Discussion of paper Servaas van Bilsen

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Outline

1. Summary
2. Habit Formation
3. Optimal Consumption
4. Discussion
Summary of paper

- Paper studies optimal consumption & portfolio choice in a utility framework
- Utility framework with multiplicative habit formation
- Analytical solution to approximate problem
- Analyse power utility and also stoch.diff.util
Habit Formation

Classical utility considers “absolute” wealth or consumption: $u(c_t)$

Humans don’t look at absolute wealth, they compare relative to a benchmark

Compare two cases:
- Receive €10
- Receive €20, and then pay €10 tax
  - Most likely, you feel less happy in second case, compare to €20

This is the idea of “habit formation”
Preferences with habit formation

- Define utility of consumption $c_t$ relative to habit level $h_t$
- Utility function: $u(c_t, h_t) = v(c_t/h_t)$
- Habit level is “moving average” of previous consumptions:
  $$dh_t = \beta(c_t - h_t) \, dt$$

- Note: consistent when $c_t \equiv h_t \equiv C$
- Interpret $c_t/h_t$ as current consumption relative to average of past consumption levels
- Remark: I like this ratio-model a lot!
- Remark: I don’t think the log-habit that BBL introduce is needed. I think the simple MA formulation used here would be more natural
- Remark: BBL’s $(\beta \ln c_t - \alpha \ln h_t)$ is only consistent for $\alpha = \beta$
Maximisation Problem: the good news

- Use [Cox and Huang, 1989] formulation of optimal consumption problem:

  \[
  \max_{c_t} \mathbb{E} \left[ \int_0^T e^{-\delta t} v \left( \frac{c_t}{h_t} \right) \, dt \right] \\
  \text{s.t. } \mathbb{E} \left[ \int_0^T M_t c_t \, dt \right] = A_0 \quad dh_t = \beta \left( c_t - h_t \right) \, dt
  \]

- Cox-Huang allows for elegant decoupling of optimal consumption and investment strategy: solvable for any utility, even stoch.diff.util

- Solve optimal investment as “delta-hedging” investment that replicates the optimal consumption pattern
Optimal Consumption

Maximisation Problem: the bad news

Optimal consumption problem:

$$\max_{c_t} \mathbb{E} \left[ \int_0^T e^{-\delta t} v \left( \frac{c_t}{h_t} \right) dt \right]$$

s.t. $$\mathbb{E} \left[ \int_0^T M_t c_t dt \right] = A_0 \quad dh_t = \beta(c_t - h_t) dt$$

- Habit $h_t$ makes problem path-dependent
- Consumption $c_t$ has impact on all future habit-levels $h_s$ with $s > t$
- This makes problem difficult to solve...
- Note: the difficulties with $h_s$ reside in the objective function
BBL introduce a change of variables: $\hat{c}_t := c_t/h_t$

Remark: they call this a “dual” formulation, but I think this is only a change of var’s

Recall: $\dot{h}_t = \beta(c_t - h_t)$

Hence: $\dot{h}_t/h_t = \beta(\hat{c}_t - 1)$

Solution: $h_t = h_0 e^{\int_0^t \beta(\hat{c}_u - 1) du}$

Also: $c_t = \hat{c}_t h_0 e^{\int_0^t \beta(\hat{c}_u - 1) du}$

Remark: “classical” utility for $\beta = 0$
Reformulate optimal consumption problem using $\hat{c}_t$:

$$\max_{\hat{c}_t} E \left[ \int_0^T e^{-\delta t} v(\hat{c}_t) dt \right]$$

s.t. $$E \left[ \int_0^T M_t \hat{c}_t h_0 e^{\int_0^t \beta (\hat{c}_u - 1) du} dt \right] = A_0$$

- This is still a path-dependent problem
- But path-dependency has now moved to the budget constraint
Constant solution

- Consider constant solution $\hat{c}_t \equiv 1$
- This is a feasible solution, when budget constraint is satisfied
- Budget constraint:
  $$\int_0^T e^{-rt} h_0 dt = A_0$$
- Note: constant payoff is deterministic, discount with risk-free rate
- Set $h_0 = A_0 / \int_0^T e^{-rt} dt$ to satisfy budget $A_0$.
- Remark: BBL do not use $h_0$, and their solution can be infeasible, and they correct for this in another way
Linearise the budget constraint

- Write general solution $\hat{c}_t = 1 + \eta_t$
- Define the functional $f(\hat{c}) := E\left[\int_0^T M_t \hat{c}_t h_t dt\right]$
- For small values of $\eta_t$ we can approximate this functional as: $f(1 + \eta) \approx f(1) + \int_0^T Df(1) \eta_t dt$ where $Df$ is the Fréchet derivative of the functional $f()$
- We also have the functional $h_t(\hat{c}) = h_0 e^{\int_0^t \beta (\hat{c}_u - 1) du}$ with $h_t(1 + \eta) \approx h_0 + \int_0^t h_0 \beta \eta_u du$
- Hence, we obtain for $f(1 + \eta)$ (using product rule):

$$f(1 + \eta) \approx A_0 + E\left[\int_0^T M_t h_0 \eta_t dt + \int_0^T M_t \int_0^t h_0 \beta \eta_u du dt\right]$$
Optimal Consumption

Linearise the budget constraint (2)

- Change order of integration in last integral:
  
  \[ f(1 + \eta) \approx A_0 + \mathbb{E} \left[ \int_0^T \left( M_t h_0 + \beta h_0 \int_t^T M_u du \right) \eta_t dt \right] \]

- Path-dependency is linearised and gives integral over all “future” \(M_u\)'s

- Bring conditional expectation \(\mathbb{E}_t[M_u] = M_t e^{-r(u-t)}\) inside integral:
  
  \[ f(1 + \eta) \approx A_0 + \mathbb{E} \left[ \int_0^T M_t h_0 \left( 1 + \beta \left( \frac{1-e^{-r(T-t)}}{r} \right) \right) \eta_t dt \right] \]

- BBL “integrate out” the path-dependency. Very clever!

- Budg.con. with adj. pricing kernel: \(f(1 + \eta) \approx A_0 + \mathbb{E} \left[ \int_0^T \hat{M}_t \eta_t dt \right] \)
Approximate Problem

• Approximate optimal consumption problem:

\[ \max_{\eta_t} \mathbb{E} \left[ \int_0^T e^{-\delta t} v(1 + \eta_t) dt \right] \]

s.t. \( \mathbb{E} \left[ \int_0^T \hat{M}_t \eta_t dt \right] = 0 \)

• This problem is now analytically solvable for all utility functions \( v() \).

• Structure of analytical optimal consumption:

\[ \hat{c}_t^* \approx 1 + \eta_t^* = I \left( \lambda e^{\delta t} \hat{M}_t \right) \]

where \( I() \) is inverse of \( v'() \) and \( \lambda \) is scale parameter for budget

• Note: obtain “classical” utility for \( \beta = 0 \)
Emphasise more the difference between “classical” optimal consumption and habit optimal

Look at “path-by-path”, average consumption paths not so interesting

Suggestion: focus more on response to unexpected shock in wealth. How does such a shock impact (expected) future consumption path?

I am worried that the expansion point $\hat{c}_t \equiv 1$ is “too far” from the optimal solution. The linearisation might not be accurate approximation of the true optimal solution.
Alternative linearisation

- Optimal consumption problem using $\hat{c}_t$:

$$\max_{\hat{c}_t} E \left[ \int_0^T e^{-\delta t} v(\hat{c}_t) dt \right]$$

s.t. $E \left[ \int_0^T M_t \hat{c}_t h_t dt \right] = A_0$

- FOC is given by:

$$e^{-\delta t} v'(\hat{c}_t) - \lambda \left( M_t h_t + \beta E_t \left[ \int_t^T M_u \hat{c}_u h_u du \right] \right) = 0$$

- Parameter $\lambda$ is Lagrange multiplier for budget constraint
- FOC is path-dependent due to integral-term
Alternative linearisation (2)

• When $\beta = 0$ then $h_t \equiv h_0$ and we obtain “classical” optimal solution:
  
  $$e^{-\delta t} v'(c_t^0) = \lambda M_t \Rightarrow c_t^0 = I\left(\lambda e^{\delta t} M_t\right)$$

  with $I()$ is inverse of $v'()$.

• For $v()$ is power-util and $M_t$ log-normal, then $c_t^0$ is also lognormal

• Consider parameter expansion around $\beta = 0$. We can compute the derivative $\partial \hat{c}_t / \partial \beta$ at $\beta = 0$ from the FOC:

  $$e^{-\delta t} v''(c_t^0) \frac{\partial \hat{c}_t}{\partial \beta} - \lambda \left( M_t \int_0^t c_u^0 \, du + \mathbb{E}_t \left[ \int_t^T M_u c_u^0 \, du \right] \right) = 0$$

• For log-normal $M_u$ and $c_u^0$ the $\mathbb{E}[]$ can be computed explicitly

• This leads to approximate solution: $c_t^* \approx c_t^0 + \beta \frac{\partial \hat{c}_t}{\partial \beta}$

• Nice interpretation as “classical” plus adjustment due to habit