Optimal Investment for Generalized Utility Functions

Thijs Kamma

Maastricht University

July 05, 2018

Overview

Introduction

- DB $\Rightarrow$ DC, individualized preferences
- What to pay? What to invest? How to invest? etc.
- Study considers terminal wealth problem
  - Non-Robust
  - Complete Market
  - Deterministic interest rates $\Leftrightarrow$ stochastic interest rates
  - Utility functions
- General framework allows for exploitation to more complex situations
- Results serve a more illustrative purpose in an applied context

Academic Research Question

Given a simplified DC setting, wherein the optimal wealth along with the delta hedges can be derived (analytically), what is the effect on these due to altered specifications in economic scenarios and/or utility functions (risk preferences).

Terminal Wealth Problem

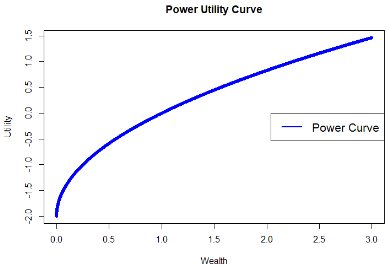
Let us introduce, merely, the following semi-mathematical notation to which we will provide the underlying interpretation subsequently

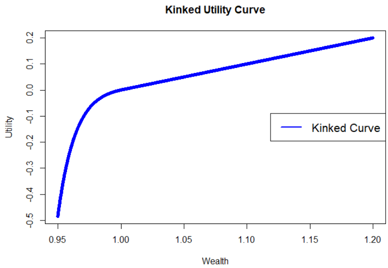
$$\begin{aligned} \max_{wealth} \quad & \text{expected happiness at T} \\ \text{s.t.} \quad & \text{expected wealth at T} = \text{initial wealth} \end{aligned} \tag{1}$$

- Maximizing expected utility over some wealth
- Constrained by a budget constraint (we cannot invest more than the yields through investing + our initial capital)
- Solution: payoff(s) at single point in time T, given risk preferences U and (some) economy via constraint
- Views: (1) participant: what to expect upon pension date; (2) pension fund: what to pay upon pension date; (3) pension fund: what/how to invest over the course of the years T

Utility Specifications

In order to model risk preferences, one generally opts for a utility function dependent on i.a. the agent's level of risk-aversion





- Constant risk-aversion specification
- Same behaviour (comparative) to wealth
- Belongs to CRRA class

- Two levels risk-aversion specification
- Behaviour comparative to (personal) benchmark level
- Belongs to CRRA class

Economic Scenarios

We assume the next two economic scenarios in order to fix ideas and study the effect of stochastic nature interest rates

Black-Scholes

- Most simple assumed economic scenario
- Trades in solely stocks (GBM) and money market account
- Stocks induce stochasticity into entire economy

Black-Scholes-Hull-White (BSHW)

- Equivalent to Black-Scholes with stochastic interest rates (Hull-White)
- Trades in stocks (GBM), non-defaultable discount bonds, and money market account
- Stocks together with interest rates induce stochasticity into entire economy

Results

The models are calibrated to real recent data with the following (additional) input, based on secondary literature

- Power utility: risk-aversion $= 0.5$ (BS) $= 2.5$ (BSHW)
- Kinked utiliity: risk-aversion $= 10$ and $= 2$, benchmark $= 1$
- Scenarios 10,000, for $T = 1$, $\rho = 0.25$

The results that we will consider today are twofold.

Optimal Wealth

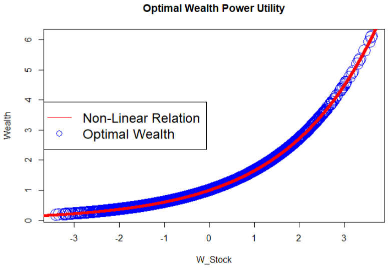
- Dynamics in a general sense; density, certainty etc.
- Sensitivity to e.g. stock or bond prices
- Answers to what to expect

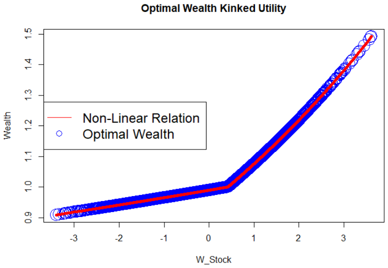
Investment Strategies

- General dynamics position (e.g. long or short)
- Measuring sensitivity to assets by definition
- Answers to what to invest

Black-Scholes Optimal Wealth

The optimal wealth associated with the power utility and kinked utility show respectively the following (plotted against uncertainty)



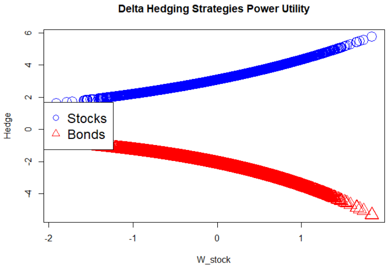


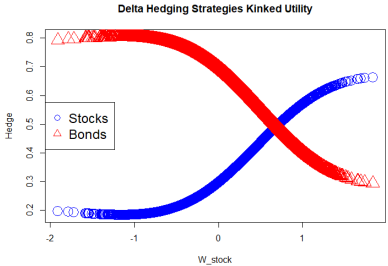
- Exponential relationship in terms of the uncertainty
- Rather widespread density of the optima
- $\mu = 1.133$, $\sigma^2 = 0.3556$

- Kinked relationship in uncertainty (prudence)
- Densely located spread of the optima
- $\mu = 1.014$, $\sigma^2 = 0.004811$

Black-Scholes Investment Strategy

The investment strategies evaluated at time $t = 0.5$, in accordance with previous graphs show us, again in terms of uncertainty





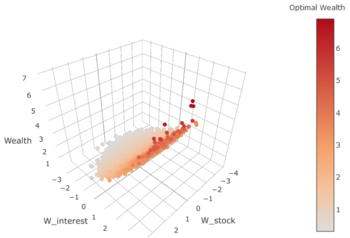
- Non-linear relationship in terms of the uncertainty
- Similar density wealth
- $\mu_S = 3.134$, $\mu_B = -2.199$, $\sigma_S^2 = 0.2909$, $\sigma_S^2 = 0.3150$

- Quasi-kinked relationship uncertainty (prudence)
- Similar density wealth
- $\mu_S = 0.3212$, $\mu_B = 0.6701$, $\sigma_S^2 = 0.01141$, $\sigma_S^2 = 0.01202$

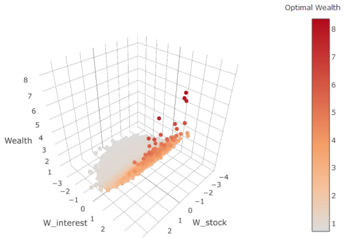
BSHW Optimal Wealth

The optimal wealth in case of a BSHW economy then shows us

Power Utility



Kinked Utility

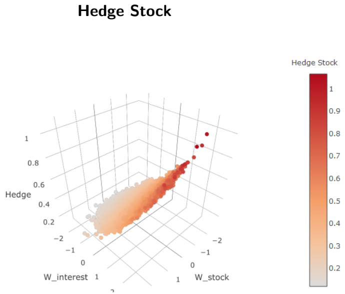


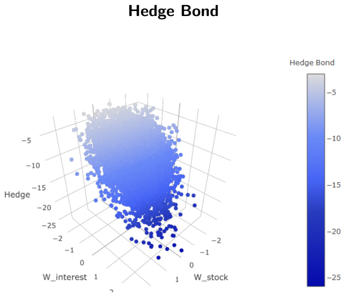
- Exponential relationship in the interest uncertainty
- Market prices of risk (explaining e.g. densities)
- $\mu = 1.587$, $\sigma = 0.4343$,

- Kinked relationship uncertainty interest rate
- Market prices of risk infer similar as power utility
- $\mu = 1.436$, $\sigma = 1.351$

BSHW Power Utility Investment Strategy

The accordingly derived investment strategies at $t = 0.5$ show us



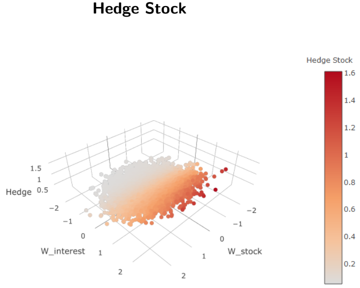


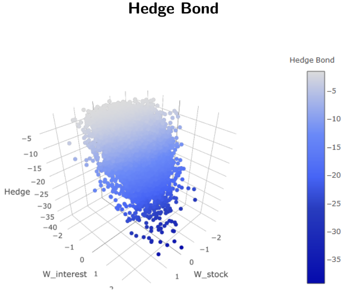
- Non-linear relationship in the interest uncertainty
- Comparatively small long position in the stock
- $\mu = 0.3615$, $\sigma = 0.01090$,

- Kinked relationship uncertainty interest rate
- Great sensitivity interest rate risk (cf. long-short diff.)
- $\mu = -9.084$, $\sigma = 7.123$

BSHW Kinked Utility Investment Strategy

Equivalently, for the kinked utility at $t = 0.5$ this then yields us





- Non-linear relationship in the interest uncertainty
- Comparatively small long position in the stock
- $\mu = 0.3191$, $\sigma = 0.0365$,

- Kinked relationship uncertainty interest rate
- Greater sensitivity interest rate risk (cf. long-short diff.)
- $\mu = -8.082$, $\sigma = 24.572$

Conclusion

- In terms of utility specifications, we observed a clear discrepancy between the prudence and risk-neutral patterns regarding the kinked utility in comparison to the power utility for both results
- Concerning the economic scenarios, the effect of inclusion of non-deterministic (non-constant) stochastic interest rates went far beyond a non-negligible effect
- General conclusion should entail a notion of the framework itself; more elaborate and rigorous settings should reproduce a real-world situation. This framework illustrates this partially by application to a rather simplified setting
- Within a DC setting, this clearly shows the possibility for an accurate way of modelling that is mathematically verified to yield optimal results

Questions?