How to Invest and Draw-Down Wealth? A Utility-Based Analysis

SERVAAS VAN BILSEN, LANS BOVENBERG, and ROGER LAEVEN*

ABSTRACT

This paper explores how Baby Boomers should invest and draw-down their accumulated wealth over the rest of their lives. To answer this question we build a consumption and portfolio choice model with multiplicative internal habit formation and stochastic differential utility. We show analytically that after a wealth shock it is optimal to adjust both the level and future growth rates of consumption, implying gradual response of consumption to financial shocks. Furthermore, fostering the ability to keep catching up with the internal habit creates upward pressure on expected consumption growth. Welfare losses associated with popular alternative investment and draw-down strategies can be large.

JEL classification: D81, D91, G11.

*Van Bilsen is with the Department of Quantitative Economics, University of Amsterdam; Bovenberg is with the Department of Economics, Tilburg University; Laeven is with the Department of Quantitative Economics, University of Amsterdam. We thank Olivia Mitchell and seminar and conference participants at Oxford, Tilburg, Wharton, CEPAR in Sydney, and the thirteenth Winter School on Mathematical Finance. This research was funded by the Netherlands Organization for Scientific Research (NWO) under grant NWO Vidi 2009 (Van Bilsen, Laeven) and the European Commission under the seventh framework program (MOPACT).

As the first wave of Baby Boomers moves into retirement, their future promises to be very different from that of their parents who enjoyed resilient Social Security and defined benefit pension plans. Retirees today face a very different challenge regarding retirement security, in that they are the first generation where retirement wealth was accumulated primarily in personal retirement accounts.[1] As their nest eggs will not automatically be annuitized, Baby Boomers thus confront the important question of how they should invest and draw-down their accumulated wealth over the rest of their lives. The objective of the present paper is to analyze this question from the perspective of a utility maximizing individual.

Financial advisors commonly recommend to split the investment portfolio into 60% risky assets and 40% risk-free assets, and to draw-down 4 to 5% of retirement wealth per year (Polyak, 2005; Whitaker, 2005). Other popular draw-down strategies include the fixed benefit approach (i.e., the individual withdraws a specified dollar amount each year until his retirement wealth is depleted), and the remaining lifetime approach (i.e., the withdrawal fraction rises with the remaining lifetime); see, e.g., Dus, Maurer, and Mitchell (2005); Horneff, Maurer, Mitchell, and Dus (2008). However, these popular draw-down strategies are arguably ad hoc, and are typically neither founded upon nor corroborated by the individual's preferences (MacDonald, Jones, Morrison, Brown, and Hardy, 2013). Alternatively, retirees can buy annuities. But while fixed annuities are usually too expensive to be an attractive financial product (especially in low interest rate regimes), variable annuities often generate volatile fluctuations in payouts (see, e.g., Chai, Horneff, Maurer, and Mitchell, 2011; Maurer, Mitchell, Rogalla, and Kartashov, 2013b).[2]

Thus, the need for a utility-based approach to analyze the investment and draw-down strategies implemented by Baby Boomers is evident. Expected utility theory with constant relative risk aversion (CRRA) is the most commonly adopted preference model to derive optimal consumption and portfolio choice. As is well-known at least since Merton (1969, 1971) and Samuelson (1969), a CRRA agent fully absorbs a wealth shock into the level (and not future growth rates) of consumption. Under CRRA, the year-on-year volatility of consumption thus matches the year-on-year volatility of wealth. However, evidence of

[1] The percentage of total U.S. retirement assets accounted for by individual retirement accounts and defined contribution pension plans rose from about 18% in 1974 to 54% in 2013 (Investment Company Institute, 2014).

[2] Insurers have more recently developed variable annuities for which surpluses earned in good years support payouts in bad years (see, e.g., Guillén, Jørgensen, and Nielsen, 2006; Maurer, Rogalla, and Siegelin, 2013a; Maurer, Mitchell, Rogalla, and Siegelin, 2014). This rapidly growing form of variable annuities is, however, opaque and difficult to value.

violations of the assumptions underlying CRRA utility – in our setting, the intertemporal independence assumption in particular – has led authors to seek for alternative models. The literature has put forward a variety of alternatives, perhaps most noticeably habit formation utility and (continuous-time) recursive utility or stochastic differential utility (SDU).[3] The present paper proposes and analyzes a model with internal habit formation and SDU, and derives the resulting investment and draw-down strategies in closed-form.

Our contribution is three-fold. First, we build a rich consumption and portfolio choice model with multiplicative habit formation, an endogenous internal habit level, and SDU. Our general model encompasses many interesting special cases such as SDU without multiplicative internal habit formation, multiplicative internal habit formation without SDU, multiplicative habit formation with an external (deterministic) habit, and CRRA utility. Second, we develop an approximation method to accurately solve our general consumption and portfolio choice problem analytically. Third, we analyze the resulting optimal investment and draw-down strategies for a Baby Boomer, and conduct a welfare analysis. We now specify each of our contributions in more detail.

We assume that the agent derives utility from the ratio between consumption and the habit level. The ratio (or multiplicative) model of habit formation, first analyzed by Abel (1990), is the only model we know of that allows consumption to fall below the habit level while simultaneously maintaining the property of constant (i.e., state-independent) relative risk aversion. A number of authors consider an agent who derives utility from the difference – rather than the ratio – between consumption and the habit level.[4] The optimal consumption choice implied by the difference (or additive) model of habit formation (Constantinides, 1990) exceeds the habit level in each economic scenario. This addictive behavior of consumption is, however, doubtful (see, e.g., Detemple and Karatzas, 2003). Indeed, empirical evidence showing significant declines in consumption levels during recessions contradicts the addictive property. Furthermore, in the difference model of habit formation, relative risk aversion depends on (surplus) wealth. This may be undesirable from a normative point of view as it leads to very low equity holdings in bad economic scenarios. Also, in our ratio model of habit formation, the portfolio

[3]The notion of SDU was introduced by Duffie and Epstein (1992) as a continuous-time limit of the preference models studied by Epstein and Zin (1989) and by Kreps and Porteus (1978). Life cycle models with Epstein-Zin preferences or internal habit formation have been widely studied in the literature. For Epstein-Zin preferences, see, e.g., Chacko and Viceira (2005); Gomes and Michaelides (2008); for the ratio model of habit formation, see, e.g., Gomes and Michaelides (2003).

[4]See, e.g., Constantinides (1990); Detemple and Zapatero (1991, 1992); Schroder and Skiadas (2002); Bodie, Detemple, Otruba, and Walter (2004); Munk (2008).

strategy is state-independent, and thus easy to implement. We assume that the habit level is a geometric (rather than an arithmetic) weighted average of the agent's own past consumption choices. Hence the habit level is internal to the agent and endogenously determined.[5] In the model of Abel (1990), the habit level depends only on consumption in the previous period.

Due to the endogeneity of the habit level, our consumption and portfolio choice problem cannot be solved in closed-form. By developing a linearization to the budget constraint, we are able to derive an analytical closed-form solution to the approximate optimization problem. Linearization of the budget constraint is not uncommon in the economic literature; see, in a different context, e.g., Campbell and Mankiw (1991); Fuhrer (2000). Our approximation method is shown to be very accurate when consumption stays close to the habit level, and when the habit level responds slowly to consumption. Indeed, our numerical results show that the approximation error (measured in terms of the relative decline in certainty equivalent consumption) is typically of order 0.01.

Our results can be summarized as follows. First, we show analytically that after a wealth shock, it is optimal to adjust both the level and the future growth rates of consumption, implying gradual response of consumption to financial shocks. This justifies a mechanism for smoothing the change in consumption due to financial shocks. The parameters in our model (i.e., the coefficient of relative risk aversion, the strength of internal habit formation, and the deprecation rate of the habit level) have clear economic interpretations, controlling the features of the optimal strategy. The coefficient of relative risk aversion determines the effect of a wealth shock on the level of consumption. The less risk averse the agent, the larger the effect of a wealth shock on the level of consumption. The strength of internal habit formation determines the effect of a wealth shock on future growth rates of consumption. The larger the strength of internal habit formation, the larger the effect of a wealth shock on future growth rates of consumption. We also show that the lower the deprecation rate of the habit level, the longer it takes to fully absorb a wealth shock into current and future consumption.

Second, as the agent adjusts both the level and future growth rates of consumption after a shock, the year-on-year volatility of consumption is less than the year-on-year volatility of wealth. Thus, a risky investment portfolio does not automatically imply

[5] Corrado and Holly (2011) show that for the ratio model of habit formation, a geometric habit specification is more desirable than an arithmetic habit specification. In particular, they prove that under the geometric habit specification, overall utility decreases as the strength of internal habit formation increases.

a high year-on-year volatility of consumption. This finding stands in sharp contrast to many popular portfolio and draw-down strategies (e.g., the remaining lifetime approach) where an increase in the risk of the investment portfolio directly translates into a higher year-on-year volatility of consumption. Furthermore, we show that the agent chooses to reduce the fraction of wealth invested in risk-bearing assets as the end of life approaches. Indeed, the agent has less time to absorb a wealth shock as he ages.

Third, we show that for a finitely-lived agent with a fixed lifetime, the expected growth rate of consumption increases with the strength of internal habit formation as well as with age.[6] If the agent were to live forever, the effects of the strength of internal habit formation and age on the expected growth rate of consumption would be absent. Indeed, in the case of an infinite horizon, the effect of a marginal change in consumption on future habit levels is independent of age, while in the more realistic case of a finite horizon, the effect of a marginal change in consumption on future habit levels decreases with age. Thus, in the finite horizon case, fostering the ability to keep catching up with the internal habit creates an upward pressure on expected consumption growth. That is, the agent prefers to postpone consumption because the utility gain of a marginal increase in consumption rises with age.

The elasticity of intertemporal substitution is in our base model, which combines CRRA utility with multiplicative internal habit formation, intimately related to the coefficient of relative risk aversion: the lower the degree of relative risk aversion, the higher the agent's willingness to engage in intertemporal substitution. In an extension of our model, we study the consumption and portfolio choice of an agent with preferences that combine SDU with multiplicative internal habit formation. This extended preference model allows us to disentangle the elasticity of intertemporal substitution from the coefficient of relative risk aversion while simultaneously maintaining the property of multiplicative internal habit formation. As a result, the change in the median growth rate of consumption following a change in the interest rate is no longer related to the coefficient of relative risk aversion. Applying our linearization of the budget constraint, we are still able to derive the agent's consumption and portfolio choice under the extended model in closed-form. A model that combines SDU with multiplicative internal habit formation has, to the best of our knowledge, not yet been studied in existing literature. The closest

[6]We note that in the case of an uncertain lifetime and the absence of longevity insurance, survival probabilities create a downward pressure on expected consumption growth (see Yaari, 1965). In that case, expected consumption growth is determined by internal habit formation as well as the shape of the survival curve.

to the current paper in this respect is Schroder and Skiadas (1999), who analytically study SDU but do not consider multiplicative internal habit formation.

Finally, we conduct a welfare analysis in order to assess the impact of pursuing alternative suboptimal investment and draw-down strategies on the agent's welfare. More specifically, we compute welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with implementing the remaining lifetime approach, the Merton approach (Merton, 1969) and the difference model of habit formation. Our results show that welfare losses can be large, especially when the agent exhibits a high degree of internal habit formation. We also show that welfare losses are typically larger for the remaining lifetime approach than for the Merton approach.

The rest of this paper is structured as follows. Section I describes the financial market, the preferences and the agent's maximization problem. Section II presents the solution method used to solve the maximization problem. This solution method involves applying a linearization to the budget constraint. Section III derives and studies the optimal consumption and portfolio policies. This section also conducts a welfare analysis. Section IV explores the consumption and portfolio choice of an individual with preferences that combine SDU with multiplicative internal habit formation. Section V studies the approximation error due to applying a linearization to the budget constraint. Finally, Section VI provides concluding remarks. Proofs are relegated to the Appendix.

I. An Internal Habit Formation Model

A. The Financial Market

We consider a Black and Scholes financial market. Let $T > 0$ be a (possibly infinite) terminal time. The financial market consists of a money market account and a risky stock, which are traded continuously. The price of the money market account, i.e., B_t, evolves according to

$$\frac{\mathrm{d}B_t}{B_t} = r\,\mathrm{d}t. \tag{1}$$

Here r stands for the interest rate. The risky stock price S_t satisfies

$$\frac{\mathrm{d}S_t}{S_t} = (r + \lambda\sigma)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \tag{2}$$

Here λ denotes the equity risk premium per unit of risk (i.e., Sharpe ratio), σ stands for the diffusion parameter (i.e., stock return volatility), and W_t corresponds to a standard Brownian motion.

It is well-known that if we exclude arbitrage opportunities in this financial market, the pricing kernel (or stochastic discount factor) M_t satisfies (see, e.g., Karatzas and Shreve, 1998)

$$\frac{\mathrm{d}M_t}{M_t} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}W_t. \tag{3}$$

In the numerical computations, we use the following financial market parameter values: the Sharpe ratio $\lambda = 20\%$, the risk-free rate $r = 1\%$, and the stock return volatility $\sigma = 20\%$. These parameter values are the same as those used by Gomes, Kotlikoff, and Viceira (2008).

B. Preferences

We represent preferences by the ratio habit model originally introduced by Abel (1990). More specifically, the instantaneous utility function is given by[7]

$$u\left(c_t, h_t\right) = v\left(\frac{c_t}{h_t}\right) = \frac{1}{1-\gamma}\left(\frac{c_t}{h_t}\right)^{1-\gamma}. \tag{4}$$

Here, c_t and h_t denote the individual's consumption choice and his habit level at time t, respectively, and $\gamma > 0$ models the individual's relative risk aversion. In the difference habit model (Constantinides, 1990), relative risk aversion depends on surplus consumption $c_t - h_t$. As a result, the optimal solution of the difference model is fundamentally different than the optimal solution of the ratio model (see Section III for further details).

Inspired by Kozicki and Tinsley (2002) and Corrado and Holly (2011), the *log* habit level $\log h_t$ satisfies the following dynamic equation:

$$\mathrm{d}\log h_t = \left(\beta \log c_t - \alpha \log h_t\right)\mathrm{d}t, \qquad \log h_0 = 0. \tag{5}$$

Here $\log h_0$ denotes the *initial* log habit level. We normalize $\log h_0$ to zero (i.e., $h_0 = 1$). We thus measure initial retirement wealth, consumption and the habit level in terms of h_0.[8] The preference parameter $\alpha \geq 0$ represents the rate at which the log habit level

[7]If $\gamma = 1$, then $u\left(c_t, h_t\right) = v\left(c_t/h_t\right) = \log\left\{c_t/h_t\right\}$.

[8]The utility function is *not* invariant to the unit of measurement. The agent should thus change the values of the preference parameters if the unit of measurement changes.

depreciates. If α is small, then the log habit level exhibits a low degree of depreciation (or, equivalently, a high degree of persistence). The preference parameter $\beta \geq 0$ indexes the extent to which the current log habit level responds to current log consumption. If β is large, then current log consumption has a large impact on the log habit level. We impose the following restriction on the agent's preference parameters:

$$\alpha \geq \beta. \tag{6}$$

The parameter restriction (6) prevents the habit level from growing exponentially over time.

The habit level h_s conditional on information available at time t is explicitly given by[9]

$$h_s = \exp\left\{\int_t^s \beta \exp\left\{-\alpha(s-u)\right\} \log c_u \,\mathrm{d}u\right\} \times \exp\left\{\exp\left\{-\alpha(s-t)\right\} \log h_t\right\}. \tag{7}$$

Equation (7) shows that we can factor the habit level h_s into two components: one dependent upon consumption choices between time t and s (i.e., the stochastic component) and the other not (i.e., the deterministic component). The parameter β indexes the importance of the stochastic component relative to the deterministic component (given α). The stochastic component becomes less important as β decreases. Indeed, if β equals zero, then

$$\exp\left\{\beta \exp\left\{-\alpha(s-u)\right\} \log c_u\right\} = 1. \tag{8}$$

The stochastic component is an exponentially weighted *product* (and not an exponentially weighted sum) of the agent's consumption choices between time t and time s. The habit level thus depends more on consumption in the recent past than it depends on consumption in the distant past.

Corrado and Holly (2011) demonstrate that for the ratio model of habit formation, a specification in which the habit level is geometric in consumption is more desirable than a specification in which the habit level is arithmetic in consumption. In particular,

[9]Equation (7) is equivalent to:

$$h_s = (h_t)^{\exp\{-\alpha(s-t)\}} \times \prod_t^s (c_u)^{\beta \exp\{-\alpha(s-u)\}\,\mathrm{d}u}.$$

Here $\prod_t^s (c_u)^{\beta \exp\{-\alpha(s-u)\}\,\mathrm{d}u} = \lim_{\Delta u \to 0} \exp\left\{\sum_{i=t/\Delta u}^{s/\Delta u} \beta \exp\left\{-\alpha\left(s - i\Delta u\right)\right\} \log c_{i\Delta u} \Delta u\right\}$ denotes the geometric integral.

they prove that under the geometric habit specification, overall utility decreases as the endogeneity parameter β increases, provided that consumption is larger than unity.[10] This property does not hold true in the arithmetic habit specification. Furthermore, the assumption of a geometric habit specification makes the agent's maximization problem analytically tractable.

C. Maximization Problem

This section formulates the agent's maximization problem. Denote by A_t the agent's retirement wealth at time t, and by π_t the fraction of wealth invested in the risky stock at time t. Wealth evolves according to

$$\mathrm{d}A_t = (r + \pi_t \lambda \sigma)\, A_t\, \mathrm{d}t - c_t\, \mathrm{d}t + \pi_t \sigma A_t\, \mathrm{d}W_t, \qquad A_0 \geq 0 \text{ given.} \tag{9}$$

Equation (9) is referred to as the agent's dynamic budget equation. This equation shows that the agent's retirement wealth equals the agent's initial retirement wealth, plus any gains from trading, minus cumulative consumption.

The agent aims to maximize expected lifetime utility

$$U_0 = \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_u\, \mathrm{d}u} v\left(\frac{c_t}{h_t}\right) \mathrm{d}t\right], \tag{10}$$

over the set of all admissible consumption and portfolio strategies subject the dynamic budget constraint (9) and the habit formation process (5).[11] Here $\mathbb{E}\left[\cdot\right]$ corresponds to the (unconditional) expectation operator, δ_u stands for the subjective rate of time preference at time u, and $v\left(c_t/h_t\right)$ represents the agent's instantaneous utility function (see equation (4)).[12]

We can, by virtue of the martingale approach (Pliska, 1986; Karatzas, Lehoczky, and Shreve, 1987; Cox and Huang, 1989, 1991), transform the agent's maximization problem

[10] Consumption is typically larger than unity because with internal habit formation, the agent has a tendency to postpone consumption (i.e., consumption tends to exhibit a positive expected growth rate); see Section III.

[11] For the definition of admissible consumption and portfolio strategies, see, e.g., Karatzas and Shreve (1998).

[12] Alternatively, we can view δ_u as the sum of the subjective rate of time preference at time u and the force of mortality at time u. More specifically, in the case of deterministic mortality risk, expected

into the following equivalent problem:

$$\begin{aligned}
&\underset{c}{\text{maximize}} \quad \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_u \, \mathrm{d}u} v\left(\frac{c_t}{h_t}\right) \mathrm{d}t\right] \\
&\text{subject to} \quad \mathbb{E}\left[\int_0^T M_t c_t \, \mathrm{d}t\right] \leq A_0, \qquad \mathrm{d}\log h_t = (\beta \log c_t - \alpha \log h_t)\, \mathrm{d}t.
\end{aligned} \tag{11}$$

The optimal portfolio choice π_t^* is determined in such a way that it finances the optimal consumption choice c_t^*.

The optimal consumption choice c_t^* (if it exists) satisfies the following first-order optimality condition:

$$e^{-\int_0^t \delta_u \, \mathrm{d}u} \frac{1}{h_t}\left(\frac{c_t}{h_t}\right)^{-\gamma} - \frac{\beta}{c_t}\mathbb{E}_t\left[\int_t^T e^{-\int_0^s \delta_u \, \mathrm{d}u} e^{-\alpha(s-t)}\left(\frac{c_s}{h_s}\right)^{1-\gamma} \mathrm{d}s\right] = yM_t. \tag{12}$$

Here $y \geq 0$ denotes the Lagrange multiplier. The left-hand side of equation (12) represents marginal utility, whereas the right-hand side denotes marginal cost. We can decompose marginal utility into two components: the first representing the effect of an increase in consumption on current instantaneous utility, and the second representing the effect of an increase in consumption on future instantaneous utilities. We cannot obtain the optimal consumption choice c_t^* in closed-form due to the presence of the conditional expectation operator $\mathbb{E}_t[\cdot]$ in the second component. The next section presents an approximate problem to problem (11) that can be solved analytically.

lifetime utility is given by (see Yaari, 1965)

$$\begin{aligned}
U_0 &= \mathbb{E}\left[\int_0^T e^{-\int_0^t \rho_u \, \mathrm{d}u} \mathbb{1}_{[t \leq D]} v\left(\frac{c_t}{h_t}\right) \mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{-\int_0^t \rho_u \, \mathrm{d}u} {}_t p_x v\left(\frac{c_t}{h_t}\right) \mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T e^{-\int_0^t \rho_u \, \mathrm{d}u} e^{-\int_0^t \mu_{x+u} \, \mathrm{d}u} v\left(\frac{c_t}{h_t}\right) \mathrm{d}t\right] = \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_u \, \mathrm{d}u} v\left(\frac{c_t}{h_t}\right) \mathrm{d}t\right].
\end{aligned}$$

Here ρ_u represents the subjective rate of time preference at time u, D is the stochastic date of death, ${}_t p_x$ denotes the probability that an agent aged x at time 0 will survive to time $x+t$, and μ_{x+u} is the deterministic force of mortality at age $x+u$.

II. The Solution Method

A. Applying a Change of Variable

By applying a change of variable, we can transform the agent's maximization problem (11) into an equivalent *dual* problem. Denote by $\widehat{c}_t$ the ratio between the agent's consumption choice and the habit level; that is,

$$\widehat{c}_t \equiv \frac{c_t}{h_t}. \tag{13}$$

We refer to $\widehat{c}_t$ as the agent's *dual consumption choice*. We can express the dynamics of the log habit level in terms of the agent's log dual consumption choice $\log \widehat{c}_t$ (substitute $\log c_t = \log h_t + \log \widehat{c}_t$ into equation (5)):

$$\begin{aligned} \mathrm{d} \log h_t &= \left(\beta \left[\log h_t + \log \widehat{c}_t\right] - \alpha \log h_t\right) \mathrm{d}t \\ &= \left(\beta \log \widehat{c}_t - [\alpha - \beta] \log h_t\right) \mathrm{d}t. \end{aligned} \tag{14}$$

Hence the agent's habit level h_s conditional on all information available at time t is explicitly given by[13]

$$\begin{aligned} h_s = \exp &\left\{ \int_t^s \beta \exp\left\{-(\alpha - \beta)(s - u)\right\} \log \widehat{c}_u \, \mathrm{d}u \right\} \\ &\times \exp\left\{\exp\left\{-(\alpha - \beta)(s - t)\right\} \log h_t\right\}. \end{aligned} \tag{15}$$

Equation (15) shows that due to the parameter restriction (6), the habit level does not grow exponentially over time. We define the dual *static* budget constraint as follows:

$$\mathbb{E}\left[\int_0^T M_t h_t \widehat{c}_t \, \mathrm{d}t\right] \leq A_0. \tag{16}$$

[13]Equation (15) is equivalent to:

$$h_s = (h_t)^{\exp\{-(\alpha-\beta)(s-t)\}} \times \prod_t^s (\widehat{c}_u)^{\beta \exp\{-(\alpha-\beta)(s-u)\}\, \mathrm{d}u}.$$

Equation (16) follows from substituting $c_t \equiv h_t \widehat{c}_t$ into the original static budget constraint. The agent's dual maximization problem is thus given by

$$\begin{aligned} &\underset{\widehat{c}_t}{\text{maximize}} \quad \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_u \,\mathrm{d}u} v\left(\widehat{c}_t\right) \mathrm{d}t\right] \\ &\text{subject to} \quad \mathbb{E}\left[\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t\right] \leq A_0, \qquad \mathrm{d}\log h_t = \left(\beta \log \widehat{c}_t - [\alpha - \beta] \log h_t\right) \mathrm{d}t. \end{aligned} \tag{17}$$

We can obtain the optimal consumption choice c_t^* from the optimal *dual* consumption choice $\widehat{c}_t^*$ as follows:

$$c_t^* = h_t^* \widehat{c}_t^*. \tag{18}$$

Here h_t^* is the optimal habit level at time t implied by substituting the agent's optimal past dual consumption choices c_u^* $(u \leq t)$ into equation (15).

To solve the agent's maximization problem (11), we can thus restrict ourselves to finding a solution to problem (17). The agent's optimal consumption choice c_t^* then follows from applying equation (18). We can however still not solve the agent's maximization problem (11) analytically because the dual static budget constraint (16) depends on the agent's dual consumption choice in a nonlinear way. Indeed, substitution of h_t into equation (16) shows that the dual static budget constraint depends on the agent's dual consumption choice $\widehat{c}_t$ nonlinearly. The next section develops a linearization to the agent's dual static budget constraint (16). After applying this linearization, we can obtain the agent's dual consumption choice in closed-form.

B. Linearizing the Budget Constraint

This section linearizes the left-hand side of the agent's dual budget constraint (16) around the consumption trajectory $\widehat{c} = c/h = 1$. We expect (and verify in Section V) that the approximation error is accurate when the agent's consumption choice c_t stays close to the habit level h_t, and when the endogeneity parameter β is small or the depreciation parameter α is large. Indeed, if the habit level is completely exogenous (i.e., $\beta = 0$ or $\alpha = \infty$), the solution to problem (17) coincides with the solution to problem (11) (see also equation (12), which shows that we can solve the first-order optimality condition analytically if $\beta = 0$ or $\alpha = \infty$). We expect the approximation to be less accurate when the agent's consumption choice c_t deviates much from the habit level h_t. Section V examines the approximation error induced by applying a linearization to the agent's dual budget constraint.

By applying a first-order Taylor series approximation, we can write the left-hand side of the agent's dual budget constraint (16) as follows (see Appendix)

$$\begin{aligned}\mathbb{E}\left[\int_0^T M_t h_t \widehat{c}_t \,\mathrm{d}t\right] &\approx \mathbb{E}\left[\int_0^T M_t \,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T M_t\left(1+\beta P_t\right)\left(\widehat{c}_t - 1\right)\mathrm{d}t\right] \\ &= -\beta\mathbb{E}\left[\int_0^T M_t P_t \,\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T \widehat{M_t}\widehat{c}_t \,\mathrm{d}t\right].\end{aligned} \tag{19}$$

Here $\widehat{M_t} \equiv M_t\left(1+\beta P_t\right)$ denotes the adjusted pricing kernel, and P_t stands for the price of a bond paying a continuous coupon, i.e.,[14]

$$P_t \equiv \mathbb{E}_t\left[\int_t^T \frac{M_s}{M_t} e^{-(\alpha-\beta)(s-t)}\,\mathrm{d}s\right] = \frac{1}{r+\alpha-\beta}\left(1 - e^{-(r+\alpha-\beta)(T-t)}\right). \tag{20}$$

C. The Approximate Problem

This section presents an approximate problem to problem (17) based on linearizing the left-hand side of the dual budget constraint (16). The approximate problem is given by

$$\begin{aligned}&\underset{\widehat{c}_t}{\text{maximize}} \quad \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_u \,\mathrm{d}u} v\left(\widehat{c}_t\right)\mathrm{d}t\right] \\ &\text{subject to} \quad \mathbb{E}\left[\int_0^T \widehat{M_t}\widehat{c}_t \,\mathrm{d}t\right] \leq \widehat{A}_0.\end{aligned} \tag{21}$$

Here $\widehat{A}_0$ denotes the *adjusted* initial wealth. We can obtain the agent's maximization problem (21) from (17) as follows. First, we replace the left-hand side of the static dual budget constraint in (17) by equation (19). Second, we eliminate the constant term

$$-\beta\mathbb{E}\left[\int_0^T M_t P_t \,\mathrm{d}t\right] \tag{22}$$

from the *new* static dual budget constraint. We are allowed to do this because the constant term (22) does not play a role in determining the first-order optimality condition. Finally, we redefine the agent's initial wealth A_0 in such a way that the optimal solution $\widehat{c}_t^\star$ is budget-feasible. That is, we determine the initial level of the agent's optimal dual

[14]We can also view P_t as the amount of wealth needed to finance the consumption stream $\log c_s / \log h_t$ if $c_s = h_s$ for every $s \geq t$.

consumption choice (i.e., the Lagrange multiplier) in such a way that

$$\mathbb{E}\left[\int_0^T M_t h_t^\star \widehat{c}_t^\star \,\mathrm{d}t\right] = A_0. \tag{23}$$

Here $h_t^\star$ is the agent's habit level at time t implied by substituting the agent's optimal past dual consumption choices $\widehat{c}_u^\star$ ($u \leq t$) into (15). Straightforward computations show that the agent's adjusted initial wealth $\widehat{A}_0$ is given by

$$\begin{aligned}\widehat{A}_0 &= A_0 + \left(\mathbb{E}\left[\int_0^T \widehat{M}_t \widehat{c}_t^\star \,\mathrm{d}t\right] - \mathbb{E}\left[\int_0^T M_t h_t^\star \widehat{c}_t^\star \,\mathrm{d}t\right]\right) \\ &= \mathbb{E}\left[\int_0^T \widehat{M}_t \widehat{c}_t^\star \,\mathrm{d}t\right].\end{aligned} \tag{24}$$

Equation (24) shows that the agent's adjusted initial wealth $\widehat{A}_0$ equals the agent's initial wealth A_0 plus the approximation error evaluated at the optimal solution. We can only compute the value of $\widehat{A}_0$ after problem (21) has been optimized.

III. Dynamic Consumption and Portfolio Choice

A. Consumption Choice

Theorem 1 below presents the optimal solution to the agent's maximization problem (21).

THEOREM 1: *Consider an agent with the utility function* (4) *and habit formation process* (5) *who solves the consumption and portfolio choice problem* (21). *Denote by $h_t^\star$ the habit level implied by substituting the agent's optimal past dual consumption choices $\widehat{c}_u^\star \equiv c_u^\star / h_u^\star$ ($u \leq t$) into equation* (15), *and by y the Lagrange multiplier associated with the static budget constraint in* (21). *Then the optimal consumption choice $c_t^\star$ is given by*

$$c_t^\star = h_t^\star \left(y e^{\int_0^t \delta_u \,\mathrm{d}u} \widehat{M}_t\right)^{-\frac{1}{\gamma}}. \tag{25}$$

The Lagrange multiplier $y \geq 0$ is determined in such a way that the agent's original budget constraint holds with equality.

A.1. Infinite Terminal Time

This section analyzes the agent's consumption choice for the case where the terminal time T equals infinity. This assumption does not necessarily imply that the agent lives forever. Indeed, if we also take into account mortality risk (see footnote 12), then T stands for the maximum age the agent can possibly reach. The Appendix shows that we can write the agent's consumption choice $c_t^\star$ (see (25) where $c_t^\star$ is expressed in terms of the state variables $h_t^\star$ and $\widehat{M}_t$) in terms of past financial shocks as follows:

$$c_t^\star = (c_0^\star)^{q_t/q_0} \exp\left\{ \int_0^t q_{t-u} \frac{1}{\gamma} \left(r + \frac{1}{2}\lambda^2 - \delta_u \right) \mathrm{d}u + \int_0^t q_{t-u} \frac{1}{\gamma} \lambda \, \mathrm{d}W_u \right\}. \tag{26}$$

The parameter q_u is defined as follows:

$$\begin{aligned} q_u &\equiv 1 + \frac{\beta}{\alpha - \beta} \left(1 - \exp\left\{-(\alpha - \beta)u\right\}\right) \\ &= q_0 + (q_\infty - q_0)\left(1 - \exp\left\{-\eta u\right\}\right). \end{aligned} \tag{27}$$

Here

$$q_0 = 1, \tag{28}$$

$$q_\infty = 1 + \frac{\beta}{\alpha - \beta}, \tag{29}$$

$$\eta \equiv \alpha - \beta. \tag{30}$$

We can view

$$\bar{q}_u \equiv q_u/\gamma \tag{31}$$

as the exposure of future log consumption $\log c_{t+u}^\star$ to a current financial shock $\lambda \, \mathrm{d}W_t$. We make the following observations. First, the risk exposure $\bar{q}_u$ increases with the horizon u: a current financial shock has a larger impact on log consumption in the distant future than it has on log consumption in the near future. This implies that consumption responds gradually to financial shocks. It provides a utility-based foundation for the existence of smoothing mechanisms in drawing-down accumulated wealth by dampening the change in consumption due to financial shocks. Second, the risk exposure of current log consumption $\log c_t^\star$ to a current financial shock $\lambda \, \mathrm{d}W_t$, i.e., $\bar{q}_0$, decreases with the coefficient of relative risk aversion γ. Hence the coefficient of relative risk aversion γ determines the effect of a current financial shock on the level of log consumption (i.e.,

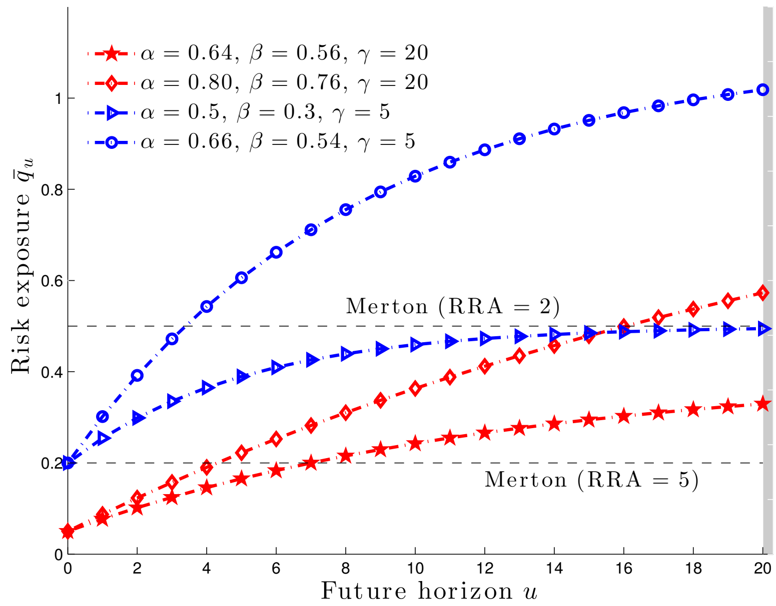

Figure 1. Illustration of the risk exposure of future log consumption to a current financial shock. The figure illustrates the risk exposure $\bar{q}_u$ (i.e., the risk exposure of future log consumption $\log c^{\star}_{t+u}$ to a current financial shock $\lambda\, \mathrm{d}W_t$) as a function of the horizon u for various sets of parameter values. The figure also shows the Merton risk exposure for RRA $= 2$ and RRA $= 5$. Here RRA stands for relative risk aversion.

current log consumption). Third, $\beta/(\alpha - \beta)$ determines the effect of a current financial shock on future growth rates of consumption. If the endogeneity parameter β is large or the depreciation parameter η is small, then a current financial shock has a large effect on future growth rates of consumption (see also equation (15)). Fourth, we can view $\eta = \alpha - \beta$ as the rate at which $\bar{q}_u$ converges to $\bar{q}_\infty$. If η is small (i.e., the habit level depreciates at a slow pace), then it takes a long time to fully absorb a financial shock into current and future consumption. Finally, the Merton consumption strategy (see Merton, 1969) emerges as a special case when $\bar{q}_u = 1/\gamma$ for all u. The risk exposure of an agent with CRRA utility is always smaller than the risk exposure of an agent with utility function (4), given γ. Figure 1 shows $\bar{q}_u$ (expressed relative to $\sigma = 20\%$) as a function of the horizon u for various sets of parameter values. We choose the parameter values such that the average risk exposure matches the risk exposure of a CRRA agent.

Equation (31) demonstrates that the parameters $\bar{q}_0$ (i.e., the exposure of current log consumption to a current shock), $\bar{q}_\infty$ (i.e., the exposure of long-term log consumption to a current financial shock), and η (i.e., the time it takes to absorb a financial shock) fully characterize the risk exposure $\bar{q}_u$. We can *uniquely* identify the agent's original preference parameters α, β and γ from $\bar{q}_0$, $\bar{q}_\infty$ and η:

$$\alpha = \frac{\bar{q}_\infty}{\bar{q}_0}\eta, \tag{32}$$

$$\beta = \frac{\bar{q}_\infty - \bar{q}_0}{\bar{q}_0}\eta, \tag{33}$$

$$\gamma = \frac{1}{\bar{q}_0}. \tag{34}$$

The Appendix shows that log consumption $\log c_t^\star$ evolves according to

$$\mathrm{d}\log c_t^\star = \log F_t^{\mathrm{d}t} + q_0\frac{1}{\gamma}\left(r + \frac{1}{2}\lambda^2 - \delta_t\right)\mathrm{d}t + q_0\frac{1}{\gamma}\lambda\,\mathrm{d}W_t. \tag{35}$$

Here

$$\begin{aligned}\log F_t^v \equiv &\int_0^t (q_{t+v-u} - q_{t-u})\frac{1}{\gamma}\left(r + \frac{1}{2}\lambda^2 - \delta_u\right)\mathrm{d}u + \left(\frac{q_{t+v}}{q_0} - \frac{q_t}{q_0}\right)\frac{1}{\gamma}\log c_0^\star \\ &+ \int_0^t (q_{t+v-u} - q_{t-u})\frac{1}{\gamma}\lambda\,\mathrm{d}W_u.\end{aligned} \tag{36}$$

The left-hand side of equation (35) consists of three terms. The first two terms denote the median growth rate of log consumption. The term $\log F_t^{\mathrm{d}t}$ represents past financial shocks that are reflected into the current median growth rate of log consumption. This term disappears if $\beta = 0$ or $\bar{q}_u = 1/\gamma$ for all u. The second term represents the desired growth rate of consumption. The median value of log consumption stays constant over time if $\beta = 0$, $\delta_u = \delta$ and $r = \delta - \frac{1}{2}\lambda^2$ for all u. Finally, the last term corresponds to current financial shocks that are absorbed into the level of log consumption. The (annualized) volatility of $\mathrm{d}\log c_t^\star$ equals $q_0/\gamma \cdot \lambda$.

Figure 2 shows the median growth rate of consumption as a function of age for various sets of parameter values. The black dashed line corresponds the case where $\delta_u = r+\frac{1}{2}\lambda^2 = 3\%$ for all u. In that case, the median growth rate of consumption is zero (i.e., median consumption stays constant over time). The other lines illustrate the median growth rate of consumption if δ changes at the age of retirement (65 years) from 3% to 2%. The parameter δ can change because of a (discretionary) change in the force of mortality (see also footnote 12). We observe that the agent reallocates consumption from the short-run to the long-run. Indeed, the agent expects to live longer so that he postpones consumption and saves more for the future. The effect of a decrease in δ on the median growth rate

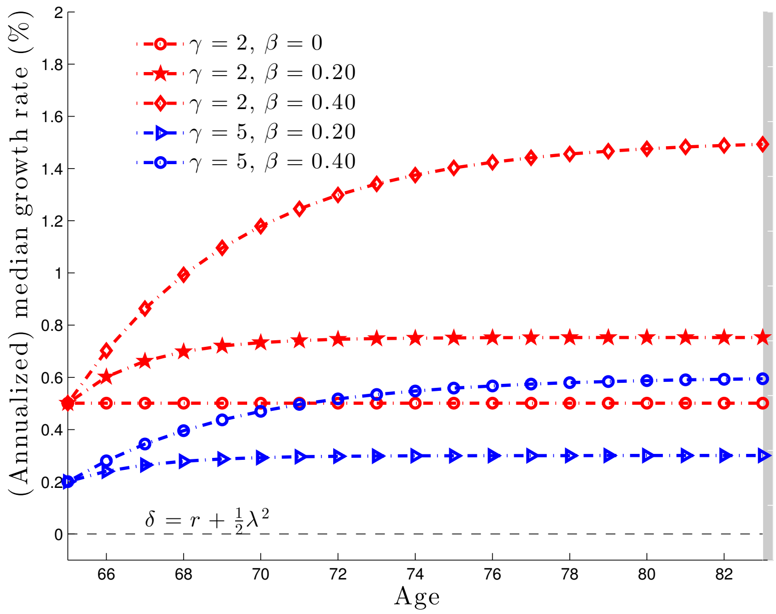

Figure 2. Illustration of the median growth rate of consumption ($T = \infty$). The figure illustrates the median growth rate of consumption as a function of age for various sets of parameter values. The depreciation parameter α is set equal to 0.6, the subjective rate of time preference δ to 0.02, and the Lagrange multiplier y to unity. The black dashed line represents the median growth of consumption in the case of $\delta = r + \frac{1}{2}\lambda^2 = 0.03$.

of consumption is more pronounced for large values of β and small values of γ. Hence, if β is large or γ is small, then the agent is more willing to substitute consumption over time. Section IV considers a utility specification in which the coefficient of relative risk aversion does not affect the agent's willingness to substitute consumption over time.

A.2. Finite Terminal Time

This section analyzes the agent's consumption choice for the case where the terminal time T is finite (i.e., $T < \infty$). The Appendix shows that we can write the agent's consumption choice $c_t^\star$ in terms of past financial shocks as follows:

$$c_t^\star = (c_0^\star)^{q_t/q_0} \exp\left\{ \int_0^t q_{t-u} \frac{1}{\gamma} \left(\widehat{r}_u + \frac{1}{2}\lambda^2 - \delta_u \right) \mathrm{d}u + \int_0^t q_{t-u} \frac{1}{\gamma} \lambda \, \mathrm{d}W_u \right\}. \tag{37}$$

Here

$$\widehat{r}_u \equiv \beta + \frac{r - \alpha\beta P_u}{1 + \beta P_u} \tag{38}$$

We can obtain equation (37) from equation (26) by replacing the interest rate r with the adjusted interest rate $\widehat{r}_u$. The Appendix proofs the following theorem.

THEOREM 2: *Let the adjusted interest rate* $\widehat{r}_u$ *be defined by equation* (38). *Then:*

1. *The adjusted interest rate* $\widehat{r}_u$ *increases as the endogeneity parameter* β *increases, given* $\eta = \alpha - \beta$.

2. *The adjusted interest rate* $\widehat{r}_u$ *decreases as the terminal time* T *increases. In particular,* $\widehat{r}_u \to r$ *if* $T \to \infty$.

Current consumption has a large impact on future habit levels if β is large. Also, the utility gain of an infinitesimal increase in consumption is smaller when the agent is (relatively) young (i.e., small t) than when the agent is (relatively) old (i.e., large t). These two facts together explain why the median consumption choice tends to go up with age if the endogeneity parameter β is large. Indeed, as already pointed out by Deaton (1992), the agent derives utility not only from consumption levels but also from consumption growth. If T equals infinity, the utility gain of an infinitesimal increase in consumption is age-independent. Hence, the agent does no longer have a desire to postpone consumption. In our model, four factors thus affect the median consumption choice (see also equation (37)): the financial market (i.e., r, λ and σ), the subjective rate of time preference, the survival curve, and the strength of internal habit formation. Figure 3 illustrates the median growth rate of consumption for various values of the endogeneity parameter β.

B. *Portfolio Choice*

This section analyzes the agent's portfolio choice $\pi_t^\star$. The Appendix shows that the replicating portfolio strategy $\pi_t^\star$ is given by

$$\pi_t^\star = \widehat{q}_t \frac{\lambda}{\sigma}. \tag{39}$$

Here $0 \leq \widehat{q}_t \leq 1$ denotes the (weighted) average risk exposure. That is,

$$\widehat{q}_t = \int_t^T \bar{q}_u \frac{V_t^u}{V_t} \, \mathrm{d}u, \tag{40}$$

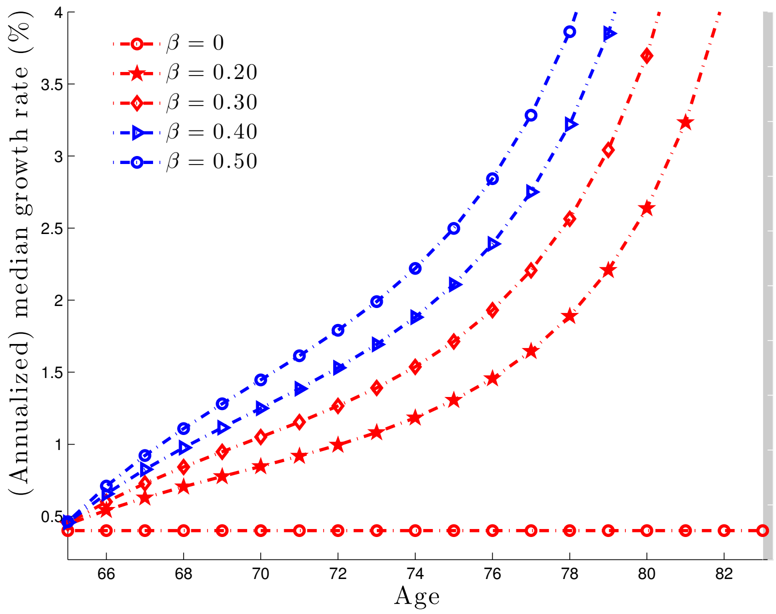

Figure 3. Illustration of median growth rate of consumption ($T < \infty$). The figure illustrates the median growth rate of consumption as a function of age for various values of the endogeneity parameter β. The parameter $\eta \equiv \alpha - \beta$ is set equal to 0.2, the coefficient of relative risk aversion γ to 5, the subjective rate of time preference δ to 0.01, the terminal time T to 20, and the Lagrange multiplier y to unity.

where $V_t \equiv \int_t^T V_t^u \, du$ and V_t^u denotes the (market) value at time t of $c_{t+u}^\star$:

$$V_t^u \equiv c_t^\star F_t^u C_t^u. \tag{41}$$

Equation (41) shows that the market value of future consumption, i.e., $V_t^u / c_t^\star$, consists of two factors. The first factor, i.e., F_t^u, represents past financial shocks that are absorbed into future growth rates of consumption. This factor equals unity if the agent directly adjusts consumption after unexpected financial shocks (i.e., $\bar{q}_u = 1/\gamma$ for all u). The horizon-dependent *annuity factor* C_t^u summarizes the impacts of desired consumption streams and future rates of return on the market value of future consumption. The Appendix provides an explicit analytical expression for the horizon-dependent annuity factor C_t^u (see equation (A3) in the Appendix).

Table I shows the median portfolio choice as a function of age for various sets of parameter values. The agent implements a life cycle investment strategy (that is, the fraction of wealth invested in the risky stock decreases on average as the agent ages),

Table I

The Agent's Median Portfolio Choice

The table reports the agent's median portfolio choice (i.e., the median fraction of assets invested in the risky stock) as a function of age for various sets of parameter values. The table also reports the Merton portfolio strategy. (1) corresponds to $\alpha = 0.64$, $\beta = 0.56$, $\gamma = 20$; (2) to $\alpha = 0.80$, $\beta = 0.76$, $\gamma = 20$; (3) to $\alpha = 0.5$, $\beta = 0.3$, $\gamma = 5$; and (4) to $\alpha = 0.66$, $\beta = 0.54$, $\gamma = 5$.

Age	(1)	(2)	(3)	(4)	Merton (RRA = 2)	Merton (RRA = 5)
65	0.20	0.27	0.41	0.65	0.50	0.20
70	0.18	0.24	0.39	0.64	0.50	0.20
75	0.15	0.19	0.36	0.56	0.50	0.20
80	0.10	0.12	0.29	0.40	0.50	0.20
85	0.05	0.05	0.20	0.20	0.50	0.20

Table II

Volatilities

The table reports the volatility of the change in consumption σ_c and the volatility of the change in wealth σ_A as a function of age for various sets of parameter values. (1) corresponds to $\alpha = 0.64$, $\beta = 0.56$, $\gamma = 20$; and (2) to $\alpha = 0.66$, $\beta = 0.54$, $\gamma = 5$. The numbers represent a percentage.

Age	(1)		(2)		Merton (RRA = 2)		Merton (RRA = 5)	
	σ_c	σ_A	σ_c	σ_A	σ_c	σ_A	σ_c	σ_A
65	1.00	3.95	4.00	12.99	10.00	10.00	4.00	4.00
70	1.00	3.56	4.00	12.81	10.00	10.00	4.00	4.00
75	1.00	2.92	4.00	11.12	10.00	10.00	4.00	4.00
80	1.00	1.99	4.00	7.90	10.00	10.00	4.00	4.00
85	1.00	1.00	4.00	4.00	10.00	10.00	4.00	4.00

even without taking human capital into account. Indeed, the agent has less time to absorb a wealth shock as he grows older.

Table II shows the (annualized) volatility of the relative change in consumption and the (annualized) volatility of the relative change in wealth. With internal habit formation, the volatility of the relative change is consumption is smaller than the volatility of the relative change in wealth. Hence the agent can take substantial stock market risk without affecting the year-on-year volatility of consumption. Indeed, the degree of internal habit formation largely determines the fraction of wealth invested in the risky stock, while the coefficient of relative risk aversion largely determines the year-on-year fluctuations in consumption.

C. *Welfare Analysis*

This section conducts a welfare analysis. More specifically, we compare a number of alternative popular draw-down and investment strategies to the draw-down and investment strategy implied by the agent's maximization problem (21). The welfare loss associated with implementing an alternative draw-down and investment strategy is computed *relative* to the agent's optimal draw-down and investment strategy. More precisely, the performance of an alternative strategy is evaluated by measuring the relative decline in certainty equivalent consumption. We define the certainty equivalent of an uncertain consumption strategy to be the constant, certain consumption level that yields indifference to the uncertain consumption strategy. Certainty equivalents are computed using the lifetime utility function (10). Due to the presence of internal habit formation, the computation of certainty equivalents is non-standard. In the welfare analysis, we consider the following alternative draw-down and investment strategies:

- The remaining lifetime approach (i.e., the $1/T$-rule): the proportion of wealth withdrawn from the agent's retirement wealth is given by

$$\frac{c_t}{A_t} = \frac{1}{T-t}. \tag{42}$$

  Here $T-t$ denotes the agent's remaining lifetime which is assumed to be non-random. We assume that the agent invests a fixed percentage (0, 20, 40, 60 or 80 percent) of wealth into the risky stock. Equation (42) shows that consumption responds directly to a financial shock.

- The Merton approach: the share of wealth withdrawn from the agent's retirement wealth is given by

$$\frac{c_t}{A_t} = \frac{x_1 - x_2}{\exp\{(x_1 - x_2)(T-t)\} - 1}, \tag{43}$$

  where

$$x_1 \equiv \frac{1-\gamma}{\gamma}\left(r + \frac{1}{2}\lambda^2\right) + \frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)^2 \lambda^2 \tag{44}$$

$$x_2 \equiv \frac{1}{\gamma}\delta. \tag{45}$$

  We assume that the coefficient of relative risk aversion equals 2, 5 or 20. Like the remaining lifetime approach, consumption is directly adjusted after a wealth shock.

- Difference model of habit formation. We assume that the agent maximizes

$$U_0 = \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_u \, \mathrm{d}u} \frac{1}{1-\gamma} \left(c_t - h_t\right)^{1-\gamma} \mathrm{d}t\right]$$

subject to the dynamic budget constraint (9) and the habit formation process

$$\mathrm{d}h_t = \left(\beta c_t - \alpha h_t\right) \mathrm{d}t.$$

The consumption strategy is given by

$$c_t = h_t + \left(\widehat{M}_t y e^{\int_0^t \delta_u \, \mathrm{d}u}\right)^{-\frac{1}{\gamma}}. \tag{46}$$

Here y is a Lagrange multiplier. Consumption (46) responds gradually to a financial shock. The investment strategy follows from replicating the consumption strategy (46). Unlike the investment strategy implied by the ratio model (see equation (39)), the investment strategy implied by the difference model depends on the habit level. Indeed, if the habit level approaches consumption, the agent reduces the fraction of wealth invested in the risky stock.

Table III reports welfare losses associated with implementing the remaining lifetime approach. The welfare losses are relatively large if the agent exhibits a significant degree of internal habit formation; see the first two rows of Table III. Table IV reports the welfare losses due to implementing the Merton strategy. The welfare losses are smaller compared to the remaining lifetime approach. Indeed, the Merton strategy emerges as a special case of our model when $\beta = 0$. However, welfare losses associated with implementing the Merton strategy may still be significant when the strength of internal habit formation is large. Finally, Table V reports the welfare losses due to implementing the difference model of habit formation. Again, the welfare loss increases as the strength of internal habit formation (i.e., β) increases. Also, the welfare losses are larger compared to the Merton approach.

Table III

Welfare Losses

The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) due to implementing the remaining lifetime approach. The table reports the welfare losses for various values of the fraction of wealth invested in the risky stock. The numbers represent a percentage. We assume that $\delta_u = \delta = 3\%$ for all u and $T = 20$.

Optimal Strategy	Fraction of Assets Invested in the Risky Stock				
	0%	20%	40%	60%	80%
$\alpha = 0.64,\ \beta = 0.56,\ \gamma = 20$	35.16	34.07	36.97	42.76	50.37
$\alpha = 0.80,\ \beta = 0.76,\ \gamma = 20$	42.93	42.35	43.84	46.97	51.45
$\alpha = 0.50,\ \beta = 0.30,\ \gamma = 5$	7.45	2.90	2.79	7.11	15.44
$\alpha = 0.66,\ \beta = 0.54,\ \gamma = 5$	8.85	3.16	1.71	4.79	11.87

Table IV

Welfare Losses

The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) due to the consumption and portfolio strategy of an agent with CRRA utility (i.e., the Merton strategy). The table reports the welfare losses for various values of the coefficient of relative risk aversion γ underlying the Merton strategy. The numbers represent a percentage. We assume that $\delta_u = \delta = 3\%$ for all u and $T = 20$.

Optimal Strategy	Relative Risk Aversion Coefficient		
	2	5	20
$\alpha = 0.64,\ \beta = 0.56,\ \gamma = 20$	23.85	5.19	2.44
$\alpha = 0.80,\ \beta = 0.76,\ \gamma = 20$	29.55	13.13	8.50
$\alpha = 0.50,\ \beta = 0.30,\ \gamma = 5$	2.95	0.63	3.93
$\alpha = 0.66,\ \beta = 0.54,\ \gamma = 5$	2.85	1.86	5.31

IV. Stochastic Differential Utility

A. Preferences and Maximization Problem

This section considers consumption and portfolio choice of an agent with preferences that combine SDU with multiplicative internal habit formation. We specify the agent's utility process $\{V_t\}_{t\in[0,T]}$ in terms of the intertemporal aggregator f. More specifically, $\{V_t\}_{t\in[0,T]}$ satisfies the following integral equation ($t \in [0, T]$):

$$V_t = \mathbb{E}_t \left[\int_t^T f\left(\frac{c_s}{h_s}, V_s, s \right) \mathrm{d}s \right]. \tag{47}$$

Table V

Welfare Losses

The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) due to implementing the difference model of habit formation. The numbers represent a percentage. We assume that $\delta_u = \delta = 3\%$ for all u and $T = 20$.

Optimal Strategy	Welfare Loss
$\alpha = 0.64,\ \beta = 0.56,\ \gamma = 20$	5.63
$\alpha = 0.80,\ \beta = 0.76,\ \gamma = 20$	9.10
$\alpha = 0.50,\ \beta = 0.30,\ \gamma = 5$	4.33
$\alpha = 0.66,\ \beta = 0.54,\ \gamma = 5$	4.31

As in the previous sections, the log habit level $\log h_t$ evolves according to equation (5). The intertemporal aggregator f is characterized by[15]

$$f\left(\frac{c_t}{h_t}, V_t, t\right) = (1+\zeta)\left[\frac{\left(\frac{c_t}{h_t}\right)^{\varphi}}{\varphi}|V_t|^{\frac{\zeta}{1+\zeta}} - \delta V_t\right]. \tag{48}$$

Here $\zeta > -1$ and $\varphi < \min\{1, 1/(1+\zeta)\}$ are preference parameters. Equation (48) is usually referred to as the Kreps-Porteus aggregator. If $\zeta = 0$ and the habit level h_t equals unity (i.e., $\alpha = \beta = 0$), then $f(c_t/h_t, V_t, t)$ reduces to

$$f\left(\frac{c_t}{h_t}, V_t, t\right) = \frac{1}{\varphi}c_t^{\varphi} - \delta V_t. \tag{49}$$

Equation (47) is then equivalent to the additive utility specification

$$V_t = \mathbb{E}_t\left[\int_t^T e^{-\delta(s-t)}\frac{1}{\varphi}c_s^{\varphi}\mathrm{d}s\right]. \tag{50}$$

The agent aims to maximize V_0 in equation (47) (at $t = 0$) with $f\left(\frac{c_t}{h_t}, V_t, t\right)$ given by equation (48) subject to the habit formation process (5) and the dynamic budget constraint (9).

We can transform this dynamic consumption and portfolio choice problem into an equivalent static consumption and portfolio choice problem (similar to what we did in Section I.C). After transforming this static problem into a dual problem and applying a

[15] If $\zeta = 0$, then $f(c_t/h_t, V_t, t) = (1+\zeta V_t)[\log\{c_t/h_t\} - (\delta/\zeta)\log\{1+\zeta V_t\}]$.

linearization to the static dual budget constraint (which takes the same form as in our base model; see Section II.B for further details), we obtain the following maximization problem:

$$\begin{aligned} &\underset{\widehat{c}_t}{\text{maximize}} \quad \mathbb{E}\left[\int_0^T f\left(\widehat{c}_t, V_t, t\right) \mathrm{d}t\right] \\ &\text{subject to} \quad \mathbb{E}\left[\int_0^T \widehat{M}_t \widehat{c}_t \,\mathrm{d}t\right] \leq \widehat{A}_0. \end{aligned} \tag{51}$$

The next section presents the optimal solution to problem (51).

B. Dynamic Consumption and Portfolio Choice

The agent's maximization problem (51) obtained in the dual model upon linearizing the dual budget constraint can be solved by invoking the approach of Schroder and Skiadas (1999). The next theorem presents the optimal consumption choice.

THEOREM 3: *Consider an agent with utility process* (47)*, intertemporal aggregator* (48) *and habit formation process* (5) *who solves the consumption and portfolio choice problem* (51)*. Let $h_t^\star$ be the agent's habit level implied by substituting the agent's optimal past dual consumption choices $\widehat{c}_u^\star \equiv c_u^\star / h_u^\star$ $(u \leq t)$ into equation* (15) *and let y be the Lagrange multiplier associated with the static budget constraint in* (51)*. Then the agent's optimal consumption choice $c_t^\star$ is given by*

$$c_t^\star = h_t^\star \exp\left\{\int_0^t \left(\psi\left[\widehat{r}_u + \frac{1}{2}\frac{\lambda^2}{\gamma} - \delta\right] + \frac{1}{2}\frac{\lambda^2}{\gamma^2}\left[\gamma - 1\right]\right) \mathrm{d}u + \psi y + \frac{\lambda}{\gamma}\int_0^t \mathrm{d}W_u\right\}, \tag{52}$$

where

$$\begin{aligned} \psi &= \frac{1}{1-\varphi}, \\ \zeta &= \frac{1-\gamma}{\varphi} - 1. \end{aligned}$$

We can write the agent's consumption choice $c_t^\star$ in terms of past financial shocks as

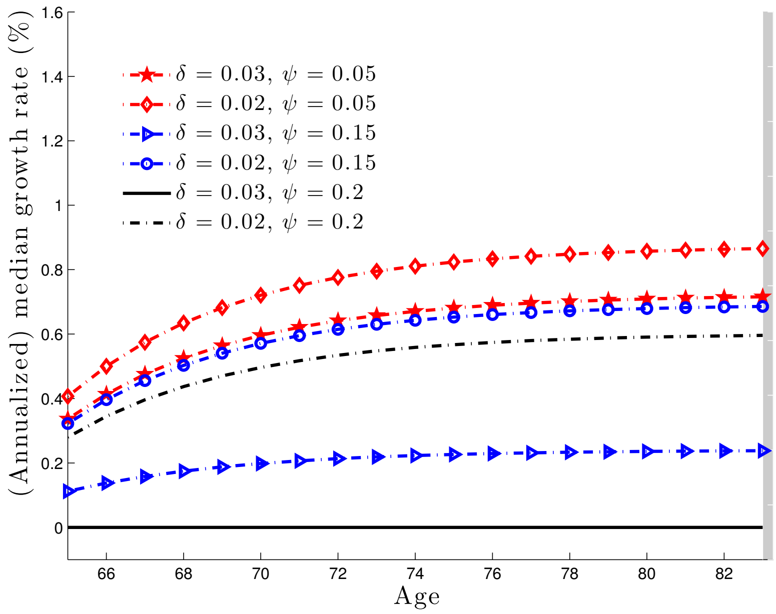

Figure 4. Illustration of median growth rate of consumption (SDU utility). The figure illustrates the median growth rate of consumption as a function of age for various values of ψ and δ. The endogeneity parameter β is set equal to 0.4, the depreciation parameter α to 0.6, the coefficient of relative risk aversion γ to 5, the terminal time T to infinity, and the Lagrange multiplier y to unity. The black solid line corresponds to the case where median consumption growth is zero.

follows:

$$\begin{aligned} c_t^\star = (c_0^\star)^{\frac{q_t}{q_0}} \exp\left\{ \int_0^t q_{t-u} \left(\psi \left[\widehat{r}_u + \frac{1}{2}\frac{\lambda^2}{\gamma} - \delta \right] + \frac{1}{2}\frac{\lambda^2}{\gamma^2}[\gamma - 1] \right) \mathrm{d}u + \right. \\ \left. + q_{t-u}\frac{1}{\gamma}\lambda \int_0^t \mathrm{d}W_u \right\}. \end{aligned} \tag{53}$$

The optimal portfolio choice $\pi_t^\star$ is the same as in Section III.B. Equation (53) shows that with SDU, the parameter ψ determines the willingness to substitute consumption over time. Relative risk aversion is thus decoupled from the elasticity of intertemporal substitution. Figure 4 shows the median growth rate of consumption as a function of age for various values of ψ and δ. We observe that the change in the median growth rate of consumption following a (permanent) change in the subjective rate of time preference δ at the age of retirement (65 years) is small if ψ is small in absolute terms.

V. The Accuracy of the Approximation Method

The consumption and portfolio strategies presented in Section III are exact only in the case of $\beta = 0$ and/or $\alpha = \infty$. In all other cases the consumption and portfolio strategies are approximate based upon linearizing the left-hand side of the agent's dual budget constraint (16) around the consumption trajectory $c/h = 1$. This section analyzes the approximation error due to applying a linearization to the dual budget constraint.[16]

We determine the optimal consumption choice c_t^* by using the method of backward induction. That is, first, we determine the optimal consumption choice at the terminal time T. Then, the optimal consumption choice at time $T-1$ is determined taking the optimal consumption choice at time T as given. We continue this process backwards in time until all optimal consumption choices have been determined. The terminal time T is set equal to three (we also consider the case where the terminal time T equals four), the time interval Δt equals unity and the underlying uncertainty is described by a binomial tree.[17] The computation of the optimal consumption choice c_t^* rapidly becomes infeasible as the number of time steps increases.

We evaluate the performance of the (sub-optimal) consumption choice $c_t^\star$ by measuring the relative decline in certainty equivalent consumption (see Section III.C for the definition of certainty equivalent consumption).[18] Tables VI – IX report our results. The first three tables show the welfare losses (in terms of the relative decline in certainty equivalent consumption) for the case where the terminal time equals three. We find that the approximation error is an increasing function of β, and an decreasing function of α and γ. Indeed, if α is large and/or β is small, the impact of an increase in consumption on future habit levels is limited. Also, if γ is large, consumption stays close to the habit level. In all cases, the approximation error is smaller than 1%. Table IX reports the approximation error for case where the terminal time T equals four. The approximation error is still very small.

[16]The Appendix linearizes the left-hand side of the dual budget constraint (16) in a multi-period, discrete-time setting.

[17]By considering a binomial tree, we can exactly compute the conditional expectations involved in the optimization technique.

[18]The certainty equivalent consumption choice $\bar{c}$ always exists if $\alpha \geq \beta$. In particular, $\frac{\partial U_0}{\partial \bar{c}} \geq 0$ if $\beta \int_0^T e^{-\alpha t}\, \mathrm{d}t \leq 1$. If T is large, then $\int_0^T e^{-\alpha t}\, \mathrm{d}t \approx \frac{1}{\alpha}$. Hence we can always compute (for any T) the certainty equivalent consumption choice $\bar{c}$ if $\frac{\beta}{\alpha} \leq 1$.

Table VI
Welfare Losses ($\gamma = 2$)**.**

The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with implementing the consumption choice $c_t^\star$. The numbers represent a percentage. We only report welfare losses for the case $\alpha \geq \beta$. The terminal time T is set equal to 3, initial wealth to 3, and the subjective rate of time preference δ to 3%.

	β				
α	0	0.15	0.2	0.3	0.6
0	0	–	–	–	–
0.15	0	0.0229	–	–	–
0.2	0	0.0178	0.0516	–	–
0.3	0	0.0149	0.03053	0.1263	–
0.6	0	0.0153	0.0293	0.0689	0.6840

VI. Concluding Remarks

In this paper, we have built a rich consumption-portfolio choice model with preferences that combine both multiplicative internal habit formation and stochastic differential utility. To solve our preference model, we have developed an approximation method based upon linearizing the agent's (dual) budget constraint. For reasonable values of the preference parameters, the approximation error induced by our method is very small. We have shown that after a wealth shock, the agent optimally chooses to adjust both the level and future growth rates of consumption, giving rise to gradual response of consumption to financial shocks. Furthermore, expected consumption tends to grow with age, and relative risk aversion does not affect the willingness to substitute consumption over time. A possible venue for future research would be to confront our preference model with actual consumption data.

Table VII

Welfare Losses ($\gamma = 5$)**.**

The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with implementing the consumption choice $c_t^\star$. The numbers represent a percentage. We only report welfare losses for the case $\alpha \geq \beta$. The terminal time T is set equal to 3, initial wealth to 3, and the subjective rate of time preference δ to 3%.

	β				
α	0	0.15	0.2	0.3	0.6
0	0	–	–	–	–
0.15	0	0.0078	–	–	–
0.2	0	0.0012	0.0101	–	–
0.3	0	0.0164	0.0021	0.0197	–
0.6	0	0.0017	0.0285	0.0096	0.0479

Table VIII

Welfare Losses ($\gamma = 20$)**.**

The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with implementing the consumption choice $c_t^\star$. The numbers represent a percentage. We only report welfare losses for the case $\alpha \geq \beta$. The terminal time T is set equal to 3, initial wealth to 3, and the subjective rate of time preference δ to 3%.

	β				
α	0	0.15	0.2	0.3	0.6
0	0	–	–	–	–
0.15	0	0.0000	–	–	–
0.2	0	0.0002	0.0003	–	–
0.3	0	0.0019	0.0008	0.0024	–
0.6	0	0.0000	0.0008	0.0012	0.0017

Table IX

Welfare Losses ($\gamma = 2$).

The table reports the welfare losses (in terms of the relative decline in certainty equivalent consumption) associated with implementing the consumption choice $c_t^\star$. The numbers represent a percentage. We only report welfare losses for the case $\alpha \geq \beta$. The terminal time T is set equal to 4, initial wealth to 4, and the subjective rate of time preference δ to 3%.

	β				
α	0	0.15	0.2	0.3	0.6
0	0	–	–	–	–
0.15	0	0.0486	–	–	–
0.2	0	0.0561	0.0989	–	–
0.3	0	0.0777	0.1025	0.2309	–
0.6	0	0.3997	0.2779	0.2081	1.3957

Appendix A. Proofs

Derivation of (19)

This appendix linearizes the left-hand side of the agent's dual static budget constraint (16) around the consumption trajectory $\widehat{c} = c/h = 1$. The partial derivative of h_s with respect to $\widehat{c}_t\,\mathrm{d}t$ is given by

$$\frac{\partial h_s}{\partial \widehat{c}_t}\frac{1}{\mathrm{d}t} = \beta \exp\left\{-(\alpha-\beta)(s-t)\right\}\frac{h_s}{\widehat{c}_t}. \tag{A1}$$

Equation (A1) follows from differentiating (15) with respect to $\widehat{c}_t\,\mathrm{d}t$. The partial derivative (A1) evaluated along the consumption trajectory $\widehat{c} = 1$ yields

$$\left.\frac{\partial h_s}{\partial \widehat{c}_t}\frac{1}{\mathrm{d}t}\right|_{\widehat{c}=1} = \beta \exp\left\{-(\alpha-\beta)(s-t)\right\}.$$

Define the function $f\left(\widehat{c}\right)$ as follows:

$$f\left(\widehat{c}\right) \equiv \mathbb{E}\left[\int_0^T M_t h_t \widehat{c}_t\,\mathrm{d}t\right].$$

Straightforward computations show

$$\begin{aligned} f(1) &= \mathbb{E}\left[\int_0^T M_t\,\mathrm{d}t\right], \\ \left.\frac{\partial f\left(\widehat{c}\right)}{\partial \widehat{c}_t}\frac{1}{\mathrm{d}t}\right|_{\widehat{c}=1} &= M_t + \beta\mathbb{E}_t\left[\int_t^T M_s \exp\left\{-(\alpha-\beta)(s-t)\right\}\mathrm{d}s\right] \\ &= M_t\left(1+\beta P_t\right). \end{aligned}$$

We can, by virtue of Taylor series expansion, approximate the dual budget constraint $f\left(\widehat{c}\right)$ by

$$f\left(\widehat{c}\right) \approx f(1) + \mathbb{E}\left[\int_0^T M_t\left(1+\beta P_t\right)\left(\widehat{c}_t - 1\right)\mathrm{d}t\right].$$

Proof of Theorem 1

The Lagrangian $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L} &= \mathbb{E}\left[\int_0^T e^{-\int_0^t \delta_u \, \mathrm{d}u} v\left(\widehat{c}_t\right) \mathrm{d}t\right] + y\left(\widehat{A}_0 - \mathbb{E}\left[\int_0^T \widehat{M}_t \widehat{c}_t \, \mathrm{d}t\right]\right) \\ &= \int_0^T \mathbb{E}\left[e^{-\int_0^t \delta_u \, \mathrm{d}u} v\left(\widehat{c}_t\right) - y\widehat{M}_t \widehat{c}_t\right] \mathrm{d}t + y\widehat{A}_0.\end{aligned}$$

Here $y \geq 0$ denotes the Lagrange multiplier associated with the static budget constraint. The agent aims to maximize $e^{-\int_0^t \delta_u \, \mathrm{d}u} v\left(\widehat{c}_t\right) - y\widehat{M}_t \widehat{c}_t$. The optimal dual consumption choice $\widehat{c}_t$ satisfies the following first-order optimality condition:

$$e^{-\int_0^t \delta_u \, \mathrm{d}u} \widehat{c}_t^{\,-\gamma} = y\widehat{M}_t.$$

After solving the first-order optimality condition, we obtain the following maximum:

$$\widehat{c}_t^\star = \left(e^{\int_0^t \delta_u \, \mathrm{d}u} y\widehat{M}_t\right)^{-\frac{1}{\gamma}}.$$

Hence (use equation (18))

$$c_t^\star = h_t^\star \left(y e^{\int_0^t \delta_u \, \mathrm{d}u} \widehat{M}_t\right)^{-\frac{1}{\gamma}}.$$

A standard verification (see, e.g., Karatzas and Shreve, 1998, p. 103) stating that the optimal solution to the Lagrangian equals the optimal solution to the dual problem completes the proof.

Derivation of (26), (35) *and* (37)

This appendix explicitly writes the agent's consumption choice $c_t^\star$ in terms of past financial shocks. We can write the adjusted pricing kernel $\widehat{M}_t \equiv M_t\left(1 + \beta P_t\right)$ as follows (this follows from applying Itô's Lemma to $\widehat{M}_t = f\left(M_t, P_t\right) = M_t\left(1 + \beta P_t\right)$):

$$\widehat{M}_t = \widehat{M}_0 \exp\left\{-\int_0^t \left(\widehat{r}_u + \frac{1}{2}\lambda^2\right) \mathrm{d}u\right\} \exp\left\{\lambda \int_0^t \mathrm{d}W_u\right\}. \tag{A2}$$

Substituting equation (A2) into equation (25) yields

$$\widehat{c}_t^\star \equiv \frac{c_t^\star}{h_t^\star} = \exp\left\{\frac{1}{\gamma}\int_0^t \left(\widehat{r}_u + \frac{1}{2}\lambda^2 - \delta_u\right) \mathrm{d}u + \frac{\bar{y}}{\gamma}\right\} \exp\left\{\frac{\lambda}{\gamma}\int_0^t \mathrm{d}W_u\right\}.$$

Here $\bar{y} \equiv -\left(\log y + \log \widehat{M}_0\right)$.

We can write the habit level as follows:

$$
\begin{aligned}
h_t^\star &= \exp\left\{\int_0^t \beta \exp\left\{-(\alpha-\beta)(t-u)\right\} \log \widehat{c}_u^\star \mathrm{d}u\right\} \\
&= \exp\left\{\int_0^t \beta \exp\left\{-(\alpha-\beta)(t-u)\right\}\right. \\
&\left.\left[\frac{1}{\gamma}\int_0^u \left(\widehat{r}_v + \frac{1}{2}\lambda^2 - \delta_v\right) \mathrm{d}v + \frac{\bar{y}}{\gamma} + \frac{\lambda}{\gamma}\int_0^u \mathrm{d}W_v\right] \mathrm{d}u\right\} \\
&= \exp\left\{\int_0^t \left(\bar{q}_{t-u} - \frac{1}{\gamma}\right)\left(\widehat{r}_u + \frac{1}{2}\lambda^2 - \delta_u\right) \mathrm{d}u\right\} \\
&\exp\left\{\left(\bar{q}_t - \frac{1}{\gamma}\right)\bar{y} + \int_0^t \left(\bar{q}_{t-u} - \frac{1}{\gamma}\right)\lambda\, \mathrm{d}W_u\right\}.
\end{aligned}
$$

Here

$$
\begin{aligned}
\bar{q}_{t-u} &\equiv \frac{1}{\gamma}\left[1 + \beta\int_u^t \exp\left\{-(\alpha-\beta)(t-v)\right\} \mathrm{d}v\right] \\
&= \frac{1}{\gamma}\left[1 + \frac{\beta}{\alpha-\beta}\left(1 - \exp\left\{-(\alpha-\beta)(t-u)\right\}\right)\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
c_t^\star &= h_t^\star \exp\left\{\frac{1}{\gamma}\int_0^t \left(\widehat{r}_u + \frac{1}{2}\lambda^2 - \delta_u\right) \mathrm{d}u + \bar{y}\right\} \exp\left\{\frac{\lambda}{\gamma}\int_0^t \mathrm{d}W_u\right\} \\
&= \exp\left\{\int_0^t \bar{q}_{t-u}\left(\widehat{r}_u + \frac{1}{2}\lambda^2 - \delta_u\right) \mathrm{d}u + \bar{q}_t\bar{y} + \int_0^t \bar{q}_{t-u}\lambda\, \mathrm{d}W_u\right\} \\
&= (c_0^\star)^{\bar{q}_t/\bar{q}_0} \exp\left\{\int_0^t \bar{q}_{t-u}\left(\widehat{r}_u + \frac{1}{2}\lambda^2 - \delta_u\right) \mathrm{d}u + \int_0^s \bar{q}_{t-u}\lambda\, \mathrm{d}W_u\right\}.
\end{aligned}
$$

Equation (26) follows from equation (37) and Theorem 2.

Dividing $\log c_{t+\Delta t}^\star$ by $\log c_t^\star$ and taking the limit $\Delta t \Rightarrow 0$ yields equation (35).

Proof of Theorem 2

We first proof that the (partial) derivate of $\widehat{r}_u$ with respect to β is positive given $\eta = \alpha - \beta$. Substituting $\alpha = \eta + \beta$ into equation (38) yields

$$
\widehat{r}_u = \beta + \frac{r - (\eta+\beta)\beta P_u}{1 + \beta P_u}.
$$

The derivate of $\widehat{r}_u$ with respect to β is given by

$$\begin{aligned}\frac{\partial \widehat{r}_u}{\partial \beta} &= 1 + \frac{-\left(1+\beta P_u\right)\left(\eta+2\beta\right)P_u - \left(r-(\eta+\beta)\beta P_u\right)P_u}{\left(1+\beta P_u\right)^2} \\ &= 1 + \frac{-\eta P_u - 2\beta P_u - \eta\beta P_u^2 - 2\left(\beta P_u\right)^2 - rP_u + \eta\beta P_u^2 + \left(\beta P_s\right)^2}{1+2\beta P_u + \left(\beta P_u\right)^2} \\ &= 1 + \frac{-\eta P_u - 2\beta P_u - \left(\beta P_u\right)^2 - rP_u}{1+2\beta P_u + \left(\beta P_u\right)^2}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial \widehat{r}_u}{\partial \beta} \geq 0 &\Leftrightarrow \frac{-\eta P_u - 2\beta P_u - \left(\beta P_u\right)^2 - rP_u}{1+2\beta P_u + \left(\beta P_u\right)^2} \geq -1 \\ &\Leftrightarrow \eta P_u + 2\beta P_u + \left(\beta P_u\right)^2 + rP_u \leq 1 + 2\beta P_u + \left(\beta P_u\right)^2 \\ &\Leftrightarrow (r+\eta)\,P_u \leq 1 \Leftrightarrow 1 - \exp\left\{-(r+\eta)(T-u)\right\} \leq 1.\end{aligned}$$

This last inequality is obviously true. Hence $\partial\widehat{r}_u/\partial\beta$ is positive (given η).

Finally, we proof that the (partial) derivate of $\widehat{r}_u$ with respect to T is negative. The derivate of $\widehat{r}_u$ with respect to T is given by

$$\frac{\partial \widehat{r}_u}{\partial T} = -r\left(1+\beta P_u\right)^{-2}\frac{\partial P_u}{\partial T} - \alpha\beta\left(1+\beta P_u\right)^{-2}\frac{\partial P_u}{\partial T}.$$

Using the fact that $\partial P_u/\partial T$ is positive, we find that $\partial\widehat{r}_u/\partial T$ is negative. Furthermore, simple algebra yields that $\widehat{r}_u = r$ if $T=\infty$ (here we use the fact that $P_u \Rightarrow 1/(r+\alpha-\beta)$ as $T\Rightarrow\infty$).

Derivation of (39)

Straightforward computations show that

$$\begin{aligned}V_t^u &= \mathbb{E}_t\left[\frac{M_{t+u}}{M_t}c^\star_{t+u}\right] \\ &= c_t^\star F_t^u \mathbb{E}_t\left[\exp\left\{-\int_0^u\left(r+\frac{1}{2}\lambda^2\right)\mathrm{d}v - \int_0^u \lambda\,\mathrm{d}W_{t+u-v}\right\}\right. \\ &\left.\exp\left\{\int_0^u q_v\frac{1}{\gamma}\left(r+\frac{1}{2}\lambda^2-\delta_{t+u-v}\right)\mathrm{d}v + \int_0^u q_v\frac{1}{\gamma}\lambda\,\mathrm{d}W_{t+u-v}\right\}\right] \\ &= c_t^\star F_t^u C_t^u,\end{aligned}$$

where

$$C_t^u \equiv \exp\left\{-\int_0^u \left([1-\bar{q}_v]\, r + \frac{1}{2}\bar{q}_v\lambda^2 + \bar{q}_v\delta_{t+u-v} - \frac{1}{2}\bar{q}_v^2\lambda^2\right) \mathrm{d}v\right\}. \tag{A3}$$

It follows from Itô's Lemma that

$$\frac{\partial \log V_t}{\partial W_t} = \frac{1}{V_t}\int_t^T \frac{\partial V_t^u}{\partial W_t}\,\mathrm{d}u = \int_t^T \bar{q}_u \frac{V_t^u}{V_t}\lambda\,\mathrm{d}u. \tag{A4}$$

We also have (this follows from applying Itô's Lemma to the dynamic budget constraint (9))

$$\frac{\partial \log A_t}{\partial W_t} = \pi_t\sigma. \tag{A5}$$

Setting equation (A5) equal to equation (A4) and solving for π_t yields (39).

Proof of Theorem 3

Schroder and Skiadas (1999) derive the optimal dual consumption choice $\widehat{c}_t^\star$. The optimal consumption choice $c_t^\star$ follows from equation (18).

Appendix B. Multi-Period Discrete-Time Model

This section linearizes the left-hand side of the agent's dual budget constraint (16) in a multi-period, discrete-time setting. Let us denote by Δt the time step (the magnitude of Δt is usually taken to be small). The habit level is given by

$$\begin{aligned} h_{n\Delta t} &= \exp\left\{\beta\left[\sum_{i=1}^{n}(1-[\alpha-\beta]\Delta t)^{n-i}\log\widehat{c}_{i\Delta t}\right]\Delta t\right\} \\ &= \prod_{i=1}^{n}\widehat{c}_{i\Delta t}^{\,\beta(1-[\alpha-\beta]\Delta t)^{n-i}\Delta t}. \end{aligned}$$

Here, $n \in \{0, ..., \lfloor T/\Delta t\rfloor - 1\}$. The agent's dual budget constraint can be written as follows:

$$\begin{aligned} \mathbb{E}\left[\sum_{n=0}^{\lfloor T/\Delta t\rfloor-1} M_{(n+1)\Delta t} h_{n\Delta t}\widehat{c}_{(n+1)\Delta t}\Delta t\right] = \mathbb{E}\Bigg[\sum_{n=0}^{\lfloor T/\Delta t\rfloor-1} & M_{(n+1)\Delta t} \\ & \left(\times\prod_{i=1}^{n}\widehat{c}_{i\Delta t}^{\,\beta(1-[\alpha-\beta]\Delta t)^{n-i}\Delta t}\right)\widehat{c}_{(n+1)\Delta t}\Delta t\Bigg]. \end{aligned}$$

Let us define the following function:

$$
\begin{aligned}
f(\widehat{c}) &\equiv f\left(\widehat{c}_{\Delta t}, ..., \widehat{c}_{\lfloor T/\Delta t\rfloor \Delta t}\right) \\
&\equiv \mathbb{E}\left[\sum_{n=0}^{\lfloor T/\Delta t\rfloor -1} M_{(n+1)\Delta t}\left(\prod_{i=1}^{n} \widehat{c}_{i\Delta t}^{\beta(1-[\alpha-\beta]\Delta t)^{n-i}\Delta t}\right)\widehat{c}_{(n+1)\Delta t}\Delta t\right].
\end{aligned}
$$

By Taylor series expansion,

$$
\begin{aligned}
f(\widehat{c}) &\approx f(1) + \mathbb{E}\left[\sum_{n=0}^{\lfloor T/\Delta t\rfloor -1} \left.\frac{\partial f(\widehat{c})}{\partial \widehat{c}_{(n+1)\Delta t}}\right|_{\widehat{c}=1}\left(\widehat{c}_{(n+1)\Delta t} - 1\right)\right] \\
&= f(1) + \mathbb{E}\left[\sum_{n=0}^{\lfloor T/\Delta t\rfloor -1} \left.\frac{\partial f(\widehat{c})}{\partial \widehat{c}_{(n+1)\Delta t}}\right|_{\widehat{c}=1}\left(\widehat{c}_{(n+1)\Delta t} - 1\right)\right].
\end{aligned}
$$

Straightforward computations show that

$$
\begin{aligned}
\left.\frac{\partial f(\widehat{c})}{\partial \widehat{c}_{(n+1)\Delta t}}\right|_{\widehat{c}=1} &= M_{(n+1)\Delta t}\Delta t \\
&\quad + \beta \mathbb{E}_{(n+1)\Delta t}\left[\sum_{i=n+2}^{\lfloor T/\Delta t\rfloor -1} M_{i\Delta t}(1-[\alpha-\beta]\Delta t)^{i-(n+2)}\Delta t\right]\Delta t \\
&= M_{(n+1)\Delta t}\left(1+\beta P_{(n+1)\Delta t}\right)\Delta t,
\end{aligned}
$$

where

$$
P_{(n+1)\Delta t} \equiv \mathbb{E}_{(n+1)\Delta t}\left[\sum_{i=n+2}^{\lfloor T/\Delta t\rfloor -1} \frac{M_{i\Delta t}}{M_{(n+1)\Delta t}}(1-[\alpha-\beta]\Delta t)^{i-(n+2)}\Delta t\right].
$$

Hence,

$$
f(\widehat{c}) \approx f(1) + \mathbb{E}\left[\sum_{n=0}^{\lfloor T/\Delta t\rfloor -1} M_{(n+1)\Delta t}\left(1+\beta P_{(n+1)\Delta t}\right)\left(\widehat{c}_{(n+1)\Delta t} - 1\right)\Delta t\right].
$$

REFERENCES

Abel, Andrew B., 1990, Asset prices under habit formation and catching up with the Joneses, *American Economic Review* 80, 38–42.

Bodie, Z., J. B. Detemple, S. Otruba, and S. Walter, 2004, Optimal consumption-portfolio choices and retirement planning, *Journal of Economic Dynamics and Control* 28, 1115–1148.

Campbell, J. Y., and N. G. Mankiw, 1991, The response of consumption to income: A cross-country investigation, *European Economic Review* 35, 723–767.

Chacko, G., and L. M. Viceira, 2005, Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets, *The Review of Financial Studies* 18, 1369–1402.

Chai, J., W. J. Horneff, R. H. Maurer, and O. S. Mitchell, 2011, Optimal portfolio choice over the life cycle with flexible work, endogenous retirement, and lifetime payouts, *Review of Finance* 15, 875–907.

Constantinides, George M., 1990, Habit formation: A resolution of the equity premium puzzle, *Journal of Political Economy* 98, 519–543.

Corrado, L., and S. Holly, 2011, Multiplicative habit formation and consumption: A note, *Economics Letters* 113, 116–119.

Cox, John C., and Chi-fu Huang, 1989, Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory* 49, 33–83.

Cox, John C., and Chi-fu Huang, 1991, A variational problem arising in financial economics, *Journal of Mathematical Economics* 20, 465–487.

Deaton, A., 1992, *Understanding Consumption* (Oxford University Press).

Detemple, J. B., and I. Karatzas, 2003, Non-addictive habits: Optimal consumption-portfolio policies, *Journal of Economic Theory* 113, 265–285.

Detemple, J. B., and F. Zapatero, 1991, Asset prices in an exchange economy with habit formation, *Econometrica* 59, 1633–1657.

Detemple, J. B., and F. Zapatero, 1992, Optimal consumption-portfolio policies with habit formation, *Mathematical Finance* 2, 251–274.

Duffie, D., and L. G. Epstein, 1992, Stochastic differential utility, *Econometrica* 60, 353–394.

Dus, I., R. Maurer, and O. S. Mitchell, 2005, Betting on death and capital markets in retirement: A shortfall risk analysis of life annuities, *Financial Services Review* 14, 169–196.

Epstein, L. G., and S. E. Zin, 1989, Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

Fuhrer, J. C., 2000, Habit formation in consumption and its implications for monetary-policy models, *American Economic Review* 90, 367–390.

Gomes, F., and A. Michaelides, 2003, Portfolio choice with internal habit formation: A life-cycle model with uninsurable labor income risk, *Review of Economic Dynamics* 6, 729–766.

Gomes, F., and A. Michaelides, 2008, Asset pricing with limited risk sharing and heterogeneous agents, *The Review of Financial Studies* 21, 415–448.

Gomes, Francisco J., Laurence J. Kotlikoff, and Luis M. Viceira, 2008, Optimal life-cycle investing with flexible labor supply: A welfare analysis of life-cycle funds, *American Economic Review* 98, 297–303.

Guillén, M., P. L. Jørgensen, and J. P. Nielsen, 2006, Return smoothing mechanisms in life and pension insurance: Path-dependent contingent claims, *Insurance: Mathematics and Economics* 38, 229–252.

Horneff, W. J., R. H. Maurer, O. S. Mitchell, and I. Dus, 2008, Following the rules: Integrating asset allocation and annuitization in retirement portfolios, *Insurance: Mathematics and Economics* 42, 396–408.

Investment Company Institute, 2014, Investment company fact book. a review of trends and activities in the u.s. investment company industry.

Karatzas, Ioannis, John P. Lehoczky, and Steven E. Shreve, 1987, Optimal consumption and portfolio decisions for a "small investor" on a finite horizon, *SIAM Journal of Control and Optimization* 25, 1557–1586.

Karatzas, Ioannis, and Steven E. Shreve, 1998, *Methods of Mathematical Finance*, volume 39 (Springer).

Kozicki, and Tinsley, 2002, Dynamic specifications in optimizing trend-deviation macro models, *Journal of Economic Dynamics and Control* 26, 1585–1611.

Kreps, D. M., and E. L. Porteus, 1978, Temporal resolution of uncertainty and dynamic choice theory, *Econometrica* 46, 185–200.

MacDonald, Bonnie-Jeanne, B. Jones, R. J. Morrison, R. L. Brown, and M. Hardy, 2013, Research and reality: A literature review on drawing down retirement financial savings, *North American Actuarial Journal* 17, 181–215.

Maurer, R., O. S. Mitchell, R. Rogalla, and I. Siegelin, 2014, Accounting and actuarial smoothing of retirement payouts in participating life annuities, NBER Working Paper.

Maurer, R., R. Rogalla, and I. Siegelin, 2013a, Participating payout annuities: Lessons from germany, *ASTIN Bulletin* 43, 159–187.

Maurer, R. H., O. S. Mitchell, R. Rogalla, and V. Kartashov, 2013b, Life cycle portfolio choice with systematic longevity risk and variable investment-linked deferred annuities, *Journal of Risk and Uncertainty* 80, 649–676.

Merton, R. C., 1971, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3, 373–413.

Merton, Robert C., 1969, Lifetime portfolio selection under uncertainty: The continuous-time case, *Review of Economics and Statistics* 51, 247–257.

Munk, C., 2008, Portfolio and consumption choice with stochastic investment opportunities and habit formation in preferences, *Journal of Economic Dynamics and Control* 32, 3560–3589.

Pliska, Stanley R., 1986, A stochastic calculus model of continuous trading: Optimal portfolios, *Mathematics of Operations Research* 11, 371–382.

Polyak, I., 2005, New advice to retirees: Spend more at first, cut back later, The New York Times.

Samuelson, P. A., 1969, Lifetime portfolio selection by dynamic stochastic programming, *The Review of Economics and Statistics* 51, 239–246.

Schroder, M., and C. Skiadas, 1999, Optimal consumption and portfolio selection with stochastic differential utility, *Journal of Economic Theory* 89, 68–126.

Schroder, Mark, and Costis Skiadas, 2002, An isomorphism between asset pricing models with and without linear habit formation, *Review of Financial Studies* 15, 1189–1221.

Whitaker, B., 2005, Managing retirement, after you really retire, The New York Times.

Yaari, M. E., 1965, Uncertain lifetime, life insurance, and the theory of the consumer, *The Review of Economic Studies* 32, 137–150.