Robust Hedging of Terminal Wealth under Interest Rate Risk and Inflation Risk

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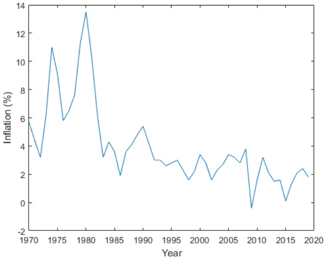
Hedging long-term investments

- Inflation risk is especially relevant on the long-term

Figure. Yearly inflation (%) in the US



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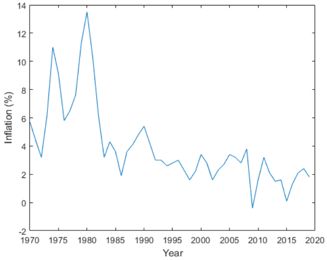

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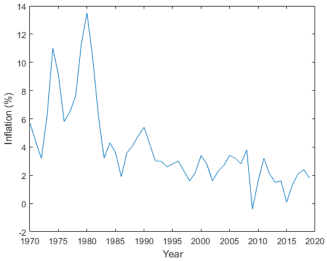

- Theoretically, both the interest rate and inflation risk can be hedged by buying Index Linked Bonds (ILBs)
- However, in practice ILBs are illiquid (Ciocyte and Westerhout (2017), Beetsma et al. (2020))
  $\rightarrow$ **How to (robustly) hedge inflation risk without ILBs?**

Literature - Sangvinatsos and Wachter strategy

- Influential study of Sangvinatsos and Wachter (2005): the **SW-strategy** hedges the nominal interest rate and inflation without ILBs
- The hedge demand turns out to be a long-short position in nominal bonds, with typically extreme weights
- As a consequence, the strategy is very sensitive to bond maturities and estimation errors in parameters

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- The hedge demand turns out to be a long-short position in nominal bonds, with typically extreme weights
- As a consequence, the strategy is very sensitive to bond maturities and estimation errors in parameters
  - E.g. uncertainty about the market price of risk (Flor and Larsen (2011), Garlappi and Uppal (2007), Martellini et al. (2015), and Feldhütter et al. (2012))

Literature - Alternative strategies

- Related strategy of Brennan and Xia (2002) (**BX-strategy**) results in similar issues
  - See e.g. Martellini et al. (2015). The proposed solution is implementing constraints on the portfolio weights, but in that case no analytical solution

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- Related strategy of Brennan and Xia (2002) (**BX-strategy**) results in similar issues
    - See e.g. Martellini et al. (2015). The proposed solution is implementing constraints on the portfolio weights, but in that case no analytical solution
- We find that one bond strategies lead to less extreme and less sensitive demands
    - Drawback: inflation risk cannot be hedged fully, e.g. (Mkaouar et al., 2017)
    - Van Bilsen, Boelaars and Bovenberg (2020) derive the optimal strategy in the BX-model (**BBB-strategy**)
    - Munk, Sørensen and Vinther (2004) (**MSV-strategy**) derive the optimal strategy in another restricted SW-model

Our contribution

- Focus on a robust real wealth hedge and the corresponding hedge demand for nominal bonds

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- Focus on a robust real wealth hedge and the corresponding hedge demand for nominal bonds
- Robust choice of bond maturities
- Sensitivity to changes in parameters
    - Unlike existing literature, we do not focus on market prices of risk
    - Relevant parameters are the mean-reversion parameters and the feedback parameter, the parameter that takes into account the impact of the inflation rate level on the nominal interest rate drift

Preview of results

- The choice of bond maturities is relevant. The ‘robust’ choice contains a mid-term bond (10Y) and a very long-term bond (50Y). This differs from commonly used bond maturities

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- Parameter uncertainty is relevant
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  - Similarly, we observe a loss of 15-95% if the investor believes that the **feedback parameter** doubles

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  - Similarly, we observe a loss of 15-95% if the investor believes that the **feedback parameter** doubles
- One bond strategies are often more robust than two bond strategies, but these fail to hedge inflation risk well

Objective

- Maximize expected utility from terminal real wealth:

$$\max_{W_T} \mathbb{E}_0 \left[u\left(\frac{W_T}{\Pi_T} \right) \right] \text{ s.t. } \mathbb{E}_0[\zeta_T W_T] = W_0$$

  - Nominal wealth W_t, price level Π_t
  - (Large) investment horizon T
  - CRRA-utility function $u(x) = \begin{cases} \frac{x^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(x) & \text{if } \gamma = 1 \end{cases}$
    with coefficient of relative risk aversion γ
  - Nominal pricing kernel ζ_t, expressed in factor loadings $\tilde{\lambda}$:

$$\frac{d\zeta_t}{\zeta_t} = -R_t^f\, dt - \tilde{\lambda}_R dB_t^R - \tilde{\lambda}_\pi dB_t^\pi$$

Risks

- Investor aims to hedge Π_t and nominal interest rate R_t
- Assume price level process $\frac{d\Pi_t}{\Pi_t} = \pi_t \, dt$
  - π_t expected inflation
- Both risk factors follow an Ornstein-Uhlenbeck process:

$$d \begin{bmatrix} R_t \\ \pi_t \end{bmatrix} = - \begin{bmatrix} \kappa_R & \kappa_{R\pi} \\ 0 & \kappa_\pi \end{bmatrix} \begin{bmatrix} R_t - \bar{R} \\ \pi_t - \bar{\pi} \end{bmatrix} dt + \begin{bmatrix} \sigma_R & 0 \\ 0 & \sigma_\pi \end{bmatrix} \begin{bmatrix} dB_t^R \\ dB_t^\pi \end{bmatrix}$$

  with mean-reversion parameters κ_R and κ_π;
  and feedback parameter $\kappa_{R\pi}$
- Assume B_t^R and B_t^π are correlated by correlation matrix ρ

Available assets

- Available assets: risk-free rate R_t, bonds i with time to maturity τ_i and price $P_t(\tau_i)$:

$$P_t(\tau_i) = e^{-a(\tau_i) - b_R(\tau_i)R_t - b_{R,\pi}(\tau_i)\pi_t}$$

  - $a(\tau_i),\ b_R(\tau_i),\ b_{R,\pi}(\tau_i)$ represent sensitivities of $P_t(\tau_i)$
    - $b_R(\tau_i)$ depends on κ_R
    - $a(\tau_i)$ and $b_{R,\pi}(\tau_i)$ depend on κ_R, κ_π, and $\kappa_{R\pi}$
  - Consequence: fully hedge π_t by short-long composition

Remaining definitions

- Let σ_X correspond to the factor volatilities and σ to the bond exposures:

$$\sigma_X = \begin{bmatrix} \sigma_R & 0 \\ 0 & \sigma_\pi \end{bmatrix} \; ; \; \sigma = \begin{bmatrix} -\sigma_R b_R(\tau_1) & -\sigma_\pi b_{R\pi}(\tau_1) \\ -\sigma_R b_R(\tau_2) & -\sigma_\pi b_{R\pi}(\tau_2) \end{bmatrix}$$

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Variable	**Definition**
x_t^*	Optimal nominal bond allocations
λ	Market prices of risk
$\Omega = \sigma\rho\sigma'$	Variance-covariance matrix
$\Lambda = \sigma\lambda$	Risk premiums
$\tilde{\lambda} = \rho^{-1}\lambda$	Factor loadings of nominal pricing kernel

Solution Sangvinatsos and Wachter (2005)

- The optimal strategy contains a speculative and a hedging demand:

$$x_t^* = \frac{1}{\gamma}(\sigma')^{-1}\rho^{-1}\lambda + \left(1 - \frac{1}{\gamma}\right)\Omega^{-1}\sigma'\rho\sigma_X B(\tau)'$$

where $B(\tau)$ contains the exposures to shocks in R_t and π_t and is dependent on κ_R, κ_π, and $\kappa_{R\pi}$

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  where $B(\tau)$ contains the exposures to shocks in R_t and π_t and is dependent on κ_R, κ_π, and $\kappa_{R\pi}$
- Two bonds: hedge demand is **not** impacted by ρ or σ_X
- Extreme values of x_t^* if $\tau_1 \approx \tau_2, \kappa_{R\pi} \approx 0$, or $\kappa_R \approx \kappa_\pi$

Conditions extreme positions

Example extreme positions

Choice bond maturities

- Approach "robust" choice $(\tau_1^{min}, \tau_2^{min})$ by minimising the sum of absolute optimal bond allocations at $t = 0$:

$$(\tau_1^{min}, \tau_2^{min}) = \underset{\tau_1, \tau_2}{\arg\min} \left(|x_{0,1}^*| + |x_{0,2}^*| \right)$$

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- Result robust choice $\forall T \in \{1, 5, 10, 20, 30\}$; $\forall \gamma \in \{1, 2, 25\}$; $\tau_1, \tau_2 \in \{1, 2, ..., 50\} \to (11,50)$

Robust to include long-term bond $\geq 30Y$

Example relevance impact bond maturities

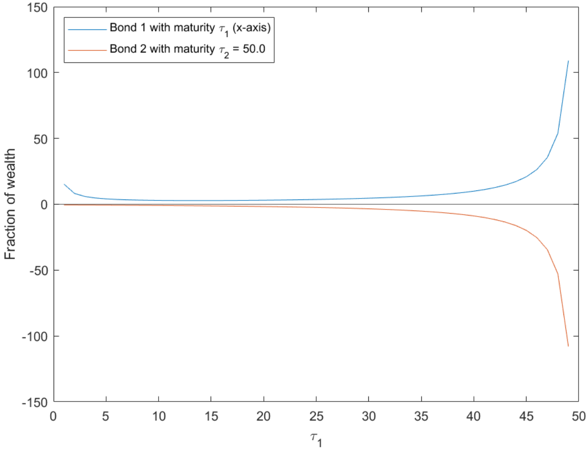

Figure. Optimal bond allocations at $t = 0$ (in fraction of wealth) for $\gamma = 25$, $\kappa_{R\pi} = -0.078$, $T = 30$, $\tau_2 = 50$, and a variable τ_1 given on the x-axis.

Initial parameters

- Parameters from Brennan and Xia (2002)

Set Θ	
$\kappa_{R\pi}$	-0.078
κ_R	0.105
σ_R	0.019
λ_R	-0.219
κ_π	0.027
σ_π	0.014
λ_π	-0.105
$\rho_{R\pi}$	0.733

Impact parameters on wealth

- Let the Data Generating Process (DGP) be based on initial parameter set Θ
- Let the perceived optimal strategy $\hat{x}_t$ be based on investor's beliefs $\hat{\Theta}$

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- Given $\hat{x}_t$, simulate W_t by Euler approximation with monthly time intervals and monthly portfolio rebalancing
- Results for $\gamma \in \{1, 2, 25\}$
- Set $T = 30$; $(\tau_1, \tau_2) = (10, 50)$; and $W_0 = 1{,}000$

Impact feedback parameter $\kappa_{R\pi}$

- Incorrect beliefs about $\kappa_{R\pi}$ are relevant for the two bond strategy

Table. Certainty Equivalent (CE) where DGP is based on $\kappa_{R\pi}$, but the investor believes it is based on $\hat{\kappa}_{R\pi}$ Support values $\kappa_{R\pi}$

	Two bond SW-strategy				*One bond SW-strategy*			
γ	*2*	*25*	*2*	*25*	*2*	*25*	*2*	*25*
$\kappa_{R\pi}$ / $\hat{\kappa}_{R\pi}$	-0.078		0		-0.078		0	
-0.117	2,694	153	1,371	0				
-0.078	**3,041**	**2,517**	1,371	0				
-0.039	2,430	109	1,371	0				
-0.001	1,398	0	1,371	0				
0								

Impact feedback parameter $\kappa_{R\pi}$

- Incorrect beliefs about $\kappa_{R\pi}$ are relevant for the two bond strategy
- For the one bond strategy less relevant, but a low CE

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$\kappa_{R\pi}$ \ $\hat{\kappa}_{R\pi}$	-0.078		0		-0.078		0	
-0.117	2,694	153	1,371	0	2,319	203	1,885	2
-0.078	**3,041**	**2,517**	1,371	0	**2,324**	**251**	1,904	4
-0.039	2,430	109	1,371	0	2,316	200	1,920	6
-0.001	1,398	0	1,371	0	2,285	105	1,927	7
0					2,283	103	**1,927**	**7**

Impact remaining parameters - SW-strategy

- Incorrect beliefs about κ_R and κ_π cause a large loss in utility from terminal real wealth x_t^* for sets 0,4,and 5 $\hat{\Theta}^k$
- For $\gamma = 25$, the CE can even decrease to 0

	CE			Loss w.r.t. set 0 (%)		
γ	1	2	25	1	2	25
Parameter set k						
0 (Initial)	**3,725**	**3,041**	**2,517**			
1 ($\kappa_R, \sigma_R \uparrow$)	1,657	571	0	-56%	-81%	-100%
2 ($\kappa_\pi, \sigma_\pi \uparrow$)	3,436	2,712	937	-8%	-11%	-63%
3 ($\kappa_R, \sigma_R \downarrow$)	2,171	1,501	4	-42%	-51%	-100%
4 ($\kappa_\pi, \sigma_\pi \downarrow$)	3,715	3,013	2,061	0%	-1%	-18%
5 ($\kappa_R, \sigma_R, \kappa_\pi, \sigma_\pi \uparrow$)	313	24	0	-92%	-99%	-100%
6 ($\kappa_R, \sigma_R, \kappa_\pi, \sigma_\pi \downarrow$)	2,126	1,445	3	-43%	-52%	-100%
7 ($\rho_{R\pi} = 0.6$)	3,574	2,979	2,511	-4%	-2%	0%
8 ($\rho_{R\pi} = 0.9$)	1,291	1,802	2,425	-65%	-41%	-4%

Table. Certainty Equivalent (CE) if the true parameter set is Θ, but the investor believes it is $\hat{\Theta}^k$, where $(\tau_1, \tau_2) = (10, 50)$, $\kappa_{R\pi} = -0.078$. Impact τ_i

Impact remaining parameters (BX/BBB/MSV)

- The BBB-strategy appears to be the most 'robust to parameter uncertainty' (best worst-case scenario over $\hat{\Theta}^k$)

Corresponding x_t^* | $k \in \{1,2,3,6\}$ | $k \in \{7,8\}$

- However, robust one bond strategies still lead to low CE in case of correct beliefs (Θ)

	Θ (Initial)			$\hat{\Theta}^5(\kappa_R, \sigma_R, \kappa_\pi, \sigma_\pi \uparrow)$		
γ	*1*	*2*	*25*	*1*	*2*	*25*
CE^{SW2}	**3,725**	**3,041**	**2,517**	**313**	**24**	**0**
CE^{BX}	3,725	3,041	2,517	1,692	1,330	26
CE^{SW1}	**3,048**	**2,324**	**251**	**3,045**	**2,271**	**70**
CE^{BBB}	3,048	2,324	251	2,970	2,259	211
CE^{MSV}	2,833	2,283	103	2,750	2,257	8

Table. Certainty Equivalent (CE) if the true parameter set is Θ, but the investor believes it is $\hat{\Theta}^k$, where $(\tau_1, \tau_2) = (10, 50)$, $\kappa_{R\pi} = -0.078$.

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  - Full inflation risk hedge
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Summary

- In the two bond SW-strategy, the bond demands are typically extreme, especially when the bond maturities lay close to each other
  - Full inflation risk hedge
  - Even for the robust choice $(\tau_1, \tau_2) = (10, 50)$, parameter uncertainty about the feedback parameter and mean-reversion parameters has a considerable impact
- We study alternative strategies
  - The BX-strategy results in similar issues
  - One bond strategies have less extreme demands which leads to the opposite results:

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  - Full inflation risk hedge
  - Even for the robust choice $(\tau_1, \tau_2) = (10, 50)$, parameter uncertainty about the feedback parameter and mean-reversion parameters has a considerable impact
- We study alternative strategies
  - The BX-strategy results in similar issues
  - One bond strategies have less extreme demands which leads to the opposite results:
    - Less impact of parameter uncertainty or non-robust choice of bond maturities
    - Possible large loss in utility due to no full inflation risk hedge

Future research

- Formal robust optimisation
  - Trade-off between parameter uncertainty and inflation hedging
  - Let the investor choose both the bond maturities and the type of strategy (two bond SW-strategy, BBB-strategy, ...)

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R_t^f-parameters vs. r_t-parameters

$$\sigma_R = \sqrt{\sigma_r^2 + \sigma_\pi^2 + 2\rho_{r\pi}\sigma_r\sigma_\pi}$$

$$\rho_{R\pi} = \frac{\rho_{r\pi}\sigma_r\sigma_\pi + \sigma_\pi^2}{\sigma_R\sigma_\pi}$$

$$\kappa_R = \frac{\sigma_R^2}{2 * \left(\frac{\sigma_r^2}{2\kappa_r} + \frac{\sigma_\pi^2}{2\kappa_\pi} + \frac{\rho_{r\pi}\sigma_r\sigma_\pi}{\sqrt{\kappa_r\kappa_\pi}}\right)}$$

$$\lambda_R = \frac{\lambda_r b_r(\tau_R)\sigma_r + \lambda_\pi c(\tau_R)\sigma_\pi}{\sqrt{(b_r(\tau_R))^2\sigma_r^2 + (c(\tau_R))^2\sigma_\pi^2 + 2 * \rho_{r\pi} * b_r(\tau_R)\sigma_r * c(\tau_R)\sigma_\pi}}$$

Explanation extreme bond allocations ▸▸ x_t^*

- The two bond SW-strategy can be simplified to:

$$x_t^* = \frac{1}{\gamma}(\sigma')^{-1}\rho^{-1}\lambda + \left(1 - \frac{1}{\gamma}\right)(A(\tau)')^{-1}B(\tau)'$$

  where $A(\tau)$ is dependent on the mean-reversion matrix and

$$\text{Det}\,(A) = \frac{\kappa_{R\pi}}{\kappa_R - \kappa_\pi}(b_R(\tau_2)c_\pi(\tau_1) - b_R(\tau_1)c_\pi(\tau_2))$$

  where $b_R(\tau_i)$ and $c_\pi(\tau_i)$ are similar functions based on κ_R and κ_π. If $\kappa_R = \kappa_\pi$, then $b_R(\tau_i) = c_\pi(\tau_i)$.
- Hence, extreme if at least one of conditions below holds:
  - $\kappa_{R\pi} \approx 0$
  - $\kappa_R \approx \kappa_\pi$
  - $\tau_1 \approx \tau_2$

Explanation extreme bond allocations

- Extreme values of x_t^* if $\kappa_R \approx \kappa_\pi, \tau_1 \approx \tau_2$, or $\kappa_{R\pi} \approx 0$ ▸▸ x_t^*

	Basis		*Bond maturities*		*Mean-reversion*		*Feedback par.*	
	$\kappa_{R\pi} = -0.078$; $\kappa_R = 0.105$; $\kappa_\pi = 0.027$		$\kappa_{R\pi} = -0.078$; $\kappa_R = 0.105$; $\kappa_\pi = 0.027$		$\kappa_{R\pi} = -0.078$; $\kappa_R = 0.105$; $\kappa_\pi = 0.090$		$\kappa_{R\pi} = -0.001$; $\kappa_R = 0.105$; $\kappa_\pi = 0.027$	
(T_1, T_2)	10	50	3	10	10	50	10	50
$(x_{0,1}^*, x_{0,2}^*)$	**2.9**	**-0.9**	**17.1**	**-5.6**	**18.1**	**-10.8**	**17,165**	**-11,118**

Table. Numerical example of x_0^* for $T = 30, \gamma = 25$. The basis scenario is based on the initial parameter set Θ

Parameter sensitivity sets

- Parameter sensitivity sets $\hat{\Theta}^k, k \in \{1, 2, ...8\}$ ▸▸ Impact $\hat{\Theta}^k$

Set k	Initial	1	2	3	4	5	6	7	8
κ_R	0.105	0.210	0.105	0.053	0.105	0.210	0.053	0.105	0.105
σ_R	0.019	0.027	0.019	0.014	0.019	0.027	0.014	0.019	0.019
κ_π	0.027	0.027	0.054	0.027	0.014	0.054	0.014	0.027	0.027
σ_π	0.014	0.014	0.020	0.014	0.010	0.020	0.010	0.014	0.014
$\rho_{R\pi}$	0.733	0.733	0.733	0.733	0.733	0.733	0.733	0.600	0.900

Impact feedback parameter $\kappa_{R\pi}$ for $j \in \{1, 2\}$

- Similar observations as for $j \in \{3, 4, 5\}$ Results $j \in \{3, 4, 5\}$
- Having incorrect beliefs about $\kappa_{R\pi}$ is more relevant for one bond strategies if $\kappa_{R\pi}$ decreases

$\kappa_{R\pi}$	-0.117			-0.078		
γ	*1*	*2*	*25*	*1*	*2*	*25*
CE^{SW2}	**3,718**	**3,165**	**2,722**	**3,725**	**3,041**	**2,517**
CE^{BX}	3,520	2,694	153	3,725	3,041	2,517
CE^{SW1}	**2,912**	**2,442**	**620**	**3,048**	**2,324**	**251**
CE^{BBB}	2,876	2,434	487	3,048	2,324	251
CE^{MSV}	2,461	2,344	57	2,833	2,283	103

Table. Certainty Equivalent (CE) for three values of $\kappa_{R\pi}$, while the investor beliefs it equals -0.078 (BX/BBB) or 0 (MSV), and $(\tau_1, \tau_2) = (10, 50)$

Explanation extreme impact on CE

- Beliefs of $\hat{\Theta}^5$ (increase in κ_R, κ_π) lead to more deviation from $x_t^*|\Theta$ than beliefs of $\hat{\Theta}^4$ (κ_π decreases) Impact on CE

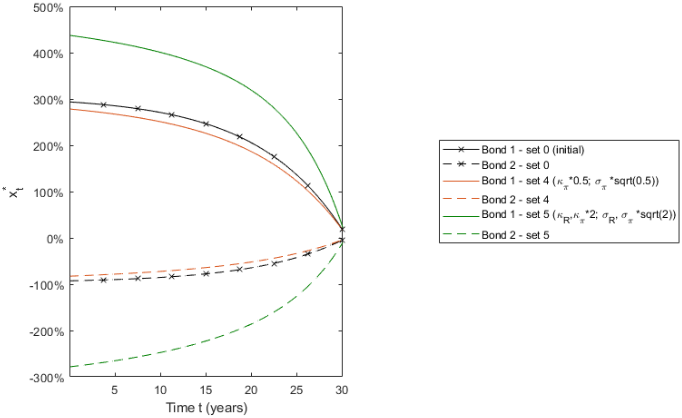

Figure. Optimal bond allocations for $\gamma = 25$, $\tau_1, \tau_2 = (10, 50)$

Impact τ_1 and τ_2 on parameter uncertainty

- The loss in wealth because of parameter uncertainty depends on the choice of bond maturities
- This loss might even become larger, but also smaller Initial (τ_1, τ_2)

(τ_1, τ_2)	(10,50)		(5,30)		(3,10)	
γ	*2*	*25*	*2*	*25*	*2*	*25*
Parameter set k						
0 (Initial, Θ)	**3,041**	**2,517**	**3,041**	**2,517**	**3,041**	**2,517**
1 $(\kappa_R, \sigma_R \uparrow)$	571	0	469	0	311	0
2 $(\kappa_\pi, \sigma_\pi \uparrow)$	2,712	937	3,028	2,346	3,017	2,296
3 $(\kappa_R, \sigma_R \downarrow)$	1,501	4	1,360	2	1,192	1
4 $(\kappa_\pi, \sigma_\pi \downarrow)$	3,013	2,061	3,038	2,476	2,993	2,431
5 $(\kappa_R, \sigma_R, \kappa_\pi, \sigma_\pi \uparrow)$	24	0	275	0	598	0
6 $(\kappa_R, \sigma_R, \kappa_\pi, \sigma_\pi \downarrow)$	1,445	3	1,373	2	1,187	1
7 $(\rho_{R\pi} = 0.6)$	2,979	2,511	2,979	2,511	2,979	2,511
8 $(\rho_{R\pi} = 0.9)$	1,802	2,425	1,802	2,425	1,802	2,425

Table. Certainty Equivalent per bond maturity combination (τ_1, τ_2) if $\kappa_{R\pi} = -0.078$, the true parameter set is Θ, but the investor believes it is $\hat{\Theta}^k$.

Robust choice when τ_1 is fixed

- We saw that if $\tau_1, \tau_2 \in \{1, 2, ..., 50\}$, then (11,50) is a robust choice for the bond maturities Robust choice
- τ_1 fixed; $\tau_2 \in \{1, 2, ..., 50\} \rightarrow$ include long-term bond $\geq$30Y:
-

τ_1	5	10	15	20	25	**30**	**35**	**40**	**45**	**50**
τ_2 (robust)	**50**	**50**	**50**	**50**	**50**	8	9	10	10	11

Support empirical values $\kappa_{R\pi}$

- For both the one- and two bond strategies, $\kappa_{R\pi} = \{-0.078, 0\}$ are relevant variables:

Strategy	**#bonds**	$\hat{\kappa}_{R\pi}$
SW2	2	$\kappa_{R\pi}$
BX	2	$\kappa_{\pi} - \kappa_{R}$
SW1	1	$\kappa_{R\pi}$
BBB	1	$\kappa_{\pi} - \kappa_{R}$
MSV	1	0

- Therefore, after including some sensitivities we investigate impact beliefs when $\hat{\kappa}_{R\pi} \in \{-0.117, -0.078, -0.039, -0.001, 0\}$

Impact feedback parameter $\kappa_{R\pi}$ on bond demands

- The very extreme positions in the two bond SW-strategy fully hedge inflation risk, even when $\kappa_{R\pi} \approx 0$.
- If $\kappa_{R\pi} \to 0$, it is an advantage that the BX-, MSV-, and BBB-strategies are independent on the actual $\kappa_{R\pi}$

Impact on CE

j	3			4			5		
$\kappa^j_{R\pi}$	-0.039			-0.001			0		
γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
x_1^*	**520%**	**516%**	**513%**	**7,632%**	**12,597%**	**17,165%**			
x_2^*	**-170%**	**-204%**	**-236%**	**-4,818%**	**-8,099%**	**-11,118%**			
x_1^{BX}	426%	357%	294%	426%	357%	294%	426%	357%	294%
x_2^{BX}	-109%	-100%	-93%	-109%	-100%	-93%	-109%	-100%	-93%
$\tilde{x}_0^*$	**160%**	**83%**	**13%**	**185%**	**77%**	**-22%**	**185%**	**77%**	**-22%**
x^{BBB}	138%	91%	48%	138%	91%	48%	138%	91%	48%
x^{MSV}	185%	77%	-22%	185%	77%	-22%	185%	77%	-22%

Table. Perceived optimal bond allocations if $\kappa^j_{R\pi}$ is the true value, but the investor believes $\kappa_{R\pi}$ equals -0.078 (BX/BBB) or 0 (MSV), $(\tau_1, \tau_2) = (10, 50)$

Impact remaining parameter beliefs for $k \in \{1,2,3,6\}$

- Similar results as for $k \in \{4,5\}$ ▸▸ $k \in \{4,5\}$ ▸▸ $k \in \{7,8\}$

	Θ(Initial)			$\hat{\Theta}^1(\kappa_R, \sigma_R \uparrow)$			$\hat{\Theta}^2(\kappa_\pi, \sigma_\pi \uparrow)$		
γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
CE^{SW2}	**3,725**	**3,041**	**2,517**	**1,657**	**571**	**0**	**3,436**	**2,712**	**937**
CE^{BX}	3,725	3,041	2,517	3,449	2,882	1,849	2,673	1,409	0
CE^{SW1}	**3,048**	**2,324**	**251**	**3,025**	**2,297**	**106**	**3,037**	**2,309**	**232**
CE^{BBB}	3,048	2,324	251	2,995	2,304	249	3,044	2,315	182
CE^{MSV}	2,833	2,283	103	2,750	2,247	20	2,833	2,299	57

	$\hat{\Theta}^3(\kappa_R, \sigma_R \downarrow)$			$\hat{\Theta}^6(\kappa_R, \sigma_R, \kappa_\pi, \sigma_\pi \downarrow)$		
γ	*1*	*2*	*25*	*1*	*2*	*25*
CE^{SW2}	**2,171**	**1,501**	**4**	**2,126**	**1,445**	**3**
CE^{BX}	2,638	1,981	30	2,454	1,847	48
CE^{SW1}	**3,046**	**2,291**	**139**	**3,014**	**2,226**	**87**
CE^{BBB}	2,851	2,284	232	2,876	2,212	200
CE^{MSV}	2,590	2,239	155	2,590	2,153	242

Table. Certainty Equivalent (CE) if the true parameter set is Θ, but the investor believes it is $\hat{\Theta}^k$, where $(\tau_1, \tau_2) = (10, 50)$, $\kappa_{R\pi} = -0.078$.

Impact remaining parameter beliefs for $k \in \{7, 8\}$

- The hedge demand is almost insensitive to changes in $\rho_{R\pi}$, except for the MSV-strategy

Return to main results for $k \in \{4, 5\}$

Show additional results for $k \in \{1, 2, 3, 6\}$

	Θ(Initial)			$\hat{\Theta}^7(\rho_{R\pi} = 0.6)$			$\hat{\Theta}^8(\rho_{R\pi} = 0.9)$		
γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
CE^{SW2}	**3,725**	**3,041**	**2,517**	**3,574**	**2,979**	**2,511**	**1,291**	**1,802**	**2,425**
CE^{BX}	3,725	3,041	2,517	3,574	2,979	2,511	1,291	1,802	2,425
CE^{SW1}	**3,048**	**2,324**	**251**	**3,042**	**2,318**	**243**	**3,041**	**2,315**	**242**
CE^{BBB}	3,048	2,324	251	3,042	2,318	243	3,041	2,315	242
CE^{MSV}	2,833	2,283	103	2,833	2,257	200	2,833	2,273	18

Table. Certainty Equivalent (CE) if the true parameter set is Θ, but the investor believes it is $\hat{\Theta}^k$, where $(\tau_1, \tau_2) = (10, 50)$, $\kappa_{R\pi} = -0.078$.

Impact remaining parameter beliefs on bond demands

- The larger robustness to parameter uncertainty of one compared to two bond strategies can be explained by the smaller (changes in) absolute values of the bond demands.

Robustness to parameter uncertainty

	Θ(Initial)			$\hat{\Theta}^4(\kappa_\pi, \sigma_\pi \downarrow)$			$\hat{\Theta}^5(\kappa_R, \sigma_R, \kappa_\pi, \sigma_\pi \uparrow)$		
γ	*1*	*2*	*25*	*1*	*2*	*25*	*1*	*2*	*25*
x_1^*	**426%**	**357%**	**294%**	**415%**	**344%**	**278%**	**565%**	**498%**	**437%**
x_2^*	**-109%**	**-100%**	**-93%**	**-102%**	**-92%**	**-83%**	**-257%**	**-268%**	**-278%**
x_1^{BX}	426%	357%	294%	401%	322%	249%	484%	358%	242%
x_2^{BX}	-109%	-100%	-93%	-92%	-77%	-64%	-186%	-145%	-107%
$\tilde{x}^*$	**138%**	**91%**	**48%**	**149%**	**103%**	**62%**	**144%**	**59%**	**-19%**
x^{BBB}	138%	91%	48%	143%	106%	71%	110%	66%	26%
x^{MSV}	185%	77%	-22%	185%	91%	4%	194%	58%	-66%

Table. Perceived optimal bond allocations if the true parameter set is Θ, but the investor believes it is $\hat{\Theta}^k$, where $(\tau_1, \tau_2) = (10, 50)$, $\kappa_{R\pi} = -0.078$. The strategies x^* and $\tilde{x}^*$ denote the two and one bond SW-strategy respectively.