ENDOWMENT ASSURANCE PRODUCTS–EFFECTIVENESS OF RISK–MINIMIZING STRATEGIES UNDER MODEL RISK

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Abstract. This paper analyzes and discusses the effects of model misspecification associated with *both* interest rate *and* mortality risk on the hedging decisions of insurance companies. We consider hedging strategies in different instruments (zero bonds) which are risk–(variance–)minimizing with respect to an assumed model. In this case, the associated expected costs and the variance of the costs are the same for all strategies. While the introduction of model risk, i.e. a deviation of assumed from true models, has the same effect on the expected costs, this is not true with respect to the variance. It turns out that the choice of hedging instruments has a crucial impact on the *robustness* of the strategies. In addition, the results of the paper can be used to emphasize the necessity to use a combined hedging model. In terms of robust hedging, a separate specification of interest rate model and mortality model is inconvenient, even in the case that interest rate and mortality are assumed to be independent.

Keywords: *Model risk, robust hedging, mortality risk, stochastic interest rates, risk–minimizing, choice of hedge instruments*

JEL–Codes: *G13, G22, G23*
Subject and Insurance Branch Codes: : *IM10, IE10, IE50, IB10*

1. Introduction

Life insurance contracts contain both, financial market and mortality risk. Typically, it is not possible to achieve a perfect hedge for the combined risk. Along the lines of Föllmer and Sondermann (1996) and Møller (1998, 2001), we consider hedging strategies which are risk–minimizing when model misspecification is neglected.[1] Without model risk, these strategies are able to match the liabilities of an insurance cohort which is *sufficiently large*, i.e. the law of large numbers applies. However, the strategies fail to be (variance) effective if the assumed model and the true model do not coincide. A misspecification of the hedging model gives rise to an additional variance term which cannot be diversified. In order to analyze this problem, we test the strategies which are risk–minimizing with respect to one particular model to their robustness according to model misspecification.

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[1] For an excellent overview of quadratic hedging approaches see Schweizer (2001).

Robust hedging strategies are well established in the financial market literature, i.e. in a setup which does not include the insurance typical risk. Without postulating completeness, we refer to the papers of Avellaneda et al. (1995), Lyons (1995), Bergman et al. (1996), Dudenhausen et al. (1998), El Karoui et al. (1998), Hobson (1998) and Mahayni (2003). Here, a strategy is called robust if it gives a superhedge in a whole class of models, c.f. also Branger and Mahayni (2006). In contrast, at the interface of insurance and finance, robustness features of risk management strategies are not widely discussed. Certainly, there are papers dealing with different scenarios of mortality risk and/or stochastic death distributions, for instance, Marocco and Pitacco (1998), Milevsky and Promislow (2001), Blake et al. (2004), Biffis and Millossovich (2004), Dahl (2004), Ballotta and Haberman (2006), Gründl et al. (2006), Miltersen and Persson (2006) and Dahl and Møller (2006). The observation that mortality rates are stochastic emphasizes the misspecification problem. The topic of model risk already enters the analysis with the question about the *true* mortality rate generating process. Assuming that the *true* model does not exist or is unknown, a meaningful approach is given by robust strategies (or robust models), i.e. strategies which satisfy certain features with respect to a large class of models or mortality scenarios. We try to fill this gap by a first step. Being aware of the distinction between true and assumed (combined) models, the purpose of this paper is the analysis and illustration of the effectiveness of risk management strategies which are derived in one particular model under combined misspecification scenarios.

Our analysis is carried out by considering a specific contract type, an endowment assurance product.[2] Here, the benefits of the contracts are given in terms of a life cover together with an investment element. The payoff is given by the maximum of a fixed amount (the sum assured) and an insurance account. The maturity date where the payoff occurs is conditioned on the time of death of the life insured. It is either given by a specified (terminal) date or the nearest future reference date after an early death. For contribution, the customer pays periodic premiums which are contingent on his mortality, too. Intuitively, the resulting risk–minimizing strategy can be explained as follows. Without the uncertainty about the random times of death, the cash flow of the benefits and contributions is deterministic. In particular, the benefits can be hedged perfectly by long–positions in bonds with matching maturities. Therefore, the most natural hedging instruments are given by the corresponding set of zero coupon bonds. Because of liquidity constraints in general or transaction costs in particular, it is not possible or convenient for the hedger to trade in all the bonds. Therefore, we also consider the case where it is only possible to hedge in a subset of zero bonds, i.e. the unavailable bonds are synthesized by using the available bonds. In this case, the number of bonds which is used for hedging purpose depends on both, the assumed mortality law and the assumed interest rate dynamic. Accordingly, the hedging performance depends on the *combined* model risk.

It is shown that, independent of interest rate risk and hedging instruments, an overestimation of the death probability implies that the insurance company stays on the safe side on average, i.e. a superhedge is achieved in the mean. Thus, dominating the

[2]Endowment assurance products are the most popular policies among all insurance plans. For example, about 75% of the life insurance contracts sold in Germany belong to this category. However, the results can also be adopted to a more geneneral contract setup.

true death probabilities can also be explained by the usage of conservative hedging strategies. However, the crucial point concerns the variance of the hedging error. We show that, under model misspecification of interest rate risk and mortality law, a subset of hedging instruments (zero coupon bonds) may result in a lower variance than the corresponding strategy in all bonds. This is explained by the combined effects of model risk on the interest rate and mortality side.

The remaining of the paper is organized as follows. Section 2 states the basic features of the insurance contract under consideration. In addition, we give a representation of fair contract specifications. Section 3 introduces the underlying interest rate model and some terminologies concerning trading and hedging decisions. In Section 4, we determine the class of strategies which are risk–minimizing with respect to the assumed model setup. In particular, we distinguish between the case that all bonds are used and the case that a subset of bonds is used. In addition, the effects of model risk on the cost processes are analyzed. Section 5 illustrates some numerical results for the cost distributions under different scenarios of model misspecification. Section 6 concludes the paper.

2. Product and Model Description

2.1. **Contract Specification.** We consider a discrete set $\underline{\underline{T}}$ of equidistant reference dates

$$\underline{\underline{T}} = \{t_0, \dots, t_{N-1}, t_N\},$$

where $\Delta t = t_{i+1} - t_i$ denotes the distance between two reference dates. One can consider that the customer will retire in t_N years. The contributions of the insured are given according to a periodic premium principle, i.e. if the insured is still alive at t_i $(i = 0, \dots, N-1)$, he pays a constant periodic premium A.

The benefits of the insured are described by the following. If the insured dies in the interval $[t_{i-1}, t_i[$ $(i = 1, \dots N)$, his benefits at t_i are

$$\bar{G}_{t_i} := \max\{h, G_{t_i}\}, \tag{1}$$

$$\text{where} \quad G_{t_i} := \tilde{A}_{t_{i-1}} e^{g\,\Delta t} \quad \text{and} \quad \tilde{A}_{t_i} := \sum_{j=0}^{i} A e^{g(t_i - t_j)}.$$

When the insured survives t_N, he receives $\bar{G}_{t_N}$ at t_N. h can be interpreted as the endowment part of the contract payoff and G as an insurance account. The insured gets back his paid premiums accrued with an interest rate guarantee $g \geq 0$.

According to the above contract specification, the benefits and contributions are deterministic if the remaining lifetime of the insured is known. The randomness is only caused by the uncertainty about the remaining lifetime τ of the insured, i.e. the mortality risk or timing risk. With respect to an insured aged x at $t_0 = 0$ where τ^x denotes his remaining lifetime, the discounted value of the contributions is given by

$$A \sum_{i=0}^{N-1} \exp\left\{-\int_{t_0}^{t_i} r(u)\,du\right\} 1_{\{\tau^x > t_i\}}, \tag{2}$$

and the discounted value of the benefits is

(3)
$$\sum_{i=0}^{N-1} \exp\left\{-\int_{t_0}^{t_{i+1}} r(u)\,du\right\} \bar{G}_{t_{i+1}}\ 1_{\{t_i<\tau^x\le t_{i+1}\}} + \exp\left\{-\int_{t_0}^{t_N} r(u)\,du\right\} \bar{G}_{t_N}\ 1_{\{\tau^x>t_N\}}.$$

where $r(t)$ denotes the continuously compounded interest rate prevailing at time t.

2.2. **Pricing and fair contract specification.** In the following, we assume that the interest rate risk and the mortality risk are independent and the initial (market) prices $D^M(t_0,t_i)$ of all zero bonds with all maturities t_i $(i=1,\dots N)$ are known. It means that in this place it is still unnecessary to specify the interest rate model. The *actuarial present value* of the contributions is therefore[3]

$$A\sum_{i=0}^{N-1} D^M(t_0,t_i)\ {}_{t_i}p_x, \tag{4}$$

where ${}_tp_x := P(\tau^x > t) = E_P[1_{\{\tau^x>t\}}]$ represents the *true* probability that the life (currently) aged x survives further t years. In contrast, ${}_t\tilde{p}_x := \tilde{P}(\tau^x > t) = E_{\tilde{P}}[1_{\{\tau^x>t\}}]$ denotes the assumed probability which is actually used by the insurance company.[4] A comment on the consequences of the distinction is postponed until at the end of this section. For the time being, we only use the *true* survival and death probabilities. With respect to the benefits, it holds

Proposition 2.1 (Contract or benefits value). *Let h be a constant such that there exists $k\in\{1,\dots,N-1\}$ with $G_{t_k} < h \le G_{t_{k+1}}$, then, in a complete arbitrage free market, the actuarial present value C_{t_0} of the benefits is given by*

$$C_{t_0} = \sum_{i=0}^{k-1} h\ D^M(t_0,t_{i+1})\ {}_{t_i|\Delta t}q_x + \sum_{i=k}^{N-1} G_{t_{i+1}}\ D^M(t_0,t_{i+1})\ {}_{t_i|\Delta t}q_x + G_{t_N}\ D^M(t_0,t_N)\ {}_{t_N}p_x$$

where ${}_{s|t}q_x := P\,(s<\tau^x\le s+t)$.

Proof: The proof is immediately given by taking the expectation of Equation (3).

□

Remark 2.2. We indeed assume that the survival/death probabilities depend on the age of the insured, only. For example, this implies that

$$\begin{aligned} {}_{s+t}p_x &= {}_sp_x\ {}_tp_{x+s} \\ {}_{s|t}q_x &= {}_sp_x\ {}_tq_{x+s} = {}_sp_x\ (1-{}_tp_{x+s}) \end{aligned}$$

where ${}_tq_x := 1 - {}_tp_x$ denotes the probability that an x-aged life dies in the following t years. Finally, the mortality law can also be stated in terms of the mortality rate μ

$$\exp\left\{-\int_0^t \mu_{x+s}\,ds\right\} := {}_tp_x.$$

[3]The *actuarial present value* is based on the assumption that the mortality risk can be diversified. The law of large numbers implies that the random time of death can be replaced by the expectation.

[4]Throughout the paper, we use ˜ to describe the assumed parameters.

Product Example

i	G_{t_i}	h	$\bar{G}_{t_i}$	$P(\tau^x \in]t_{i-1}, t_i])$	$D(t_0, t_{i+1})$
1	523.01	22491.7	22491.7	0.00178031	0.951229
2	1070.1	22491.7	22491.7	0.00190781	0.904837
...					
...					
23	20625.0	22491.7	22491.7	0.01029050	0.316637
24	22097.4	22491.7	22491.7	0.01116730	0.286505
25	23637.5	22491.7	23637.5	0.01210890	0.272532
...					
...					
≥ 30	32469.8	22491.7	32469.8	0.807003	0.22313

TABLE 1. Insurance account G and death dependent payoff $\bar{G}$ for an insurance contract maturing in $t_N = 30$ years, a guaranteed rate $g = 0.045$ and $h = 22491.7$ and a life aged $x = 35$.

Now, we consider the question how to specify a *fair* contract, i.e., how to specify the *fair* contract parameters h^* and g^* for a given periodic premium A. The so–called equivalence principle states that a contract is fair if the present value of the contributions is equal to the present value of the benefits, i.e.

COROLLARY 2.3.

$$h^*(g) = \frac{A \sum_{i=0}^{N-1} D^M(t_0, t_i)\ {}_{t_i}p_x - \sum_{i=k}^{N-1} G_{t_{i+1}}\ D^M(t_0, t_{i+1})\ {}_{t_i|\Delta t}q_x - G_{t_N}\ D^M(t_0, t_N)\ {}_{t_N}p_x}{\sum_{i=0}^{k-1} D^M(t_0, t_{i+1})\ {}_{t_i|\Delta t}q_x}.$$

PROOF: The result follows immediately from Proposition 2.1 and Equation (4).

□

Notice that h is a decreasing function of g in view of fair contract analysis. As g goes up, G_T increases and so does $\bar{G}_T$. A rise in h leads to an increase in $\bar{G}_T$ as well. Customers of such a contract benefit from both a higher h and a higher g.

2.3. **Example.** A product example is given in Table 1. In the last column, the prices of zero bonds are given for a flat term structure with $r = 0.05$. The death distribution is assumed to follow a Makeham law where

$$\mu_{x+t} = H + Kc^{x+t}. \tag{5}$$

Along the lines of Delbaen (1990), we use the parameter constellation

$$H = 0.0005075787,\ K = 0.000039342435 \text{ and } c = 1.10291509.$$

Observe that a high h^*–value must be provided to the customer if the offered minimum interest rate guarantee is much lower than the spot rate. This highlights the possibility that $h < G_{t_N}$ is incompatible with a fair contract specification. This feature is illustrated in Figure 1 where for small values of g, the fair value h^* is larger

Fair parameter combinations (g^*, h^*)

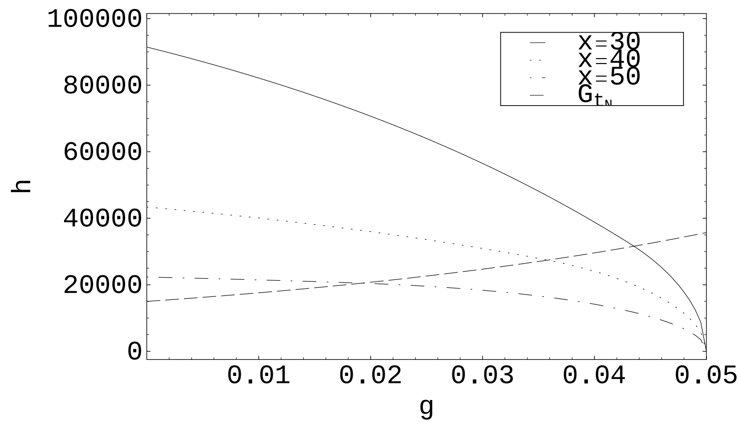

FIGURE 1. Fair parameter combinations (g^*, h^*) where the same parameters as Table 1 are used.

than G_{t_N}.[5] The interesting case, i.e. the case where $h^* < G_{t_N}$, results from a minimum interest rate guarantee g which is (slightly) smaller than the instantaneous risk free rate of interest.

REMARK 2.4. According to the meaning of *fair*, a fair contract is given with respect to the *true* death/survival probabilities. Obviously, the insurance company is, purposely or not, using assumed probabilities which deviate from the true ones. As a consequence, the insurance company uses an assumed, not necessarily fair, contract value

$$\tilde{C}_{t_0} = \sum_{i=0}^{k-1} h\ D^M(t_0, t_{i+1})\ {}_{t_i|\Delta t}\tilde{q}_x + \sum_{i=k}^{N-1} G_{t_{i+1}}\ D^M(t_0, t_{i+1})\ {}_{t_i|\Delta t}\tilde{q}_x + G_{t_N}\ D^M(t_0, t_N)\ {}_{t_N}\tilde{p}_x$$

such that the contract parameters which are actually offered to the insurance taker deviate from the fair ones.

3. Interest Rate Model Setup and Hedging

In the considered endowment assurance contracts, interest rate risk is the exclusive financial market risk. This section aims at describing the interest rate dynamics and introducing some terminologies concerning trading and hedging decisions.

3.1. **Interest rate model.** We consider a complete and arbitrage free financial market model where the dynamic of the zero coupon bonds $D(., \bar{t})$ paying one monetary unit at maturity $\bar{t} \in [0, T^*]$ is lognormal. In particular, trading terminates at time $T^* > 0$.Thus, the interest rate dynamic is given by a Gauss–Markov Heath, Jarrow and Morton (1992) model, i.e.

$$dD(t, \bar{t}) = D(t, \bar{t})\,(r(t)dt + \sigma_{\bar{t}}(t)dW^*(t)) \tag{6}$$

[5]In reality most contracts contain additional bonus features which reduce the fair h^*–value substantially.

where W^* denotes a d–dimensional Brownian motion, and the volatility of the zero coupon bond $\sigma_{\bar{t}}(t)$ is a time-dependent function with $\sigma_{\bar{t}}(\bar{t}) = 0$. In particular, it holds

$$D(t,\bar{t}) = E_{P^*}\left[\exp\left(-\int_t^{\bar{t}} r(u)\,du\right)\Big|\mathcal{F}_t\right]. \tag{7}$$

In order to simplify the expositions, we restrict ourselves to one–factor models, i.e. W^* is a one–dimensional Brownian motion. In particular, all examples are given in the context of a Hull and White (1990) model where the dynamic of the short rate r is given by

$$dr(t) = (\theta(t) - a\,r(t))\,dt + \sigma^r\,dW^*(t) \quad \text{where } r(t_0) = const. \tag{8}$$

where $a, \sigma^r > 0$ and θ are chosen to fit the term structure of interest rates being currently observed in the market. In particular, it holds[6]

$$\begin{aligned} \theta(t) &= \frac{\partial f^M(t_0,t)}{\partial t} + a f^M(t_0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-t_0)}\right) \\ \text{with} \quad f^M(t_0,T) &:= -\frac{\partial \ln D^M(t_0,T)}{\partial T}. \end{aligned} \tag{9}$$

Again $D^M(t_0,T)$ denotes the price of a zero coupon bond with maturity T which is observed in the financial market at t_0.[7]

Model risk can be described either by the discrepancy between the true and assumed model or by the discrepancy between the true and assumed model parameters. The present paper focuses on the model risk resulting from parameter misspecification. The "true" interest rate parameters are a, σ^r and θ such that the true interest model satisfies Equation (8). In contrast, the assumed short rate model is given by

$$d\tilde{r}(t) = \left(\tilde{\theta}(t) - \tilde{a}\tilde{r}(t)\right)\,dt + \tilde{\sigma}^r\,dW^*(t) \quad \text{where } \tilde{r}(t_0) = r(t_0). \tag{10}$$

$\tilde{\theta}$ is used to calibrate the zero bond prices at t_0. It is worth emphasizing that a consequence of fitting the term structure of interest at t_0 is that

$$\begin{aligned} D(t_0,\bar{t}) &= D^M(t_0,\bar{t}) = \tilde{D}(t_0,\bar{t}), \\ \text{i.e. } E_{P^*}\left[\exp\left(-\int_{t_0}^{\bar{t}} r(u)\,du\right)\Big|\mathcal{F}_{t_0}\right] &= E_{\tilde{P}^*}\left[\exp\left(-\int_{t_0}^{\bar{t}} \tilde{r}(u)\,du\right)\Big|\mathcal{F}_{t_0}\right]. \end{aligned}$$

3.2. **Hedging.** Not like mortality risk which is diversifiable due to a known deterministic mortality rate setup, interest rate risk can be eliminated by trading or hedging on the financial market. This subsection introduces some basic hedging terminologies.

Definition 3.1 (Trading strategy, value process, duplication). Let $D = (D^{(1)}, \ldots, D^{(N)})$ be the price processes of underlying assets (zero bonds, respectively). A trading strategy $\phi = (\phi^{(1)}, \ldots, \phi^{(N)})$ in these assets is given by a $\mathbb{R}^N$–valued, predictable process

[6] C.f. for example Brigo and Mercurio (2007) p. 73.

[7] Notice that in the case of a flat term structure of interest with yield $f^M(t_0,M) = r_0$, we simply have $\theta(t) = ar_0 + \frac{\sigma^2}{2a}(1 - e^{-2at})$.

which is integrable with respect to D. The value process $V(\phi)$ associated with ϕ is defined by

$$V_t(\phi) = \sum_{i=1}^{N} \phi_t^{(i)} D_t^{(i)}.$$

Let C be a contingent claim with maturity $T \in [0, T^*]$ which is a $\mathcal{F}_T$–measurable random payoff received at time T, then ϕ duplicates C iff

$$V_T(\phi) = C, \quad P\text{–a.s.}.$$

The deviation of the terminal value of the strategy from the payoff is called duplication cost L^{dup}, i.e.,

$$L_T^{\mathrm{dup}} := C - V_T(\phi).$$

Definition 3.2 (Gain process). If ϕ is a trading strategy in the assets $D^{(1)}, \ldots, D^{(N)}$, the gain process $(I_t(\phi))_{t\in[0,T]}$ associated with ϕ is defined as follows:

$$I_t(\phi) := \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} \, dD_u^{(i)}.$$

Definition 3.3 (Rebalancing cost process). If ϕ is a trading strategy, the cost process $L^{reb}(\phi)$ associated with ϕ is defined as follows:

$$L_t^{reb}(\phi) := V_t(\phi) - V_0(\phi) - I_t(\phi).$$

Notice that the above definition of cost process neglects the timing of the costs, i.e. the rebalancing costs at two different trading dates are simply added. In order to take account of the time value, a numeraire is used, i.e., all costs are measured in terms of one reference date. Unless mentioned otherwise, we use the money account as numeraire and denote the discounted versions of D, V, L^{reb} and L^{dup} with a superscript $*$, e.g $D_t^* = e^{-\int_0^t r(u)\,du}\, D_t$.

Definition 3.4 ((Discounted) Total Cost). The (discounted) total costs of a hedging strategy ϕ for a claim C are described as the sum of (discounted) rebalancing and duplication cost.

$$L_t^{tot}(\phi) \;=\; L_t^{reb}(\phi) + L_t^{dup}(\phi), \qquad L_t^{tot,*}(\phi) = L_t^{reb,*}(\phi) + L_t^{dup,*}(\phi).$$

Definition 3.5 (Super– and Subhedge). A hedging strategy ϕ for the claim C is called superhedge (subhedge) iff $L_t^{tot}(\phi) \le 0$ ($L_t^{tot}(\phi) \ge 0$) for all $t \in [0, T]$. In particular, a strategy which is a superhedge and a subhedge at the same time is called perfect hedge.

It is noticed that super– and subhedge in the mean can be defined similarly, when the expectation of the total cost is considered. A strategy which is super– and subhedge in the mean at the same time is called mean–self–financing.

LEMMA 3.6. *The total hedging costs* L_T^{tot} *and* $L_T^{tot,*}$ *(at time* T*) can be reformulated as*

$$L_T^{tot}(\phi) = C_T - (V_0(\phi) + I_T(\phi)), \qquad L_T^{tot,*}(\phi) = C_T^* - (V_0^*(\phi) + I_T^*(\phi)).$$

PROOF: According to the above definitions we have

$$L_T^{tot} = L_T^{reb} + L_t^{dup} = V_T - (V_0 + I_T) + C_T - V_T = C_T - (V_0 + I_T)$$

□

For the late use which requires only a subset of zero coupon bonds for hedging, we introduce the following lemma which is dedicated to the duplication or synthesization of a (non–available) bond in the framework of a one–factor short rate model. We assume that the bond to be synthesized matures prior to the maturity of the bonds used for hedging.

LEMMA 3.7 (Duplication of bonds). *Let* $t_i, t_{N-1}, t_N \in \mathbb{R}$ $(t_i < t_{N-1} < t_N)$ *be the maturities of three zero coupon bonds. If the dynamics of these bonds satisfy Equation (6), then it holds: The strategy* $\bar{\phi}^{(i)} = \left(\bar{\phi}^{(i,N-1)}, \bar{\phi}^{(i,N)}\right)$ *with the value process* $\left\{\bar{\phi}_t^{(i,N-1)} D(t,t_{N-1}) + \bar{\phi}_t^{(i,N)} D(t,t_N)\right\}$ *synthesizes the zero bond with maturity* t_i *iff*

$$\bar{\phi}_t^{(i,N-1)} = \frac{D(t,t_i)}{D(t,t_{N-1})}\lambda_t^{(i)} \tag{11}$$

$$\bar{\phi}_t^{(i,N)} = \frac{D(t,t_i)}{D(t,t_N)}(1-\lambda_t^{(i)}) \tag{12}$$

and

$$\lambda_t^{(i)} := \lambda(t,t_i,t_{N-1},t_N) = \frac{\sigma_{t_N}(t) - \sigma_{t_i}(t)}{\sigma_{t_N}(t) - \sigma_{t_{N-1}}(t)}. \tag{13}$$

In particular, in the Hull–White (1990) model (c.f. Equation (8)), we obtain

$$\lambda_t^{(i)} = \frac{e^{-at_i} - e^{-at_N}}{e^{-at_{N-1}} - e^{-at_i}}. \tag{14}$$

PROOF: A straightforward generalization of the rank condition in the classical Black–Scholes setting gives that a market consisting of d zero coupon bonds is complete if the affine subspace generated by the bond volatilities is of rank d, c.f. for example Proposition 4.3 of Dudenhausen and Schlögl (2002). In the simple case of a one–factor short rate model, the market is already complete in two bonds. There exists a uniquely defined self–financing strategy $\bar{\phi}^{(i)} = \left(\bar{\phi}^{(i,N-1)}, \bar{\phi}^{(i,N)}\right)$ with the value process $\left\{\bar{\phi}_t^{(i,N-1)} D(t,t_{N-1}) + \bar{\phi}_t^{(i,N)} D(t,t_N)\right\} = \{D(t,t_i)\}$, which is called synthesizing or duplication strategy for the zero bond with maturity t_i $(t_i < t_{N-1} < t_N)$. It can be shown, c.f. for example Proposition 4.1 of Dudenhausen and Schlögl (2002), that the portfolio weights λ of the strategy are determined by the condition that the weighted volatilities of the hedging instruments coincide with the volatility of the asset to be synthesized, i.e.

$$\lambda_t^{(i)}\sigma_{t_{N-1}}(t) + (1-\lambda_t^{(i)})\sigma_{t_N}(t) = \sigma_{t_i}(t). \tag{15}$$

The synthesizing strategy is given by

$$\bar{\phi}_t^{(i,N-1)} = \frac{D(t,t_i)}{D(t,t_{N-1})}\lambda_t^{(i)}, \;\; \bar{\phi}_t^{(i,N)} = \frac{D(t,t_i)}{D(t,t_N)}(1-\lambda_t^{(i)}).$$

In a Hull–White (1990) model which belongs to the class of affine term structure models, zero coupon bonds can be expressed as the following exponential functions:[8]

$$(16)\qquad \begin{aligned} D(t,\bar{t}) &= \mathcal{A}(t,\bar{t})e^{-\mathcal{B}(t,\bar{t})r(t)} \\ \mathcal{B}(t,\bar{t}) &= \frac{1}{a}\left(1-e^{-a(\bar{t}-t)}\right) \\ \mathcal{A}(t,\bar{t}) &= \frac{D^M(t_0,\bar{t})}{D^M(t_0,t)}\exp\left\{\mathcal{B}(t,\bar{t})f^M(t_0,t) - \frac{\sigma^2}{4a}\mathcal{B}(t,\bar{t})^2\right\}. \end{aligned}$$

An application of Ito's lemma to Equation (16) results in

$$(17)\qquad dD(t,\bar{t}) = D(t,\bar{t})\left(r(t)\,dt + \sigma^r\mathcal{B}(t,\bar{t})\,dW^*(t)\right).$$

In other words, the bond volatility $\sigma_{\bar{t}}(t)$ takes the form of

$$(18)\qquad \sigma_{\bar{t}}(t) = \sigma^r\mathcal{B}(t,\bar{t}) = \frac{\sigma^r}{a}\left(1-e^{-a(\bar{t}-t)}\right).$$

Inserting Equation (18) into Equation (15) gives the result.

□

Notice that the synthesizing concept depends on the assumption that the "true" model is known. If the strategy is constructed according to an assumed model which does not necessarily coincide with the true one, there is a tracking error. In the following, we use the notation $\bar{\phi}^{(i)}(\tilde{\lambda})$ instead of $\bar{\phi}^{(i)}(\lambda)$ to emphasize the dependence on the model assumptions.

Lemma 3.8. *If the t_i–bond synthesizing strategy $\phi^{(i)}(\tilde{\lambda})$ defined in Lemma 3.7 is implemented according to an assumed bond volatility structure $\tilde{\sigma}$ (not necessarily coinciding with the true volatility structure σ), then the hedging (tracking) error is given by*

$$(19)\qquad \begin{aligned} L_{t_i}^{tot,*}(\bar{\phi}^{(i)}) &= L_{t_i}^{reb,*}(\bar{\phi}^{(i)}) \\ &= \int_0^{t_i} D^*(u,t_i)g^{(i)}(u)\,dW^*(u) \end{aligned}$$

where

$$(20)\qquad \begin{aligned} g^{(i)}(t) &:= \sigma_{t_i}(t) - \left(\frac{\tilde{\sigma}_{t_N}(t)-\tilde{\sigma}_{t_i}(t)}{\tilde{\sigma}_{t_N}(t)-\tilde{\sigma}_{t_{N-1}}(t)}\sigma_{t_{N-1}}(t) + \frac{\tilde{\sigma}_{t_i}(t)-\tilde{\sigma}_{t_{N-1}}(t)}{\tilde{\sigma}_{t_N}(t)-\tilde{\sigma}_{t_{N-1}}(t)}\sigma_{t_N}(t)\right) \\ &= \left(\tilde{\lambda}^{(i)}(t)-\lambda^{(i)}(t)\right)\left(\sigma_{t_N}(t)-\sigma_{t_{N-1}}(t)\right). \end{aligned}$$

In particular, the variance of the hedging error is

$$(21)\qquad Var_{P^*}[L_{t_i}^{tot,*}(\bar{\phi})] = (D(t_0,t_i))^2\int_0^{t_i}\exp\left\{\int_0^u \sigma_{t_i}^2(s)\,ds\right\}\left(g^{(i)}(u)\right)^2 du.$$

[8] C.f. for example Brigo and Mercurio (2007) p.75.

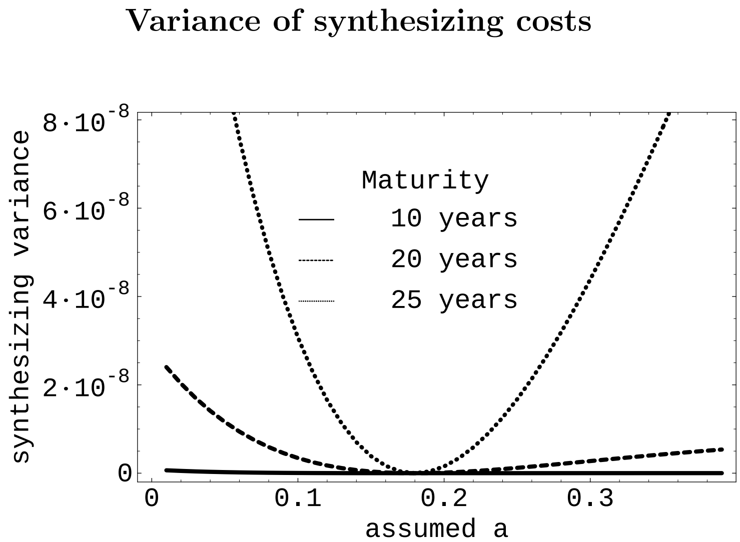

FIGURE 2. Variance of synthesizing costs for a bond maturing in $t_i = 10$ (20 and 25 years respectively) depending on the assumed parameter $\tilde{a}$ when $a = 0.18$, $\sigma^r = 0.02$. The hedging instruments are given by the bonds with maturity $t_{N-1} = 29$ and $t_N = 30$ years.

PROOF: According to Definition 3.3, we have

$$dL_t^{reb,*}(\bar{\phi}^{(i)}) = dD^*(t,t_i) - \left(\bar{\phi}_t^{(i,N-1)}\, dD^*(t,t_{N-1}) + \bar{\phi}_t^{(i,N)}\, dD^*(t,t_N)\right).$$

With respect to the (true) P^*–dynamic of a bond with maturity $\bar{t}$ it holds

$$dD^*(t,\bar{t}) = D^*(t,\bar{t})\sigma_{\bar{t}}(t)\, dW^*(t).$$

Together with Lemma 3.7 where λ is replaced by the assumed $\tilde{\lambda}$ it immediately follows

$$dL_t^{reb,*}(\bar{\phi}^{(i)}) = D^*(t,t_i)\left[\sigma_{t_i}(t) - \left(\tilde{\lambda}_t^{(i)}\sigma_{t_{N-1}}(t) + (1-\tilde{\lambda}_t^{(i)})\sigma_{t_N}(t)\right)\right] dW^*(t).$$

The first part of the above lemma follows with

$$\begin{aligned} & \sigma_{t_i}(t) - \left(\tilde{\lambda}_t^{(i)}\sigma_{t_{N-1}}(t) + (1-\tilde{\lambda}_t^{(i)})\sigma_{t_N}(t)\right) \\ = & \left(\lambda_t^{(i)}\sigma_{t_{N-1}}(t) + (1-\lambda_t^{(i)})\sigma_{t_N}(t)\right) - \left(\tilde{\lambda}_t^{(i)}\sigma_{t_{N-1}}(t) + (1-\tilde{\lambda}_t^{(i)})\sigma_{t_N}(t)\right) \\ = & \left(\tilde{\lambda}^{(i)}(t) - \lambda^{(i)}(t)\right)\left(\sigma_{t_N}(t) - \sigma_{t_{N-1}}(t)\right). \end{aligned}$$

Finally, notice that

$$Var_{P^*}[L_{t_i}^{tot,*}(\bar{\phi})] = \int_0^{t_i} E_{P^*}\left[(D(u,t_i))^2\right]\left(g^{(i)}(u)\right)^2 du.$$

□

REMARK 3.9. Assume that the "true" and "assumed" model are each given by a Hull and White model, i.e. the "true" model satisfies Equation (8) and the assumed one is given by Equation (10), i.e. the model risk is described by a misspecification of the model parameters. Notice that, in the case that both models are calibrated to the same zero bond curve at t_0, the synthesizing costs, given in Lemma 3.8, are independent of the assumed spot rate volatility $\tilde{\sigma}^r$, i.e.

$$g^{(i)}(u) = \frac{\sigma^r}{a}\left(\frac{e^{-\tilde{a}t_i} - e^{-\tilde{a}t_N}}{e^{-\tilde{a}t_{N-1}} - e^{\tilde{a}t_i}} - \frac{e^{-at_i} - e^{-at_N}}{e^{-at_{N-1}} - e^{at_i}}\right) e^{au}\left(e^{-at_N} - e^{-at_{N-1}}\right).$$

This is explained by the assumption that the bond volatilities which are assumed by the hedger are proportional in the spot rate volatility $\tilde{\sigma}^r$. In consequence, the weighting function $\tilde{\lambda}$ does not depend on $\tilde{\sigma}^r$, c.f Equation (14), such that the model risk is given in terms of the assumed speed of mean reversion parameter $\tilde{a}$. Figure 2 illustrates the variance of the synthesizing costs.[9] Observe that the variance is a convex function in $\tilde{a}$ where the minimum (of zero) is obtained for $\tilde{a} = a$. Finally, it is worth emphasizing that $g^{(i)} > 0$ ($g^{(i)} < 0$) for $\tilde{a} > a$ ($\tilde{a} < a$) and $g^{(i)} = 0$ iff there is no misspecification of the interest rate model.[10]

4. Hedging with subsets of Bonds

Insurance risk implies that no perfect hedge (a self–financing and at the same time duplicating strategy) can be found to hedge the considered contract. A non–vanishing cost process arises. Furthermore, we discuss how the cost process evolves when model risk is taken into consideration.

Model risk associated with the insurance risk can be interpreted as the risk that the number of the insured who survive a specified time horizon differs from that calculated for the risk management purposes. This might happen because of an insufficient number of homogenous insurance takers, i.e. the law of large numbers does not apply to small cohorts. Further, it might result from a wrong choice of mortality law, either on purpose or due to model uncertainty. Please note that the payment structure is based on the realized time of death/survival of the insured. Thus, the effectiveness of a (fixed income) hedging strategy depends on the assumption that the realized structure coincides with the assumed one. Apparently, a deviation causes a an extra hedging error.

Another factor which influences the effectiveness of a hedging strategy is the availability of the hedging instruments. Also the set of admissible strategies depends on the available instruments. Hedging is trivial if the hedging instruments include the claim to be hedged. For instance, a deterministic payoff can be statically hedged if the corresponding zero bonds are traded. In fact, in the ideal case of a large insurance cohort combined with a known mortality law, the payoff structure of the contract under consideration can be hedged in a model–independent way if there is a zero bond for every payment date available. In the following, we call such a set of zero bonds the natural hedging instruments, i.e. the set of zero coupon bonds with maturities $t_1, \ldots, t_N$, $\{D(., t_1), \ldots, D(., t_N)\}$.[11] Φ is used to denote the set of hedging

[9] With respect to the computation, it is convenient to notice that

$$\int_0^u \sigma_{t_i}(s)\sigma_{t_j}(s)\,ds = \left(\frac{\sigma^r}{a}\right)^2\left(u - \frac{e^{-at_i} + e^{-at_j}}{a}\left(e^{au} - 1\right) + \frac{e^{-a(t_i+t_j)}}{2a}\left(e^{2au} - 1\right)\right).$$

[10] Notice that $\sigma_{t_N}(u) - \sigma_{t_{N-1}}(u)$ is positive and $\lambda(a)$ is increasing in a.

[11] This is motivated by the contract value given in Proposition 2.1.

strategies associated with these natural hedging instruments:

$$\Phi = \left\{ \phi = (\phi^{(1)}, \ldots, \phi^{(N)}) \middle| \phi \text{ is trading strategy with } V(\phi) = \sum_{j=1}^{N} \phi^{(i)} D(., t_i) \right\}.$$

However, due to liquidity constraints in general or transaction costs in particular, it is not possible or convenient to use all bonds for the hedging purpose. It is necessary to restrict the class of strategies Φ and the relevant subset is denoted by $\Psi \subset \Phi$. Obviously, independent of the optimality criterion which is used to construct the hedging strategy, the effectiveness of the optimal strategy $\psi^* \in \Psi$ can be improved if there are additional hedging instruments available. We stick to the one–factor term structure models and set

$$\Psi = \left\{ \psi \in \Phi \middle| \psi = (0, \ldots, 0, \psi^{(N-1)}, \psi^{(N)}) \right\}.$$

Remark 4.1. Recall that the assumption of a one–factor term structure model implies that two bonds are enough to synthesize any bond with maturity $\{t_1, \ldots, t_N\}$. Since the bonds cease to exist as time goes by, it is simply convenient to use the two bonds with the longest time to maturity. It might be more practical to use two hedging instruments which differ much from each other, e.g., two bonds whose maturities are not very close, like t_1 and t_N–bond. However, which two bonds to choose is not the concern of the paper and those who are interested in this topic please refer to Dudenhausen and Schlögl (2002).

4.1. **Assumed Risk–Minimizing Strategies.** Unless there is a perfect hedge, hedging strategies can only be compared according to meaningful optimality criteria. First, we consider the mean–self–financing requirement. Obviously, this (and all the other optimality criteria) depend on the measure under consideration. Abstracting from model risk together with the simplifying assumption that the real world measure P and P^* coincide, the criteria are formulated with respect to P^*. The following proposition indicates that the mean–self–financing feature is not enough to give a meaningful strategy.

Proposition 4.2. *For $\phi \in \Phi$ and a claim with payoff $C_T = \bar{G}_T$ at the random time $T = \min\left\{t_N, t_{n^*(\tau^x)+1}\right\}$, it holds*

$$E_{P^*}\left[L_T^{tot,*}(\phi)\right] = C_{t_0} - V_{t_0}(\phi)$$

where C_{t_0} is given as in Proposition 2.1.

Proof: Due to the fact that T is bounded above by t_N and that C^* and I^* are P^*–martingales, Lemma 3.6 together with optional stopping theorem leads to

$$E_{P^*}\left[L_T^{tot,*}(\phi)\right] = E_{P^*}[C_T^*] - (V_{t_0}^*(\phi) + E_{P^*}[I_T^*(\phi)]) = C_{t_0}^* - V_{t_0}^*(\phi).$$

□

The above proposition states that any strategy where the initial investment coincides with the price of the claim to be hedged is mean–self–financing. Therefore, it is necessary to use an additional optimality criterion. In the following, we consider a conventional hedging criterion used in the incomplete market, i.e., the considered hedging strategies are risk–minimizing if model risk is neglected. Since a strategy which is risk–minimizing with respect to the measure P is also P–mean–self–financing, risk–minimizing feature contains mean–self–financing feature. In the analysis of risk–minimizing hedging, we look for an admissible strategy which minimizes the variance of the future costs at any time $t \in [0,T]$. Along the lines of Møller (1998), we derive the risk–minimizing hedging strategy for both cases: the class of natural hedging instruments Φ and the subset Ψ. In fact, the derivation of the strategies is given with respect to the (assumed) measure $\tilde{P}^*$. In a second step, the efficiency with respect to the true measure P^* is analyzed.

The construction of the strategies ϕ (strategy in all bonds) and ψ (strategy in the last two bonds) is based on the assumed contract value of the claim to be hedged.

Proposition 4.3 (Assumed contract value). *In our arbitrage–free model setup, the contract value at time $t \in [0,\tau^x]$ is given by*

$$\tilde{C}_t = \left[\sum_{j=n^*(t)+1}^{N-1} \bar{G}_{t_j}\, D(t,t_j)\ {}_{t_{j-1}-t|\Delta t}\tilde{q}_{x+t} + \bar{G}_{t_N}\, D(t,t_N)\ \underbrace{\left({}_{t_{N-1}-t|\Delta t}\tilde{q}_{x+t} + {}_{t_N-t}\tilde{p}_{x+t}\right)}_{={}_{t_{N-1}-t}\tilde{p}_{x+t}} \right].$$

Proof: Using standard theory of pricing by no arbitrage implies that the (assumed) contract value at t $(0 \le t < T)$ is given by the expected discounted payoff under the martingale measure $\tilde{P}^*$, i.e.,

$$\tilde{C}_t = E_{\tilde{P}^*}[e^{-\int_t^T r_u du}\bar{G}_T|\mathcal{F}_t].$$

In particular, the above proposition is a straightforward generalization of Proposition 2.1.

□

The above proposition immediately motivates a duplication strategy on the set $\{t \le \tau^x\}$. Prior to the death time τ^x, the contract value (at time t) can be synthesized by a trading strategy which consists of bonds with maturities t_i $(i = n^*(t)+1,\ldots,N)$.

Assumption 4.4. *Throughout the following, we assume that the insurance company notices the death of the customer only when no further premium is paid by the insured.*

By this, we indeed assume that the strategy proceeds on the time set $t \in]\tau^x, T]$ in the same way as on the set $t \in [0,\tau^x]$.

Proposition 4.5. *Let ϕ be the $\tilde{P}^*$–risk– (variance–) minimizing trading strategy in Φ and ϕ_t $(t \in [0,T])$ takes the form of:*

$$\begin{aligned} \phi_t^{(i)} &= 1_{\{t \le t_i\}}\, \bar{G}_{t_i}\ {}_{t_{i-1}-t|\Delta t}\tilde{q}_{x+t} \qquad i = 1,\cdots,N-1 \\ \phi_t^{(N)} &= \bar{G}_{t_N}\ {}_{t_{N-1}-t}\tilde{p}_{x+t}. \end{aligned}$$

Proof: It is easily seen that $V_{t_0}(\phi)$ and the (assumed) contract value $\tilde{C}_{t_0}$ which is given in Proposition 4.3 coincide. Together with Proposition 4.3 it follows that ϕ is $\tilde{P}^*$–mean–self–financing. Since the stochastic interest rate risk can be eliminated by trading in all "natural" zero coupon bonds, Møller's (1998) results (where the independence of mortality and market risk is assumed) can be adopted here. Combining Theorems 4.4 and 4.9 of Møller (1998) gives the risk–minimizing hedging strategy for an endowment insurance which is a mixture of a pure endowment and a term insurance contract.

□

It is noticed that the number of bonds decreases as time goes by. At time t, only bonds with maturities later than $n^*(t)$ are traded, i.e., the hedger buys $\bar{G}_{t_i} \cdot {}_{t_{i-1}-t|\Delta t}\tilde{q}_{x+t}$ units of $D(t,t_i)$ and $\bar{G}_{t_N}\, {}_{t_{N-1}-t}\tilde{p}_{x+t}$ units of $D(t,t_N)$.

Remark 4.6. The advantage of the strategy given in Proposition 4.5 is that it does not depend on the assumed interest rate model but only on the observed bond prices. In particular, the strategy is also P^*–risk–minimizing if there is no model risk concerning the mortality law, i.e. if ${}_{t_{i-1}-t|\Delta t}\tilde{q}_{x+t} = {}_{t_{i-1}-t|\Delta t}q_{x+t}$ for all $i = 1, \dots, N-1$ and ${}_{t_{N-1}-t}\tilde{p}_{x+t} = {}_{t_{N-1}-t}p_{x+t}$.

Due to the reason named above, it is not unrealistic that only a subset of bonds are available for hedging purposes and the following proposition gives the risk–minimizing strategy associated with the last two bonds.

Proposition 4.7. *Let ψ denote the $\tilde{P}^*$–risk– (variance–) minimizing trading strategy with respect to the set of trading strategies $\Psi \subset \Phi$, then it holds (for $t \in [0,T]$)*

$$\begin{aligned} \psi_t^{(N-1)} &= 1_{\{\tau^x \geq t\}} \left(1_{\{t \leq t_{N-2}\}} \sum_{i=n^*(t)+1}^{N-2} \phi_t^{(i)} \bar{\phi}_t^{(i,N-1)}(\tilde{\lambda}) + 1_{\{t \leq t_{N-1}\}} \phi^{(N-1)} \right) \\ \psi_t^{(N)} &= 1_{\{\tau^x \geq t\}} \left(1_{\{t \leq t_{N-2}\}} \sum_{i=n^*(t)+1}^{N-2} \phi_t^{(i)} \bar{\phi}_t^{(i,N)}(\tilde{\lambda}) + \phi^{(N)} \right) \end{aligned}$$

where ϕ is defined according to Proposition 4.5 and $\bar{\phi}$ is defined along the lines of Lemma 3.7.

Proof: The value of the strategy ϕ specified in Proposition 4.5 is described by

$$V_t(\phi) = \sum_{i=1}^{N} \phi_t^{(i)} D(t,t_i) = \sum_{i=1}^{N-2} \phi_t^{(i)} D(t,t_i) + \phi_t^{(N-1)} D(t,t_{N-1}) + \phi_t^{(N)} D(t,t_N).$$

Along the lines of Lemma 3.7, the positions in all bonds with maturity t_i ($i = 1, \dots, N-2$) can be replaced by positions $\bar{\phi} = \left(\bar{\phi}^{(N-1)}, \bar{\phi}^{(N)}\right)$, c.f. Equations (11) and (12), in the bonds with maturity t_{N-1} and t_N such that

$$\begin{aligned} V_t(\phi) &= \left(\sum_{i=n^*(t)+1}^{N-2} \phi^{(i)} \frac{D(t,t_i)}{D(t,t_{N-1})} \tilde{\lambda}_1^{(i)}(t) + \phi_t^{(N-1)} \right) D(t,t_{N-1}) \\ &\quad + \left(\sum_{i=n^*(t)+1}^{N-2} \phi^{(i)} \frac{D(t,t_i)}{D(t,t_N)} \tilde{\lambda}_2^{(i)}(t) + \phi_t^{(N)} \right) D(t,t_N), \end{aligned}$$

where $\tilde{\lambda}_1^{(i)}(t) := \tilde{\lambda}(t,t_i,t_{N-1},t_N)$ and $\tilde{\lambda}_2^{(i)}(t) := 1 - \tilde{\lambda}(t,t_i,t_{N-1},t_N)$ are defined according to Equation (15). Notice that $V_t\left(\bar{\phi}\right) = D(t,t_i)$ $\tilde{P}$–almost surely ($\forall \tilde{P}$ equivalent to $\tilde{P}^*$) implies $Var_{\tilde{P}^*}[L_T^*(\psi)] = Var_{\tilde{P}^*}[L_T^*(\phi)]$ (alternatively, this can be deducted from Proposition 4.10). This together with $\Psi \subset \Phi$ ends the proof.

The above proposition states that ψ is constructed as follows. The first step is the same as in Proposition 4.5, i.e. one considers the risk-minimizing strategy ϕ where all the bonds are traded, without the introduction of model risk on the mortality side. In the second step, the unavailable hedging instruments $D(.,t_1),\dots,D(.,t_{N-2})$ are each replaced by a strategy consisting of the traded zero bonds $D(.,t_{N-1})$ and $D(.,t_N)$, c.f. Lemma 3.7. The underlying synthesizing strategy relies on the assumed distribution for the remaining life and the assumed model for the interest rate. If the assumed dynamics deviate from the true ones, the resulting strategy is only risk–minimizing with respect to the measure $\tilde{P}^*$ but not under the (true) measure P^*.

4.2. **Effectiveness under model risk.** Before we give the expected costs and the variance of the costs under the (true) measure P^*, it is necessary to consider (additional) costs which are caused by using a periodic premium principle instead of a single premium paid today. Notice that premia are paid as long as the insured lives, i.e. periodically, while the hedging strategy concerning the asset side is implemented today. As a consequence, the future (expected) premium inflows are to be pre–financed. To be more precise, the implementation of the above strategies is based on taking a credit at t_0. Since the initial value of the hedging strategies is given by the (assumed) expected value of the premium inflows, the insurer must borrow the amount $\sum_{i=1}^{N-1} A\,{}_{t_i}\tilde{p}_x\, D(t_0,t_i)$. The underpinning strategy for this is to sell $A\,{}_{t_i}\tilde{p}_x$ bonds with maturity t_i $(i=1,\dots,t_{N-1})$. Obviously, the premium amount which is actually paid by the insured does not necessarily coincide with the amount which is needed to pay back the credit. This difference leads to costs which shall be added to the costs resulting from the asset side. Formally, these costs can be represented as a sequence of cash flows. The insurer must, at each time t_i $(i=1,\dots,t_{N-1})$, pay back the amount $A\,{}_{t_i}\tilde{p}_x$, i.e. independent of whether the insured survives. In summary, the additional discounted costs $L_T^{add,*}$ associated with the *borrowing strategy* are

$$L_T^{add,*} \;=\; \sum_{i=1}^{N-1} e^{-\int_0^{t_i} r_u\,du} A\left({}_{t_i}\tilde{p}_x - 1_{\{\tau^x > t_i\}}\right). \tag{22}$$

Proposition 4.8 (Expected total discounted hedging costs). *Let L_T^* denote the discounted total costs from both the asset and the liability side, i.e. $L_T^* = L_T^{tot,*} + L_T^{add,*}$. ϕ (ψ) denotes the strategy given in Proposition 4.5 (4.7). For $w \in \{\phi,\psi\}$ it holds (under model risk)*

$$E_{P^*}[L_T^*(w)] = E_{P^*}[L_T^{tot,*}(w)] + E_{P^*}[L_T^{add,*}(w)]$$

where

$$E_{P^*}[L_T^{tot,*}(w)] \;=\; D(t_0,t_N)\bar{G}_{t_N}({}_{t_N}p_x - {}_{t_N}\tilde{p}_x) + \sum_{j=1}^{N-1}({}_{t_{j-1}|\Delta t}q_x - {}_{t_{j-1}|\Delta t}\tilde{q}_x)D(t_0,t_j)\bar{G}_{t_j}$$

and $$E_{P^*}[L_T^{add,*}(w)] \;=\; \sum_{i=1}^{N-1} D(t_0,t_i)A\left({}_{t_i}\tilde{p}_x - {}_{t_i}p_x\right).$$

Proof: This proposition is an immediate consequence of Propositions 4.2, i.e.

$$\begin{aligned} E_{P^*}[L_T^{tot,*}(w)] &= C_{t_0} - V_{t_0}(w) \\ &= D(t_0,t_N)\bar{G}_{t_N}\,{}_{t_N}p_x + \sum_{j=1}^{N-1} {}_{t_{j-1}|\Delta t}q_x D(t_0,t_j)\bar{G}_{t_j} - V_{t_0}(w). \end{aligned}$$

Notice that

$$V_{t_0}(\phi) = \sum_{i=1}^{N} \phi^{(i)} D(t_0, t_i) = \sum_{i=N-1}^{N} \psi^{(i)} D(t_0, t_i) = V_{t_0}(\psi) = \tilde{C}_{t_0}$$

where $\tilde{C}_{t_0}$ is given in Proposition 4.3.

□

It is noted that the same value results for the expected discounted costs of two strategies with the same initial investment under the martingale measure P^*. This implies that the expected discounted hedging costs do not depend on the set of bonds used for hedging purpose. In addition, Proposition 4.8 states that, the sign of the expected value depends on the relation of assumed and true mortality parameters but not on the interest rate parameters. Notice that a negative sign implies a superhedge in the mean, while positive costs refer to a subhedge in the mean, c.f. Definition 3.5. However, the optimality of a strategy is measured by the risk–minimization criterion, i.e. it is interesting to analyze how the $\tilde{P}^*$–variance–minimizing strategies ϕ and ψ react to model risk. Therefore, a performance measure which presents itself is the variance of the strategies under the (true) measure P. Recall that this measure is assumed to coincide with the martingale measure P^*, i.e. the risk premium is assumed to be zero. The following proposition gives the additional variance which is caused by a deviation from the set of natural hedging instruments. First, consider the following useful lemma:

Lemma 4.9. *Let ϕ and ψ be given as in Proposition 4.5 and 4.7, then it holds*

$$I_t^*(\psi) - I_t^*(\phi) = -\sum_{i=1}^{N-2} \int_0^{\min\{t,t_i\}} \phi_u^{(i)} D^*(u, t_i)\, g^{(i)}(u)\, dW_u^*, \tag{23}$$

where $g^{(i)}$ is defined according to Proposition 3.8, c.f. Equation (20). In particular, without model misspecification, it holds $I_t^*(\psi) = I_t^*(\phi)$.

Proof: The proof is given in Appendix A.1.

□

Notice that the difference in the gain processes $I_t^*(\psi) - I_t^*(\phi)$ coincides with the (weighted) sum of the synthesizing costs. For $i = 1, \ldots, N-2$, $\phi^{(i)}$ gives the number of zero bonds with maturity t_i which are synthesized by positions in the zero bonds with maturity t_{N-1} and t_N. Accordingly, the above lemma results in the weighted synthesizing costs of Lemma 3.7.

Proposition 4.10 (Variance difference). *Let $AV_T^{tot,*}$ denote the difference between the discounted cost variances associated with the strategy ϕ in all bonds and the strategy ψ in the restricted set of bonds, i.e.*

$$AV_T^{tot,*} := Var_{P^*}[L_T^{tot,*}(\psi)] - Var_{P^*}[L_T^{tot,*}(\phi)],$$

then it holds

$$AV_T^{tot,*} = V_T^{tot,*} + 2CV_T^{tot,*} \tag{24}$$

$$\text{where } V_T^{tot,*} = \sum_{i=1}^{N} a_i \; E_{P^*}[\big(I_{t_i}^*(\psi) - I_{t_i}^*(\phi)\big)^2],$$

$$CV_T^{tot,*} = \sum_{i=1}^{N} a_i \; E_{P^*}\left[\big(I_{t_i}^*(\phi) - \bar{G}_{t_i}^*\big)\big(I_{t_i}^*(\psi) - I_{t_i}^*(\phi)\big)\right]$$

and $a_i := {}_{t_{i-1}|\Delta t}q_x$ *(*$i = 1, \ldots, N-1$*),* $a_N := {}_{t_{N-1}|\Delta t}q_x + {}_{t_N}p_x$.

Proof: Notice that

$$L_T^*(\psi) = \sum_{i=1}^{N} 1_{\{t_{i-1}<\tau^x\leq t_i\}}\left(\bar{G}_{t_i}^* - V_{t_0}^*(\psi) + I_{t_i}^*(\psi)\right) + 1_{\{\tau^x\geq t_N\}}\left(\bar{G}_{t_N}^* - V_{t_0}^*(\psi) + I_{t_N}^*(\psi)\right).$$

In particular, we have

$$E_{P^*}\left[\left(L_T^{tot,*}(\psi)\right)^2\right] = \sum_{i=1}^{N} a_i E_{P^*}\left[\left(\bar{G}_{t_i}^* - V_{t_0}^*(\psi) + I_{t_i}^*(\psi)\right)^2\right]$$

The rest of the proof is immediately obtained by the observations $V_{t_0}(\psi) = V_{t_0}(\phi)$ and $E_{P^*}\left[L_T^{tot,*}(\psi)\right] = E_{P^*}\left[L_T^{tot,*}(\phi)\right]$, i.e.

$$\mathrm{Var}_{P^*}[L_T^{tot,*}(\psi)] - \mathrm{Var}_{P^*}[L_T^{tot,*}(\phi)] = E_{P^*}\left[\left(L_T^{tot,*}(\psi)\right)^2\right] - E_{P^*}\left[\left(L_T^{tot,*}(\phi)\right)^2\right]$$

□

The above proposition states that the variance of the strategy ψ is given in terms of the variance of ϕ plus the variance of the (sum of weighted) synthesizing costs and a covariance part. The covariance part itself consists of two components, i.e. the covariance of the trading gains of ϕ and the synthesizing costs and the covariance of $\bar{G}$ and the synthesizing costs. It is important to note that the last part is not equal to zero because of a time lag between the payoff $\bar{G}_{t_i}^*$ and the synthesizing costs of bonds which mature prior to t_i, c.f. Lemma 4.9. Again, an explicit calculation is given with respect to a Gauss–Markov zero bond model, c.f. Equation (6).

Lemma 4.11. *If the the interest rate dynamics are given by a Gauss–Markov Heath, Jarrow and Morton (1992) model, c.f. Equation (6), then* $AV_T^{tot,*}$ *is given by Equation (24) with*

$$V_T^{tot,*} = \sum_{i=1}^{N} a_i \sum_{j=1}^{N-2}\sum_{k=1}^{N-2}\int_0^{\min\{t_i,t_j,t_k\}} \phi_u^{(j)}\phi_u^{(k)}\, g^{(j)}(u)\, g^{(k)}(u) E_{P^*}\left[D^*(u,t_j)D^*(u,t_k)\right] du$$

$$CV_T^{tot,*} = \sum_{i=1}^{N} a_i\, \bar{G}_{t_i} 1_{\{t_i\leq t_{N-2}\}} \sum_{j=1}^{\min\{i-1,N-2\}} \int_0^{t_j} \phi_u^{(j)}\sigma_{t_i}(u) g^{(j)}(u) E_{P^*}\left[D^*(u,t_i)D^*(u,t_j)\right] du$$

$$- \sum_{i=1}^{N} a_i \sum_{j=1}^{N}\sum_{k=1}^{N-2}\int_0^{\min\{t_i,t_j,t_k\}} \phi_u^{(j)}\phi_u^{(k)}\sigma_{t_j}(u)\, g^{(k)}(u) E_{P^*}\left[D^*(u,t_j)D^*(u,t_k)\right] du$$

Proof: The proof is given in the appendix, c.f. A.2.

□

Intuitively, it is clear that the variances of both strategies coincide if the assumed and true interest models coincide. Formally, this is due to the features of the function g, c.f. Remark 3.9. In particular, it holds $g^{(i)} = 0$ if there exists no model risk concerning the interest rate dynamic. With respect to the Hull–White context, the sign of g is solely determined by the sign of the difference between the true and assumed speed of mean reversion parameters a and $\tilde{a}$. However, the sign of the covariance part crucially depends on the relation of the true and assumed mortality law. So does the sign of AV. Unfortunately, it is not possible to isolate the effect for arbitrary contract designs and mortality laws. [12] However, the next section gives some illustrative examples of the combined effects of mortality and interest rate misspecifications.

5. Illustration of results

As stated in the introduction, mortality *misspecification* is often deliberately introduced by the insurance company to achieve safety margins. If exposed to longevity risk, premiums are often calculated according to a life table based on higher survival probabilities as the table used for products with exposure to mortality risk. A straightforward illustration of shifting the life–expectancies can be achieved by shifting the age. The assumed force of mortality $\tilde{\mu}$ results from the true force of mortality by shifting the age, i.e. the true force of mortality is μ_{x+s} and the assumed one is

$$\tilde{\mu}_{x+s} = \mu_{\tilde{x}+s} \quad \forall x, s \geq 0.$$

The mortality rate μ is chosen according to the Makeham law where the parameters are given as in the example of Sec. 1.

With respect to the true and assumed interest rate models, we stay in a one–factor Hull–White model. In particular, the initial term structure of interest is assumed to be flat with $r_0 = 0.05$.

5.1. **Expected total costs.** Figures 3 and 4 demonstrate how the death and survival probability, i.e., ${}_{t_{j-1}|\Delta t}q_x$ and ${}_tp_x$ change with the age x. With the change of x, the death and survival probability demonstrate a parallel shift. If the true age of the customer is 40, then an assumed age of 50 leads to an overestimation of the death probability and an assumed age of 30 results in an underestimation of the death probability. Of course the survival probability has exactly a reversed trend.

How the expected discounted total costs with the assumed age $\tilde{x}$ for varying t_N is depicted in Figure 5. It is noticed that, for a given t_N, the expected discounted total cost exhibits a negative relation in $\tilde{x}$. The higher $\tilde{x}$, the lower the expected total costs. It is observed that, independent of the set of hedging instruments (bonds), the hedger achieves profits in mean (negative expected discounted cost) if he overestimates the death probabilities. Hence, negative expected discounted costs result when true x is smaller than the assumed one. Reversed effects are observed when the insurer underestimates the death probability. Here, a real age of 35 is taken and it is observed that for $\tilde{x} = 40$, 45, the expected costs have negative values, and for $\tilde{x} = 25$, 30, the expected costs exhibit positive values. When the true age coincides

[12] Recall that the number of bonds which are held for hedging purpose depends on the payoff structure, the assumed mortality law and the assumed interest rate dynamic.

Death and Survival Probabilities for Varying x Values

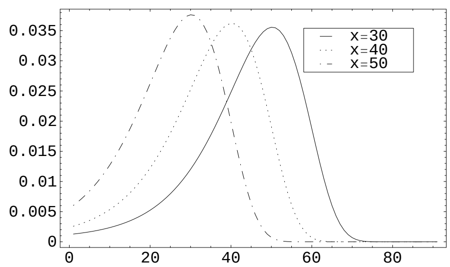

FIGURE 3. ${}_{t_{j-1}|\Delta t}q_x$ for $x = 30$, 40, 50.

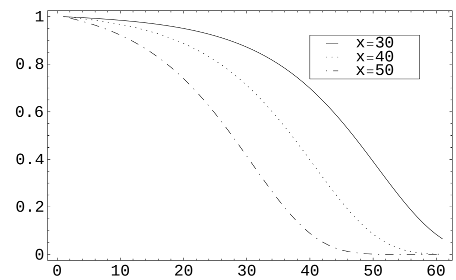

FIGURE 4. ${}_tp_x$ for $x = 30$, 40, 50.

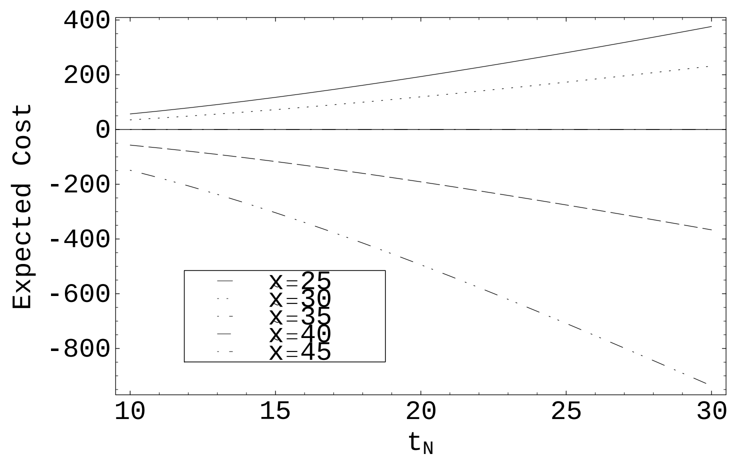

FIGURE 5. Expected cost for varying t_N with $\tilde{x} = 25$, 30, 35, 40, 45 and the real $x = 35$. The rest parameters are the same as in Table 1.

with the assumed one, the considered strategy is mean–self–financing because the expected discounted cost equals zero. Moreover, the longer the period of the contract t_N, the more substantial effects the mortality risk has on the expected discounted total costs. It is quite obvious because a longer time period leads to more uncertainty concerning the mortality risk.

5.2. **Variance difference $AV_T^{tot,*}$.** According to Proposition 4.10, the variance difference $AV_T^{tot,*}$ can be decomposed into two parts, a variance part V and two times a covariance part CV. For illustration purpose, it is even convenient to consider three parts, i.e. to split the covariance CV into $CV1$ and $CV2$. In summary,

$$AV = V + 2\,(CV1 + CV2) \tag{25}$$

where V is the variance of synthesizing costs, $CV1$ the covariance between the trading gains of ϕ and the synthesizing costs, and $CV2$ is minus times the covariance between $\bar{G}$ and the synthesizing costs. Table 2 illustrates the results achieved in our illustration example.

First, consider the variance of synthesizing costs. Obviously, the lowest value is zero. This value is only achieved in the case without misspecification of the interest rate model. With respect to the Hull White model setup, this case is given if the true and assumed speed of mean reversion levels coincide, i.e. $\tilde{a} = a$. The larger the extent of misspecification, i.e. the higher the distance $|\tilde{a} - a|$, the higher is the variance of the synthesizing costs. However, the magnitude of this variance is pretty small, in particular in relation to the total variance difference. Therefore, the variance of synthesizing costs does not play a crucial role for $AV_T^{tot,*}$.

Second, both the covariance components can demonstrate either positive or negative sign. This is due to the sign of $g^{(i)}$. Along the lines of Lemma 4.11, $CV2$ has the same sign as $g^{(i)}$, whereas $CV1$ owns an opposite sign as $g^{(i)}$. Hence, these two covariance components always own opposite signs. More specifically, a $\tilde{a}$ which is smaller than a leads to a negative $g^{(i)}$, and consequently a negative value for $CV2$ and a positive one for $CV1$.[13] Observe that $CV2$ plays a deciding role in the magnitude of $AV_T^{tot,*}$, i.e. the sign of $AV_T^{tot,*}$ coincides with that of $CV2$. In the case that $\tilde{a} > a$ (an underestimation of the volatilities of the zero coupon bonds), strategy ψ owns a higher variance than strategy ϕ, whereas in the case that $\tilde{a} < a$ (an overestimation of the volatilities of the zero coupon bonds), strategy ψ can even achieve a smaller variance than ϕ. For $\tilde{a} = a$ (the case of no model risk associated with the interest rate), strategy ϕ and ψ make no difference to the variance of the total cost.

Third, mortality risk starts to have impacts on $AV_T^{tot,*}$ when there exists model risk associated with the interest rate. A positive relation between $|AV_T^{tot,*}|$ and $\tilde{x}$ is observed. In other words, compared to the case that $\tilde{x} < x$, more sophisticated hedging decisions are required in the case of overestimation of death probability ($\tilde{x} > x$). This is due to the more pronounced effects of $\tilde{x}$ on $AV_T^{tot,*}$. For instance, the insurer can find a much less riskier strategy ψ (then ϕ), when he combines an overestimation of the death probability ($\tilde{x} > x$) with an overestimation of the volatilities of the zero coupon bonds ($\tilde{a} < a$). On the contrary, if the insurer invests in the strategy ψ under a scenario $\tilde{x} > x$ combined with $\tilde{a} > a$, he indeed involves himself in a much riskier financial position. This exactly highlights the importance of a *combined* (parameter) choice of the interest rate and mortality model. How the variance difference $AV_T^{tot,*}$ evolves with the $\tilde{a}$ for difference $\tilde{x}$-values is plotted in Figure 6. The higher the $\tilde{x}$, the more pronounced effects of $\tilde{a}$ on the $AV_T^{tot,*}$ can be observed.

6. Conclusion

The risk management of an insurance company must take into account model risk. The uncertainty about the true model concerns the insurance typical risk and the market risk. Both mortality and interest rate misspecification are unavoidable in practice. In comparison with the market risk, the uncertainty about the life expectancy is low. However, it turns out that, in combination with interest rate risk, even a small difference between assumed and true mortality law has a great

[13] See also Remark 3.9.

Decomposition of variance difference AV_T

$\tilde{a}$	$\tilde{x}=25$				$\tilde{x}=35$				$\tilde{x}=45$			
	V	$CV1$	$CV2$	AV_T	V	$CV1$	$CV2$	AV_T	V	$CV1$	$CV2$	AV_T
0.150	0.04132	0.859164	-364.508	-727.256	0.247439	2.06094	-915.446	-1826.52	1.447670	4.97346	-2343.96	-4676.52
0.155	0.02842	0.708994	-301.794	-602.141	0.170192	1.70098	-757.966	-1512.36	0.995767	4.10459	-1940.29	-3871.38
0.160	0.01802	0.561690	-239.882	-478.623	0.107886	1.34779	-602.490	-1202.18	0.631244	3.25213	-1541.95	-3076.76
0.165	0.01004	0.417194	-178.760	-356.675	0.060109	1.00122	-448.987	-895.911	0.351714	2.41575	-1148.83	-2292.48
0.170	0.00442	0.275451	-118.413	-236.272	0.026462	0.66115	-297.425	-593.501	0.154842	1.59514	-760.856	-1518.37
0.175	0.00110	0.136405	-58.8311	-117.388	0.006553	0.32745	-147.773	-294.884	0.038346	0.78999	-377.941	-754.264
0.180	0	0	0	0	0	0	0	0	0	0	0	0
0.185	0.00107	-0.13382	58.0922	115.918	0.006430	-0.32133	145.924	291.212	0.037630	-0.77514	373.050	744.587
0.190	0.00425	-0.26510	115.458	230.390	0.025480	-0.63667	290.029	578.810	0.149113	-1.53572	741.290	1479.66
0.195	0.00948	-0.39390	172.109	343.439	0.056790	-0.94612	432.344	862.853	0.332377	-2.28205	1104.80	2205.37
0.200	0.01669	-0.52027	228.056	455.089	0.100023	-1.24982	572.900	1143.40	0.585400	-3.01440	1463.60	2921.88
0.205	0.02583	-0.64426	283.313	565.363	0.154832	-1.54788	711.723	1420.51	0.906208	-3.73306	1817.95	3629.34
0.210	0.03684	-0.76592	337.890	674.285	0.220890	-1.84041	848.844	1694.23	1.292870	-4.43832	2167.75	4327.91

TABLE 2. Decomposition of variance differences AV_T for varying $\tilde{a}$ and $\tilde{x}$ with true $a = 0.18$ and $x = 35$, where V stands for the variance of synthesizing costs, $CV1$ for the covariance between the trading gains of ϕ and the synthesizing costs and $CV2$ for the covariance between $\bar{G}$ and the synthesizing costs, i.e. $AV_T = V + 2CV1 + 2CV2$. The rest parameters are the same as in Table 1.

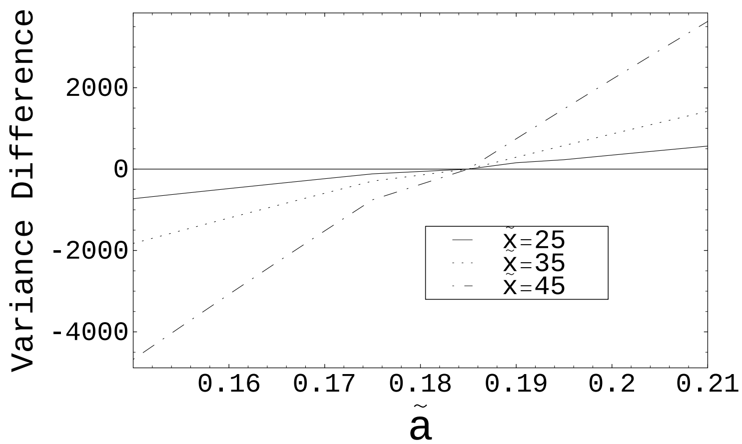

FIGURE 6. Variance difference for varying $\tilde{a}$ with $\tilde{x} = 25, 35, 45$ and the real $x = 35$. The rest parameters are the same as in Table 1.

impact on the hedging performance. The problem is caused by an interdependence of model risk concerning the interest rate dynamic and the mortality distribution. The impacts are especially severe if the insurance company restricts its risk management strategy to a subset of zero bonds. The risk implied by the restriction of hedging instruments can be measured by calculating the variance difference of the hedging costs, i.e. the difference caused by comparing the strategy using the restricted set of hedging instruments with the one using the *entire term structure*. It is shown that the sign of the variance difference depends on the combined misspecification effects, i.e. an analysis of the separate effects resulting from interest rate risk and mortality risk is not enough. Therefore, it is necessary to take into account the combined effects. In consequence, a *robust* hedging strategy can only be specified by a suitable specification of the combined interest rate and mortality model. Particularly, this argument holds also even when interest rate and mortality risk are assumed to be independent.

Appendix A. Proofs

A.1. **Proof of Lemma 4.9.** According to Proposition 4.7, it holds

$$\begin{aligned}\psi^{(N-1)} &= \sum_{i=1}^{N-2} \phi^{(i)}\bar{\phi}^{(i,N-1)} + \phi^{(N-1)}\\ \psi^{(N)} &= \sum_{i=1}^{N-2} \phi^{(i)}\bar{\phi}^{(i,N)} + \phi^{(N)}\end{aligned}$$

where $\bar{\phi}^{(i)} = \left(\bar{\phi}^{(i,N-1)}, \bar{\phi}^{(i,N)}\right)$ is given in lemma 3.7. Together with the definition of a gain process, c.f. Definition 3.2, i.e.

$$\begin{aligned}I_t^*(\phi) &= \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} dD^*(u,t_i)\\ \text{and } I_t^*(\psi) &= \int_0^t \psi_u^{(N-1)} dD^*(u,t_{N-1}) + \int_0^t \psi_u^{(N)} dD^*(u,t_N)\\ &= \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)}\bar{\phi}_u^{(i,N-1)} dD^*(u,t_{N-1}) + \int_0^t \phi_u^{(N-1)} dD^*(u,t_{N-1})\\ &\quad + \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)}\bar{\phi}^{(i,N)} dD^*(u,t_N) + \int_0^t \phi_u^{(N)} dD^*(u,t_N).\end{aligned}$$

it follows

$$\begin{aligned}&I_t^*(\psi) - I_t^*(\phi)\\ =\ &\sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)}\bar{\phi}_u^{(i,N-1)} dD^*(u,t_{N-1}) + \sum_{i=1}^{N-2} \int_0^{\min\{t,t_i\}} \phi_u^{(i)}\bar{\phi}_u^{(i,N)} dD^*(u,t_N) - \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} dD^*(u,t_i)\end{aligned}$$

Inserting $\bar{\phi}^{(i)} = \left(\bar{\phi}^{(i,N-1)}, \bar{\phi}^{(i,N)}\right)$, c.f. Lemma 3.7, where λ is replace by the assumed $\tilde{\lambda}$ gives

$$\begin{aligned}I_t^*(\psi) - I_t^*(\phi) &= \sum_{i=1}^{N-2} \int_0^{\min\{t,t_i\}} \phi_u^{(i)} \Bigg(\tilde{\lambda}^{(i)}(u) \frac{D(u,t_i)}{D(u,t_{N-1})} dD^*(u,t_{N-1})\\ &\quad + (1-\tilde{\lambda}^{(i)}(u)) \frac{D(u,t_i)}{D(u,t_N)} dD^*(u,t_N) - dD^*(u,t_i) \Bigg).\end{aligned}$$

Using the that the (true) dynamic of the discounted zero bonds is a P^*–martingale, i.e.

$$dD^*(u,t_i) = D^*(u,t_i)\sigma_{t_i}(u) dW_u^*.$$

finally gives

$$I_t^*(\psi) - I_t^*(\phi) = \sum_{i=1}^{N-2} \int_0^{\min\{t,t_i\}} \phi_u^{(i)} D^*(u,t_i) \left(\tilde{\lambda}^{(i)}(u)\sigma_{t_{N-1}}(u) + (1-\tilde{\lambda}^{(i)}(u))\sigma_{t_N}(u) - \sigma_{t_i}(u) \right) dW_u^*.$$

A.2. **Proof of Lemma 4.11.** First, notice that Lemma 4.9, immediately gives

$$\begin{aligned} & E_{P^*}[\big(I^*_{t_i}(\psi) - I^*_{t_i}(\phi)\big)^2] \\ = \;& E_{P^*}\left[\left(\sum_{j=1}^{N-2}\int_0^{\min\{t_i,t_j\}} \phi^{(j)}_u D^*(u,t_j)\, g^{(j)}(u)\, dW^*_u\right)^2\right] \\ = \;& \sum_{j=1}^{N-2}\sum_{k=1}^{N-2}\int_0^{\min\{t_i,t_j,t_k\}} \phi^{(j)}_u \phi^{(k)}_u\, g^{(j)}(u)\, g^{(k)}(u) E_{P^*}\left[D^*(u,t_j)D^*(u,t_k)\right]\, du \end{aligned}$$

Then, consider

$$\begin{aligned} & E_{P^*}[I^*_{t_i}(\phi)(I^*_{t_i}(\psi) - I^*_{t_i}(\phi))] \\ = \;& -E_{P^*}\left[\sum_{j=1}^{N}\int_0^{\min\{t_i,t_j\}} \phi^{(j)}_u\, dD^*(u,t_j) \sum_{j=1}^{N-2}\int_0^{\min\{t_i,t_j\}} \phi^{(j)}_u D^*(u,t_j)\, g^{(j)}(u)\, dW^*_u\right] \\ = \;& -\sum_{j=1}^{N}\sum_{k=1}^{N-2}\int_0^{\min\{t_i,t_j,t_k\}} \phi^{(j)}_u \phi^{(k)}_u \sigma_{t_j}(u)\, g^{(k)}(u) E_{P^*}\left[D^*(u,t_j)D^*(u,t_k)\right]\, du \end{aligned}$$

Finally, consider

$$\begin{aligned} & E_{P^*}\left[\bar{G}^*_{t_i}\left(I^*_{t_i}(\psi) - I^*_{t_i}(\phi)\right)\right] \\ = \;& -\bar{G}_{t_i} E_{P^*}\left[exp\left\{-\int_0^{t_i} r_u\, du\right\} \sum_{j=1}^{N-2}\int_0^{\min\{t_i,t_j\}} \phi^{(j)}_u D^*(u,t_j)\, g^{(j)}(u)\, dW^*_u\right] \\ = \;& -\bar{G}_{t_i} 1_{\{t_i \le t_{N-2}\}} E_{P^*}\left[exp\left\{-\int_0^{t_i} r_u\, du\right\} \sum_{j=1}^{\min\{i-1,N-2\}}\int_0^{t_j} \phi^{(j)}_u D^*(u,t_j)\, g^{(j)}(u)\, dW^*_u\right] \\ = \;& -\bar{G}_{t_i} 1_{\{t_i \le t_{N-2}\}} \sum_{j=1}^{\min\{i-1,N-2\}} E_{P^*}\left[exp\left\{-\int_0^{t_j} r_u\, du\right\} E\left[exp\left\{-\int_{t_j}^{t_i} r_u\, du\right\}\middle| \mathcal{F}_{t_j}\right]\right. \\ & \left.\int_0^{t_j} \phi^{(j)}_u D^*(u,t_j)\, g^{(j)}(u)\, dW^*_u\right] \\ = \;& -\bar{G}_{t_i} 1_{\{t_i \le t_{N-2}\}} \sum_{j=1}^{\min\{i-1,N-2\}} E_{P^*}\left[D^*(t_j,t_i)\int_0^{t_j} \phi^{(j)}_u D^*(u,t_j)\, g^{(j)}(u)\, dW^*_u\right] \end{aligned}$$

Using

$$\frac{D^*(t_j,t_i)}{D(0,t_i)} = \exp\left\{-\frac{1}{2}\int_0^{t_j} \sigma^2_{t_i}(s)\, ds + \int_0^{t_j} \sigma_{t_i}(s)\, dW^*(s)\right\} =: Z_{t_j}$$

gives

$$\begin{aligned} & E_{P^*}\left[\bar{G}^*_{t_i}\left(I^*_{t_i}(\psi) - I^*_{t_i}(\phi)\right)\right] \\ = \;& -\bar{G}_{t_i} D(0,t_i) 1_{\{t_i \le t_{N-2}\}} \sum_{j=1}^{\min\{i-1,N-2\}} E_{P^*}\left[Z_{t_j}\int_0^{t_j} \phi^{(j)}_u D^*(u,t_j)\, g^{(j)}(u)\, dW^*_u\right]. \end{aligned}$$

Finally, notice that with Girsanov's theorem we have

$$Z_{t_j} = \left(\frac{dP^{t_i}}{dP^*} \right)_{t_j}$$

such that W^{t_i} is a Brownian motion with respect to P^{t_i} and

$$dW^{t_i}(u) = dW^*(u) - \sigma_{t_i}(u)\, du.$$

In particular, it follows

$$\begin{aligned} & E_{P^*}\left[\bar{G}^*_{t_i}\left(I^*_{t_i}(\psi) - I^*_{t_i}(\phi)\right)\right] \\ = \;& -\bar{G}_{t_i} D(0,t_i) 1_{\{t_i \le t_{N-2}\}} \sum_{j=1}^{\min\{i-1,N-2\}} \int_0^{t_j} \phi_u^{(j)} E_{P^{t_i}}\left[D^*(u,t_j)\right] g^{(j)}(u) \sigma_{t_i}(u)\, du. \end{aligned}$$

where

$$E_{P^{t_i}}\left[D^*(u,t_j)\right] = D(0,t_j) \exp\left\{ \int_0^u \sigma_{t_i}(s)\sigma_{t_j}(s)\, ds \right\} = \frac{1}{D(t_0,t_i)} E_{P^*}\left[D^*(u,t_i) D^*(u,t_j)\right],$$

i.e.

$$\begin{aligned} & E_{P^*}\left[\bar{G}^*_{t_i}\left(I^*_{t_i}(\psi) - I^*_{t_i}(\phi)\right)\right] \\ = \;& -\bar{G}_{t_i} 1_{\{t_i \le t_{N-2}\}} \sum_{j=1}^{\min\{i-1,N-2\}} \int_0^{t_j} \phi_u^{(j)} \sigma_{t_i}(u) g^{(j)}(u) E_{P^*}\left[D^*(u,t_i) D^*(u,t_j)\right] du. \end{aligned}$$

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