_D}$$

with $\gamma_{i1} = 5$. Due to this formulation the risk aversion parameter decreases over time, resulting in an increase in the amount of wealth of an MV-agent invested in the risky assets. This specification is chosen by Adriaens to help the MV-agents to keep up with the fundamentalists. These agents would otherwise take over the market. Because their predictions of the dividend process exhibit an exponential trend, resulting in a growing fundamental value of the risky assets, and hence an increasing allocation of wealth to risky assets by fundamentalists.

We will run each simulation for 5000 periods (weeks), and predict the last 1000 periods. The first 4000 periods are used to start up the economy and clear
away effects of the startup conditions. These are the same specifications as used by Adriaens. This way the results of this benchmark simulation and the simulations in the next chapter are not only comparable to each other, but also to Adriaens’ study (Adriaens, 2008).

### 3.2.1 Fundamentalists

In this section we will simulate an market with only fundamentalist agents to see the behavior of fundamentalists in the artificial financial market. First, we will describe the simulation results. Second, we will zoom in on some issues regarding fundamentalists in the artificial financial market.

#### Simulation of a Fundamentalists Only Economy

In the economy with fundamentalist agents only we will use all settings as described in the beginning of this section. Next to those values, we will base the portfolio adjustment rule on the parameters $\alpha = 3.4$ and $\theta = 1.3$. These parameter values are in line with the values used by Adriaens (2008), who has done simulations with multiple values for $\alpha$ and $\theta$. Those simulations point out that the values given above minimize the autocorrelations in the returns.

Table 3.1 gives an arbitrary outcome of the artificial financial market with only fundamentalist agents. The mean return of asset 1 is 0.076% per week, which means the yearly return on asset 1 is 3.95%. The kurtosis is larger than 3 for all assets and the values for the Jaque-Bera tests are very high compared to the 5% critical value for the Jaque-Bera test of 5.99. This implies that the asset returns are not normally distributed. The excess kurtosis also implies that heavy tales are present in the distribution of asset returns. Furthermore, figure 3.2 shows that there is little autocorrelation present in the asset returns, as well as in the squared and absolute returns.

In comparison to the research by Cont (2001), some of the stylized facts are present in this simulation. For instance, heavy tales and the absence of autocorrelation
Table 3.1: Summary statistics for the fundamentalists only economy.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean (%)</th>
<th>Covariance matrix (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.076</td>
<td>0.593 0.010 −0.032</td>
</tr>
<tr>
<td>2</td>
<td>0.094</td>
<td>0.010 0.587 −0.016</td>
</tr>
<tr>
<td>3</td>
<td>0.078</td>
<td>−0.032 −0.016 0.697</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Asset</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Jarque-Bera</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>−0.014</td>
<td>4.339</td>
<td>74.702</td>
<td>$\rho_{12} = 0.017$</td>
</tr>
<tr>
<td>2</td>
<td>0.124</td>
<td>4.913</td>
<td>155.116</td>
<td>$\rho_{13} = −0.050$</td>
</tr>
<tr>
<td>3</td>
<td>0.261</td>
<td>5.674</td>
<td>309.185</td>
<td>$\rho_{23} = −0.025$</td>
</tr>
</tbody>
</table>

Figure 3.1: Prices and returns (in %) for fundamentalists only economy; assets in rows, prices in column 1, returns in column 2.

Changes in Speed Adjustment and Responsiveness

As mentioned above, the value of the speed adjustment parameter $\alpha = 3.4$ is chosen to minimize autocorrelation. Changing this value has an effect on the economy. When $\alpha$ is decreased (the adjustment speed of agents becomes slower), the covariance matrix as well as kurtosis decreases. This is caused by the fact that the price adjustments
Figure 3.2: Autocorrelations for fundamentalists only economy; assets in rows, return in column 1, squared returns in column 2, absolute returns in column 3. The dashed lines indicate the 5% critical value for the test of whether or not the autocorrelations are equal to 0.

will be smaller and hence the shocks in returns will also be smaller. Next to that, the slower adjustment speed prevents the asset prices to return to their fundamental value in one period, causing significant positive first order autocorrelation.

When $\alpha$ is taken larger (the adjustment speed of agents becomes faster). The covariance matrix as well as the kurtosis become larger. This is caused by more rapid portfolio adjustments by agents, when the asset prices become larger than the fundamental value. This also leads to a significant negative autocorrelation.

The responsiveness parameter $\theta$ influences the demand of fundamentalists in the following way. Recall from equation 3.1 that the relation between $\theta$ and the demand of asset $j$ by agent $i$ is $(|S_{f,i,j,t} - S_{j,t-1}| / S_{f,i,j,t})^\theta$. In this part of equation 3.1 $|S_{f,i,j,t} - S_{j,t-1}| / S_{f,i,j,t}$ is usually between 0 and 1. This means that if the responsiveness, $\theta$, of an agent is increased (responsiveness becomes slower) the demand adjustment of that agent decreases. And the other way around, if $\theta$ decreases the demand adjustment becomes larger. In the later case there will be an overshooting of the fundamental price and therefore a negative first order autocorrelation in asset returns. This effect is in the similar direction as the effect of increasing $\alpha$.

Because there is an opposite effect of increasing $\alpha$ and $\theta$ on the autocor-
relation of returns. There might be a combination of $\alpha$ and $\theta$ that minimizes autocorrelation in returns. By definition of Adriaens (2008), the trading size ($TS$) is according

$$TS = \alpha \left( \frac{|S_{f,i,j,t} - S_{j,t-1}|}{S_{f,i,j,t}} \right)^\theta S_{f,i,j,t}$$

or approximately in terms of logarithms

$$\log(TS) \approx \log(\alpha) + \theta \log \left( \frac{|S_{f,i,j,t} - S_{j,t-1}|}{S_{f,i,j,t}} \right) + \log(S_{f,i,j,t})$$

If $TS$ is taken constant this leads to a relation between $\alpha$ and $\theta$ of:

$$\theta \approx a + b \log(\alpha)$$

Adriaens (2008) finds after numerous experiments that the parameters $\hat{a} = 0.9874$ and $\hat{b} = 0.2570$ minimize the autocorrelation. Because Cont mentions absence of autocorrelation as one of the stylized facts of asset returns (Cont, 2001). Next to that, the higher value for $\alpha$, the higher the kurtosis of the individual asset returns. Because a high kurtosis implies heavy tales which is also one of Cont’s stylized facts, a relative high value of $\alpha$ might capture this stylized fact better.

### 3.2.2 MV-agents

In this section we will simulate an market with only mean-variance efficient agents to see the behavior of MV agents in the artificial financial market. First, we will describe the simulation results. Second, we will zoom in on some issues regarding MV-agents in the artificial financial market.

**Simulation of a MV Agents Only Economy**

In this section we present a simulation with only MV agents. The specification of the market is the same as described in the beginning of section 3.2. Next to that, the MV agents use the sample mean and covariance matrix of their entire memory to predict next period asset returns is described in section 3.1.2. An overview of this simulation is given in figure 3.3

The MV agents only economy is very unstable and always results in a crash after an number of periods. Figure 3.4 shows a possible cause. The initial returns used
Figure 3.3: Prices and returns (in %) for a simulation of the MV-only economy; assets in rows, prices in column 1, returns in column 2.

Figure 3.4: Returns of the individual assets including the 1000 generated returns used for forecasting for a simulation of the MV-only economy

by agents to estimate their expected returns and covariances are on average lower and more volatile than the realized returns. This causes the expected return to increase as lower returns drop from the agents memory and the higher realized returns enter. At the same time the less volatile realized returns that enter the memory of the agents decrease the variance of the asset returns. As a result the prices rise even further. The realized returns of asset 2 are more volatile than those of asset 1 and 3 and therefore this asset becomes less attractive for agents, resulting in a crash of asset 2. To prevent this the regulator intervenes as is shown in figure 3.5. However this will not prevent agents from willing to sell asset 2 resulting in the further decrease of asset 2. When the simulation continues at a certain time agents all start to buy the now very cheap
3 Benchmark Artificial Financial Market

Figure 3.5: Regulator intervention during the simulation of the MV-only economy

asset 2 and sell assets 1 and 3 resulting in an crash of those two assets.

This simulation reveals that there is not enough heterogeneity among MV agents to let the MV only market become stable. To create more heterogeneity either MV agents should be more different in there forecast rules or other types of agents should be included.

3.3 Economy 0: The Benchmark Model

The first economy that we will discuss is the same economy as described by Ardiaens in section 4.2.4 of his Ph.D. thesis (Adriaens, 2008). This model will serve as a benchmark model that will be used as a reference for further research. Economy 0 consists of 100 fundamentalist agents and 100 MV agents. We use $\alpha = 4.5$ and $\theta = 1.1$ to minimize autocorrelations in returns.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean (%)</th>
<th>Covariance matrix (%)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.029</td>
<td>1.743 $\rho$</td>
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<tr>
<td>2</td>
<td>0.107</td>
<td>-0.007 1.885 0.058</td>
</tr>
<tr>
<td>3</td>
<td>0.124</td>
<td>0.064    0.058 1.953</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Asset</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Jarque-Bera</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.006</td>
<td>3.580</td>
<td>14.004</td>
<td>$\rho_{12} = -0.004$</td>
</tr>
<tr>
<td>2</td>
<td>0.116</td>
<td>3.535</td>
<td>14.187</td>
<td>$\rho_{13} = 0.035$</td>
</tr>
<tr>
<td>3</td>
<td>0.107</td>
<td>3.426</td>
<td>9.468</td>
<td>$\rho_{23} = 0.030$</td>
</tr>
</tbody>
</table>

Table 3.2: Summary statistics of economy 0 with 100 fundamentalist agents and 100 MV agents
Table 3.2 shows the summary statistics for an arbitrary run of economy 0. The Kurtosis as well as the Jarque-Bera tests show that the individual asset returns are not likely to be normally distributed. The excess kurtosis shows that the distribution of the individual asset returns has heavy tails compared to a normal distribution. Figure 3.6 shows that, despite that the $\alpha_i$ and $\theta_i$ parameter are chosen to minimize autocorrelation, there is still some autocorrelation. Mainly in the first and second lag. This might be caused by the simplicity of the model, which has only three risky assets and 200 agents trading them.

Adriaens (2008) investigates different extensions to this model. Among other things he finds that using different econometric techniques in determining the expected dividend by fundamentalists does not have a large influence in the simulation results, therefore we will use only the “last dividend rule” explained in section 3.1.2 in our research. Adriaens (2008) also finds that increasing the econometric techniques from which MV-agents can choose to predict their covariance matrix, will positively influence the possibility to replicate the stylized fact “aggregational Gaussianity”. MV-agents are only used in the last specification of our research we use the most simple specification of the beliefs of MV agents to make their influence easily identifiable in this economy where trend followers are introduced to the benchmark model.
4 A Market with Fundamentalists and Trend Followers

Using Agent Based Computational Economics it is possible to design a market where agents have heterogeneous beliefs of future asset payoffs. This means that different groups of agents have different beliefs about future asset prices. Most models with heterogeneous agents consider two type of agents\(^1\). The first type being the *fundamentalist* who has rational expectations, believing that the EMH holds and prices of assets are solely determined by the fundamental values derived from this EMH. The second type of agent being a type of *noise traders* who does not believe prices to be fully determined my the fundamental value of the EMH. A commonly used type of noise traders is the chartist or technical annalist who believes prices can be predicted using simple rules based on patterns in historical data series on the asset prices and are therefore called *trend followers*.

In the recent literature there are numerous models trying to create a heterogeneous financial market consisting of both fundamentalists and trend followers\(^2\). Many models claim to exhibit many of the stylized facts (Cont, 2001).

In this chapter trend followers will be introduced to the artificial financial market to investigate the effects of this type of agents on the artificial economy and the ability of this new economy to capture the stylized facts of asset returns. Trend followers are agents who believe that all relevant informations that determines future asset payoffs is represented in the price of the asset. Trend followers believe for instance that if prices of an asset go up, they also tend to go up in the next period.

In the first section of this chapter, trend followers are introduced. In the

\(^1\)for an overview see LeBaron (2007)
\(^2\)for instance Gaunersdorfer and Hommes (2007), and Hi and Li (2007)
next three sections, trend followers with different beliefs about the way to determine a trend are introduced. In the fifth section, an economy with more heterogeneity among agents is discussed. In the sixth section, some agents are allowed to switch between beliefs behaving as trend followers or as fundamentalists. In the last section, trend followers are introduced to the benchmark economy of section 3.3 to see their influence on this economy.

4.1 Trend Followers

Trend followers believe that all relevant informations that determines future asset payoffs is represented in the price of the asset and that assets which were past winners will continue to be winners in the future. Trend followers will increase their demand for an asset when the price of that asset in the previous period is higher than a moving average of historical returns. They will decrease their demand when the price in the previous period is lower than the moving average of historical returns.

The portfolio adjustments of trend followers are modeled in the following way:

\[ u_{i,j,t} = u_{i,j,t-1} + \beta_i (S_{j,t-1} - S_{MA,i,j,t}) \]  

(4.1)

where \( \beta_i > 0 \) is a adjustment parameter. \( S_{MA,i,j,t} \) is a moving average. When \( S_{MA,i,j,t} < S_{j,t-1} \) the trend follower will increase his demand for asset \( j \) in period \( t + 1 \). When on the other hand \( S_{MA,i,j,t} > S_{j,t-1} \) he will decrease his demand.

In the simulations in this chapter, all parameters except for the speed adjustment parameters and the responsiveness parameter for fundamentalists are the same as the ones in the benchmark model discussed in the previous chapter. Furthermore, the same realization of the dividend process that is used in the benchmark model is used in all simulations to make the results better comparable.

There are numerous possible specifications for the moving average process \( S_{MA,i,j,t} \). In the next three subsections three different moving average processes are evaluated. In the first section, the simple moving average is considered. In the second section, a weighted moving average, and in the third section an exponential moving average is used.
4.2 Economy 1: Trends Based on a Simple Moving Average

A simple moving average is defined as:

$$S_{MA,i,j,t} = \frac{S_{j,t-1} + \ldots + S_{j,t-n_i}}{n_i}$$

This moving average is the average over the length of the memory of agent $i$ ($n_i$) with equal weights. In economy 1 there are 100 Fundamentalists and 100 Trend Followers using simple moving averages to determine the presence of a long term trend in the asset prices, with a memory length that is uniformly distributed between 75 and 100 periods (weeks). To reduce computation time the number of possible memory lengths is reduced to 20 in the same way that is used by Adriaens (2008) to model the memory length of MV-agents in his research project.

The speed adjustment ($\alpha_i$) and responsiveness parameter ($\theta_i$) of fundamentalists, as well as the adjustment parameter of trend followers ($\beta_i$), are chosen in such a way that autocorrelation is minimized and that the kurtosis is significantly higher than 3. This is done to capture the no-autocorrelation and heavy tails stylized fact. The parameters used are the same for all agents $\alpha_i = 7$, $\theta_i = 1.3$ and $\beta_i = 0.1$. The influence of changing the adjustment parameters is discussed in a subsequent paragraph below.

The results for the last 1000 periods of a simulation with 5000 periods are provided in table 4.1. If we compare the results to the results of the simulation with fundamentalists only in table 3.1, we see that the mean returns are almost equal. However, the variances and covariances are a lot lower. The model is negative skewed in contrast to the benchmark. The kurtosis of this model is lower than that of the fundamentalists only economy. Although, still larger than 3 and hence pointing at heavy tails. The J-B test still rejects normality.

Figure 4.1 shows very little autocorrelation. the returns only show a significant autocorrelation at a 95% confidence level in first lag of asset 1 and 2.
Table 4.1: Summary Statistics for economy 1; 100 fundamentalist agents and 100 trend followers using a simple moving average to examine the presence of a trend and parameters $\alpha_i = 7$, $\theta_i = 1.3$ and $\beta_i = 0.1$

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean (%)</th>
<th>Covariance matrix (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.024</td>
<td>0.682 $-$0.009 0.034</td>
</tr>
<tr>
<td>2</td>
<td>0.102</td>
<td>$-$0.009 0.682 0.018</td>
</tr>
<tr>
<td>3</td>
<td>0.116</td>
<td>0.034 0.018 0.590</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Asset</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Jarque-Bera</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-$0.107</td>
<td>4.413</td>
<td>85.113</td>
<td>$\rho_{12} = -0.014$</td>
</tr>
<tr>
<td>2</td>
<td>$-$0.062</td>
<td>3.802</td>
<td>27.457</td>
<td>$\rho_{13} = 0.053$</td>
</tr>
<tr>
<td>3</td>
<td>$-$0.079</td>
<td>4.230</td>
<td>64.027</td>
<td>$\rho_{23} = 0.028$</td>
</tr>
</tbody>
</table>

Changing the Parameters $\alpha_i$, $\theta_i$ and $\beta_i$

As we have seen in section 3.2.1 not all possible values of the speed adjustment and responsiveness parameters will result in an economy which resembles the stylized facts (Cont, 2001). Different values will result in different autocorrelation patterns. One of the goals of this artificial financial market is to mimic as many of the stylized facts (Cont, 2001) as possible, $\alpha_i$ and $\theta_i$ are chosen in such a way that autocorrelation is minimized. Next to that, $\alpha_i$ is chosen sufficiently large so the assets have excess kurtosis, resulting in heavy tales.

When trend followers are added to the economy, the number of adjustment parameters that can be used to minimize the autocorrelation process raises from two to three. Due to the influence of the parameter $\beta_i$ the autocorrelation is no longer minimized when $\alpha_i$ and $\theta_i$ are chosen following the relation of equation 3.2.1. In this paragraph the influence of changing the $\alpha_i$, $\theta_i$ and $\beta_i$ parameters will be discussed in relation to the chosen values for the parameters in the simulation above. Table 4.7 in the appendix at the end of this chapter shows the summary statistics of all specifications discussed below.

Changes to $\alpha_i$ and $\theta_i$ will to a large extent have the same influence as we have seen in section 3.2.1 where the fundamentalists only economy was discussed. If $\theta_i$ is decreased from 1.3 to 1.2 the mean, variance, covariance, kurtosis, and correlation all increase substantially. The skewness becomes positive for asset 1 and
2 and the autocorrelations become significant in the first few lags. Such skewness and autocorrelations are not desirable when trying to mimic stylized facts. When increasing $\theta_i$ from 1.3 to 1.4, all statistics decrease. However, the magnitude of the decrease is very small. Next to that, the first few lags of the autocorrelation function exhibit a significant positive and fast decaying autocorrelation. As mentioned in section 3.2.1 to reduce this significant positive autocorrelation $\alpha_i$ can be increased. We will see below that increasing $\beta_i$ will also result in significant, positive, fast decaying autocorrelation as increasing $\theta_i$ does. So, if we want to increase $\theta_i$ and still minimize autocorrelation in the economy, we have to increase $\alpha_i$ relative to $\beta_i$ resulting in less influence of trend followers on the economy. Numerous simulation runs point out that $\theta_i = 1.3$ is the lowest value of $\theta_i$ for which there is no or little significant autocorrelation. If $\theta_i$ is decreased the simulation runs always exhibit a significant, alternating autocorrelation in the first couple of lags. So, for $\theta_i = 1.3$ the size of adjustments of trend followers compared to the size of adjustments of fundamentalists is highest.

If $\alpha_i$ is decreased from 7 to 6.5 all summary statistics decrease slightly and there is a significant positive autocorrelation in the first few lags. When $\alpha_i$ is increased all statistics increase slightly but there is still no significant autocorrelation at a 95%
confidence level. If $\alpha_i$ is increased even further negative autocorrelation in the first few lags will arise. So, changing $\alpha_i$ will influence the economy in the same way as the parameter did in the fundamentalists-only economy.

Decreasing the adjustment parameter for trend followers $\beta_i$ from 0.1 to 0.05 has no effect on the mean returns, but the variance and covariance decrease. The skewness increases and becomes positive for asset 2 and 3. If $\beta_i$ is increased from 0.1 to 0.2, the mean returns as well as the variance and covariance increase, whereas the skewness and kurtosis decrease. If $\beta_i$ is increased positive autocorrelation arises.

As stated above, the effect of increasing $\theta_i$ and $\beta_i$ on the autocorrelation function both result in significant, positive, fast decaying autocorrelation. This positive autocorrelation can be compensated by increasing $\alpha_i$ to minimize the autocorrelation in the economy. However, a much larger increase of $\alpha_i$ is required to compensate the positive autocorrelation caused by an increase in $\beta_i$, thereby decreasing the influence of trend followers on the economy. Hence, the specification above minimizes the autocorrelation while maximizing the influence of trend followers in the economy.

### 4.3 Economy 2: Trends Based on a Weighted Moving Average

In economy 2 trend followers use a weighted moving average to investigate the presence of a trend in the asset prices. The weighted moving average is modeled as:

$$S_{MA,i,j,t} = \frac{n_i S_{j,t-1} + (n_i - 1) S_{j,t-2} + \ldots + 2 S_{j,t-n_i+1} + S_{j,t-n_i}}{n_i + (n_i - 1) + \ldots + 2 + 1} \quad (4.2)$$

Trend followers find prices in the near past more important than prices longer ago. All parameters as well as the distribution of the memories used by agents to determine the trend are the same as in the previous economy. The results of the last 1000 observations of a simulation of 5000 periods are provided in table 4.2.

When comparing the means, variances, covariances and kurtosis of economy 2 with those of economy 1, we see that there is little difference. Except for the skewness of this economy, which is remarkably more negative. When changing $\alpha_i$, $\theta_i$ and $\beta_i$, this economy responds in the same way as the economy where trend followers
<table>
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<th>Asset</th>
<th>Mean (%)</th>
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<td>0.034</td>
</tr>
<tr>
<td>2</td>
<td>0.102</td>
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</tr>
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<td>0.116</td>
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<td>0.587</td>
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<th>Jarque-Bera</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>89.586</td>
<td>( \rho_{12} = -0.013 )</td>
</tr>
<tr>
<td>2</td>
<td>−0.030</td>
<td>3.810</td>
<td>27.487</td>
<td>( \rho_{13} = 0.054 )</td>
</tr>
<tr>
<td>3</td>
<td>−0.021</td>
<td>4.242</td>
<td>64.312</td>
<td>( \rho_{23} = 0.028 )</td>
</tr>
</tbody>
</table>

Table 4.2: Summary Statistics for economy 2; 100 fundamentalist agents and 100 trend followers using a weighted moving average to examine the presence of a trend and parameters \( \alpha_i = 7 \), \( \theta_i = 1.3 \) and \( \beta_i = 0.1 \).

use a simple moving average as discussed in the previous section. The only difference in the results is that the weighted moving average accounts for a slightly lower (more negative) skewness. The summary statistics for simulations with different parameter settings are reported in table 4.8 in the appendix to this chapter.

### 4.4 Economy 3: Trends Based on an Exponential Moving Average

In economy 3 trend followers update their beliefs using an exponential moving average given by:

\[
\overline{S}_{MA,i,j,t} = \gamma_i S_{j,t-1} + (1 - \gamma_i) \overline{S}_{MA,i,j,t-1}
\]  

Trend followers update \( \overline{S}_{MA,i,j,t} \) using a weighted average over the last period’s price and the last period’s moving average. They apply a weighting factor \( \gamma_i \) that decreases exponentially\(^3\). If \( \gamma_i \) is chosen 1 only the previous period’s price is used. If \( \gamma_i \) is chosen close to 0 the moving average is highly dependent on the moving average of the previous period and the demand of agents is almost not influenced by the previous period’s price. In this specification agents can be heterogeneous in the way they discount older prices. If \( \gamma_i \) is high the agent tries to profit from short period trends. If

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\(^3\)see the appendix to this chapter for a derivation of the moving average equation that shows why past data is discounted exponentially.
\[ S_{MA,i,j,0} = \frac{S_{j,-h} + \ldots + S_{j,0}}{h} \]

where \( h \) is the length of the vector of historical prices available to all agents at the beginning of the simulation at \( t = 1 \).

In this economy we use the following parameter settings \( \alpha_i = 7, \theta_i = 1.3, \beta_i = 0.1 \) and \( \gamma_i = 0.9 \) for all \( I \) agents. The results for the last 1000 of 5000 observations are provided in table 4.3.

The results are very similar to the results of economy 2 summarized in table 4.2. Except for the skewness, which is slightly more negative. In economy 1 and 2 trend followers are heterogeneous in terms of the length of the historical prices used to determine the presence of a trend. In economy 3 all trend followers are assumed to discount prices using the same weighting factor \( \gamma_i \). In the next subsection we will first
Table 4.3: Summary Statistics for economy 3; 100 fundamentalists and 100 trend followers using an exponential moving average to examine the presence of a trend and parameters $\alpha_i = 7$, $\theta_i = 1.3$, $\beta_i = 0.1$ and $\gamma_i = 0.9$ describe the influence of changing $\gamma_i$ on the economy. Subsequently in the section that follows, we will describe an economy where trend followers use an exponential moving average and are heterogeneous among $\gamma_i$.

**Changing the Parameters $\alpha_i$, $\theta_i$, $\beta_i$ and $\gamma_i$**

This economy responds in approximately the same way to changes in the parameters $\alpha_i$, $\theta_i$ and $\beta_i$ as the economy using a simple moving average to determine the demand for trend followers, which is described in section 4.2. The summary statistics for simulations with different parameter settings are reported in table 4.9 in the appendix to this chapter.

Decreasing $\gamma_i$ from 0.9 to 0.7 has the following effect on the statistics in table 4.3: the mean and covariance stay approximately the same, whereas the variance increases substantially. The kurtosis decreases and the skewness becomes more negative. The Jarque-Bera test still highly rejects normality. The autocorrelation process shows a significant negative autocorrelation in the first lag. Decreasing $\gamma_i$ even further to 0.5 shows changes of the statistics in the same direction except for the skewness which becomes less negative again. When we use different parameter settings for $\alpha_i$, $\theta_i$ and $\beta_i$ (not trying to minimize autocorrelation), we see that in general a low $\gamma_i$ leads to negative first leg autocorrelation, whereas a high $\gamma_i$ leads to positive first order autocorrelation.
4.5 Economy 4: More Heterogeneity among Agents

In this economy we create more heterogeneity by using different parameter values of some parameters for each agent $i$. In contrast to economy 3 where the same values are used for each agent of the same type. Fundamentalists are assumed to use a speed adjustment parameter $\alpha_i$ drawn from a uniform distribution between 5 and 7. Furthermore, trend followers are assumed to discount the importance of older prices at a different rate by using different values of $\gamma_i$. It is assumed that this weighting factor is drawn from a uniform distribution between 0.5 and 0.9. All other parameter values are the same as in the previous economies. Results of this simulation for the last 1000 of 5000 periods are provided in table 4.4 and figure 4.4.

When we compare the statistics of economy 4 to those of economy 3 in table 4.3, we see some changes. The mean returns are the same for both simulations. There is some difference in the covariances and the heterogeneity results in higher variances for all three assets. The skewness is much more negative. The kurtosis is a bit lower, but still significantly larger than three. This means that heavy tails are still
Table 4.4: Summary Statistics for economy 4; 100 fundamentalists and 100 trend followers using a exponential moving average with heterogeneous weighting factors to examine the presence of a trend and parameters $\alpha_i \sim U(5;7)$, $\theta_i = 1.3$, $\beta_i = 0.1$ and $\gamma_i \sim U(0.5;0.9)$.

In comparison to the Fundamentalists only simulation described in section 3.2.1 and table 3.1, we see very little difference between the mean of the returns. The variances as well as the covariances are all slightly lower. The same holds for the kurtosis. However, there is a large difference in the skewness. In the fundamentalists-only economy the skewness of asset 2 and 3 is positive and the skewness of asset 1 slightly negative. In the economy with also heterogeneous trend followers the skewness of the returns of all assets is negative and the magnitude of the skewness is much bigger. The correlation between the returns of the 3 assets decreases and normality is still strongly rejected by the Jarque-Bera test.

In comparison to teh summary statistics of economy 0 in table 3.2, the returns of economy 4 are all a little lower. There is a very large difference in variance between the two settings. Economy 0 has almost twice the variance of economy 4. The skewness in the two economies is also very different, being positive in economy 0 and higher in magnitude and negative in economy 4. The kurtosis and correlations in economy 0 are also much lower than in economy 4. Furthermore, there is a significant autocorrelation in the lower lags present in economy 0 (see figure 3.6) that is much larger than the autocorrelations for economy 4 in figure 4.4.

The difference in the results of economy 4 and economy 0 show that it might
Figure 4.4: Autocorrelations for economy 4; assets in rows, return in column 1, squared returns in column 2, absolute returns in column 3. The dashed lines indicate the 5% critical value for testing whether or not the autocorrelations are equal to 0.

be possible to capture some of the stylized facts in this simulation that were not captured by Adriaens (2008), but some others might be less present.

4.6 Economy 5: Type Switching

An artificial financial market can model an evolutionary system of heterogeneous agents, in which agents can learn and adept their beliefs based on their experience. One of the first and simplest ways to capture such evolutionary dynamics are Adaptive Believe Systems presented by Brock and Hommes (1998). In an adaptive believe system, each agent can choose from a finite set of different beliefs or predictors of the future asset price. The selection of the belief that is used in the next investment period, is based upon a fitness of a performance measure such as past realized profit. The choice is (boundedly) rational in the sense that, at each point in time, agents choose the predictor generating the highest past performance. An important feature of this approach is that irregular switching is triggered by a rational choice between simple prediction strategies.

In our artificial financial market agents might switch between being a trend
follower and a fundamentalist based on past performance of both strategies. If prices deviate from their fundamental value trend following might seem more profitable to an economic agent than trading based on the fundamental value. The agent might therefore switch to become a trend follower. This is called herding behavior. If prices revert towards their fundamental value, becoming a fundamentalist might seem more profitable.

For instance Hi and Li (2007) and Gaunersdorfer and Hommes (2007) use type switching for agents between fundamentalist and trend follower. These models exhibit volatility clustering and an autocorrelation structure that comes close to real asset data (Franke, 2008). However, much of volatility clustering is caused by the incorporation of a noise term in the model to determine the fraction of fundamentalists versus the fraction of trend followers as is pointed out by Franke (2008). The goal of this subsection is to incorporate type switching without a noise term in economy 5 to see how our artificial financial market model behaves in terms of stylized facts when type switching is allowed.

### 4.6.1 Type Switching and Prediction Rules

A variant of the adaptive believe system, that is used in the literature is the Exponentially Weighted Mean Squared Prediction Error (EWMSPE) following LeBaron (2002). Using this approach, in every period, agent computes the performance of each of his prediction rule . Here is the set of rules used by agent to calculate the expected return of asset . Thus, consists either of the rule to predict returns for a fundamentalist approach, the rule to predict returns for a trend following approach, or both rules. The performance is updated every period using the following rule.

\[ v_{l,i,j,t} = (1 - \delta) v_{l,i,j,t-1} + \delta \left( R_{jt} - \mathbb{E}_{t-1}^l (R_{jt}) \right)^2 \]

with \( \delta > 0 \) representing a weighting parameter, \( \mathbb{E}_{t-1}^l (R_{jt}) \) the return on asset in period using prediction rule and all data up to period \( t - 1 \), and \( R_{jt} \) is the realized return on asset in period \( t \).

EWMSPE can be interpreted in terms of costs following Adriaens (2008). If

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4for instance, Adriaens (2008) and LeBaron (2002)
the EWMSPE is relatively high, this indicates a rule that forecasts badly, and hence possibly into suboptimal investment decisions. Next to these costs, there might also be additional costs, for instance, for data collection and econometric analysis. The total costs of rule \( l \in L_i \) for asset \( j \) of agent \( i \) in period \( t \), \( V_{l,i,j,t} \), becomes:

\[
V_{l,i,j,t} = \phi_0^l + \phi_1^l v_{l,i,j,t} + \epsilon^l_{i,j,t}
\]

where \( \phi_0^l \) donated the constant costs for rule \( l \), \( \phi_1^l \) the variable costs and \( \epsilon^l_{i,j,t} \) a noise term that is \( N \left( 0, \left( \sigma^l \right)^2 \right) \) distributed. Agent \( i \) then selects the rule with the lowest costs in the next period for every asset \( j \) individually.

In our model we use a slightly different approach that is inspired by the EWMSPE\(^5\). We assume that agents do not use costs to calculate the performance of different beliefs and adapt their beliefs to minimize costs. Instead, we assume agents are trying to maximize the return on their portfolio of risky assets, they use the excess predicted return time \( t \) over the realized return in period \( t \). If agents see that their strategy is lagging in terms of excess returns compared to the other strategy they become uncertain about their strategy because the agent thinks he could make a better return using the different strategy. If the current strategy of the agent keeps performing less than the other strategy for a considerable time the agent will switch to the other strategy.

We assume the agent \( i \) to calculate the performance \( v \) of each of his \( l \in L_i \) prediction rules for asset \( j \) in period \( t \) as follows:

\[
v_{l,i,j,t} = (1 - \delta) v_{l,i,j,t-1} + \delta \left( E_{l,t-1}^j (R_{jt}) - R_{jt} \right)
\]

with \( \delta > 0 \) representing a weighting parameter, \( E_{l,t-1}^j (R_{jt}) \) the predicted return on asset \( j \) in period \( t \) using prediction rule \( l \) and all data available up to period \( t - 1 \), and \( R_{jt} \) is the realized return on asset \( j \) in period \( t \).

Agents are assumed to use the same strategy for all risky assets. This is the prediction rule \( l \) following from: \( \arg \max_l (\sum_{j=1}^J v_{l,i,j,t}) \). In the next two subsections the way in which the two different types of agents predict their future returns will be discussed.

---

\(^5\)We tested the EWMSPE rule with multiple parameter settings but this rule does not result in type switching. Dependent on the parameter settings for \( \alpha_i \), \( \theta_i \), \( \beta_i \), and \( \gamma_i \) all investors either prefer trend following or the fundamental approach during the entire simulation.
Predicting Returns for Fundamentalists

Fundamentalists use the adjustment rule in equation 3.1 to determine their demand for period $t$. This decision is based on a forecast of the fundamental price of an asset, which is based on a prediction of the dividend paid on asset $j$ in period $t$. This is in line with Adriaens (2008) and is assumed to be the same as the last known dividend that is paid (see section 3.1.2). Fundamentalists do not forecast a return, but a fundamental price. They use on this fundamental price in predicting $E_{i,t-1}(R_{jt})$ used in equation 4.4. It is assumed that fundamentalists make the return prediction after period $t-1$ prices and dividends are observed.

Fundamentalists predict their returns using the formula:

$$R_{f,i,j,t} = \frac{S_{f,i,j,t} + E_{i,t-1}(D_{jt}) - S_{j,t-1}}{S_{j,t-1}}$$

Here $E_{i,t-1}(D_{jt})$ is the expected dividend payment in period $t$ according to the model used by agent $i$ using all information up to period $t-1$. In our case: $E_{i,t-1}(D_{jt}) = D_{j,t-1}$. As mentioned earlier, $S_{j,t-1}$ is the realized price of asset $j$ in period $t-1$ and $S_{f,i,j,t}$ is the calculated fundamental value of asset $j$ by agent $i$ at time $t$.

Also in line with Adriaens (2008), fundamentalists might be aware that it is possible that this fundamental price will not realize immediately, but rather in the long run. Therefore, they will not apply this returns directly for computing the forecast errors and making decisions. Instead, they estimate the following regression for each asset $j$:

$$R_{j\tau} = \beta_{i0} + \beta_{i1}R_{f,i,j,\tau} + \epsilon_{f,j,\tau} \quad \tau = t - M_i, \ldots, t - 1$$

using $M_i$ periods of information, under the assumption of mean-independence: $\mathbb{E}(\epsilon_{f,j,\tau}) = \mathbb{E}(\epsilon_{f,j,\tau}|\Omega_\tau) = 0$, with $\Omega_\tau$ is the set of all information available at time $\tau$. Based on this regression fundamentalists predict the next periods return as:

$$E_{i,t-1}(R_{jt}) = \hat{\beta}_{i0} + \hat{\beta}_{i1}R_{f,i,j,t}.$$

Predicting Returns for Trend Followers

Trend followers use the adjustment formula in equation 4.1 to determine their demand in period $t$. They assume a trend to be present in the asset prices and hence expect the
future price of the asset to follow that trend. Their formulation of the return on asset $j$ is assumed to be as follows:

$$R_{tf,i,j,t} = \frac{\mathbb{E}_{i,t-1}^*(X_{tf,i,j,t}) - S_{j,t-1}}{S_{j,t-1}}$$

where $\mathbb{E}_{i,t-1}^*(X_{tf,i,j,t})$ is the expected payoff by trend following agent $i$ of asset $j$ in period $t$ estimated in period $t-1$, and $S_{j,t-1}$ is the realized price of asset $j$ using all information up to period $t-1$. $\mathbb{E}_{i,t-1}^*(X_{tf,i,j,t})$ is assumed to be predicted using a trend in the moving average. The trend follower uses a regression to predict the price of asset $j$ in period $t$ based on the moving average $\bar{S}_{MA,i,j,t}$, which is calculated at time $t$ before $S_{j,t}$ is known based on past values of $S_j$. They use the regression:

$$X_{tf,i,j,\tau} = \alpha_{i0} + \alpha_{i1}\bar{S}_{MA,i,j,\tau} + \epsilon_{tf,j,\tau}$$

$\tau = t - M_i, \ldots, t - 1$

using $M_i$ periods of information, under the assumption that $\mathbb{E}(\epsilon_{tf,j,\tau}) = \mathbb{E}(\epsilon_{tf,j,\tau}|\Omega_\tau) = 0$, where $\Omega_\tau$ is the set of all information available at time $\tau$. Based on this regression the forecast for the next periods return can be calculated using:

$$\mathbb{E}_{i,t-1}^*(X_{tf,i,j,t}) = \hat{\alpha}_{i0} + \hat{\alpha}_{i1}\bar{S}_{MA,i,j,t}.$$  

Concerning the realization of their predicted price trend followers have the same insight as fundamentalists, namely that the behavior of other agents might influence their predicted returns in the short run. They assume a trend in returns following its moving average in the long run. Therefore, they also do not apply the calculated returns directly when computing forecast errors and making decisions. Instead, they estimate the following regression for each asset $j$:

$$R_{jt} = \gamma_{j0} + \gamma_{j1}R_{tf,i,j,\tau} + \eta_{tf,j,\tau} \quad \tau = t - M_i, \ldots, t - 1$$

using $M_i$ periods of information, under the assumption that $\mathbb{E}(\epsilon_{tf,j,\tau}) = \mathbb{E}(\epsilon_{tf,j,\tau}|\Omega_\tau) = 0$, where $\Omega_\tau$ is the set of all information available at time $\tau$. Based on this regression they predict the next period’s return used in equation equation 4.4 using:

$$\mathbb{E}_{i,t-1}^*(R_{jt}) = \hat{\gamma}_{j0} + \hat{\gamma}_{j1}R_{tf,i,j,t}.$$
### 4.6.2 Simulation Results

In this economy we use 50 fundamentalists, 50 trend followers, and 100 agents who may change between the two types. The 100 agents who may change start as fundamentalist but that can change their beliefs by becoming trend follower if that strategy consistently predicts better returns and vice versa. To create a large difference between agents we use the parameter settings of economy 4. So, fundamentalists use the parameters $\theta_i = 1.3$ and $\alpha_i \sim U(5, 7)$. Trend followers use $\beta_i = 0.1$ and $\gamma_i \sim U(0.5, 0.9)$. The memory that each agent uses for predicting their returns is drawn from a uniform distribution between 100 and 800 periods. This wide interval is chosen to increase the heterogeneity. The agent’s memory length affects the agent’s return prediction and hence also the perceived performance of the two strategies by the agent. This will result in a large difference in the predicted optimal strategy for agents in the switching group in some time periods. $\delta = \frac{1}{20}$ is used by all agents as a weighting factor in their performance prediction for rule $l$. This way past performance of the rule is heavily dependent on the past performance compared to the last period prediction. The rest of the parameters is chosen to be the same as in the previous economies.

The results for the last 1000 of 5000 periods of a simulation using the same dividend process as is used in the previous economies are presented in Table 4.5, figure

---

**Table 4.5:** Summary Statistics for economy 5; 50 Fundamentalists and 50 trend followers, and 100 Type switchers using an exponential moving average with heterogeneous weighting factors to examine the presence of a trend and parameters $\alpha_i \sim U(5, 8)$, $\theta_i = 1.3$, $\beta_i = 0.1$ and $\gamma_i \sim U(0.5; 0.9)$. Memory lengths of agents are $M_i \sim U(100, 800)$ and $\delta = 1/20$.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean (%)</th>
<th>Covariance matrix (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.023</td>
<td>0.617 0.016 0.010</td>
</tr>
<tr>
<td>2</td>
<td>0.099</td>
<td>0.016 0.640 0.019</td>
</tr>
<tr>
<td>3</td>
<td>0.114</td>
<td>0.010 0.019 0.545</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Asset</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Jarque-Bera</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.603</td>
<td>4.137</td>
<td>114.432</td>
<td>$\rho_{12} = 0.026$</td>
</tr>
<tr>
<td>2</td>
<td>-0.262</td>
<td>4.205</td>
<td>71.999</td>
<td>$\rho_{13} = 0.018$</td>
</tr>
<tr>
<td>3</td>
<td>-0.560</td>
<td>4.906</td>
<td>203.600</td>
<td>$\rho_{23} = 0.033$</td>
</tr>
</tbody>
</table>

---

"Not having this insight will result in unrealistically large return predictions."
Figure 4.5: Autocorrelations for economy 5; assets in rows, return in column 1, squared returns in column 2, absolute returns in column 3. The dashed lines indicate the 5% critical value for the testing whether or not the autocorrelations are equal to 0.

4.5, and figure 4.6. The fraction of fundamentalists in the economy is shown in figure 4.7.

Figure 4.5 shows little autocorrelation in the returns for this simulation. The autocorrelation is only significant at a 95% confidence level in some of the higher lags. The autocorrelation in squared and absolute returns are significantly positive for a lot of lags at a 95% confidence level. Figure 4.6 shows that the volatility in the returns is much higher in the middle section of the simulation and at the very end. We can see in figure 4.7 that during these periods the number of fundamentalists in the economy is much larger than the number of trend followers. In the periods where volatility is lower, the number of trend followers is larger. Furthermore, we can see in figure 4.7 that the simulation can be divided into four periods in which eventually all agents who have the possibility to switch choose the same strategy.

When we compare economy 5 to economy 4 we see that the mean returns and the variances are a bit lower in economy 5. The skewness is different but still large and negative. The kurtosis is a bit higher for economy 5. But, the largest difference in in the autocorrelations of the absolute and squared returns.

When we compare economy 5 to economy 0, we see lower returns, variances
Figure 4.6: Prices and returns (in %) of economy 5; assets in rows, prices in column 1, returns in column 2.

Figure 4.7: Fraction of trend followers in economy 5 from time period 4000 until time period 5000; assets in rows.

and covariances for economy 5. We also see a much lower skewness and a higher kurtosis. The autocorrelation patterns in figures 3.6 and 4.5 show very different autocorrelations for both economies. In economy 0 the autocorrelation of returns is highly significant in the lower lags. In economy 5 this autocorrelation is much lower and not significant in the lower lags. For the squared and absolute returns of asset 1 and 2 the opposite is true. This autocorrelation is significantly positive in economy 5, whereas it is much lower and insignificant for almost all lags in economy 0.
4.7 Economy 6: Integrating Trend Followers in the Benchmark Model

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean (%)</th>
<th>Covariance matrix (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.018</td>
<td>0.875 0.004 0.032</td>
</tr>
<tr>
<td>2</td>
<td>0.096</td>
<td>0.004 0.926 0.020</td>
</tr>
<tr>
<td>3</td>
<td>0.110</td>
<td>0.032 0.020 0.893</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Asset</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Jarque-Bera</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.136</td>
<td>3.165</td>
<td>4.202</td>
<td>(\rho_{12} = 0.004)</td>
</tr>
<tr>
<td>2</td>
<td>-0.061</td>
<td>2.811</td>
<td>2.123</td>
<td>(\rho_{13} = 0.036)</td>
</tr>
<tr>
<td>3</td>
<td>-0.036</td>
<td>2.943</td>
<td>0.347</td>
<td>(\rho_{23} = 0.022)</td>
</tr>
</tbody>
</table>

Table 4.6: Summary Statistics for economy 6; 100 fundamentalists and 100 trend followers using an exponential moving average with heterogeneous weighting factors to examine the presence of a trend and parameters \(\alpha_i = 4, \theta_i = 1.7, \beta_i = 2\) and \(\gamma_i \sim U(0.7; 0.9)\)

![Autocorrelations](image)

Figure 4.8: Autocorrelations for economy 6; assets in rows, return in column 1, squared returns in column 2, absolute returns in column 3.

The final economy that we will investigate is an economy where trend followers with heterogeneous beliefs using an exponential moving average are added to the benchmark economy described in section 3.3. In this economy there are three types of agents. There are 100 fundamentalists, 50 MV-agents and 50 Trend Followers. The MV-agents use a memory length drawn from a uniform distribution between 600
and 800 periods. The Fundamentalists use the parameter values \( \alpha_i = 4 \) and \( \theta_i = 1.7 \). The trend followers use \( \beta_i = 2 \) and \( \gamma_i \) drawn form a uniform distribution between 0.7 and 0.9. The values of the parameters for fundamentalists and trend followers are chosen in such a way that autocorrelation in returns is minimized. The summary statistics for simulations with different parameter settings are reported in table 4.10 in the appendix to this chapter.

The results of a the last 1000 of 5000 periods of a simulation with the same realization of the dividend process as in the previous simulations are provided in table 4.6 and figure 4.8. In this table we see that the kurtosis for asset 2 and 3 is smaller than 3 indicating no heavy tales present in the return distribution. Furthermore, the Jarque-Bera test fails to reject normality in the returns of asset 2 and 3. Figure 4.8 shows a very highly significant negative autocorrelation for asset 2 and 3 as well as very high significant autocorrelation in the lower lags of the squared and absolute returns of asset 3.

When comparing economy 6 to economy 4 (without MV-agents), we see that the mean returns are slightly lower for economy 6. The variances are all much higher. The skewness is much less negative and the kurtosis is much lower.

When comparing economy 6 to economy 0 (without trend followers), we also see lower mean returns. However the variances are approximately twice as high for economy 0 as for economy 6. Economy 6 is negatively skewed and economy 0 is positively skewed. The kurtosis for economy 6 is much lower, as well as the results from the Jarque-Bera test. The autocorrelations in figure 4.8 are less in magnitude than those in figure 3.6, but the largest autocorrelations are approximately in the lower lags.

Comparing the simulations of the tree economies mentioned above altogether, we see that the existence of trend followers in an economy might result in a lower (negative) skewness and might reduce the amount of autocorrelation in the model. It might also reduce the variance. In the next chapter we will use statistical tests to see till what extend the stylized facts of Cont (2001) are present in each of the economies.
4. A Market with Fundamentalists and Trend Followers

4.A Appendix

Equation 4.3:
Suppose \( \gamma_i \in [0, 1] \) is constant over time and \( S_{MA,i,j,0} \) is known (for instance determined by averaging historical data). For some agent \( i \) and asset \( j \), at time \( t = 0 \):

\[
S_{MA,i,j,1} = \gamma_i S_{j,0} + (1 - \gamma_i) S_{MA,i,j,0}
\]

At time \( t = 1 \)

\[
S_{MA,i,j,2} = \gamma_i S_{j,1} + (1 - \gamma_i) S_{MA,i,j,1} = \gamma_i S_{j,1} + (1 - \gamma_i)(\gamma_i S_{j,0} + (1 - \gamma_i) S_{MA,i,j,0}) = \gamma_i S_{j,1} + \gamma_i (1 - \gamma_i) S_{j,0} + (1 - \gamma_i)^2 S_{MA,i,j,0}
\]

At time \( t = 2 \)

\[
S_{MA,i,j,3} = \gamma_i S_{j,2} + (1 - \gamma_i) S_{MA,i,j,0} = \gamma_i S_{j,2} + (1 - \gamma_i)(\gamma_i S_{j,1} + \gamma_i (1 - \gamma_i) S_{j,0} + (1 - \gamma_i)^2 S_{MA,i,j,0}) = \gamma_i S_{j,2} + \gamma_i (1 - \gamma_i) S_{j,1} + \gamma_i (1 - \gamma_i)^2 S_{j,0} + (1 - \gamma_i)^3 S_{MA,i,j,0}
\]

When we continue this iteration we get:

\[
S_{MA,i,j,t} = \sum_{k=1}^{t} \gamma_i (1 - \gamma_i)^{t-k} S_{j,t-k} + (1 - \gamma_i)^t S_{MA,i,j,0}
\]

Because \( \gamma_i \in [0, 1] \) the influence of past prices on \( S_{MA,i,j,t} \) decreases exponentially over time.

Tables:
<table>
<thead>
<tr>
<th>$\alpha_i$</th>
<th>$\theta_i$</th>
<th>$\beta_i$</th>
<th>Mean (%)</th>
<th>Covariance Matrix (%)</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Jarque-Bera</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1.3</td>
<td>0.1</td>
<td>asset 1</td>
<td>0.024</td>
<td>0.682</td>
<td>-0.009</td>
<td>0.034</td>
<td>-0.107</td>
</tr>
<tr>
<td>7</td>
<td>1.3</td>
<td>0.05</td>
<td>asset 1</td>
<td>0.025</td>
<td>0.666</td>
<td>-0.009</td>
<td>0.034</td>
<td>-0.133</td>
</tr>
<tr>
<td>7</td>
<td>1.3</td>
<td>0.2</td>
<td>asset 1</td>
<td>0.025</td>
<td>0.731</td>
<td>-0.006</td>
<td>0.034</td>
<td>-0.048</td>
</tr>
<tr>
<td>7</td>
<td>1.2</td>
<td>0.1</td>
<td>asset 1</td>
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<td>asset 1</td>
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<td>0.025</td>
<td>0.666</td>
<td>-0.009</td>
<td>0.034</td>
<td>-0.133</td>
</tr>
<tr>
<td>7</td>
<td>1.3</td>
<td>0.2</td>
<td>asset 1</td>
<td>0.025</td>
<td>0.731</td>
<td>-0.006</td>
<td>0.034</td>
<td>-0.048</td>
</tr>
</tbody>
</table>

**Table 4.7:** Sensitivity analysis for economy 1: the first part of the table provides the summary statistics of the economy described in table 4.1. The second part of the table describes the summary statistics for a simulation with a higher and lower value for $\alpha_i$, respectively. The third part of the table describes the summary statistics for a simulation with a higher and lower value for $\theta_i$, respectively. The bottom part of the table describes the summary statistics of a simulation with a higher and lower value for $\beta_i$, respectively.
<table>
<thead>
<tr>
<th>$\alpha_i$</th>
<th>$\theta_i$</th>
<th>$\beta_i$</th>
<th>Mean (%)</th>
<th>Covariance Matrix (%)</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Jarque-Bera</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1.3</td>
<td>0.1</td>
<td>asset 1</td>
<td>0.025</td>
<td>0.682</td>
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<tr>
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<td>0.773</td>
<td>0.019</td>
<td>-0.008</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>asset 3</td>
<td>0.116</td>
<td>0.041</td>
<td>0.019</td>
<td>0.676</td>
<td>-0.037</td>
</tr>
<tr>
<td>7</td>
<td>1.2</td>
<td>0.1</td>
<td>asset 1</td>
<td>0.031</td>
<td>2.039</td>
<td>-0.180</td>
<td>0.148</td>
<td>0.104</td>
</tr>
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<td></td>
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</tr>
<tr>
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Table 4.8: Sensitivity analysis for economy 2: the first part of the table provides the summary statistics of the economy described in table 4.2. The second part of the table describes the summary statistics for a simulation with a higher and lower value for $\alpha_i$ respectively. The third part of the table describes the summary statistics for a simulation with a higher and lower value for $\theta_i$ respectively. The bottom part of the table describes the summary statistics of a simulation with a higher and lower value for $\beta_i$ respectively.
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Table 4.9: Sensitivity analysis for economy 3: the first part of the table provides the summary statistics of the economy described in table 4.3. The second part of the table describes the summary statistics for a simulation with a higher and lower value for $\alpha_i$ respectively. The third part of the table describes the summary statistics for a simulation with a higher and lower value for $\theta_i$ respectively. The fourth part of the table describes the summary statistics of a simulation with a higher and lower value for $\beta_i$ respectively. The bottom part of the table describes the summary statistics of a simulation with a higher and lower values for $\gamma_i$. 

4. A Market with Fundamentalists and Trend Followers
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Table 4.10: Sensitivity Analysis for economy 6: the first part of the table provides the summary statistics of the economy described in table 4.6. The second part of the table describes the summary statistics for a simulation with a higher and lower value for $\alpha_i$ respectively. The third part of the table describes the summary statistics for a simulation with a higher and lower value for $\theta_i$ respectively. The third part of the table describes the summary statistics for a simulation with a higher and lower value for $\beta_i$ respectively. The fourth part of the table describes the summary statistics of a simulation with a higher and lower values for $\gamma_i$ respectively. The bottom part of the table describes the summary statistics of a simulation with a higher and lower values for $\gamma_i$ respectively.
5 Comparison of the Simulated Economies

In this chapter the economies that are discussed in chapter 4 are compared to each other and to the benchmark economy. We will use econometric methods to investigate to what extent multiple simulation runs of the different economies exhibit characteristics of the stylized facts (Cont, 2001). In the first part of this chapter the added value of comparing multiple runs compared to a single simulation run will be discussed. After that, the stylized facts and econometric methods used to measure those facts are discussed, and an overview of the performance of the different economies is provided.

5.1 Sampling Inaccuracy and Simulation Inaccuracy

A typical way to investigate the performance of simulation models like agent based financial markets is running a single or few realizations of the simulation model and analyze whether the outcomes of the variables of interest (in our model among others returns) share more or less the same data patterns as actual date (for instance the S&P500). However, realizations of a simulation model are subject to two types of inaccuracy: sampling inaccuracy, and simulation inaccuracy. Sampling inaccuracy is caused by observing a limited sample instead of a whole population; in our model a limited number of time periods. Simulation inaccuracy is caused by observing a limited number of simulations instead of all possible realizations of the model. In the previous chapters always one arbitrary realization of the simulation model was discussed.
A systematic procedure to investigate the difference between two simulation models is proposed by Li et al. (2009). Those authors point out that the distributional characteristics of an simulated economy can be retrieved with an arbitrary level of precision by running many independent simulations of the model. This possibility allows for the quantification of simulation errors in a straightforward way, making comparison between different simulation models possible based on standard econometric techniques. Exploiting the extra dimension available in simulation models, the combined sampling and simulation inaccuracy can be made negligibly small (when compared to actual outcomes), by having the number of simulations sufficiently large.

### 5.1.1 Definitions and Assumptions

Before discussing ways to compare different simulation models, this section will first explain the set up and discuss the assumptions underlying the systematic approach to compare simulation models by Li et al. (2009). All notations and definitions of variables below are taken the same as in their paper.

#### Set up

A simulation model will be denoted by \( m \). This model \( m \) is part of the class of simulation models denoted by \( \mathcal{M} \). The simulation models \( m \) generate an output observed on variables, \( X \). (in our model: stock prices, stock returns, dividends, and holdings of individual agents).

Given a simulation model class \( \mathcal{M} \), we assume a transformation \( d_T \) that assigns to each \( m \in \mathcal{M} \) the corresponding distribution function \( d_T(m) \). We assume \( d_T(m) \in \mathcal{D}_T \), thus, \( d_T : \mathcal{M} \mapsto \mathcal{D}_T \). Given \( m \) we generate \( N \) simulations, which we assume to be independent. Each simulation results in \( T \) observations of the \( k \) variables, one observation per time period. We denote the observations for model \( m \) in simulation run \( j \) for each time period \( t \) to be the vector \( X_{t}^{m, j} \in \mathbb{R}^k \). We now have \( (X_1^{m, j}, \ldots, X_T^{m, j}) \overset{i.i.d.}{\sim} d_T(m), \) for \( j = 1, \ldots, N \).

In order to compare distributions, distribution characteristics are used (for instance, means or higher moments). The set of those distribution characteristics
is denoted by $\mathcal{E}$. To work with those distribution characteristics we assume the presence of a transformation $\psi_T$ that assigns to each distribution function $\delta_T \in \mathcal{D}_T$ the corresponding characteristics $\varphi_T$. So, $\varphi_T = \psi_T(\delta_T) \in \mathcal{E}$. Next we define $\varphi_{m,T} = \psi_T(d_t(m))$ as the distributional characteristics of model $m$.

Using the above formulations we are able to compare $d_T(m)$ for different models $m \in \mathcal{M}$, by using the distribution characteristics $\varphi_{m,T}$ for model $m$. We can estimate $d_T(m)$ by the empirical distribution function based on $N$ i.i.d simulations. We denote this empirical distribution function by $\hat{d}_{m,T,N}$, and use this distribution to estimate the corresponding $\varphi_{m,T}$ by $\hat{\varphi}_{m,T,N} = \psi_T(\hat{d}_{m,T,N})$. Under appropriate regularity conditions, this estimator will be consistent for $N \to \infty$ and we can use this estimator to quantify its estimation and simulation accuracy. Using standard econometric techniques we will be able to compare the distribution functions of two simulation models $d_T(m_1)$ and $d_T(m_2)$ by means of comparing $\varphi_{m_1,T}$ and $\varphi_{m_2,T}$, on the basis of the corresponding estimators $\hat{\varphi}_{m_1,T,N}$ and $\hat{\varphi}_{m_2,T,N}$.

**Stationarity**

The various comparisons between models will be made under the (strict) stationarity assumption. This assumption becomes more or less realistic after appropriate data transformations, like transforming prices in returns.

We first consider stationarity with short range distributional characteristics (like the mean at time $t$). Due to the (strict) stationarity assumption the distribution of an arbitrary number of observations ($k_1$ till $k_l$), defined as $X_{t+k_1}^{m,j}, \ldots, X_{t+k_l}^{m,j}$, does not depend on $t$. So we can write:

$$(X_{t+k_1}^{m,j}, \ldots, X_{t+k_l}^{m,j}) \sim d_{(k_1, \ldots, k_l)}(m)$$

with $d_{(k_1, \ldots, k_l)}(m)$ the distribution function of the observations of a simulation run $j$ using model $m$.

We denote the set of induced $(k \times l)$-dimensional distribution function functions (to which $d_{(k_1, \ldots, k_l)}(m)$ belong) by $\mathcal{D}_{k_1, \ldots, k_l}$ and the subset generated by the model class $\mathcal{M}$ by $d_{(k_1, \ldots, k_l)}(\mathcal{M})$. Short range dependent distribution characteristics are transformations of distribution functions in $\mathcal{D}_{k_1, \ldots, k_l}$, for a sufficiently large but finite $k_l$. Thus, if we let $\psi : \mathcal{D}_{(k_1, \ldots, k_l)} \mapsto \mathcal{E}$ be such a transformation, that assigns
each distribution \( \delta_{(k_1,\ldots,k_l)} \in D_{(k_1,\ldots,k_l)} \) the corresponding distribution characteristics of interest in \( \mathcal{E} \). We than have for the distribution characteristics of the simulation models:

\[
\varphi_{m,(k_1,\ldots,k_l)} = \psi \left( d_{(k_1,\ldots,k_l)}(m) \right) \in \mathcal{E}
\]

### 5.1.2 Estimators for the Distribution Characteristics

Assuming asymptotic independence we can estimate the characteristics of \( \varphi_{m,(k_1,\ldots,k_l)} = \psi \left( d_{(k_1,\ldots,k_l)}(m) \right) \) Due to the two dimensions \( T \) and \( N \), Li et al. (2009) point out that here are five alternatives to estimate \( \varphi_{m,(k_1,\ldots,k_l)} \):

1. Estimate \( \varphi_{m,(k_1,\ldots,k_l)} \) consistently by \( \hat{\varphi}_{m,(k_1,\ldots,k_l),T} = \psi \left( \hat{d}_{m,(k_1,\ldots,k_l),T} \right) \), for \( T \to \infty \), where \( \hat{d}_{m,(k_1,\ldots,k_l),T} \) is the empirical distribution of \( d_{(k_1,\ldots,k_l)}(m) \) using the time series of the \( j \)-th simulation.

2. Estimate \( \varphi_{m,(k_1,\ldots,k_l)} \) consistently by \( \hat{\varphi}_{m,(k_1,\ldots,k_l),N} = \psi \left( \hat{d}_{m,(k_1,\ldots,k_l),N} \right) \), for \( N \to \infty \), where \( \hat{d}_{m,(k_1,\ldots,k_l),N} \) is the empirical distribution of \( d_{(k_1,\ldots,k_l)}(m) \) using the \( N \) simulations for some given \( t \). Note, that this approach does not need the asymptotic independence and stationarity assumptions to yield a consistent estimator for the given \( t \), but only \( N \) i.i.d distributed simulations.

3. Estimate \( \varphi_{m,(k_1,\ldots,k_l)} \) consistently by averaging \( \hat{\varphi}_{m,(k_1,\ldots,k_l),T} \) over the \( N \) simulations: \( \tilde{\varphi}_{m,(k_1,\ldots,k_l),T} = \frac{1}{N} \sum_{j=1}^{N} \hat{\varphi}_{m,(k_1,\ldots,k_l),T} \). Note that this estimator is consistent, at least, if we take the sequential limit \( T \to \infty \) possibly followed by \( N \to \infty \).

4. Estimate \( \varphi_{m,(k_1,\ldots,k_l)} \) consistently by averaging \( \hat{\varphi}_{m,(k_1,\ldots,k_l),N} \) over the \( T \) periods: \( \tilde{\varphi}_{m,(k_1,\ldots,k_l),N} = \frac{1}{T} \sum_{t=1}^{T} \hat{\varphi}_{m,(k_1,\ldots,k_l),N} \). Note that this estimator is consistent, at least, if we take the sequential limit \( N \to \infty \) possibly followed by \( T \to \infty \).

5. Estimate \( \varphi_{m,(k_1,\ldots,k_l)} \) consistently by \( \hat{\varphi}_{m,(k_1,\ldots,k_l),N,T} = \psi \left( \hat{d}_{m,(k_1,\ldots,k_l),N,T} \right) \), for \( N, T \to \infty \), where \( \hat{d}_{m,(k_1,\ldots,k_l),N,T} \) is the empirical distribution of \( d_{(k_1,\ldots,k_l)}(m) \) using all observations and taking the limits \( N, T \to \infty \).

Under appropriate additional regularity conditions we are able to quantify the (asymptotic) estimation/simulation inaccuracy and characterize the asymptotic
distribution of the estimators. In particular estimation alternatives 3. - 5. (where two limits are taken) satisfy:

\[ \sqrt{NT} \left( \hat{\varphi}_{m,(k_1, \ldots, k_l)} - \varphi_{m,(k_1, \ldots, k_l)} \right) \overset{d}{\to} N \left( 0, \Sigma_{\varphi_{m,(k_1, \ldots, k_l)}} \right) \]

, with \( \Sigma_{\varphi_{m,(k_1, \ldots, k_l)}} \) is the same asymptotic covariance matrix for all three alternatives.

### 5.1.3 Comparing Multiple Simulation Models

A comparison between two models \( m_1 \) and \( m_2 \) in \( M \) corresponds to the null hypothesis:

\[ H_0 : \varphi_{m_1,(k_1, \ldots, k_l)} = \varphi_{m_2,(k_1, \ldots, k_l)} \]

versus the alternative hypothesis:

\[ H_0 : \varphi_{m_1,(k_1, \ldots, k_l)} \neq \varphi_{m_2,(k_1, \ldots, k_l)} \]

To test this hypothesis we can use any of the five alternatives described above for estimating \( \varphi_{m_1,(k_1, \ldots, k_l)} \) and \( \varphi_{m_2,(k_1, \ldots, k_l)} \). The standard Wald-test can be used as test procedure. In case we take \( N, T \to \infty \) the test statistic is:

\[ NT \left( \hat{\varphi}_{m_1,(k_1, \ldots, k_l)} - \hat{\varphi}_{m_2,(k_1, \ldots, k_l)} \right)' \left( \hat{\Sigma}_{\varphi_{m_1,(k_1, \ldots, k_l)}} + \hat{\Sigma}_{\varphi_{m_2,(k_1, \ldots, k_l)}} \right)^{-1} \left( \hat{\varphi}_{m_1,(k_1, \ldots, k_l)} - \hat{\varphi}_{m_2,(k_1, \ldots, k_l)} \right) \overset{d}{\to} \chi_\nu^2 \]

where, for \( a \in 1, 2 \), \( \hat{\varphi}_{ma,(k_1, \ldots, k_l)} \) is any of the three alternative estimators 3. -5. described above, and \( \hat{\Sigma}_{\varphi_{ma,(k_1, \ldots, k_l)}} \) is a consistent estimator for \( \Sigma_{\varphi_{ma,(k_1, \ldots, k_l)}} \), and the degrees of freedom \( \nu \) are typically equal to the number of elements in \( \hat{\varphi}_{ma,(k_1, \ldots, k_l)} \).

### 5.1.4 Implications

In the remainder of this chapter we will compare simulations of the six economies described in the previous chapter and the benchmark economy from 3.3. To reduce the simulation inaccuracy multiple simulation runs will be used in this comparison. A drawback of the use of more simulations is that additional computation time is required. As a trade off between simulation accuracy and simulation time 25 simulation runs of each economy are compared. As an estimator of \( \varphi_{m,(k_1, \ldots, k_l)} \) we will report the average of \( \varphi_{m,(k_1, \ldots, k_l)} \) over the \( N \) simulations:

\[ \bar{\varphi}_{m,(k_1, \ldots, k_l),N} = \frac{1}{N} \sum_{t=1}^{N} \varphi_{m,(k_1, \ldots, k_l),t} \]
Comparison of the Simulated Economies (alternative 3. of Li et al. (2009)). The length of each simulation will be 5000 periods. We will each time use the last 1000 of those 5000 periods for our analysis. The other parameters are the same as described in the previous two chapters.

5.2 Stylized Facts

This section describes an overview of the most important stylized facts that are apparent in the time series date of asset returns and provides a comparison of the economies investigated in the previous two chapters based on those stylized facts. The stylized facts (Cont, 2001) are used in the literature, for instance also by Adriaens (2008). We will investigate the presence of the same stylized facts as investigated by Adriaens, because in this way this research project is comparable to his research project. For this reason we will use the same econometric methods to test these stylized facts as were used by Adriaens. A summary of the result is reported in tables 5.2 and 5.3 in the appendix to this chapter. We will investigate the following stylized facts:

Absence of autocorrelation — (Linear) autocorrelation of asset returns are often insignificant. To test the absence of autocorrelation in the simulated asset returns we report the maximum absolute autocorrelation of the asset returns of the first 30 lags. We report the average maximum absolute autocorrelation for the 25 simulation runs in table 5.3. In determining the critical value for these tests we take into account the Bonferroni correction to take multiple tests into account. Bonferroni (1936) states that testing \( n \) dependent or independent hypotheses on a set of data, one way of maintaining the family-wise error rate is to test each individual hypothesis at a statistical significance level of \( 1/n \) times what it would be if only one hypothesis were tested. For our tests we use a sample size of 1000 periods, so the 5\% critical values become 0.0994. In table 5.2 we report the fraction of replications of the simulation for which we fail to reject the null hypothesis of no autocorrelation based on this critical value.

Heavy tails — The (unconditional) distribution of returns seems to display a power-law or Pareto like tail. To test the presence of heavy tails in comparison to a normal distribution, the presence of excess kurtosis in the asset return series are
tested. We report the average kurtosis over the 25 simulation runs in table 5.3. The kurtosis should be significantly larger than 3 for heavy tails. In table 5.2 we report the fraction of replications of the simulation for which the kurtosis of the empirical distribution of asset returns in significantly larger than 3 at a 95% confidence level.

Gain/Loss asymmetry — One observes larger downwards movements in stock prices and stock index values than upwards movements. To test for the presence of gain/loss asymmetry, we compare the average of positive returns with the average of negative returns. Again we report the average values over 25 simulations in table 5.3. The fraction of replications of the simulation with significant higher returns for losses than for gains at a 95% confidence level are reported in table 5.2.

Aggregate Gaussianity — As one increases the time scale over which returns are evaluated, their distribution looks more and more like a normal distribution. In particular the shape of the distribution is not the same at different time scales. We test the presence of aggregational Gaussianity by comparing the Jarque-Bera statistics for returns computed over a time period of 1 week, 4 weeks and 12 weeks. The 5% critical value for rejecting the null hypothesis of a normal distribution is 5.99. We report the average Jarque-Bera over the 25 simulation runs in table 5.3. the fraction of rejections of the Jraque-Bera test for teh 25 replications is reported in table 5.2.

Volatility clustering — Different measures of volatility display a positive autocorrelation over several days, which quantifies the fact that high volatility events tend to cluster over time. To test the presence of volatility clustering we use an autoregressive conditional heteroskedasticity (ARCH) model. In this model the variance of an error term in an autoregressive model is considered to be a function of the variances of the previous time periods’ error terms. We use 1 lag in the ARCH equation. The test statistic has a $\chi^2$ - distribution with 1 degree of freedom. So the 5% critical value is 3.84. We report the the average ARCH effect over the 25 simulations in table 5.3. the fraction of rejections of null hypothesis of no ARCH effect in the 25 replications of the simulation are reported in table 5.2.
Slow decay of autocorrelation in absolute returns (long memory) — The autocorrelation function of absolute returns decays slowly as a function of the time lag. This is sometimes interpreted as a sign of long range dependence. To test the decay of autocorrelation in absolute returns the FIGARCH(1, d, 1) model is used¹ and we test whether this parameter is significantly different from 0 and 1. We report the average long range parameter \( d \) over the 25 simulation replications in table 5.3 and report the fraction of replications for which the parameter \( d \) is significantly different from 0 or 1 at a 95% confidence level in table 5.2.

Leverage effect — Most measures of volatility of an asset are negatively correlated with the returns of that asset. We test this stylized fact by computing the volatility using a rolling window of 100 periods. We report the correlation with returns. The 5% critical values for testing whether this relation is significantly positive or negative are 0.055. The fraction of the 25 replications with a significant negative correlation is reported in table 5.2

Volume/volatility correlation — Trading volume is correlated with all measures of volatility. We test the presence of this stylized fact by computing the volatility using a rolling window of 100 periods, and the trading volume from the consecutive portfolios of the agents. The critical values for testing whether or not there is significant volume/volatility correlation is again 0.055 (5%). The fraction of replications with a significant positive correlations is reported in table 5.2

Table 5.1 gives an overview of the reproduction of the stylized facts by the different economies. A ‘+’ in this table denotes that the econometric test described above shows that the stylized fact was present in all or almost all of the replications of the simulation and for all or almost all of the assets. A ‘0’ denotes that the test shows that the stylized fact was significantly apparent in more than half of number of replications of the simulation in average over the three assets. A ‘−’ sign denotes that the stylized fact was not significant in more than half of the replications.

¹We use the GarchKit 1.2 MatLab toolbox, developed at Ohio State University to estimate the parameter \( d \).
Comparison of the Simulated Economies

<table>
<thead>
<tr>
<th>Economies</th>
<th>0</th>
<th>1</th>
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<th>3</th>
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<td>+</td>
<td>+</td>
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<td>–</td>
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<tr>
<td>Heavy Tails</td>
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<td>+</td>
<td>+</td>
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<td>+</td>
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<td>–</td>
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<td>+</td>
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<td>–</td>
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<td>0</td>
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</tbody>
</table>

Table 5.1: Reproduction of stylized facts. ‘+’ Indicates that the stylized fact is significant in all or almost all replications of the simulation. ‘0’ indicates that the stylized fact is significant in the majority of the replications, but that there are still replications for which the pretense of the stylized fact is insignificant. A ‘−’ sign denotes that the stylized fact is insignificant in the majority of the replications of the simulation. Economies: 0 - benchmark (3.3). 1 - trend followers with simple moving average (4.2). 2 - Trend followers with weighted moving average (4.3). 3 - trend followers with exponential moving average (4.5). 4 - trend followers with exponential moving average and more heterogeneity (4.5). 5 - type switching (4.6). 6 - benchmark model plus trend followers (4.7)

First of all, if we compare the results of the benchmark, economy 0, to the results found by Adriaens for a single replication of this economy\(^2\), we see that where slow decay of autocorrelation in absolute returns, and volume/volatility correlation are significantly present in the single replication that Adriaens tests, those stylized facts are insignificant in some replications tested for this research. On the other hand, aggregational Gaussianity that is insignificant in the test of Adriaens for his simulation, this stylized fact is present in some of the replications of our simulation. As is shown in table 5.2

If we look at the stylized facts one at a time, we see that absence of autocorrelation, which was not explained by economy 0 is present in some of the replications

\(^2\)see Adriaens (2008)
of economies 1, 2 and 5 and in almost all replications of economies 3 and 4. In economies 3 and 4 there is a constant number of trend followers using an exponential moving average process to estimate a trend in asset prices. From all the economies we tested this specification looks like the most promising to minimize autocorrelations in the returns. In economy 6 where those type of trend followers are added to Adriaens benchmark model, table 5.3 shows that the average maximum absolute autocorrelation is approximately half of that of economy 0, but still highly significant in almost all replications.

The second stylized fact, heavy tails seems to be present in almost all of the replications of economies 0, 1, 2, 3 and 5 but is insignificantly in some of the replications for economy 4 (see table 5.2) and for many of the replications of economy 6 (also see table 5.2). In economy 4 more heterogeneity is added by varying parameters for individual agents. It seems that this has in influence on the tails of the empirical distribution of asset returns.

The next stylized Fact, gain/loss asymmetry was not reproduced by the economy 0. Table 5.3 shows that this relation is on average in the opposite direction. The same holds for economies 2 and 3 with trend followers that use respectively a simple and a weighted moving average to predict a trend. The economies with trend followers that use a exponential moving average (economy 3, 4 and 5) perform much better in reproducing this stylized fact. In economy 3 gain/loss asymmetry is significant for almost all replications. Table 5.2 shows that in average over the assets 88% of the replications of economy 4 show significant higher losses than gains. Table 5.2 shows only 41% of the replications of economy 5 show significant more gains than losses.

Aggregational Gaussianity seems to be present in part of the replications for all economies. Table 5.3 shows the Jarque-Bera test statistic for a 1, 4, and 12 week time interval. The results of the 4 and 12 week intervals show that the average returns are less than the critical value of 5.99 for all assets in all economies, implying that in both situations the return distributions behaves like a normal distribution. The difference between the Jarque-Bera statistics for 1 week and 4 weeks is much larger in magnitude than the difference between 4 and 12 weeks.
Volatility clustering is significant in all replications of the simulations of all economies. This is in contrast to the stylized fact “slow decay of autocorrelation in absolute returns”. The value for $d$ in the FIGARCH($1, d, 1$) model\(^3\) for which the average over all replications is reported in table 5.3 is 0 for a number of simulation runs. None of the economies is capable of replicating slow decay of autocorrelation in average returns in all or almost all of the replications, but economy 5 gives much better results than the other economies. In this economy on average over the assets 80% of the replications show a $d$ significantly different from 0 or 1.

Looking at the last two stylized facts, we see that the leverage effect is not significant in almost all replications of the simulations for all economies. Volume volatility correlation is also not significant in most of the replications. Economy 0 and economy 5 have slightly more replications showing a significant positive correlation between trading volume and volatility.

\[\text{Figure 5.1: Fraction of replications where the stylized fact is significantly present. 0 - benchmark (3.3). 1 - trend followers with simple moving average (4.2). 2 - Trend followers with weighted moving average (4.3). 3 - trend followers with exponential moving average (4.5). 4 - trend followers with exponential moving average and more heterogeneity (4.5). 5 - type switching (4.6). 6 - benchmark model plus trend followers (4.7)}\]

Figure 5.1 shows the a plot of the results in table 5.2 in the appendix. From this figure we see that economy 4 captures most of the stylized facts. However, slow decay of autocorrelation in absolute returns and volume/ volatility clustering

\(^3\)see (Li et al., 2009) for a description of the FIGARCH process
are captured in more replications of economy 5. Unfortunately, this comes at a cost. Economy 5 is less good at reproducing absence of autocorrelation, gain/loss asymmetry and aggregational Gaussianity. Furthermore, economy 6 does not perform well in replicating stylized facts.

We can conclude that the model with fundamentalists and trend followers perform quite well in replicating stylized facts. With this model we are able to capture some stylized facts that we were unable to capture with the benchmark model consisting of fundamentalists and MV-agents. Most notably are absence of autocorrelation and gain/loss asymmetry. Those two stylized facts are not reproduced by the benchmark model, and are reproduced in almost all replications of economies 3 and 4. Unfortunately, when we introduce trend followers to the benchmark model in economy 6, we are not able to reproduce many of the stylized facts in that model. So, we are not able to maintain stylized facts that are reproduced in the model with trend followers by introducing trend followers to the model of Adriaens (2008). The model with fundamentalists and trend followers can be seen as a variation on the artificial financial market model of Adriaens that is able to capture some different stylized facts. In the next chapter we will conclude and give recommendations for further research.
## 5.A Appendix

<table>
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<td>0.16</td>
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<td>0.36</td>
<td>0.12</td>
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**Table 5.2:** Fraction of the 25 replications of the simulation models for which the econometric test suggest a significant presence of the stylized fact. Economies: 0 - benchmark (3.3). 1 - trend followers with simple moving average (4.2). 2 - trend followers with weighted moving average (4.3). 3 - trend followers with exponential moving average (4.5). 4 - trend followers with exponential moving average and more heterogeneity (4.5). 5 - type switching (4.6). 6 - benchmark model plus trend followers (4.7)
Table 5.3: Average of test statistics for the stylized facts over 25 simulations. Standard deviation are between brackets.
6 Conclusions, Discussion and Recommendations

In the previous chapter we have seen that the artificial financial market designed by Adriaens (2008) performs quite well in reproducing stylized facts when agents participating in this market are assumed to be either fundamentalist or trend follower. We are able to consistently reproduce the stylized facts “no autocorrelation in returns” and “gain/loss asymmetry” that are not reproduced if we assume the agents in the artificial financial market to be either fundamentalists or MV-agents. However, we have seen in chapter 4 that we are limited in the choices of the values of the speed adjustment parameter and responsiveness to change parameter for fundamentalists, and the adjustment parameter and exponential weighting factor for trend followers. We chose these parameters to be in a range to minimize autocorrelation and did not try to base these values on trading strategies of agents in real financial markets. This choice was made because the goal of this research project was to mimic the stylized facts. Additional research would be required to see how this model performs when compared to actual data on financial markets. This can be done in a way described by, for instance Li et al. (2009).

If we look at the literature on artificial financial markets with fundamentalists and trend followers we see that some of those models perform quite well, for instance, the models of Hi and Li (2007), and of Gaunersdorfer and Hommes (2007). Both contributions use a model with agents that behave as fundamentalist or trend follower and are able to switch between the two strategies if they believe changing will result in higher returns. This type switching allows both models to perform quite well in replicating the slow decay of autocorrelations in absolute returns. However, the formulation of the price functions used in both contributions is recently heavily criticized by Franke (2008). Franke provides evidence that the fractions of fundamentalists and
trend followers at each point in time in both models are ascribed to the specification of a stochastic term in the shock price equation. He argues that the slow decay of autocorrelations in absolute returns is only caused by this stochastic term. When he respecifies the models to normalize the price shocks he finds that the slow decay of autocorrelations in absolute returns becomes insignificant.

In economy 5 we try to introduce type switching in our artificial financial market based on an adaptive believe system than does not take into account a stochastic noise term. In figure 5.1 and table 5.2 we see that the ability to reproduce the stylized fact “slow decay of autocorrelations in absolute returns” increases substantially. Although at a cost of the replication of other stylized facts. But the results suggest that it can be possible to capture “slow decay of autocorrelations in absolute returns” in an artificial financial market without using a stochastic noise term. Additional research into the formulation of an adaptive believe system, without noise term, that is able to replicate slow decay of autocorrelations in absolute returns is however necessary.

Comparing the performance of different specifications of the model based on multiple replications of the simulation has provided additional insights to the earlier research of Adriaens (2008). We have seen that some of the stylized facts are significant or insignificant in almost all simulations, but that others are significant in only part of the simulations.

Regarding Agent-based artificial financial markets in general we have seen that the research field has made rapid developments over the past years to become an interesting alternative to traditional analytical financial models in order to cope with investment decisions of individual economic agents that do not all behave in the same rational way. The flexibility of agent-based artificial financial markets allows for the modeling of many of the characteristics of actual financial markets. However, the flexibility of Agent-based artificial financial markets comes at a cost. LeBaron (2007) summarizes the four most common points of criticism: the many parameters in the models; the stability of the model when adding new trading strategies; the small number of assets and agents; and the timing of decisions and trades.

There are numerous interesting ways for further research of extending the artificial financial market with trend followers and fundamentalists. We already mentioned the gap between actual data on financial markets and the artificial financial
market, as well as the issue of parameter estimation. When regarding the stylized facts (Cont, 2001) we have seen that economy 4 is capable of reproducing many of the stylized facts, and that type switching increases slow decay in autocorrelation in absolute returns. A model of type switching that is capable of also replicating the stylized facts that are replicated by economy 4 consistently, can become very powerful in reproducing the stylized facts. Additionally, one could look into ways to model more correlation between volatility and trading volume and negative correlation between returns and volatility (leverage effect).

With increasing computer power it becomes possible to specify more complex artificial financial markets. An interesting extension to the artificial financial market of Adriaens would be to increase the number of assets and investigate into ways to loose the assumption that agents have a position in all assets. It is also interesting to model asymmetry in wealth levels of economic agents. In real financial markets large institutional investors like pension funds and insurance companies account for a large fraction of wealth in the market. These investors unusually have very defensive trading strategies with a long investment horizon, often based on exogenous variables like the among others the fundamental value. Next to those investors, there is a large group of agents with much smaller individual wealth aiming to make a profit mostly on a shorter investment horizon, sometimes using a momentum or technical trading strategy like trend following.
Bibliography


