Discussion of paper Boon-Werker

Antoon Pelsser[1]

[1]Maastricht University & Netspar
Email: a.pelsser@maastrichtuniversity.nl

13 October 2017
International Pension Day – Utrecht

Outline

Summary of paper

- Investigate Variable Annuity for optimal consumption problem
- Use CRRA utility and Black-Scholes-Vašíček model setup
- Analyse optimal solution via “lockbox” approach
- Quantify welfare losses due to sub-optimal VA strategy

Optimal Consumption Problem

- General problem formulation:

$$\max_{\{C_t\}} \mathbb{E}^{\mathbb{P}}\left[\int_0^T U(t, C_t)\, dt\right]$$
$$\text{s.t. } \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{C_t}{N_t}\, dt\right] = \frac{W_0}{N_0}$$

- Maximise $\mathbb{P}$-expected utility of consumption
- Under a $\mathbb{Q}$ budget-constraint (w.r.t. numéraire N_t)
- $\mathbb{Q}$ is unique for complete and arbitrage-free economy (given N_t)
- Note: general formulation!

Optimal Consumption Problem 2

- Use $\mathbb{P}$-expectations everywhere:

$$\max_{\{C_t\}} \mathbb{E}^{\mathbb{P}}\left[\int_0^T U(t, C_t)\,dt\right]$$

$$\text{s.t. } \mathbb{E}^{\mathbb{P}}\left[\int_0^T C_t \frac{Q_t}{N_t}\,dt\right] = W_0 \frac{Q_0}{N_0}$$

- The process Q_t is the Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$.
- Note: Q_t/N_t is the *unique* stochastic discount factor for the economy (irrespective of choice for numéraire N_t)

Optimal Consumption Problem 3

- Use stochastic discount factor $M_t := Q_t/N_t$:

$$\max_{\{C_t\}} \mathbb{E}^{\mathbb{P}}\left[\int_0^T U(t, C_t)\, dt\right]$$
$$\text{s.t. } \mathbb{E}^{\mathbb{P}}\left[\int_0^T C_t M_t\, dt\right] = W_0 M_0$$

- M_t is the unique stochastic discount factor for the economy
- Note: this is the generalised version of the Boon-Werker setup

Optimal Consumption Solution

- Build Lagrange function $\mathcal{L}(\{C_t\}, \lambda)$:

$$\mathcal{L}(\{C_t\}, \lambda) := \mathbb{E}^{\mathbb{P}}\left[\int_0^T U(t, C_t)\, dt\right] - \lambda\left(\mathbb{E}^{\mathbb{P}}\left[\int_0^T C_t M_t\, dt\right] - W_0 M_0\right)$$

- Simplify as:

$$\mathcal{L}(\{C_t\}, \lambda) := \mathbb{E}^{\mathbb{P}}\left[\int_0^T \Big(U(t, C_t) - \lambda C_t M_t\Big) dt\right] + \lambda W_0 M_0$$

Optimal Consumption Solution 2

- Consider impact of small change in consumption plan $\{C_t + h_t\}$:

$$\mathcal{L}(\{C_t + h_t\}, \lambda) - \mathcal{L}(\{C_t\}, \lambda) \approx \mathbb{E}^{\mathbb{P}}\left[\int_0^T \Big(U_C(t, C_t) - \lambda M_t\Big) h_t \, dt\right]$$

- For *optimal* consumption path C_t^*, value function $\mathcal{L}()$ is not affected by *any* perturbation $\{h_t\}$, hence: $U_C(t, C_t^*) - \lambda M_t \equiv 0 \quad \forall t$
- Express solution as:

$$C_t^* = I(t, \lambda M_t) \quad \forall t,$$

where $I(t, .)$ is the inverse function of $U_C(t, .)$

Optimal Consumption Solution 3

- Consider optimal solution:

$$C_t^* = I(t, \lambda M_t) \quad \forall t$$

- Note: (almost) explicit solution to general formulation of problem!
- Utility function $U(t, C_t)$ reflected in $I(t, .)$
- Economy reflected in stochastic discount factor M_t

- Optimal consumption C_t^* at time t is determined by $I(t, .)$ and λM_t
- Lagrange parameter λ "scales" whole consumption path
- Solve budget equation, which is non-linear for λ

Optimal Consumption for CRRA utility

- CRRA: $U(t,C) = e^{-\beta t}\, C^{1-\gamma}/(1-\gamma)$, with $U_C(t,C) = e^{-\beta t}\, C^{-\gamma}$
- Solving $e^{-\beta t}\, C^{-\gamma} = \lambda M$ gives: $C_t^* = e^{-(\beta/\gamma)t}(\lambda M_t)^{-(1/\gamma)}$
- For CRRA we can solve budget constraint explicitly for λ:

$$\lambda^{-(1/\gamma)}\mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-(\beta/\gamma)t} M_t^{-(1/\gamma)} M_t\, dt\right] = W_0 M_0$$

- Hence, CRRA allows for fully explicit solution for C_t^*:

$$C_t^* = e^{-(\beta/\gamma)t}\frac{W_0 M_0}{\mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-(\beta/\gamma)t} M_t^{1-(1/\gamma)}\, dt\right]} M_t^{-(1/\gamma)}$$

- Note: product structure is *very specific* for CRRA!

Two-Stage Lockbox Formulation

- Authors discuss "Lockbox Separation" of Sharpe (2007)
- Separation into series of static strategies that finances consumption at each date

- What is added value of lockbox separation?
- Follows immediately from general solution:

$$C_t^* = I(t, \lambda M_t) \quad \forall t$$

- Lockbox feels like shoehorning general result into CRRA product-form
- Delete from the paper! But, perhaps I completely missed the point...

Variable Annuity

- Variable Annuity (VA) product has a reference investment portfolio that finances a consumption stream
- I like this! VA has qualitatively the right structure

- We don't know "true" utility-function and "true" model for economy
- But we do know it is definitely not CRRA and BS-Vašíček
- I don't like that you shoehorn VA into CRRA-BSV setup
- Optimal solution in CRRA-BSV is "more accurately computing the wrong answer"
- I don't care about welfare loss relative to CRRA-BSV setup...

Variable Annuity – Robustness

- VA has qualitatively the right structure
- Extend with time-dependent (life-cycle) investment strategy and AIR
- For which combinations of $U()$ and M_t would (extended) VA strategy be optimal, given general solution:

$$C_t^{\text{VA}} = I(t, \lambda M_t) \quad \forall t$$

- Can we then frame this into a "robustness" perspective?
- When is VA "approximately the right answer"?
- Could we always find a $U()$ and M_t combination that is "close" to a specific benchmark model? Then, VA would be a "robust" strategy!

Robust VA – Sketch

- For constant MPR's κ the stoch.disc.factor has a log-normal form

$$dM = -rM\,dt + M\,\kappa'\cdot dW$$

- For generalised power-util: $I(t, X) = h(t)X^{-1/\gamma(t)}$ and

$$C_t^{\mathrm{VA}} = h(t)(\lambda M_t)^{-1/\gamma(t)}$$

- Hence, C_t^{VA} still has log-normal form, which can be replicated by (extended) VA reference portfolio

- Would this setup be "sufficiently close" to any benchmark model?