How to Invest and Draw-Down Wealth? A Utility-Based Analysis

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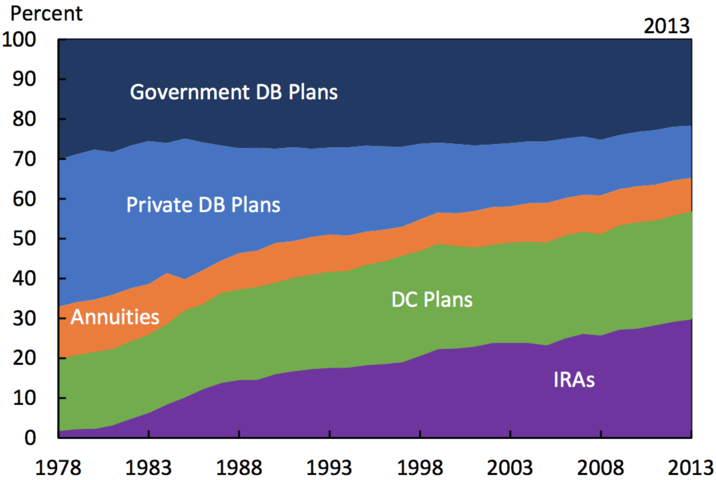
Motivation [1]



Source: Investment Company Institute

Motivation [2]

- Key question:

  How to Invest and Draw-Down Wealth in Retirement?

- Ingredients of a pension plan:
  - Expected Pension Stream (e.g., flat).
  - Investment Policy (e.g., 20% equities).
  - Response of Consumption to Investment Shocks (e.g., 5 year smoothing period).
- Can current pension proposals be justified on the basis of utility theory?
- What can we learn from utility theory? Policy advise?
- Preferences that combine:
  - ratio habit model (Abel, 1990) and
  - stochastic differential utility (Duffie and Epstein, 1992).

Contributions

- Methodological
  - We obtain the optimal pension plan in closed-form using a linearization of the budget constraint.
  - Approximation error is small.
- Economic
  - We derive the optimal smoothing mechanism.
  - Relative risk aversion controls the change in the level of consumption (i.e., year-on-year volatility in consumption).
  - Strength of habits controls the change in future growth rates of consumption.
  - Relative risk aversion and strength of habits control investment policy.

Outline of My Talk

1. Model Setting
2. Solution Technique
3. Analysis of the Life Cycle Strategies
4. Stochastic Differential Utility
5. Accuracy of the Approximation Method

Model Setting: Preferences

- The instantaneous utility function is given by

$$u(c_t, h_t) = v\left(\frac{c_t}{h_t}\right) = \frac{1}{1-\gamma}\left(\frac{c_t}{h_t}\right)^{1-\gamma}.$$

- The log habit level $\log h_t$ satisfies the following dynamic equation:

$$\mathrm{d}\log h_t = (\beta \log c_t - \alpha \log h_t)\,\mathrm{d}t, \quad \log h_0 = 0.$$

- Habit level depends on past consumption choices.
  - β: strength of habits
  - α: persistence of habits

Model Setting: Financial Market

- We consider a Black and Scholes financial market.
- The price of the money market account B_t evolves according to

$$\frac{\mathrm{d}B_t}{B_t} = r\,\mathrm{d}t.$$

- The risky stock price S_t satisfies

$$\frac{\mathrm{d}S_t}{S_t} = (r + \lambda\sigma)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t.$$

Model Setting: Optimization Problem

- The individual aims to maximize expected lifetime utility

$$U_0 = \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_s \,\mathrm{d}s} v\left(\frac{c_t}{h_t}\right)\mathrm{d}t\right]$$

subject to

$$\begin{aligned}
\mathrm{d}A_t &= (r + \pi_t \lambda \sigma) A_t \,\mathrm{d}t - c_t \,\mathrm{d}t + \pi_t \sigma A_t \,\mathrm{d}W_t, \\
\mathrm{d}\log h_t &= (\beta \log c_t - \alpha \log h_t)\,\mathrm{d}t.
\end{aligned}$$

- Difficult to solve! To the best of my knowledge, model has not been solved yet.

Solution Method: Change of Variable

- Define

$$\widehat{c}_t = \frac{c_t}{h_t}.$$

- Equivalent optimization problem:

$$\begin{aligned}
\max_{\widehat{c}_t : 0 \le t \le T} \quad & \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_s \, \mathrm{d}s} v\left(\widehat{c}_t\right) \mathrm{d}t\right] \\
\text{subject to} \quad & \mathbb{E}\left[\int_0^T M_t h_t \widehat{c}_t \, \mathrm{d}t\right] \le A_0, \\
& \mathrm{d} \log h_t = \left(\beta \log \widehat{c}_t - [\alpha - \beta] \log h_t\right) \mathrm{d}t.
\end{aligned}$$

Here, M_t is the stochastic discount factor at time t.

- Objective function is independent of past choices, but budget constraint depends on past choices.

Solution Method: Linearization (trick)

- Denote by P_t the price of a bond paying a continuous coupon.
- Approximation of the left-hand side of the budget constraint around $\widehat{c} = c/h = 1$ yields

$$\mathbb{E}\left[\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t\right] \approx -\beta \mathbb{E}\left[\int_0^T M_t P_t \,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T \widehat{M}_t \widehat{c}_t \,\mathrm{d}t\right].$$

Here, $\widehat{M}_t = M_t (1 + \beta P_t)$ denotes the adjusted stochastic discount factor at time t.

Solution Method: Approximate Problem

◇ The approximate problem is given by

$$\max_{\widehat{c}_t : 0 \le t \le T} \quad \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_s \, \mathrm{d}s} v\left(\widehat{c}_t\right) \mathrm{d}t\right]$$

$$\text{s.t.} \quad \mathbb{E}\left[\int_0^T \widehat{M}_t \widehat{c}_t \, \mathrm{d}t\right] \le \widehat{A}_0.$$

Here, $\widehat{A}_0$ denotes the adjusted initial wealth.

◇ We denote by $\widehat{c}_t^*$ the optimal solution to the approximate problem.

◇ We determine $\widehat{A}_0$ such that entire wealth A_0 is spent on the consumption strategy $h_t^* \widehat{c}_t^*$.

Analysis of the Life Cycle Strategies: 'Optimal' Consumption

- The individual's optimal consumption choice c_t^* is given by

$$c_t^* = h_t^* \left(y e^{\int_0^t \delta_s \, \mathrm{d}s} \widehat{M}_t \right)^{-\frac{1}{\gamma}} .$$

- The Lagrange multiplier y is determined such that the original budget constraint holds with equality.

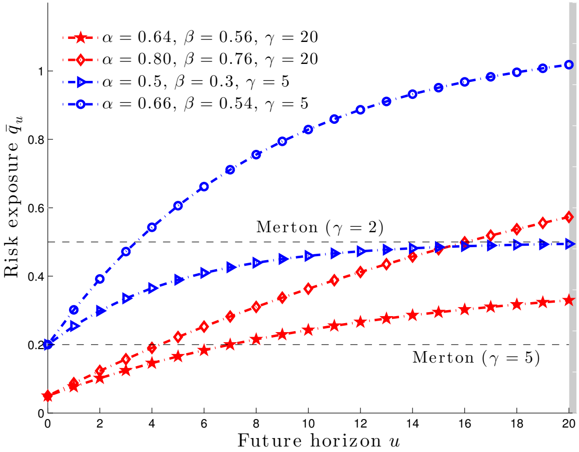
Analysis of the Life Cycle Strategies: 'Optimal' Risk Exposure

- The exposure of future log consumption $\log c^*_{t+u}$ to a current financial shock $\lambda\,\mathrm{d}W_t$ is given by

$$\bar{q}_u = \frac{\partial \log c^*_{t+u}}{\partial \lambda\,\mathrm{d}W_t} = \frac{1}{\gamma}\left[1 + \frac{\beta}{\alpha - \beta}\left(1 - \exp\{-(\alpha - \beta)u\}\right)\right].$$

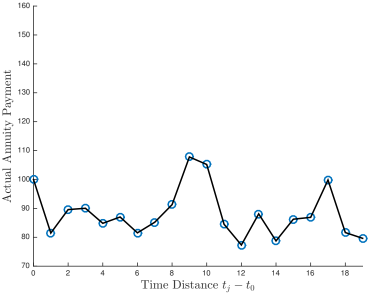
- The risk exposure $\bar{q}_u$ **increases** with the horizon u.
  - Consumption cuts are delayed following a financial shock.
- γ: effect of a current shock on the **level** of log consumption. New interpretation!
- $\beta/(\alpha - \beta)$: effect of a current shock on future **growth rates** of consumption.
- $\alpha - \beta$: the rate at which $\bar{q}_u \Rightarrow \bar{q}_\infty$ (**smoothing period**).

Analysis of the Life Cycle Strategies: ‘Optimal’ Risk Exposure [2]

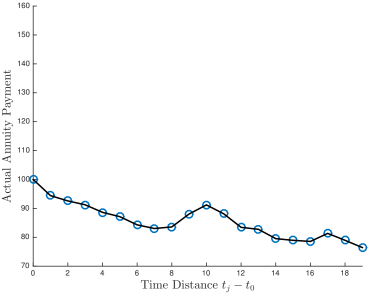


Analysis of the Life Cycle Strategies: Consumption Pattern

Same expected pension streams; same prices



(a) No Smoothing



(b) Smoothing

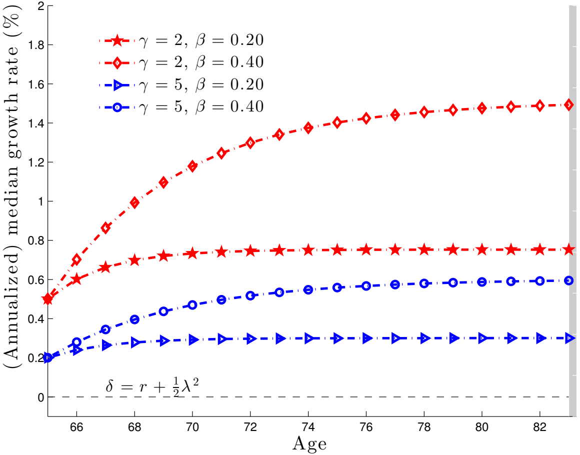
Analysis of the Life Cycle Strategies: Consumption Dynamics ($T = \infty$)

- Log consumption $\log c_t^*$ evolves according to

$$\mathrm{d} \log c_t^* = \log F_t^0 + \bar{q}_0 \left(r + \frac{1}{2}\lambda^2 - \delta_t \right) \mathrm{d}t + \bar{q}_0 \lambda \, \mathrm{d}W_t.$$

- $\log F_t^0$: past shocks that are reflected into the current median growth rate of log consumption.
- The second term represents the aspirational growth rate of consumption.
- The last term corresponds to current shocks that are absorbed into the level of log consumption.

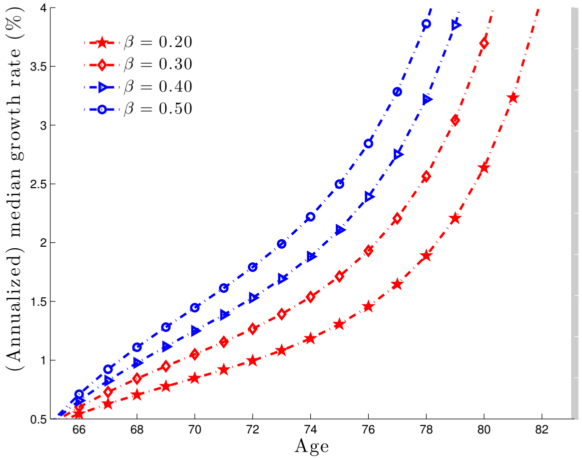
Analysis of the Life Cycle Strategies: Consumption Dynamics ($T = \infty$) [2]



Analysis of the Life Cycle Strategies: Consumption Dynamics ($T < \infty$)

- The expected growth rate of consumption increases as the 'strength' parameter β increases given $\eta = \alpha - \beta$.
- The expected growth rate of consumption increases with age.
  - Undesirable? $\Rightarrow$ Stochastic Differential Utility.

Analysis of the Life Cycle Strategies: Consumption Dynamics ($T < \infty$) [2]



Analysis of the Life Cycle Strategies: 'Optimal' Investment

- The replicating portfolio strategy is given by

$$\pi_t^* = \widehat{q}_t \frac{\lambda}{\sigma}.$$

- Here $0 \leq \widehat{q}_t \leq 1$ denotes the (weighted) average risk exposure. That is,

$$\widehat{q}_t = \int_t^T \frac{1}{\gamma} \left[1 + \frac{\beta}{\alpha - \beta} \left(1 - \exp\left\{ -(\alpha - \beta) u \right\} \right) \right] \frac{V_t^u}{V_t} \, \mathrm{d}u,$$

where $V_t = \int_t^T V_t^u \, \mathrm{d}u$ and V_t^u is the time-t value of c_{t+u}^*.

- γ determines year-on-year volatility of consumption.
- γ and β determine investment policy.

Analysis of the Life Cycle Strategies: Life Cycle Investment

Age	(1)	(2)	Merton ($\gamma = 2$)	Merton ($\gamma = 5$)
65	0.20	0.41	0.50	0.20
70	0.18	0.39	0.50	0.20
75	0.15	0.36	0.50	0.20
80	0.10	0.29	0.50	0.20
85	0.05	0.20	0.50	0.20

Notes: (1) corresponds to $\alpha = 0.64$, $\beta = 0.56$, $\gamma = 20$; and (2) to $\alpha = 0.5$, $\beta = 0.3$, $\gamma = 5$.

Analysis of the Life Cycle Strategies: Wealth Volatility (σ_A) vs. Payout Volatility (σ_c)

Age	(1)		(2)		Merton ($\gamma = 2$)	
	σ_c	σ_A	σ_c	σ_A	σ_c	σ_A
65	1.00	3.95	4.00	12.99	10.00	10.00
70	1.00	3.56	4.00	12.81	10.00	10.00
75	1.00	2.92	4.00	11.12	10.00	10.00
80	1.00	1.99	4.00	7.90	10.00	10.00
85	1.00	1.00	4.00	4.00	10.00	10.00

Notes: (1) corresponds to $\alpha = 0.64$, $\beta = 0.56$, $\gamma = 20$; and (2) to $\alpha = 0.66$, $\beta = 0.54$, $\gamma = 5$.

Analysis of the Life Cycle Strategies: Welfare Analysis

Twoalternative investment and draw-down strategies:

1. The Merton approach
2. The difference habit model
   - $u\left(c_t - h_t\right) = \frac{1}{1-\gamma}\left(c_t - h_t\right)^{1-\gamma}$.

Analysis of the Life Cycle Strategies: Welfare Losses (Merton Model)

Optimal Strategy	Risk Aversion Coefficient (γ)		
	2	5	20
$\alpha = 0.64$, $\beta = 0.56$, $\gamma = 20$	23.85	5.19	2.44
$\alpha = 0.80$, $\beta = 0.76$, $\gamma = 20$	29.55	13.13	8.50
$\alpha = 0.50$, $\beta = 0.30$, $\gamma = 5$	2.95	0.63	3.93
$\alpha = 0.66$, $\beta = 0.54$, $\gamma = 5$	2.85	1.86	5.31

Future Research: Minimum Welfare Losses! Optimize over γ.

Analysis of the Life Cycle Strategies: Welfare Losses (Difference Habit Model)

Parameters	Welfare Loss
$\alpha = 0.64$, $\beta = 0.56$, $\gamma = 20$	5.63
$\alpha = 0.80$, $\beta = 0.76$, $\gamma = 20$	9.10
$\alpha = 0.50$, $\beta = 0.30$, $\gamma = 5$	4.33
$\alpha = 0.66$, $\beta = 0.54$, $\gamma = 5$	4.31

Stochastic Differential Utility: Problem

- The individual aims to maximize

$$\underset{\widehat{c}_t : 0 \le t \le T}{\text{maximize}} \quad V_0 = \mathbb{E}_0 \left[\int_0^T f\left(\widehat{c}_t, V_t, t\right) \mathrm{d}t \right]$$

$$\text{s.t.} \qquad \mathbb{E}\left[\int_0^T \widehat{M}_t \widehat{c}_t \, \mathrm{d}t \right] \le \widehat{A}_0 .$$

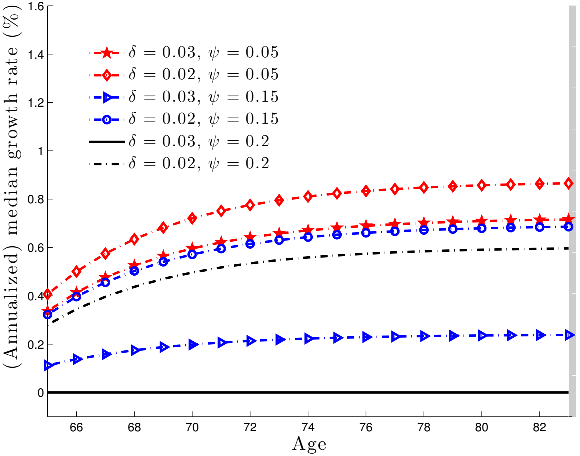
- Here, the intertemporal aggregator is given by

$$f\left(\widehat{c}_t, V_t, t\right) = (1+\zeta) \left[\frac{\left(\widehat{c}_t\right)^{\varphi}}{\varphi} |V_t|^{\frac{\zeta}{1+\zeta}} - \delta V_t \right] .$$

- Intertemporal rate of substitution: $\psi = \frac{1}{1-\varphi}$.

▸ Solution

Stochastic Differential Utility: Median Growth



Accuracy of the Approximation Method

Losses: in terms of decline in certainty equivalent consumption

	β				
α	0	0.15	0.2	0.3	0.6
0	0	–	–	–	–
0.15	0	0.0229	–	–	–
0.2	0	0.0178	0.0516	–	–
0.3	0	0.0149	0.03053	0.1263	–
0.6	0	0.0153	0.0293	0.0689	0.6840

Conclusion & Contributions

- We have build a rich consumption-portfolio choice model with ratio habit model and SDU.
- We have developed a solution technique to solve our model.
  - Approximation error is small.
- We have analyzed the optimal consumption and portfolio choice.
  - Gradual response of consumption to shocks $\Rightarrow$ justification for current smoothing schemes
- Future research: calibration exercise.

Stochastic Differential Utility: Optimal Solution

$$c_t^* = (c_0^*)^{\frac{q_t}{q_0}} \exp\left\{\int_0^t q_{t-u}\left(\psi\left[\widehat{r}_u + \frac{1}{2}\frac{\lambda^2}{\gamma} - \delta_u\right]\right)\mathrm{d}u\right\}$$
$$\exp\left\{\int_0^t \frac{1}{2}\frac{\lambda^2}{\gamma^2}[\gamma - 1]\,\mathrm{d}u + q_{t-u}\frac{1}{\gamma}\lambda\int_0^t \mathrm{d}W_u\right\}.$$

Go Back