Discussion of paper Kristy Jansen

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Outline

1. Summary
2. Illiquidity
3. Change of Variables
4. Interpretation of $p$
Summary of paper

- Paper studies optimal asset allocation with illiquid assets
- Illiquidity defined as periods of “non-trading”
- Re-write optimisation problem in terms of total wealth \((W_t + X_t)\) and fraction in illiquid assets \(\xi_t\)
- Assume CRRA utility
- Study impact of illiquid asset on optimal investment strategy
Definition Illiquid

We have different definitions of “illiquid” in the literature:

- Buy/sell illiquid asset at worse price. Like transaction costs.
- Restriction on traded quantity of illiquid asset.
- Restriction on trading time of illiquid asset.

All three definitions could be unified into one concept: price-function depends on traded quantity and on time. With numerical computation, all three cases can be computed.
Price-function

![Price-function graph](image-url)
Why do you choose the “non-traded” definition of illiquidity?

Provide more motivation for your choice!
You perform a change of variables from $(W_t, X_t)$ to $(W_t + X_t, \xi_t)$ where $\xi_t = X_t / (W_t + X_t)$

You then factor the value-function as

$$V(t, W_t + X_t, \xi_t) = (W_t + X_t)^{1-\gamma} \cdot H(t, \xi_t)$$

which you motivate by citing [Merton, 1971].

However, I cannot find this factorisation in Merton...
You do a proof by induction that this factorisation holds for all time-points

Core idea of proof:

\[ V(t - h) = \mathbb{E}_{t-h}[V(t)] = \mathbb{E}_{t-h}[(W_t + X_t)^{1-\gamma} H(t, \xi_t)] \]

\[ = (W_{t-h} + X_{t-h})^{1-\gamma} \mathbb{E}_{t-h} \left[ \left( \frac{W_t + X_t}{W_{t-h} + X_{t-h}} \right)^{1-\gamma} H(t, \xi_t) \right] \]

\[ = (W_{t-h} + X_{t-h})^{1-\gamma} \mathbb{E}_{t-h}^*[H(t, \xi_t)] \]

\[ = (W_{t-h} + X_{t-h})^{1-\gamma} H(t - h, \xi_{t-h}) \]
However...

You take conditional expectations given $\mathcal{F}_{t-h}$

$$
\mathbb{E}^*_{t-h}[H(t, \xi_t)] = \mathbb{E}^*[H(t, \xi_t) \mid W_{t-h} + X_{t-h}, \xi_{t-h}]
$$

You must rule out the case: $H(t - h, W_{t-h} + X_{t-h}, \xi_{t-h})$

In your current proof you achieve this by imposing that the control policies are functions of $\xi_t$ only, e.g. $\theta(t, \xi)$

But you are now using a smaller class of control policies compared to the original problem: $\theta(t, W, X)$. This may lead to sub-optimal control policies.

How do you compute these conditional expectations numerically? Your description in Section 2.2.2 is a bit vague...
Interpretation of $p$

- At the horizon $T$ you have a probability $p$ that the asset $X$ is liquid, and $(1 - p)$ that the asset sells at the lower value $d X_T$
- At all earlier time-steps you also have these probabilities
- What is the economic interpretation of this?
- What happens with $p$ when you make smaller time-steps? What happens in the limit for $h \to 0$?
Case $p = 0$

- We do have an interesting case for $p = 0$: asset $X$ cannot be traded at all in $[0, T]$. Economically reasonable case.
- Obtain initial position $X_0$ at $t = 0$ and then hold position until time $T$. Liquidate position for $dX_T$
- Economically viable when excess return on $X$ compensates for non-trading and discount $d$
- I believe we can solve this case in “semi explicit” form using [Cox and Huang, 1989] approach
- Also benchmark for your numerical computations
Solving case \( p = 0 \) with Cox-Huang

- For example, consider non-consumption case for \( p = 0 \), then we have

\[
\max_{X_0, W_T} \mathbb{E}
\left[
U(W_T + X_0(d e^{R_T^X}))
\right]
\]

subject to

\[
e^{-rT} \mathbb{E}^Q[W_T] + X_0 = w_0
\]

- Notice the “static” formulation, where we choose initial illiquid position \( X_0 \) and final liquid wealth \( W_T \).
- In complete market we can replicate every possible \( W_T \).
- We can “integrate out” the non-tradeable return \( R_T^X \) to obtain the modified utility: \( \hat{U}(W_T, X_0) := \mathbb{E}[U(W_T + X_0(d e^{R_T^X}))|\mathcal{F}_T^S] \)
- \( \hat{U}() \) will have higher risk-aversion due to “background risk” \( X_T \)
- Solve optimal liquid \( W_T \) for \( \hat{U}() \) with reduced budget \( w_0 - X_0 \) using [Cox and Huang, 1989]
Explicit solution for $p = 0$ and quadratic util

- **Example**: quadratic util, $U(x) = -(x - K)^2$ with saturation point $K$
- Log-normal $e^{R^X_T}$ with drift $\nu$ and vol $\tau$ and assume uncorrelated

$$
\hat{U}(W_T, X_0) = \mathbb{E}[-(W_T + X_0(de^{R^X_T}) - K)^2|\mathcal{F}^S_T] = 
-(W_T - K)^2 - 2(W_T - K)X_0de^{\nu T} - X_0^2d^2e^{(2\nu + \tau^2)T}
$$

- **FOC**: $\partial \hat{U}()/\partial W_T = -2(W_T - K) - 2X_0de^{\nu T} = 2\lambda S_T^{-(\mu - r)/\sigma}$
- **Optimal wealth**: $W_T^* = K - X_0de^{\nu T} - \lambda(X_0) S_T^{-(\mu - r)/\sigma}$ where $\lambda(X_0)$ is solved to satisfy budget constraint for each $X_0$
- **Final step**: find the optimal $X_0$ that maximises utility
- **Note**: optimal wealth $W_t^*$ is *not* a function of $\xi_t$ in this case