Discussion of paper Servaas van Bilsen

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20 January 2017
International Pension Workshop – Leiden

Outline

1. Summary
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Summary of paper

- Paper studies optimal consumption & portfolio choice in a utility framework
- Utility framework with multiplicative habit formation
- Analytical solution to approximate problem
- Analyse power utility and also stoch.diff.util

Habit Formation

- Classical utility considers "absolute" wealth or consumption: $u(c_t)$
- Humans don't look at absolute wealth, they compare relative to a benchmark
- Compare two cases:
  - Receive €10
  - Receive €20, and then pay €10 tax
  - Most likely, you feel less happy in second case, compare to €20
- This is the idea of "habit formation"

Preferences with habit formation

- Define utility of consumption c_t relative to habit level h_t
- Utility function: $u(c_t, h_t) = v(c_t/h_t)$
- Habit level is "moving average" of previous consumptions:

$$dh_t = \beta(c_t - h_t)\,dt$$

- Note: consistent when $c_t \equiv h_t \equiv C$
- Interpret c_t/h_t as current consumption relative to average of past consumption levels
- Remark: I like this ratio-model a lot!
- Remark: I don't think the log-habit that BBL introduce is needed. I think the simple MA formulation used here would be more natural
- Remark: BBL's $(\beta \ln c_t - \alpha \ln h_t)$ is only consistent for $\alpha = \beta$

Maximisation Problem: the good news

- Use [Cox and Huang, 1989] formulation of optimal consumption problem:

$$\max_{c_t} \mathbb{E}\left[\int_0^T e^{-\delta t} v\left(\frac{c_t}{h_t}\right) dt\right]$$

$$\text{s.t. } \mathbb{E}\left[\int_0^T M_t c_t dt\right] = A_0 \qquad dh_t = \beta(c_t - h_t)\, dt$$

- Cox-Huang allows for elegant decoupling of optimal consumption and investment strategy: solvable for any utility, even stoch.diff.util
- Solve optimal investment as "delta-hedging" investment that replicates the optimal consumption pattern

Maximisation Problem: the bad news

- Optimal consumption problem:

$$\max_{c_t} \mathbb{E}\left[\int_0^T e^{-\delta t} v\left(\frac{c_t}{h_t}\right) dt\right]$$

$$\text{s.t. } \mathbb{E}\left[\int_0^T M_t c_t dt\right] = A_0 \qquad dh_t = \beta(c_t - h_t)\, dt$$

- Habit h_t makes problem path-dependent
- Consumption c_t has impact on all future habit-levels h_s with $s > t$
- This makes problem difficult to solve...
- Note: the difficulties with h_s reside in the objective function

Change of variables

- BBL introduce a change of variables: $\hat{c}_t := c_t/h_t$
- Remark: they call this a "dual" formulation, but I think this is only a change of var's
- Recall: $\dot{h}_t = \beta(c_t - h_t)$
- Hence: $\dot{h}_t/h_t = \beta(\hat{c}_t - 1)$
- Solution: $h_t = h_0 e^{\int_0^t \beta(\hat{c}_u - 1)du}$
- Also: $c_t = \hat{c}_t h_0 e^{\int_0^t \beta(\hat{c}_u - 1)du}$
- Remark: "classical" utility for $\beta = 0$

Change of variables (2)

- Reformulate optimal consumption problem using $\hat{c}_t$:

$$\max_{\hat{c}_t} \mathbb{E}\left[\int_0^T e^{-\delta t} v(\hat{c}_t) dt\right]$$

$$\text{s.t. } \mathbb{E}\left[\int_0^T M_t \hat{c}_t h_0 e^{\int_0^t \beta(\hat{c}_u - 1) du} dt\right] = A_0$$

- This is still a path-dependent problem
- But path-dependency has now moved to the budget constraint

Constant solution

- Consider constant solution $\hat{c}_t \equiv 1$
- This is a feasible solution, when budget constraint is satisfied
- Budget constraint:

$$\int_0^T e^{-rt} h_0 \, dt = A_0$$

- Note: constant payoff is deterministic, discount with risk-free rate
- Set $h_0 = A_0 / \int_0^T e^{-rt} dt$ to satisfy budget A_0.
- Remark: BBL do not use h_0, and their solution can be infeasible, and they correct for this in another way

Linearise the budget constraint

- Write general solution $\hat{c}_t = 1 + \eta_t$
- Define the functional $f(\hat{c}) := \mathbb{E}\left[\int_0^T M_t \hat{c}_t h_t dt\right]$
- For small values of η_t we can approximate this functional as: $f(1+\eta) \approx f(1) + \int_0^T Df(1)\eta_t dt$ where Df is the Fréchet derivative of the functional $f()$
- We also have the functional $h_t(\hat{c}) = h_0 e^{\int_0^t \beta(\hat{c}_u - 1)du}$ with $h_t(1+\eta) \approx h_0 + \int_0^t h_0 \beta \eta_u du$
- Hence, we obtain for $f(1+\eta)$ (using product rule):

$$f(1+\eta) \approx A_0 + \mathbb{E}\left[\int_0^T M_t h_0 \eta_t dt + \int_0^T M_t \int_0^t h_0 \beta \eta_u du\, dt\right]$$

Linearise the budget constraint (2)

- Change order of integration in last integral:

$$f(1+\eta) \approx A_0 + \mathbb{E}\left[\int_0^T \left(M_t h_0 + \beta h_0 \int_t^T M_u du\right) \eta_t dt\right]$$

- Path-dependency is linearised and gives integral over all "future" M_u's
- Bring conditional expectation $\mathbb{E}_t[M_u] = M_t e^{-r(u-t)}$ inside integral:

$$f(1+\eta) \approx A_0 + \mathbb{E}\left[\int_0^T M_t h_0 \left(1 + \beta\left(\frac{1-e^{-r(T-t)}}{r}\right)\right) \eta_t dt\right]$$

- BBL "integrate out" the path-dependency. Very clever!
- Budg.con. with adj. pricing kernel: $f(1+\eta) \approx A_0 + \mathbb{E}\left[\int_0^T \hat{M}_t \eta_t dt\right]$

Approximate Problem

- Approximate optimal consumption problem:

$$\max_{\eta_t} \mathbb{E}\left[\int_0^T e^{-\delta t} v(1+\eta_t) dt\right]$$
$$\text{s.t. } \mathbb{E}\left[\int_0^T \hat{M}_t \eta_t dt\right] = 0$$

- This problem is now analytically solvable for all utility functions $v()$.
- Structure of analytical optimal consumption:

$$\hat{c}_t^* \approx 1 + \eta_t^* = I\left(\lambda e^{\delta t} \hat{M}_t\right)$$

where $I()$ is inverse of $v'()$ and λ is scale parameter for budget

- Note: obtain "classical" utility for $\beta = 0$

Discussion

- Emphasise more the difference between "classical" optimal consumption and habit optimal
- Look at "path-by-path", average consumption paths not so interesting
- Suggestion: focus more on response to unexpected shock in wealth. How does such a shock impact (expected) future consumption path?
- I am worried that the expansion point $\hat{c}_t \equiv 1$ is "too far" from the optimal solution. The linearisation might not be accurate approximation of the true optimal solution.

Alternative linearisation

- Optimal consumption problem using $\hat{c}_t$:

$$\max_{\hat{c}_t} \mathbb{E}\left[\int_0^T e^{-\delta t} v(\hat{c}_t) dt\right]$$
$$\text{s.t. } \mathbb{E}\left[\int_0^T M_t \hat{c}_t h_t dt\right] = A_0$$

- FOC is given by:

$$e^{-\delta t} v'(\hat{c}_t) - \lambda \left(M_t h_t + \beta \mathbb{E}_t \left[\int_t^T M_u \hat{c}_u h_u \, du\right]\right) = 0$$

- Parameter λ is Lagrange multiplier for budget constraint
- FOC is path-dependent due to integral-term

Alternative linearisation (2)

- When $\beta = 0$ then $h_t \equiv h_0$ and we obtain "classical" optimal solution:

$$e^{-\delta t} v'(c_t^0) = \lambda M_t \quad \Rightarrow \quad c_t^0 = I\left(\lambda e^{\delta t} M_t\right)$$

  with $I()$ is inverse of $v'()$.
- For $v()$ is power-util and M_t log-normal, then c_t^0 is also lognormal
- Consider *parameter expansion* around $\beta = 0$. We can compute the derivative $\partial \hat{c}_t / \partial \beta$ at $\beta = 0$ from the FOC:

$$e^{-\delta t} v''(c_t^0) \frac{\partial \hat{c}_t}{\partial \beta} - \lambda \left(M_t \int_0^t c_u^0 \, du + \mathbb{E}_t \left[\int_t^T M_u c_u^0 \, du \right] \right) = 0$$

- For log-normal M_u and c_u^0 the $\mathbb{E}[]$ can be computed explicitly
- This leads to approximate solution: $c_t^* \approx c_t^0 + \beta \frac{\partial \hat{c}_t}{\partial \beta}$
- Nice interpretation as "classical" plus adjustment due to habit

References I

Cox, J. and Huang, C. (1989).
Optimal consumption and portfolio policies when asset prices follow a diffusion process.
Journal of Economic Theory, 49(1):33–83.