An Empirical Investigation of Affine Term Structure Model Uncertainty

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Motivation

- Practitioners prefer simple models for tractability reason.
- Model uncertainty might be significant, if using simple (nominal) models.
  - nominal model (NOM): the model you use.
  - model uncertainty: all uncertainty that causes your model to fail to capture the true DGP.
- Affine term structure models of different estimation approaches and factors choices.

Research Question: How does model uncertainty affect asset pricing using Affine Term Structure Models (ATSM).
Literature

• Hansen and Sargent [2007] v.s. Schneider and Schweizer [2015]
  - possibly complicated nominal models v.s. potentially simple ones.
  - parametric model uncertainty v.s. model misspecification uncertainty.
• Glasserman and Xu [2014] v.s. Perez-Cruz [2008]
  - Divergence calculation by definition v.s. by empirical approach
• Adrian et al. [2013] easy-to-implement estimation for ATSM
  - focus on the different structures of factor models
This Paper

- Analyzes the **uncertainty impacts** of expected yield curves.
  - uncertainty impacts: the outcomes of the best case and the worse case.
- Compares the impacts of **chosen** nominal ATSM.
- Evaluates the impacts by **two empirical approaches** based on different divergence calculations.
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Model Uncertainty Evaluation Theory

Introduction

Model Uncertainty Illustration I

- Based on data, select a model confidence set \((\mathcal{P}_C)\) from a big collection of models with various structures and variables, deemed as a set of empirically indistinguishable models that captures the true DGP [Hansen et al., 2011].

- Quantify the model uncertainty of the NOM by \(\kappa^*\), the maximal KL divergence from the \(\mathcal{P}_C\), and use it to construct the uncertainty set \(\mathcal{P}_U\).

- Consider the impacts as the best outcome and the worst outcome, which are obtained from the alternative models in \(\mathcal{P}_U\) by a change of measure from the NOM [Glasserman and Xu, 2014].
Model Uncertainty Illustration II

\( \mathcal{P}_U \) (\( \mathcal{P}_{\kappa^*} = \mathcal{P}_{\kappa_{\theta^*}} \))

\( \mathcal{P}_C \)

NOM

\( \kappa^* \): the true DGP.

\( \kappa_{\theta^*} \): the maximum divergence from \( \mathcal{P}_C \).

\( \kappa_{\theta^*} \): equals to \( \kappa^* \); the divergence from the alternative model.

\( \tau \) (maturity)

\( y \) (bond yield)

worst case

best case

parameter uncertainty v.s. model misspecification uncertainty.
Application: Affine Term Structure Models

- The bond yield $y$ with maturity of $\tau$ at time $t$ is given by
  \[ y_t^{(\tau)} = A_\tau + B_\tau' X_{t,j} + u_t^{(\tau)}, \]
  (1)
  where parameters $A_\tau$ and $B_\tau$ are estimated by Adrian et al. [2013], and $X_{t,j}$ is the pricing-factor vector of model $j$ following a VAR(1) process.

- A model $j$ is defined as a conditional probability $p_j(y|X)$.
- We are interested in the expected pricing outcomes from $\mathcal{P}_U$
  - $E_j(y)$ under $p_j(y)$
  - the best case $\text{sup } E_j(y)$ under $p_{j,\text{sup}}(y)$
  - the worst case $\text{inf } E_j(y)$ under $p_{j,\text{inf}}(y)$
Evaluation of the Impacts I

Consider the optimization problem

$$\sup_{m \in \mathcal{P}_\kappa} \mathbb{E}_{\text{NOM}} (c \cdot m \cdot y),$$

(2)

where $m$ is the Radon-Nikodym derivative equal to $\frac{p_j(y)}{p_{\text{NOM}}(y)}$. $c = 1$ is for the best case, and $c = -1$ is for the worst case.

- Quantify model uncertainty by KL divergence, defined by $\mathbb{E} (m \log m)$.
- $m_j \in \mathcal{P}_\kappa$ implies $\mathbb{E} (m \log m) \leq \kappa$. 
Evaluation of the Impacts II

• Form the dual optimization problem,

$$\inf_{\theta > 0} \sup_m \mathbb{E}_{\text{NOM}} \left[ c \cdot m \cdot y - \frac{1}{\theta} (m \log m - \kappa) \right],$$

The optimal solution is

$$m^*_\theta = \frac{\exp (c\theta \cdot y)}{\mathbb{E}_{\text{NOM}} \{\exp [c\theta \cdot y]\}}, \quad \theta > 0,$$  \hspace{0.5cm} (3)

By change of measure, the probability measure of the best case or the worst case is

$$p_{\theta, \text{NOM}}(y) = m^*_\theta \cdot p_{\text{NOM}}(y).$$  \hspace{0.5cm} (4)
Approach 1–Evaluation of the Impacts

- The impacts are evaluated by

\[ \mathbb{E}_j(y) = \mathbb{E}_{NOM}(m_{\theta_1}^* \cdot y). \quad (5) \]

- Measure the KL divergence by

\[ \kappa_{\theta_1} = \mathbb{E}(m_{\theta_1}^* \log m_{\theta_1}^*), \quad (6) \]

and calibrate \( \theta_1 \) such that \( \kappa_{\theta_1} = \kappa^* \), giving \( \theta_1^* \).
Approach 2-Evaluation of the Impacts 1

- Given $p_{NOM}(y) \sim \mathcal{N}(\mu_y, \Sigma_y)$, then
  
  $$p_{\theta,NOM}(y) \sim \mathcal{N}(\mu_y + c\theta_2 \Sigma_y, \Sigma_y)$$
  
  by $p_{\theta,NOM}(y) = m_{\theta_2}^* \cdot p_{NOM}(y)$.

- The impacts are evaluated by
  
  $$\mathbb{E}_j(y) = \mathbb{E}_{NOM}(y + c\theta_2 \Sigma_y) \quad (7)$$
Approach 2—Evaluation of the Impacts II

- KL divergence is **empirically calculated** by the $k$NN approach [Perez-Cruz, 2008]

$$
\kappa_{\theta_2} = \frac{d}{N_2} \sum_{n_2=1}^{N_2} \log \frac{r_k(\tilde{y}_{n_2})}{s_{k+1}(\tilde{y}_{n_2})} + \log \frac{N_1}{N_2 - 1},
$$

using i.i.d samples $\tilde{y} = \{\tilde{y}_{n_2}\}_{n_2=1}^{N_2}$ and $\hat{y} = \{\hat{y}_{n_1}\}_{n_1=1}^{N_1}$ drawn from $p_{\theta,\text{NOM}}(y)$ and $p_{\text{NOM}}(y)$ respectively.

- $r_k(\tilde{y}_{n_2})$: the Euclidean distance of the $k$-th nearest-neighbour of $\tilde{y}_{n_2}$ in $\hat{y}$;
- $s_{k+1}(\tilde{y}_{n_2})$ the Euclidean distance of the $(k + 1)$-th nearest-neighbour of $\tilde{y}_{n_2}$ in $\tilde{y}$.

- $\theta_2^*$ is **calibrated** such that $\kappa_{\theta_2} = \kappa^*$
## Data Description I

### Panel A: The Estimation Errors of the NSS Approach

| Maturity (month) | $|\text{mean}|$ (%) | std (%) | obs |
|------------------|--------------|---------|------|
| 1                | 0.0020       | 0.0066  | 150  |
| 3                | 0.00002      | 0.0035  | 384  |
| 6                | 0.0012       | 0.0033  | 384  |
| 12               | 0.0005       | 0.0027  | 661  |
| 24               | 0.00002      | 0.0022  | 451  |
| 36               | 0.0010       | 0.0020  | 661  |
| 60               | 0.0004       | 0.0015  | 661  |
| 84               | 0.0004       | 0.0015  | 534  |
| 120              | 0.0007       | 0.0017  | 661  |
| 240              | 0.0005       | 0.0027  | 661  |
| 360              | 0.0013       | 0.0034  | 443  |

### Panel B: Data descriptions of NSS estimates of bond yields.

<table>
<thead>
<tr>
<th>Maturity (month)</th>
<th>mean (%)</th>
<th>std (%)</th>
<th>minimum (%)</th>
<th>maximum (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1371</td>
<td>0.0634</td>
<td>0.0004</td>
<td>0.2574</td>
</tr>
<tr>
<td>3</td>
<td>0.1437</td>
<td>0.0606</td>
<td>0.0022</td>
<td>0.2586</td>
</tr>
<tr>
<td>6</td>
<td>0.1486</td>
<td>0.0596</td>
<td>0.0016</td>
<td>0.2638</td>
</tr>
<tr>
<td>12</td>
<td>0.1540</td>
<td>0.0584</td>
<td>0.0033</td>
<td>0.2679</td>
</tr>
<tr>
<td>24</td>
<td>0.1605</td>
<td>0.0541</td>
<td>0.0146</td>
<td>0.2710</td>
</tr>
<tr>
<td>36</td>
<td>0.1648</td>
<td>0.0497</td>
<td>0.0257</td>
<td>0.2702</td>
</tr>
<tr>
<td>60</td>
<td>0.1699</td>
<td>0.0430</td>
<td>0.0443</td>
<td>0.2650</td>
</tr>
<tr>
<td>84</td>
<td>0.1728</td>
<td>0.0389</td>
<td>0.0600</td>
<td>0.2617</td>
</tr>
<tr>
<td>120</td>
<td>0.1752</td>
<td>0.0353</td>
<td>0.0771</td>
<td>0.2607</td>
</tr>
<tr>
<td>240</td>
<td>0.1793</td>
<td>0.0319</td>
<td>0.1032</td>
<td>0.2600</td>
</tr>
<tr>
<td>360</td>
<td>0.1828</td>
<td>0.0324</td>
<td>0.1123</td>
<td>0.2656</td>
</tr>
</tbody>
</table>

- Original data of bond yields are **unbalanced**.
- Apply **Nelson-Siegel-Svensson (NSS)** approach to get data balanced.
- Compare the estimates with the original data. NSS performs well.
Data Description II

Use **NSS estimates** as the data input for the following studies.
Model Confidence Set I

- The Great Candidate Set

\[ \mathcal{M}_0 = \{[1], [6], [9], [24], [60], [84], [120], [1, 6], [6, 9], [9, 12], [12, 36], [12, 60], [36, 84], [60, 120], [1, 6, 9], [6, 9, 12], [12, 24, 60], [36, 60, 84], [12, 60, 120], [1, 6, 9, 12], [6, 9, 12, 24], [24, 36, 60, 84], [6, 9, 12, 24, 36], [6, 24, 36, 60, 84], [1, 6, 12, 24], [6, 12, 24, 36, 60, 120], [9, 12, 36, 60, 84, 120], [1, 6, 9, 12, 24, 36, 60], [12, 24, 36, 60, 84, 120], [6, 9, 12, 24, 36, 60, 84, 120] \}.

- A collection of ATSM based on factor models.
- Factors are bond yields with \( \tau \)-month maturities as the numbers indicate.
- The selection procedure first ranks them according to a loss function, and then tests whether the worst has to be eliminated.
- Set significant level \( \alpha = 5\% \).
Model Confidence Set II

- For instance, pricing the 30-month bond gives the ranked set

\[ \mathcal{M}_r = \{[1], [120], [84], [6], [1, 6], [60], [9], [6, 9], [1, 6, 9], [60, 120], [9, 12], [24], [6, 9, 12], [1, 6, 9, 12], [12, 60], [36, 84], [36, 60, 84], [12, 36], [12, 60, 120], [24, 36, 60, 84], [12, 24, 60], [6, 9, 12, 24], [1, 6, 9, 12, 24], [9, 12, 36, 60, 84, 120], [12, 24, 36, 60, 84, 120], [6, 24, 36, 60, 84], [6, 9, 12, 24, 36], [6, 12, 24, 36, 60, 120], [1, 6, 9, 12, 24, 36, 60, 84, 120]\}.

The \( p \)-value for each \( r \) round of test collected correspondingly in the set

\[ p_{r-val}(\mathcal{M}_r) = \{0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.01\%, 0.01\%, 0.01\%, 0.01\%, 0.01\%, 0.01\%, 0.01\%, 1.01\%, 1.22\%, 1.80\%, 5.68\%, 6.24\%, 56.54\%, 100\%\}. \]
A General Picture of Pricing Results from models in $\mathcal{M}_0$
Model Uncertainty Investigation

- Investigate the bond pricing with maturities of 10-month, 30-month, 40-month, 50-month, 60-month, 80-month, 90-month and 110-month.
- $X_{NOM}$s for comparison
  
  \[
  NOM_1 = [1]; \quad NOM_2 = [120]; \quad NOM_3 = [12, 36]; \\
  NOM_4 = [12, 60]; \quad NOM_5 = [6, 9, 12]; \quad NOM_6 = [12, 60, 120]; \\
  NOM_7 = [1, 6, 9, 12, 24].
  \]

- The uncertainty impacts will form "uncertainty bands" across maturity.
κ* obtained by the kNN approach

Table: The KL divergence κ*

<table>
<thead>
<tr>
<th></th>
<th>τ</th>
<th>10</th>
<th>30</th>
<th>60</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOM1</td>
<td>10</td>
<td>0.0467</td>
<td>0.1854</td>
<td>0.1506</td>
<td>0.1696</td>
</tr>
<tr>
<td>NOM3</td>
<td>30</td>
<td>0.0008</td>
<td>0.0007</td>
<td>0.0009</td>
<td>0.0093</td>
</tr>
<tr>
<td>NOM5</td>
<td>60</td>
<td>—</td>
<td>0.0006</td>
<td>0.0021</td>
<td>0.0144</td>
</tr>
<tr>
<td>NOM6</td>
<td>110</td>
<td>0.0007</td>
<td>0.0005</td>
<td>0.0006</td>
<td>0.0001*</td>
</tr>
<tr>
<td>NOM7</td>
<td>284</td>
<td>—</td>
<td>0.0004</td>
<td>0.0007</td>
<td>0.0019</td>
</tr>
</tbody>
</table>

- NOM5 and NOM7 are in the MCS when pricing 10-month bond, cases not considered in this paper.
**θ**\(^*\) obtained by Approach 1 and Approach 2

**Table:** The **θ**\(^*\) for the best cases, based on calibrations by Approach 1 and Approach 2.

<table>
<thead>
<tr>
<th><strong>θ</strong>(^*)</th>
<th>(X_{NOM})</th>
<th>10</th>
<th>30</th>
<th>60</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>θ</strong>(_1^*)</td>
<td>NOM(_1)</td>
<td>6.2800</td>
<td>12.1700</td>
<td>12.1680</td>
<td>14.1700</td>
</tr>
<tr>
<td></td>
<td>NOM(_3)</td>
<td>0.6970</td>
<td>0.7130</td>
<td>0.9620</td>
<td>3.7700</td>
</tr>
<tr>
<td></td>
<td>NOM(_5)</td>
<td>—</td>
<td>0.6850</td>
<td>1.5250</td>
<td>4.7000</td>
</tr>
<tr>
<td></td>
<td>NOM(_6)</td>
<td>0.6344</td>
<td>0.6835</td>
<td>0.8250</td>
<td>0.2813</td>
</tr>
<tr>
<td></td>
<td>NOM(_7)</td>
<td>—</td>
<td>0.5219</td>
<td>0.8500</td>
<td>1.7000</td>
</tr>
<tr>
<td><strong>θ</strong>(_2^*)</td>
<td>NOM(_1)</td>
<td>20.9330</td>
<td>14.8469</td>
<td>12.4250</td>
<td>14.22</td>
</tr>
<tr>
<td></td>
<td>NOM(_3)</td>
<td>919.00</td>
<td>683.00</td>
<td>108.18</td>
<td>35.1200</td>
</tr>
<tr>
<td></td>
<td>NOM(_5)</td>
<td>—</td>
<td>111.60</td>
<td>24.00</td>
<td>24.10</td>
</tr>
<tr>
<td></td>
<td>NOM(_6)</td>
<td>1300.00</td>
<td>747.00</td>
<td>5006.00</td>
<td>3650.00</td>
</tr>
<tr>
<td></td>
<td>NOM(_7)</td>
<td>—</td>
<td>12060.00</td>
<td>66.00</td>
<td>18.00</td>
</tr>
</tbody>
</table>
A conjecture: $\theta_2$ decreases in $\Sigma_y$

Recall Approach 2

<table>
<thead>
<tr>
<th>$\Sigma_y$</th>
<th>$\theta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 10^{-7}$</td>
<td>$2.73 \times 10^{-4}$</td>
</tr>
<tr>
<td>$5 \times 10^{-7}$</td>
<td>$5.65 \times 10^{-3}$</td>
</tr>
<tr>
<td>$1 \times 10^{-6}$</td>
<td>$2.78 \times 10^{-3}$</td>
</tr>
<tr>
<td>$5 \times 10^{-6}$</td>
<td>548</td>
</tr>
<tr>
<td>$1 \times 10^{-5}$</td>
<td>275</td>
</tr>
<tr>
<td>$5 \times 10^{-5}$</td>
<td>56.2</td>
</tr>
<tr>
<td>$1 \times 10^{-4}$</td>
<td>28</td>
</tr>
<tr>
<td>$5 \times 10^{-4}$</td>
<td>6.3</td>
</tr>
<tr>
<td>$1 \times 10^{-3}$</td>
<td>3.5</td>
</tr>
<tr>
<td>$5 \times 10^{-3}$</td>
<td>1.115</td>
</tr>
</tbody>
</table>

Figure: $\theta_2$ and $\Sigma_y$ for a same $\kappa^* = 0.0021$ in pricing 60-month bond using $NOM_5$. The table on the right lists the values plotted in the figure.
Uncertainty Bands of Yield Curves

\begin{align*}
\text{NOM}_1 &= [1] \\
\text{NOM}_2 &= [120] \\
\text{NOM}_3 &= [12, 36] \\
\text{NOM}_4 &= [12, 60] \\
\text{NOM}_5 &= [6, 9, 12] \\
\text{NOM}_6 &= [12, 60, 120] \\
\text{NOM}_7 &= [1, 6, 9, 12, 24]
\end{align*}
Summary & Conclusions

- Evaluation of the impacts by two empirical approaches. Impacts by both approaches are the same. But $\theta^*$ are not, conjectured resulting from $\theta_2$ decreasing in $\Sigma_Y$.
- Comparison of the impacts of chosen nominal ATSM. All uncertainty bands satisfy to include the outcomes from $P_C$.
- Financial interpretations
  - The largest impacts are in the case using a single-factor model with a short rate.
  - The minimal impacts are in the case using three-factor model with short, medium and long rates.
Topics of Future Research

- Extend by considering more advanced ATSM.
- Study the empirical model uncertainty impacts on institutional investments.
- Develop advanced ATSM incorporating model uncertainty.
References I


Thank You!
### Table: The expected sample variances $\hat{\Sigma}_y$

<table>
<thead>
<tr>
<th>$X_{NOM}$</th>
<th>10</th>
<th>30</th>
<th>60</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NOM_1$</td>
<td>0.0007</td>
<td>0.0021</td>
<td>0.0020</td>
<td>0.0017</td>
</tr>
<tr>
<td>$NOM_3$</td>
<td>2.64 · 10$^{-6}$</td>
<td>2.87 · 10$^{-6}$</td>
<td>1.62 · 10$^{-5}$</td>
<td>1.40 · 10$^{-4}$</td>
</tr>
<tr>
<td>$NOM_5$</td>
<td>—</td>
<td>1.68 · 10$^{-5}$</td>
<td>1.16 · 10$^{-4}$</td>
<td>2.54 · 10$^{-4}$</td>
</tr>
<tr>
<td>$NOM_6$</td>
<td>1.69 · 10$^{-6}$</td>
<td>2.43 · 10$^{-6}$</td>
<td>3.21 · 10$^{-7}$</td>
<td>1.03 · 10$^{-7}$</td>
</tr>
<tr>
<td>$NOM_7$</td>
<td>—</td>
<td>1.19 · 10$^{-7}$</td>
<td>2.33 · 10$^{-5}$</td>
<td>1.27 · 10$^{-4}$</td>
</tr>
</tbody>
</table>