Score-Driven Nelson Siegel: Hedging Long-Term Liabilities.

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Abstract

Due to its affine structure the Nelson-Siegel model for yield curves can be transformed to a factor model for excess bond returns. Hedging interest rate risk in this framework amounts to eliminating the factor exposure and minimizing the residual risk. Fitting the model directly on excess returns with constant factor loadings leads to large hedging errors caused by substantial and persistent time-variation in the shape parameter of the Nelson-Siegel factor loadings. To capture this variation we develop a Dynamic Conditional Score (DCS) model for the shape parameter. This dynamic model offers superior hedging performance and reduces the hedging error standard deviation by almost 50% during the financial crisis. Much of the improvement is due to the model for the shape parameter with some further reduction achieved by a GARCH model for the residual risk.

Keywords: Dynamic Conditional Score, Risk Management, Term Structure.

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1. Introduction

Hedging interest rate risk requires a model of how prices of bonds with different maturities react to shocks. There is a large literature that models both the time series and the cross-section of the term structure of bond returns and yields. As the aim is to forecast a large cross-section of maturities, the literature imposes a factor structure on the yield curve to reduce the dimensionality of the problem. Most popular is the class of affine term structure models (Duffie and Kan, 1996; Dai and Singleton, 2000; Duffee, 2002), which is both tractable from an econometric perspective as well as consistent with no-arbitrage conditions from finance theory.

Nelson and Siegel (1987) propose a purely statistical model which has good empirical fit. The model has received considerable attention since Diebold et al. (2006b) proposed a dynamic version of the standard model, which describes the dynamics of the yield curve over time as a three factor model. The three factors are the well-known level, slope and curve factors, whose exact shape is governed by a single parameter λ.[1] As a further extension Christensen et al. (2011) showed that a simple additive constant term makes the model arbitrage free and a member of the affine class.

In this paper we use the Nelson-Siegel (NS) model for the purpose of risk management and specifically to hedge the risk of a long-term liability using shorter-term bonds. The NS model is particularly suited to this problem because it offers an additional parameter beyond duration hedging, and its parsimony allows tracking of time-variation in the covariance structure. The empirical problem of hedging long-term liabilities is of great interest to institutions like pension funds, who often have difficulty hedging long-term obligations. To this purpose, we transform the Nelson-Siegel to a factor model on excess bond returns. Since the NS model imposes a factor structure, hedging amounts to forming a portfolio with the same factor exposure as the liability, and as such, for a successful hedge, a correct specification of the factor structure is vital.

In the Nelson-Siegel model, the factor structure is fully controlled by a single parameter

[1] For a textbook description, see Diebold and Rudebusch (2013).

denoted by λ. This parameter has mostly been disregarded, and is often not even estimated, but set to a value chosen by the users. We focus on the λ and propose a Dynamic Conditional Score (DCS) version of the model which provides an intuitive way to model time-variation in λ. Crucially, if λ is time-varying, the true factor exposure cannot be adequately hedged by the standard NS model. Few studies allow for time-variation of λ. Koopman et al. (2010) estimate time-variation using an extended Kalman filter, while Creal et al. (2008) allow for time-variation as a function of the time-varying factor exposures. Both impose more restrictions on the potential dynamics of the shape parameter. Hevia et al. (2015) allow the λ parameter to change using a two regime Markov switching model. These studies focus on predicting the NS factors, whereas we aim at finding asset-liability positions that are not exposed to the factor risk at all.

Additionally, we allow for heteroskedasticity in the residuals. Properly scaling the residuals controls the magnitude of the adjustments to λ_t. Similar to Koopman et al. (2010) we use a univariate GARCH(1,1) model that drives volatility dynamics of the residuals for all maturities. Many studies, such as Bianchi et al. (2009), Hautsch and Ou (2012) and Koopman et al. (2010), concentrate on modeling time-variation in the variances of the NS factors. Our focus, again, is different. We treat the factors as fixed effects, implying that we are not making assumptions on their dynamics and volatility.

The time-variation in λ and the residual variance are interconnected as the scaled residuals directly influence the updates of λ. We find considerable time-variation in λ and show the importance of the heteroskedasticity for updating λ_t. Importantly, when not taking into account the time-variation of the variance, the factor structure breaks down during the financial crisis of 2008: λ converges to zero, effectively making it a two-factor model. Taking into account the volatility, λ actually increases to levels not seen before. If not modeled, the large variation in returns simply obscures the factor structure. The relation between the factor loadings and volatility in our model prevents the factor structure from breaking down.

We use the model to hedge long-term liabilities using short-term assets. Standard duration hedging amounts to hedging the level factor in the Nelson-Siegel model, which only takes care

of parallel shifts in the yield curve. We show that including the other two factors greatly improves hedging performance. However, in contrast to duration hedging, it is vital to properly capture the differences in factor loadings across maturities. In our out-of-sample set-up we show that the factor exposure of the hedge portfolio constructed using the Nelson Siegel with time-varying λ and volatility is reduced by almost 50% compared to the standard Nelson-Siegel model where both are constant. The gains mainly stem from the time-variation in λ, but also allowing for heteroskedasticity improves the hedging results further.

The remainder of the paper is organized as follows. In Section 2 we recap the Nelson-Siegel model and introduce how it can be used for hedging. Section 3 introduced the Dynamic Conditional Score version of the NS model with time-varying λ. Section 5 gives in-sample estimation results of the various models, which are used in an out-of-sample hedging exercise in Section 4.

2. Nelson Siegel model

Denote by $y_t(\tau)$ the yield on a discount bond at time t for maturity τ. At any time, the yield curve is some smooth function representing the yields as a function of maturity τ. Nelson and Siegel (1987) provide a parsimonious factor description of these yields. Diebold and Li (2006) formulate the model as

$$y_t(\tau) = B(\tau)\tilde{f}_t, \tag{1}$$

where

$$B(\tau) = \begin{pmatrix} 1 \\ \frac{1-\mathrm{e}^{-\lambda\tau}}{\lambda\tau} \\ \frac{1-\mathrm{e}^{-\lambda\tau}}{\lambda\tau} - \mathrm{e}^{-\lambda\tau} \end{pmatrix}' \tag{2}$$

depends on the shape parameter $\lambda > 0$. The NS model is a three-factor model where the three factors are often interpreted as level, slope and curvature. The first component has constant factor loadings of 1. As such, it influences short and long term yields equally, and can be considered as the overall level. The second component converges to one as $\tau \downarrow 0$ and

converges to zero as $\tau \to \infty$. This component mostly affects short-term rates. The third factor converges to zero as $\tau \downarrow 0$ and $\tau \to \infty$, but is concave in τ. This component therefore mainly influences the medium-term rates. Long maturities are essentially only affected by the level factor.

Given a value for λ the factors are usually estimated by the cross-sectional regression model

$$y_t(\tau_i) = B(\tau_i)' \tilde{f}_t + \tilde{\epsilon}_t(\tau_i), \qquad i = 1, \ldots, N, \tag{3}$$

At each t ($t = 1, \ldots, T$) the data for the regression consist of yields for N different maturities τ_i. The time series of estimated 'parameters' $\tilde{f}_t$ have empirically been shown to be strongly correlated over time. As such, Diebold and Li (2006) treat the estimates of $\tilde{f}_t$ as time series and use a VAR to forecast future values $\tilde{f}_{t+\ell}$, and thus the yield-curve. Diebold et al. (2006b) and Koopman et al. (2010) put the model into a state-space framework, and estimate the dynamics using a Kalman filter.

For hedging purposes, the dynamics of $\tilde{f}_t$ are less important. The aim of hedging is to obtain portfolios weights such that the return on a portfolio of assets and liabilities is not exposed to the factors. That means our interest is not in predicting future values of the factors, but in estimating factor loadings. The NS model implies discount factors of the form

$$P_t(\tau) = \exp(-b(\tau)\tilde{f}_t). \tag{4}$$

where $b(\tau) = \tau B(\tau)$. Since we are interested in the risk of a bond portfolio, ultimately we are interested in modeling bond returns instead of their price level. We therefore transform the Nelson-Siegel model to a specification that relates bond returns to factor returns. Importantly, excess returns have the same factor structure as yields. To see this, recall that log-prices are defined as $p_t(\tau) = -\tau y_t(\tau) = -b(\tau)\tilde{f}_t$, such that excess returns over a period

of length h follow as

$$\begin{aligned} r_{t+h}(\tau) &= p_{t+h}(\tau) - p_t(\tau + h) + p_t(h) \\ &= -b(\tau)\tilde{f}_{t+h} + (b(\tau + h) - b(h))\tilde{f}_t. \end{aligned} \tag{5}$$

Using the property

$$b(\tau + h) - b(h) = b(\tau)A, \tag{6}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-\lambda h} & \lambda h\,\mathrm{e}^{-\lambda h} \\ 0 & 0 & \mathrm{e}^{-\lambda h} \end{pmatrix} \tag{7}$$

does not depend on maturity τ, excess returns can be written as

$$r_{t+h}(\tau) = b(\tau)f_{t+h}, \tag{8}$$

with $f_{t+h} = -(\tilde{f}_{t+h} - A\tilde{f}_t)$. Since the diagonal elements of A are close to one, the transformed factors will have almost zero autocorrelation, unlike the original $\tilde{f}_t$. The returns standardized by their maturity have the same factor structure as yields

$$\rho_t(\tau) \equiv \frac{r_t(\tau)}{\tau} = B(\tau)f_t. \tag{9}$$

The standardization by τ has the additional benefit of taking away a large part of the cross-sectional heteroskedasticity, since volatility of bond returns increases with maturity. We estimate all our models on these standardized returns. The fact that excess returns and yields have the same factor structure, is not just a coincidence for the NS model, but a general property of all affine term structure models. Although the transformation leaves the factor structure unchanged, it will affect the properties of the error term $\tilde{\epsilon}_t$ of equation (3).

After the transformation the model reads

$$\rho_t(\tau) = B(\tau) f_t + \epsilon_t(\tau) \tag{10}$$

with transformed errors

$$\epsilon_{t+h}(\tau) = -\left(\tilde{\epsilon}_{t+h}(\tau) - \tilde{\epsilon}_t(\tau + h)\right) - \frac{h}{\tau}\left(\tilde{\epsilon}_t(\tau + h) - \tilde{\epsilon}_t(h)\right) \tag{11}$$

When estimated on yield levels, the errors $\tilde{\epsilon}_t(\tau)$ are usually strongly autocorrelated. In our estimation we will use daily data and consider maturities of one year and longer. For such small h the errors of the model in returns are almost first differences of the level errors and will most likely not exhibit significant autocorrelation. We will interpret $\epsilon_t(\tau)$ as idiosyncratic noise.

3. DCS-NS Model

In contrast to the factors, λ has not gained much attention. Generally, it is set to a constant value to maximize the curve at a certain maturity, without any estimation. However, λ is a crucial parameter, as it governs the exponential decay of the second and third term, and as such exposure to the different factors across maturities. Small values of λ produce slow decay, and vice versa. Few studies attempt to estimate λ allowing for time-variation. Koopman et al. (2010) propose a non-linear state-space framework, Creal et al. (2008) let it evolve as a linear combination of the $\tilde{f}_t$ and Hevia et al. (2015) allow for variation through a two-state markov switching model.

In this paper our main interest is in the time-variation of λ. As a preliminary indication of the importance of modeling λ as a time-varying parameter we estimate the parameter on a rolling-window of 100 days using non-linear least squares. The data is described in detail in Section 5. Results are plotted in Figure 1. The figure shows there is great variability in λ with typical values ranging between 0.1 and 0.8, whereas a full sample estimate yields $\lambda = 0.38$. The factor structure appears to largely break down during the financial crisis. Not

adequately modeling this leads to suboptimal fit where hedge portfolios will have unintended residual factor exposure.

[Figure 1 about here.]

Setting up a dynamic model for λ_t enables tracking the time variation in the factor loadings $B(\tau)$. We propose a version of the Nelson-Siegel model with time-variation in λ_t, in which its dynamics are governed by a Dynamic Conditional Score (DCS) model. DCS models were first proposed in their most general form in Creal et al. (2013). The class of models is now often used empirically, and theoretical results are derived in, amongst others, Blasques et al. (2014a) and Blasques et al. (2014b).

In the class of DCS models, the dynamics of parameters are driven by the score of the likelihood with respect to that parameter and provide a natural way to update a wide variety of (latent) parameters, when a functional form for their dynamics is not directly apparent. The DCS principle turns out to nest a lot of the models commonly used in time-series analysis. For instance, when assuming a gaussian likelihood, dynamics in the mean are given by an ARMA model, and the dynamics of volatility are given by the well-known GARCH model.

Let ρ_t be an N-vector of standardized excess returns with different maturities. Consider the conditional observation density of returns $p(\rho_t|\lambda_t)$, and let ∇_t the score with respect to λ_t, $\nabla_t = \frac{\partial \log p(\rho_t|\lambda_t)}{\partial \lambda_t}$. Then the DCS model is defined as

$$\lambda_t = \phi_0(1 - \phi_1) + \phi_1 \lambda_{t-1} + \phi_2 s_{t-1}, \tag{12}$$

where we have normalised the length of an observation interval to $h = 1$ day, and where $s_t = S_t \nabla_t$ is the score times an appropriate scaling function. Time-variation of the parameter is driven by the scaled score of the parameter and as such the dynamics of the parameters are linked to the shape of the density. Intuitively, when the score is negative the likelihood is improved when the parameter is decreased, and the DCS updates the parameter in that direction. The NS-DCS model nests a model of constant λ.

The choice of S_t delivers flexibility in how the score ∇_t updates the parameter. Creal

et al. (2013) discuss several options of the form $S_t = \mathcal{I}_{t|t-1}^{-\alpha}$, where $\mathcal{I}_{t|t-1} = E_{t-1}\left[\nabla_t^2\right]$. Three natural choices for α are 1 to obtain the variance matrix of the score, 1/2 to obtain the square root matrix and 0 to obtain the identity matrix. We choose $S_t = \mathcal{I}_{t|t-1}^{-1/2}$.

We use the DCS model to introduce time-variation in λ_t in the NS model. Let $B_t = B(\lambda_t)$ denote a $(N \times 3)$ time-varying matrix with as rows time-varying versions of $B(\tau)$, where the t subscript relates to time-variation in λ_t. We consider the following specification, which we call NS-DCS:

$$\begin{aligned} \rho_t &= B_t f_t + \epsilon_t \\ \lambda_t &= \phi_0(1-\phi_1) + \phi_1 \lambda_{t-1} + \phi_2 s_{t-1}, \end{aligned} \tag{13}$$

where we assume $\epsilon_t \sim N(0, \Sigma_t)$. The $(N \times N)$ error covariance matrix will be specified below. For this model we have $\nabla_t = \epsilon_t' \Sigma_t^{-1} G_t f_t$ and $\mathcal{I}_{t|t-1} = f_t' G_t' \Sigma_t^{-1} G_t f_t$, where $G_t \equiv \frac{\partial B_t}{\partial \lambda_t}$. The log-likelihood function is

$$\mathcal{L}_N = -\frac{1}{2} \ln |\Sigma_t| - \frac{1}{2} \epsilon_t' \Sigma_t^{-1} \epsilon_t, \tag{14}$$

Importantly, we do not model the time-variation in factors f_t. We simply define $f_t = (B_t' B_t)^{-1} B_t' \rho_t$. The reason for this is two-fold. First, as noted before the f_t are defined on returns, for which we do not expect much predictability. Second, for hedging and forecasting purposes, B_t is of interest and not f_t. By concentrating f_t out of the likelihood we do not have to consider the dependence between λ_t and f_t in the observation density and in the specification of the DCS model. This greatly simplifies the model, and makes it far easier to estimate. Moreover, it makes our analysis robust against any misspecification in the time-series process of f_t.

Finally, we want to discuss the choice for a DCS model. An alternative way to model time-variation in the latent parameter is estimation by Kalman Filter of a state space model. The two are fundamentally different, and their difference is similar to the choice between Stochastic Volatility models and GARCH-type models, where DCS models are similar to GARCH. For our purpose we prefer the DCS method. The main reason is that in the DCS model, λ_t is known conditional on $\mathcal{F}_{t-1}$ as it is a deterministic function. This makes forecasting straightforward. Moreover, the likelihood is known analytically, making estimation

straightforward. For a state-space model, λ_t is not deterministic conditional on the past, it remains unobserved. Evaluation of the log-likelihood therefore requires integrating over the path space of λ_t, and is much more computationally demanding.

3.1. Time-varying volatility

Since excess returns are known to be heteroskedastic, it seems important to allow for GARCH effects in the idiosyncratic risks. Moreover, the 'innovation' in the DCS model, s_t, is a function of Σ_t, the covariance matrix of ϵ_t. Keeping the covariance matrix constant or time-varying is similar to OLS versus GLS. When we allow for dynamics in Σ_t, the score is down-weighted more in high volatility times, reducing the impact of large idiosyncratic errors. As such we consider two versions of the DCS. In the base case we consider $\Sigma = \sigma^2 I_N$ as a constant. In the more general model we allow for a common GARCH-type process. In order to have a parsimonious specification we let $\Sigma_t = \sigma_t^2 I_N$, where

$$\sigma_t^2 = \omega + \alpha\sigma_{t-1}^2 + \beta\Gamma'(\epsilon_{t-1} \circ \epsilon_{t-1}), \tag{15}$$

with $\circ$ the Hadamard product and Γ a loading vector of the cross-sectional dimension such that $\Gamma'\Gamma = 1$. The loading vector may be estimated or, as we do, set to equal weights for each maturity, i.e. $\Gamma = \iota/\sqrt{N}$.[2] The volatility dynamics of all maturities are thus governed by a single GARCH process. Any full multivariate model would quickly suffer from the curse of dimensionality as the cross-sectional dimension typically exceeds ten or fifteen. With time-varying Σ_t we have two self-explanatory additional models, the NS-GARCH and NS-DCS-GARCH.

[2] More general specifications for Γ are possible. An obvious parsimonious alternative would be to make the weights linear in maturity, i.e. $\Gamma_\tau = a + b(\tau - \bar{\tau})$, where $\bar{\tau}$ is the average maturity. However, the simple equal weighting cannot be rejected by the data.

4. Hedging long-term liabilities

We consider an investor with a stream of future liabilities with cashflows $-Z_t(\tau)$. Using the current term structure the value of the liabilities is

$$L_t = \int_0^{\mathcal{T}} Z_t(\tau) P_t(\tau)\, \mathrm{d}\tau \tag{16}$$

The return (in excess of the riskfree rate) on the liability portfolio is then

$$\begin{aligned} r_{t+h}(L) &= \int_0^{\mathcal{T}} w_{L,t}(\tau) r_{t+h}(\tau)\, \mathrm{d}\tau \\ &= \left(\int_0^{\mathcal{T}} w_{L,t}(\tau) b_t(\tau)\, \mathrm{d}\tau \right) f_{t+h} \equiv b_t^* f_{t+h} \end{aligned} \tag{17}$$

where the (negative) liability weights are given by $w_{L,t}(\tau) = -Z_t(\tau)P_t(\tau)/L_t$ and b_t^* is a (3×1) vector. The sensitivity of the liability returns are labelled as 'generalized duration' in Diebold et al. (2006a). As the first column of $B(\tau)$ equals 1, the first element in b_t^* is the duration of the liabilities. Hedging the level factor in Nelson-Siegel is therefore equivalent to duration hedging. The second and third element are exposures to the slope and curvature factors.

In practice hedging typically focuses on the first factor only, ignoring the slope and curve factor, despite wide acceptance of the presence of at least three factors in interest rate curves. In our sample, the first factor explains about 82% of the variation in returns. With a second factor a total of 93% of variation is explained, and a third factor raises this to over 96%. Addressing only the first factor therefore inherently limits the hedging potential. Portfolios can easily be formed to hedge exposure to all three factors. If the factor loadings are accurately measured or predicted, the risk of the hedge portfolio should be significantly reduced.

If the liabilities have a factor exposure b_t^*, then a bond portfolio with returns that have the same factor exposure would only leave some idiosyncratic risk. When the parameter λ in the factor loadings changes over time, factor loadings $b_t(\tau)$ will also change over time. Knowing λ_t an investor can locally hedge by constructing a portfolio factor exposure b_t^*. With time-varying factor loadings the hedge needs to be rebalanced if factor loadings change.

Our focus is on hedging a liability with a very long maturity. More specifically we consider a 50-year maturity. The hedging of these long-term liabilities is of great interest for practitioners such as pension funds and it highlights the importance of accurately modeling the factor structure. Hedging long-term liabilities is a much more intricate problem than the hedging of short to medium term liabilities. To hedge a ten-year bond, taking an opposite position in a number of bonds with maturities close to 10 years will work quite well as their factor loadings are similar to the ten year bond. For long-term bonds, such as the fifty year bond, no liquid bonds with close maturity exist, making this infeasible. Since the market for these very long-term maturities is less liquid, the investor needs to hedge the liability using traded bonds with a maturities that are more liquid. In our formal model we assume that the investor starts with a balanced position with the assets equal to liabilities. He considers investing in bonds with maturities τ_i. Let $w(\tau_i)$ $(i = 1, \ldots, N)$ denote the portfolio weights and let $w(\tau_0)$ be the weight of the single long-term liabilitiy with $\tau_0 = 50$ years. Then the hedging problem at time t is written as

$$w(\tau_0) = -1 \tag{18a}$$

$$\sum_{i=0}^{N} w(\tau_i) = 0 \tag{18b}$$

$$\sum_{i=0}^{N} \tau_i w(\tau_i) b(\tau_i) = \mathbf{0} \tag{18c}$$

$$\min w(\tau_i)^2 \tau_i^2. \tag{18d}$$

Line a shows a short position in the long-maturity bond. Line b requires the portfolio to be self-financing, and line c implies zero factor exposure of the portfolio. If the factor loadings are forecasted correctly, that is, λ_t is equal to the true λ_t, the only remaining risk is the idiosyncratic risk of the individual bonds. Finally, line d chooses the portfolio with minimum idiosyncratic risk out of the infinite set of portfolios satisfying the first three constraints. Note that Σ_t plays no role. All idiosyncratic risks have a variance scaled by the scalar σ_t^2 due to the assumption that all covariance between returns is determined by joint factor exposures

through $b(\tau_i)$. The portfolio weights in the minimization are multiplied by τ_i because of the normalization by τ in the definition of the scaled returns $\rho_t(\tau)$.

The 50-year bond needs to be hedged using large positions in bonds with much shorter maturities. In terms of the Nelson-Siegel model, the 50-year maturity bond essentially only loads on the level factor. However, the slope and curve factor play a crucial role in hedging the long-term liability, since the portfolio of bonds used to hedge the long-position loads significantly on the slope and curve factor. These shorter bonds have to be chosen carefully, since the overall hedge portfolio also needs to be neutral to these factors.

5. Empirical Analysis

5.1. Data

We use daily data on euro swap interest rates with maturities 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, 30, 40 and 50 years from Datastream. Data is available from January 1999, except for the 40 and 50 years rates, which are available form March 2005 onwards. All rates are ask rates quoted at the close of London business. Euro swap rates are quoted on an annual bond 30/360 basis against 6 months Euribor with the exception of the 1 year rate which is quoted against 3 months Euribor. The main advantage of using swaps is that they are available and relatively liquid until long maturities, which allows us to verify the accuracy of the model for long-term liabilities.[3] The series are interpolated using a local cubic interpolation to obtain rates with annual intervals between maturities. We then use a bootstrap method to construct the yield curve. The resulting panel of yields is plotted in Figure 2.

[Figure 2 about here.]

The next step is to compute returns. We first obtain the price from the yields as

$$P_t(\tau) = \exp(-\tau y_t(\tau)) \tag{19}$$

[3]On April 1st 2004 there is almost surely a data error for the 6Y rate, which is exactly equal to the 5Y rate. This observation distorts all estimations and is replaced with an interpolated value.

Table 1: In-Sample Parameter Estimates

	Full Cross-Section				Limited Cross-Section			
	NS	NS-DCS	NS-GARCH	NS-DCS-GARCH	NS	NS-DCS	NS-GARCH	NS-DCS-GARCH
λ	0.3834		0.5502		0.5742		0.6372	
	(0.0446)		(0.0204)		(0.0228)		(0.0233)	
ϕ_0		0.4108		0.4161		0.5799		0.6531
		(0.0262)		(0.0221)		(0.0130)		(0.0278)
ϕ_1		0.9888		0.9859		0.9659		0.9614
		(0.0955)		(0.0759)		(0.0759)		(0.0716)
ϕ_2		0.0345		0.0208		0.0288		0.0145
		(0.0188)		(0.0102)		(0.0082)		(0.0045)
$\sigma^2(\times 10^5)$	6.7973	6.3411			5.0114	4.9419		
	(0.6929)	(0.6809)			(0.5587)	(0.5349)		
$\omega(\times 10^7)$			6.1422	6.5799			7.0818	5.0953
			(8.3806)	(5.2184)			(6.2002)	(4.8337)
α			0.8737	0.8760			0.8794	0.8616
			(0.1053)	(0.0841)			(0.1067)	(0.0612)
β			0.1226	0.1193			0.1191	0.1346
			(0.0987)	(0.0755)			(0.0935)	(0.0602)
LL$(\times 10^{-5})$	2.2720	2.2954	2.4945	2.5029	1.8158	1.8194	1.9748	1.9998

Note: This table provides in-sample parameter estimates for the four different models, both on the full cross-section of all seventeen maturities, as well as the model estimated on the limited cross-section which only includes bonds up until maturity of twenty years. Newey-West standard errors (22 lags) in brackets.

and obtain returns as 100 times the log-difference over time. We subtract the risk-free rate defined as $r_t^f = y_t(1)/252$, the shortest-term yield.

In the remainder of the paper we will estimate the model on two different samples. First we use the full cross-section of seventeen maturities, and second we use a limited cross-section, with only the maturities up until twenty years. The most important reason for excluding maturities longer than twenty years, is the general assumption that the so-called 'last liquid point' of the market rates is at the twenty year maturity.[4] The prices of bonds with maturities greater than this are affected by their lower liquidity, which significantly impacts the yield curve and therefore estimates of the NS-model.

5.2. In-sample Results

Estimation results are reported in Table 1. The results of the full and limited cross-section are qualitatively similar, apart from the limited cross-section having higher estimates of λ, which puts more exposure on short maturities. On the full cross-section, the shape parameter is estimated at $\lambda = 0.3834$ in the standard NS model, putting the maximum of the curve factor between 4 and 5 years. Due to the presence of heteroskedasticity, the NS-GARCH has a far larger estimate, 0.5502, which maxes the curve out between 3 and 4 years. A quick look at the rolling window estimates of Figure 1 explains the difference between the constant and GARCH volatility models, as λ is lowest during the financial crisis when volatility was high.

Next consider the DCS models. The DCS models improve over their standard counterparts: both the NS-DCS and NS-DCS-GARCH have a much larger value for the log-likelihood. The λ's are highly persistent with autoregressive coefficient over 0.98. Importantly, the coefficient of the score innovation, ϕ_2, is significantly different from zero. The models therefore produce significant dynamics in λ.

To illustrate the dynamics of the DCS model we plot the estimated λ_t in Figure 3 for the different models on the full cross-section. The main series of interest are the λ_t estimates of the NS-DCS and NS-DCS-GARCH. Interestingly, up until the second half of 2008, volatility does not play a large role and the fitted λ_t of both models are very similar, and moreover, variation is limited. However, during the crisis period, the two models diverge. The NS-DCS' estimates of λ go down, with an absolute minimum at 0.04, which effectively reduces the model to two factors. Interestingly, the λ_t implied by the NS-DCS-GARCH actually increases to levels not seen in the the first half of our sample. After most of the volatility has died down, the two DCS models converge again.

[Figure 3 about here.]

Next, in order to illustrate the impact of allowing λ_t to vary over time we show a snapshot of the models' fit in Figure 4. We plot the standardized return vector ρ_t across maturities

[4]This is the point assumed by the European Insurance and Occupational Pensions Authority (EIOPA)

along with the Nelson-Siegel fit $B_t f_t$ for the NS and NS-DCS models. As the impact of GARCH effects is difficult to illustrate, we consider a period in which the NS-DCS and NS-DCS-GARCH appear to agree on the level of λ_t, in the beginning of 2009. On this particular day, the NS-DCS estimates $\lambda_t = 0.15$ versus the 0.38 of the constant λ model, resulting in a slower decaying slope factor, and the curve factor maxing out at a later maturity. The effect on fit is large and clear. While the NS-DCS almost perfectly captures the returns across yields, the standard NS model fails to adequately capture the structure of the data, which can lead to large hedging errors.

[Figure 4 about here.]

5.3. Hedging Results

To illustrate the three-factor hedge for short and long-term liabilities we plot the empirical distribution of the ten- and fifty-year hedge portfolios in Figure 5. The top graphs depict the weights in the ten-year hedge portfolio for both duration hedging and the portfolios that hedge all three NS factors, where λ_t is derived form the NCS-DS model. The portfolios do not differ much, as the factor loadings change gradually over the neighboring maturities, and neutralizing the slope and level factor requires a marginal change in the portfolios. Moreover, there is very little time-variation as changes in λ only affect the second and third factor, which have limited impact on portfolio composition.

[Figure 5 about here.]

On the other hand, the dynamics in the 50-year bond hedge are far more extreme. First the duration hedging portfolio takes a large positive position in the bond with closest maturity (20 in this case), but spreads out the weight over different maturities to diversify away as much of the idiosyncratic risk as possible. The three-factor hedge for the 50-year maturity shows that portfolio composition is in fact largely determined by the need to neutralize the second and third factor. Moreover, there is a lot more time-variation in the portfolio composition, as the changing lambda also affects the slope and curve factors, which play a limited role for short-term hedges.

We use the four models highlighted in the previous section to obtain daily forecasts of the next day's λ_t. We contrast the hedging performance of each of the four models with that of simple duration hedging, in which we only hedge the level factor.

We set up two out-of-sample forecasting exercises. In both we re-estimate the parameters of the model every Friday using a rolling window of 1000 observations, and use these parameters to make daily forecasts of λ_t. These forecasts are used to solve the system of equations of the minimum risk hedge portfolio given by Equations (21a-d). However, in one set we use the full cross-section of the 17 maturities including the 50 year maturity itself. In the second exercise we use only the 13 maturities up until and including 20 years. This is an economically important restriction, as bonds with longer maturities are not as liquid, and the market is simply not deep enough to accommodate the needs of pension funds. Here the 50 year maturity is out-of-sample both in terms of the time-series and the cross-sectional dimensional.

We report the mean absolute hedging error (MAE) as well as the variance of the hedging error (Var). The better the hedge, the smaller these two quantities will be. We also report the mean hedging error (Mean). If the mean is positive, the hedging portfolio is earning a higher average return than the target asset. The results are reported in Table 2. We report results for the full out-of-sample period, as well as three sub-periods, corresponding to before, during and after the financial crisis. The top half shows the results for the hedging portfolios constructed using λ forecasts based on the sample including the maturities over twenty years, whereas the bottom panel is based on forecasts using only data with maturities up to twenty years.

The results for the full and partial cross-section are qualitatively similar. Models estimated on the full cross-section perform slightly better. All models have average return close to zero, so none of the models has systematic bias. Therefore, what matters for hedging purposes is the variance of the hedging errors. The MAE and portfolio return variance show differences across the different models. For instance, looking at the full cross-section on the full sample, we see that the variance of hedging errors is 0.58 for duration hedging, 0.37 for

Table 2: Daily hedging error descriptives

	Duration Hedging	NS	NS-DCS	NS-GARCH	NS-DCS-GARCH
			Full Cross-Section		
			Full Sample		
Mean	-0.0024	0.0007	-0.0021	0.0004	0.0020
MAE	0.4053	0.3091	0.2009	0.2964	0.1997
Var	0.5876	0.3691	0.2570	0.3366	0.2215
			2003-2007		
Mean	0.0061	0.0039	0.0039	0.0035	0.0086
MAE	0.1796	0.1595	0.1259	0.1537	0.1211
Var	0.1577	0.1654	0.1279	0.1563	0.1064
			2008-2009		
Mean	0.0165	0.0097	-0.0062	0.0140	0.0049
MAE	0.8106	0.5531	0.3673	0.5829	0.2927
Var	1.7106	0.8066	0.5052	0.8873	0.4167
			2010-2014		
Mean	-0.0207	-0.0070	-0.0070	-0.0093	-0.0068
MAE	0.4817	0.3708	0.2652	0.3830	0.3035
Var	0.5770	0.4067	0.2848	0.4262	0.3736
			Limited Cross-Section		
			Full Sample		
Mean	0.0031	0.0004	0.0042	0.0003	0.0019
MAE	0.4284	0.3174	0.2165	0.3193	0.2333
Var	0.6234	0.3892	0.3281	0.3947	0.3217
			2003-2007		
Mean	0.0056	0.0035	0.0036	0.0034	0.0085
MAE	0.1976	0.1529	0.1238	0.1511	0.1282
Var	0.1899	0.1553	0.1250	0.1533	0.1047
			2008-2009		
Mean	0.0211	0.0145	0.0031	0.0154	0.0116
MAE	0.8618	0.5867	0.3646	0.5944	0.3204
Var	1.8961	0.8992	0.6519	0.9245	0.5141
			2010-2014		
Mean	-0.0196	-0.0095	0.0054	-0.0101	-0.0100
MAE	0.5247	0.3847	0.2913	0.3883	0.3163
Var	0.6110	0.4288	0.3516	0.4341	0.3967

Note: This table gives descriptives of the hedging performance of the different models in sub-periods. The Mean is the average hedging error, the MAE is the mean absolute hedging error, and the Var is the variance of the hedge portfolio returns. The top panel shows results when the models are estimated on the full cross-section of maturities, and all are used to hedge the 50-year maturity, the bottom panel shows the results when optimization occurs over assets with a maximum maturity of 20 years.

the standard Nelson-Siegel and 0.26 for the DCS version. To put things into perspective, the actual variance of the 50-year maturity return is 5.30 over this period. Duration hedging therefore takes out 89% of the variance and the NS improves by hedging 93% of the variance. Importantly, the NS-DCS has additional non-trivial improvements, and hedges 95% of the variance.

For the standard NS, the inclusion of time-varying volatility does not necessarily lead to superior hedging performance. The NS-GARCH performs better in the early sample, whereas the NS model performs better in the later part of the sample. This is in line with the value of λ implied by our DCS models in Figure 3. There the DCS-NS shows that λ is high in early sample, and decreases during the crisis. The NS-GARCH estimate of λ is typically higher than the NS estimate, and more in line with the dynamic estimates in the beginning of the sample.

The DCS models improve on the standard models without exception. Overall, the NS-DCS-(GARCH) lower the MAE and variance by approximately 30% compared to the NS(-GARCH). This is clearly an economically significant improvement. The NS-DCS-GARCH performs slightly better in the early sample and during the financial crisis, while the NS-DCS beats the time-varying volatility variant in the final part of our sample. A possible explanation for the latter result is that yields have hit the zero-lower bound in recent years, and variance is artificially low.

In the first part of the sample DCS improvements are only about 20%, while during the crisis improvements are 40% to 50%, and they are still about 30% after the crisis. As would be expected, variation in λ is greater in highly dynamic markets, and the DCS models offer the greatest improvements over the constant λ models. However, even in tranquil times the DCS offers improvements over the constant model.

Although the in-sample likelihood improvement of time-varying volatility is large, for our empirical application the gains are small. The reverse is true for the time-variation in λ, where the DCS model offers smaller likelihood improvements, but the empirical applications shows that benefits can be very large.

6. Conclusion

We have proposed a Dynamic Conditional Score version of the Nelson-Siegel in which time-variation of the shape-parameter λ is modeled. The parameter is of great importance as it completely specifies the factor loadings of bonds at different maturities. We show that the time-variation in the parameter is large, and assuming it constant is a restriction rejected by the data. In-sample estimation shows the parameter varies between 0.1 and 0.9, and even within a year the parameter has a range of up to 0.6. We document that the parameter is relatively stable in the beginning of the sample which starts in 1999, but becomes highly volatile during the financial crisis of 2008.

In our empirical application, we use the DCS version of the NS model to hedge long-term liabilities using euro-swap data. The shape parameter is of great importance as hedging a liability amounts to forming a portfolio with opposite factor loadings, which are a function of λ. We find that allowing for time-variation in λ leads to significant improvements in hedging performance, greatly reducing the mean absolute hedging error and hedge portfolio variance. Specifically, hedging error variance is reduced by an additional 50% compared to the NS model's improvement over duration hedging. The variation in the shape parameter is of far greater importance than taking into account time-varying volatility, which only offers small improvements, both in the constant lambda and DCS setting.

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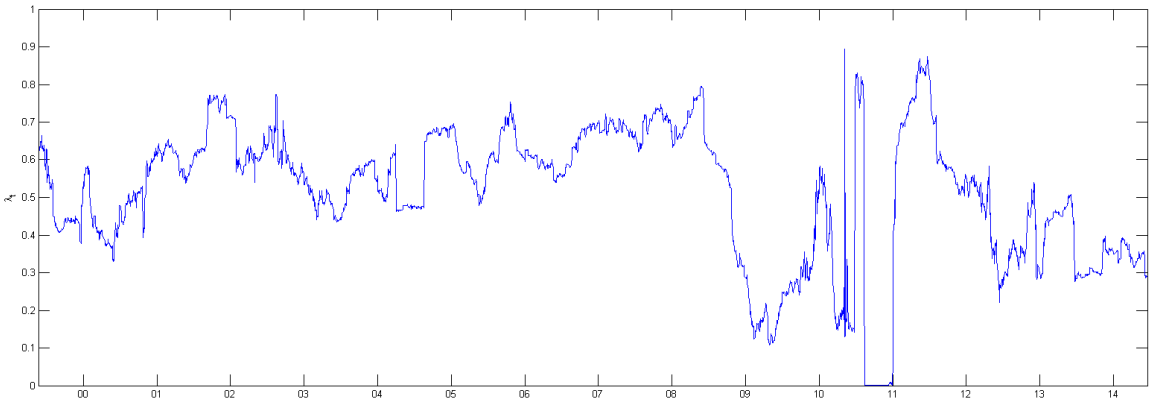
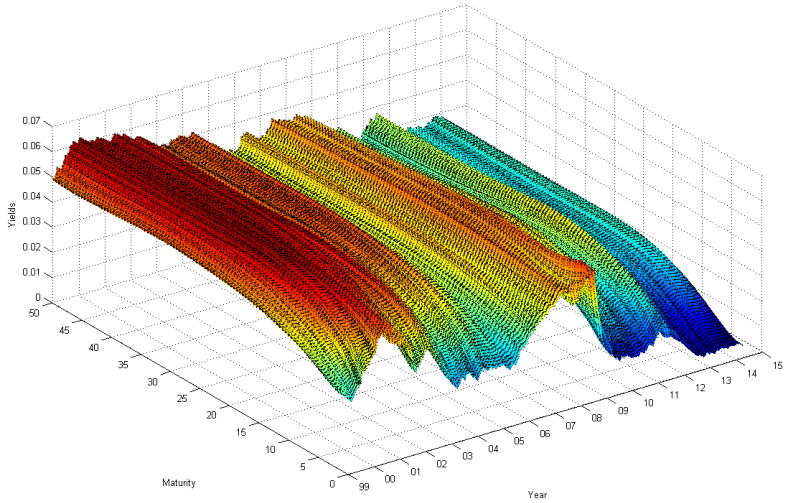
Figure 1: Time-variation in λ



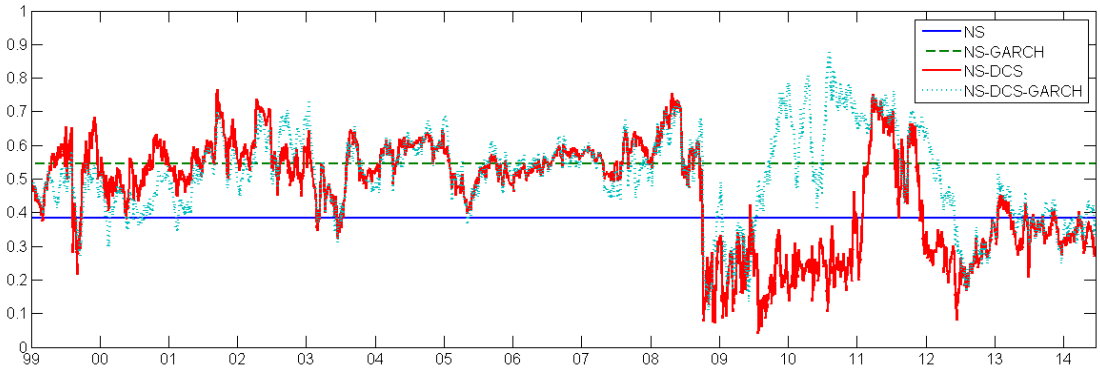
Note: This graph plots the rolling-window estimates of λ. For each day t the sum of squares $\sum_{s=1}^{L}\sum_{i=1}^{N}\epsilon_{t-L+s}(\tau_i)^2$ in (10) is minimised over f_{t-L+s} (linear given λ) and λ (nonlinear) using $L = 100$ days of data and $N = 17$ maturities.

Figure 2: Yield Curves



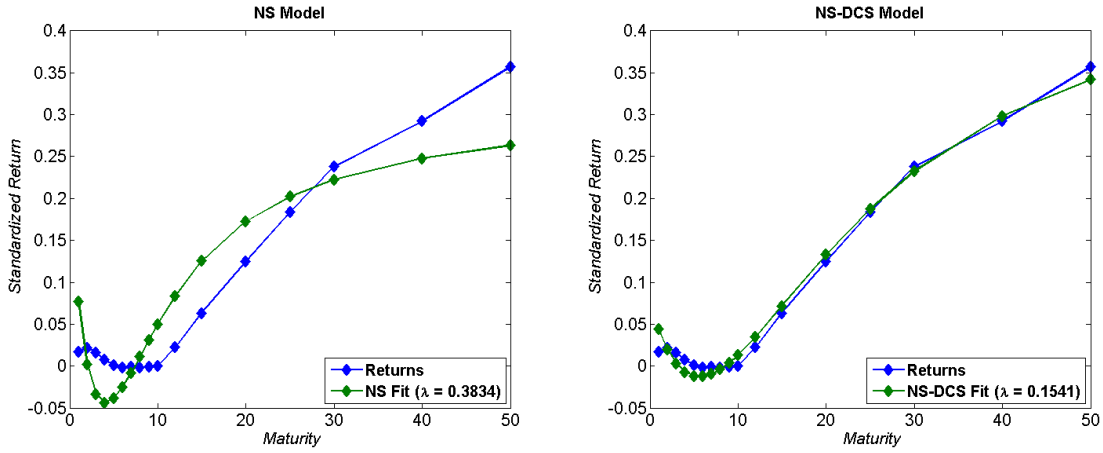
Note: This figure plots the yield curve of our euroswap data evolving over time for the different maturities.

Figure 3: Time Series of λ_t



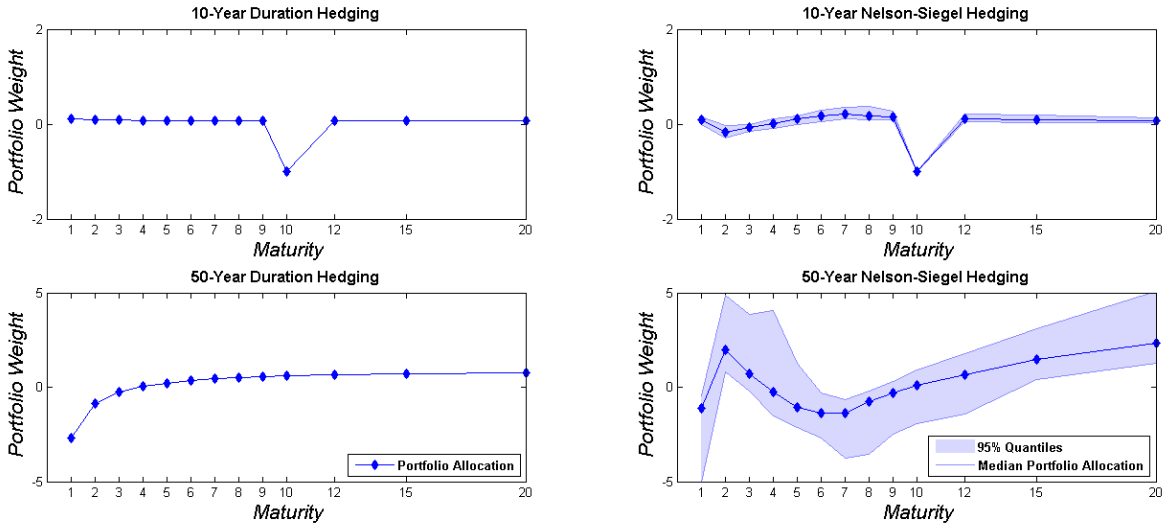
Note: This figure plots in-sample filtered path of λ_t for the four different models.

Figure 4: NS and NS-DCS Fit on January 29th, 2009.



Note: The two figures plot the maturity-standardized returns ρ_t against the models' fitted values $B_t f_t$ for the constant λ Nelson-Siegel model, as well as the dynamic score-driven NS' λ_t.

Figure 5: Hedging Long-Term Liabilities



Note: The graphs depict the importance of hedging all factors for long-term hedging. The top graphs show the portfolio allocation for hedging a bond with 10-year maturity, and the bottom graphs show the portfolio allocation for hedging a 50-year bond. The two graphs on the left show Duration hedging and the right graphs show the results of hedging all three factors using λ_t from the NCS-DS model. For the three-factor hedge we show the empirical distribution of the portfolio weights over time.

Appendix A. Derivations of the Score and Information

In this section we show the derivations of the score and information which are used for the DCS model in (13). Let $\epsilon_t = y_t - B_t f_t$. Assume $\epsilon_t \sim N(0, \Sigma_t)$. Then the likelihood is

$$\mathcal{L}_N = -\frac{1}{2}\ln|\Sigma_t| - \frac{1}{2}\epsilon_t'\Sigma_t^{-1}\epsilon_t. \tag{A.1}$$

Defining $G_t \equiv \frac{\partial B_t}{\partial \lambda_t} = \begin{pmatrix} 0 & \frac{-b_{t,3}}{\lambda_t} & \tau b_{t,2} - (\tau + \frac{1}{\lambda_t})b_{t,3} \end{pmatrix}$ and $H_t \equiv \frac{\partial^2 B_t}{\partial \lambda_t^2}$ the scores are

$$\begin{aligned} \nabla_t(\lambda_t) = \frac{\partial \mathcal{L}_N}{\partial \lambda_t} &= r_t'\Sigma^{-1}G_t f_t - f' B_t'\Sigma^{-1}G_t f_t \\ &= \epsilon_t \Sigma^{-1} G_t f_t. \end{aligned} \tag{A.2}$$

The second order derivatives are

$$\begin{aligned} \frac{\partial^2 \mathcal{L}_N}{\partial \lambda_t^2} &= r_t\Sigma^{-1}H_t f_t - f_t'G_t'\Sigma^{-1}G_t f_t - f_t'B_t'\Sigma^{-1}H_t f_t \\ &= \epsilon_t\Sigma^{-1}H_t f_t - f_t'G_t\Sigma^{-1}G_t f_t \end{aligned} \tag{A.3}$$

from which we find the information matrix as

$$\begin{aligned} \mathcal{I}_{t|t-1}(\lambda_t) &= -\mathbb{E}\left(\frac{\partial^2 \mathcal{L}_N}{\partial \lambda_t^2}\right) \\ &= f_t'G_t'\Sigma^{-1}G_t f_t. \end{aligned} \tag{A.4}$$

and thus

$$\mathcal{J}_{t|t-1}(\lambda_t) = \sqrt{\frac{1}{f_t'G_t'\Sigma^{-1}G_t f_t}}. \tag{A.5}$$