

Quantifying ambiguity bounds through hypothetical statistical testing

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Outline

- Literature
- Model
 - Application I: Robust Investment
 - Deterministic
 - Rectangularity
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- Application II: Stochastic Example
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Preview

- Robustness
- Set of alternative models
- Optimisation
- Research on quantification of uncertainty
 - Characteristics of indistinguishable set
 - Link with Type I and II error, test horizon
 - All deviations $|\lambda(t, \omega)| \leq \frac{\Phi^{-1}(1-\beta) - \Phi^{-1}(\alpha)}{\sqrt{T}}$

Literature Uncertainty

- Risk versus uncertainty
- The Ellsberg paradox (Ellsberg, 1961)
- Bayesian prior, posterior (Thomas Bayes, 1701 1761)
- Multiple prior model from Gilboa and Schmeidler (1989)
- Extension of the standard multiple prior approach Garlappi et al. (2007)

Literature Uncertainty

- Robustness: Hansen, Sargent and Tallarini (1999); Hansen, Sargent and Turmuhambetova (2006); Hansen, Sargent and Wang (2002); Hansen and Sargent (2008); Hansen and Sargent (2015)
- ϕ -Divergence: Ben-Tal, Den Hertog, De Waegenaere, Melenberg and Rennen (2013)
- Model Confidence Set: Hansen, Lunde and Nason (2011)
- Confidence Interval for Parameters

Literature Uncertainty

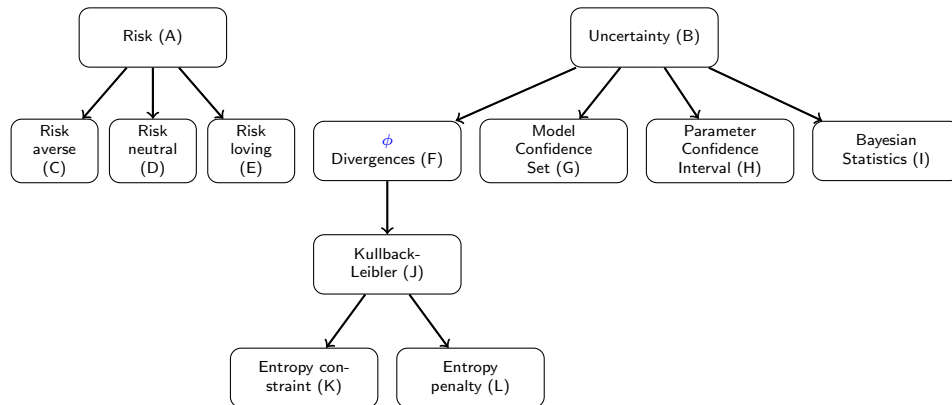


Figure: Overview of Uncertainty Sets

Robustness

- Gilboa and Schmeidler (1989) maxmin:

$$\max_{\pi} \min_{\mathbb{L} \in \mathcal{L}} \mathbb{E}^{\mathbb{L}} [U(X)]. \quad (1)$$

- Maccheroni et al. (2006) variational preferences

$$\max_{\pi} \min_{\mathbb{L}} \mathbb{E}^{\mathbb{L}} [U(X)] + c(\mathbb{L}|\mathbb{P}). \quad (2)$$

- Hansen and Sargent (2001) relative entropy: $c(\mathbb{L}|\mathbb{P}) = \theta D_{KL}(\mathbb{L}|\mathbb{P})$
- Klibanoff et al. (2005) smooth ambiguity model:

$$\max_{\pi} \mathbb{E}^{P^{\mathbb{L}}} [\phi (\mathbb{E}^{\mathbb{L}} [U(X)])] \quad (3)$$

- Bayesian: support of prior

Motivation

- Goal: we would like to identify and characterise the set of alternative models *ex ante*
- Independent from objective problem
- Consider a large class of alternatives
- Statistically indistinguishable models:
 - Based on Type I and II error and test horizon
- Outline
 - Model
 - Stochastic example
 - Deterministic

Model

- SDEs of form

$$dX = \mu(t, \omega) dt + \sigma(t, \omega) dW(t)$$

- Possible alternative models $dW_t + \lambda(t, \omega) dt$
- Ex ante ($t = 0$)
- Those $\lambda(t, \omega)$ indistinguishable from $\lambda = 0$
- $dW_t + \lambda(t, \omega) dt$ not only adjusting the mean of the probability distribution
- Stochastic example

Robust Investment

- Merton (1969): Agent generates utility from terminal wealth

$$\max_{\pi} \min_{\mathbb{L} \in \mathcal{L}} \mathbb{E}^{\mathbb{L}}[u(X(T))]$$

- Allocate π to risky asset S with

$$dS_t = \mu S_t dt + \sigma S_t (dW_t + \lambda dt)$$

- Rest on bank account B with riskfree rate r
- Equity premium puzzle
- Robust optimal investment strategy subject to $\lambda^2 \leq k^2$

$$\pi^* = \max\left(\frac{\mu - r - \sigma k}{\gamma \sigma^2}, 0\right)$$

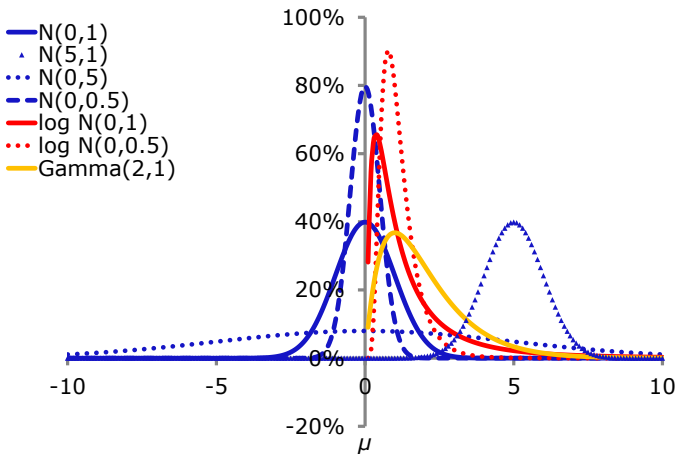
Strategies

Table: Robust Optimal Investments

γ	π^M	$\pi_{T=100}^C$	$\pi_{T=200}^C$
0.5	469%	159%	250%
1	234%	79.4%	125%
3	78.1%	26.5%	41.6%
5	46.9%	15.9%	25.0%
10	23.4%	7.94%	12.5%

For $k = \frac{2.48}{\sqrt{T}} = 0.248, 0.175$ and
 $\mu - r = 6\%$.

Alternatives



Stochastic Example

- Mean repelling process: $\lambda(t, \omega) = a \tanh(aW(t))$
- Alternative fatter tails
- Let

$$L(\mathcal{T}) = \frac{1}{2} \left(e^{-\frac{1}{2}a^2\mathcal{T} + aW(\mathcal{T})} + e^{-\frac{1}{2}a^2\mathcal{T} - aW(\mathcal{T})} \right)$$

- Under \mathbb{L}^1 a mixture of $N(a\mathcal{T}, \mathcal{T})$ and $N(-a\mathcal{T}, \mathcal{T})$, together not normal, mean 0 and variance $\mathcal{T} + (a\mathcal{T})^2$
- Under \mathbb{P} always $W(\mathcal{T}) \sim N(0, \mathcal{T})$

¹ \mathbb{P} and \mathbb{L} stand for baseline and alternative.

Model (2)

- Test $H_0 : \mathbb{P}$ versus $H_A : \mathbb{L}$
- Hence, $L(\mathcal{T})$ equals the likelihood ratio test statistic
- Radon-Nikodym derivative (Girsanov)

$$L(\mathcal{T}) = \exp \left\{ -\frac{1}{2} \int_0^{\mathcal{T}} \lambda(t, \omega)^2 dt + \int_0^{\mathcal{T}} \lambda(t, \omega) dW^{\mathbb{P}}(t, \omega) \right\}$$

- Value $L(\mathcal{T}, \omega)$ determined by realisation ω
- Test if model \mathbb{P} should be rejected in favour of model \mathbb{L}
- Two simple hypotheses, Neyman-Pearson Lemma most powerful test is likelihood ratio test

Model (3)

- Type I error: incorrectly rejecting model \mathbb{P}

$$\mathbb{P}[L(\mathcal{T}) \geq \gamma] = \alpha$$

- Type II error: incorrectly rejecting model \mathbb{L}

$$\mathbb{L}[L(\mathcal{T}) < \gamma] = \beta$$

- Power: probability of accepting model \mathbb{L}
when model \mathbb{L} is the true model

$$\begin{aligned} \mathbb{L}[L(\mathcal{T}) \geq \gamma] &= 1 - \beta \\ &= \mathbb{E}^{\mathbb{P}} [L(\mathcal{T}) \mathbb{1}(L(\mathcal{T}) \geq \gamma)] \end{aligned}$$

Deterministic

- Radon-Nikodym, without ω , log normal
- E.g. $\alpha = 0.05$, then $\Phi^{-1}(\alpha) = -1.64$
- If we take $\beta = 0.20$ then power is 0.80 and we have $\Phi^{-1}(0.80) = 0.84$
- Hence, the class of all indistinguishable models is then given by all models that satisfy

$$\left(\int_0^T \lambda(t)^2 dt \right)^{\frac{1}{2}} \leq \Phi^{-1}(1 - \beta) - \Phi^{-1}(\alpha) = 0.84 - (-1.64) = 2.48$$

Test horizon

 \mathcal{T}

- Future moment in time at which test would *hypothetically* be performed
- Extra amount of data that one would take into consideration
- Time period during which model would remain the same; no structural break

Rectangularity

- Rectangularity \Leftrightarrow time-consistency \Leftrightarrow m-stability \Leftrightarrow BSDEs
- Power \Leftrightarrow CVaR \Leftrightarrow Coherent risk measure
- For time-consistent coherent risk measures (Barrieu and El Karoui (2007))
 - $|\lambda(t, \omega)| \leq k$
- Optimal solution at any time-point t does not depend on history between $[0, t]$
- Optimal policy devised at time 0 for $t > 0$ is still valid at time t given information \mathcal{F}_t
- Intersect classes

Optimisation

- Recall

$$L(\mathcal{T}) = \exp \left\{ -\frac{1}{2} \int_0^{\mathcal{T}} \lambda(t, \omega)^2 dt + \int_0^{\mathcal{T}} \lambda(t, \omega) dW^{\mathbb{P}}(t, \omega) \right\}$$

- Optimisation problem

$$\max_{\gamma, |\lambda(t, \omega)| \leq k} \mathbb{E} [L(\mathcal{T}) \mathbb{1}(L(\mathcal{T}) \geq \gamma)] \quad (\text{MP})$$

$$\text{s.t. } \mathbb{E} [\mathbb{1}(L(\mathcal{T}) \geq \gamma)] = \alpha$$

$$dL = \lambda(t, \omega) L dW, L_0 = 1$$

- Maximum for $|\lambda(t, \omega)| \equiv k \Rightarrow \log \text{ Normal}$

Theorem (Rectangular and Coherent Sets of Indistinguishable Models)

Consider a baseline model $dX(t) = \mu(t, \omega)dt + \sigma(t, \omega)dW(t)$. The set of all models with $dW(t) + \lambda(t, \omega)dt$ and $|\lambda(t, \omega)| \leq k$ is rectangular and coherent, where

$$k = \frac{\Phi^{-1}(1 - \beta) - \Phi^{-1}(\alpha)}{\sqrt{\mathcal{T}}} \quad (4)$$

forms an indistinguishable set for a Type I error of α , a Type II error of β and a test horizon \mathcal{T} .

Bounds on Divergences

- ϕ -Divergences/ f -Divergences (non-symmetric distance measures)
- Robust results in optimisation problems
- Continuous ϕ -divergence

$$D_{\phi}(L(\mathcal{T}, \omega)) = \mathbb{E}^{\mathbb{L}} \left[\phi \left(\frac{1}{L(\mathcal{T})} \right) \right] = \mathbb{E}^{\mathbb{P}} \left[L(\mathcal{T}) \phi \left(\frac{1}{L(\mathcal{T})} \right) \right]$$

- For each measure $\phi(\cdot)$ given and convex
- Size of the uncertainty quantified by c

$$D_{\phi}(L(\mathcal{T}, \omega)) \leq c$$

- The divergences are expressed in terms of the ratio $x = \frac{1}{L(\mathcal{T}, \omega)}$ under the measure \mathbb{L}

Bounds on Divergences

Divergence	$\phi(x)$	$c \Leftrightarrow \lambda(t, \omega) \equiv k$	$k\sqrt{\mathcal{T}} = 2.48$
Kullback-Leibler	$x \ln x - x + 1$	$\frac{1}{2}k^2\mathcal{T}$	3.08
Burg entropy	$-\ln x + x - 1$	$\frac{1}{2}k^2\mathcal{T}$	3.08
J-divergence	$(x - 1) \ln x$	$k^2\mathcal{T}$	6.15
χ^2 -divergence	$\frac{1}{x}(x - 1)^2$	$e^{k^2\mathcal{T}} - 1$	467.90
Modified χ^2 -divergence	$(x - 1)^2$	$e^{k^2\mathcal{T}} - 1$	467.90
Hellinger distance	$(\sqrt{x} - 1)^2$	$2 - 2e^{-\frac{1}{8}k^2\mathcal{T}}$	1.07
Variation distance	$ x - 1 $	$4N(\frac{1}{2}k\sqrt{\mathcal{T}}) - 2$	1.57
χ -divergence of order $\vartheta > 1$	$ x - 1 ^\vartheta$	<i>numerical</i>	<i>Table 2</i>
Cressie-Read $\vartheta \neq 0, 1$	$\frac{1 - \vartheta + \vartheta x - x^\vartheta}{\vartheta(1 - \vartheta)}$	<i>expr</i>	<i>Table 2</i>

Divergence \ ϑ	1.5	2.0	2.5	3.0
χ -divergence of order ϑ	10.40	467.90	1.02×10^6	1.03×10^8
Cressie-Read	12.05	233.95	2.72×10^4	1.72×10^7

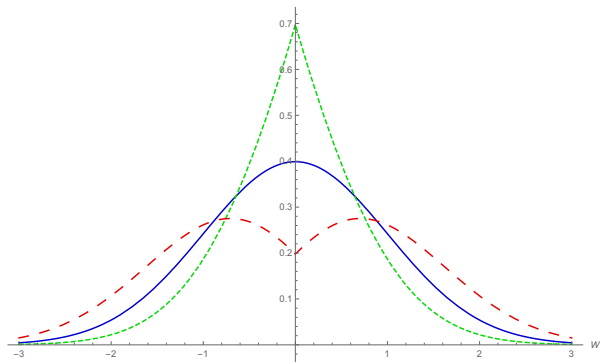
Stochastic example

- Baseline model: Brownian motion
- Stochastic deviation $\lambda(t, \omega) = a \cdot \text{sgn}(W(t, \omega))$
- $a > 0$
 - Mean repelling process
 - Increase the variance of $X(T)$ under model \mathbb{L}
- $a < 0$
 - Mean reverting process
 - Decrease the variance of $X(T)$ under model \mathbb{L}

Stochastic example

- Under \mathbb{P}
 - $dX(t) = dW(t)$
 - Model X normal distributed
- Under \mathbb{L}
 - $dX(t) = a \cdot \text{sgn}(X(t))dt + dW(t)$
 - Model X unknown distribution and variance changed
- Distribution
 - $\ln L(\mathcal{T}) \sim^{\mathbb{P}} N\left(-\frac{1}{2}a^2\mathcal{T}, a^2\mathcal{T}\right)$
 - $\ln L(\mathcal{T}) \sim^{\mathbb{L}} N\left(\frac{1}{2}a^2\mathcal{T}, a^2\mathcal{T}\right)$
- Explicit bound $|a| \leq \frac{\Phi^{-1}(1-\beta) - \Phi^{-1}(\alpha)}{\sqrt{\mathcal{T}}}$
- Does satisfy rectangular and coherence axioms

Stochastic example



Maenhout

	1891-1994	1947.2-1996.3
$\mu - r$	6.258%	7.852%
r	1.955%	0.785%
σ	0.18534	0.15218
θ	14	237
γ	7	10
k	0.2251	0.4951
$(\alpha, 1 - \beta, \mathcal{T})$	(0.05, 0.80, 122)	(0.05, 0.80, 25)
$(\alpha, 1 - \beta, \mathcal{T})$	(0.05, 0.73, 100)	(0.05, 0.9995, 100)

Conclusion

- Quantify uncertainty
- Most powerful test
- Ex ante
- For given size and power
- Stochastic deviation from the drift

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