How much should life-cycle investors adapt their behavior when confronted with model uncertainty?

by Sally Shen

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Natspar International Pension Workshop

1 Introduction
- Motivation
- Research Question
- Contribution and finding
- Literature Review

2 Model
- Finance Model
- Measure Model Uncertainty
- Robust Optimization Problem
- Solution Techniques

3 Theorem
- Robust Investment Strategy
- Optimal Drift Distortion

4 Calibrating Ambiguity Aversion Parameter
- Estimation Theory
- Calibrate θ

5 Numerical Results
- Robust Optimal Investment Strategy

6 Conclusion

Motivation

- US inflation becomes harder to forecast. (Stock and Watson 2007, JMCB)

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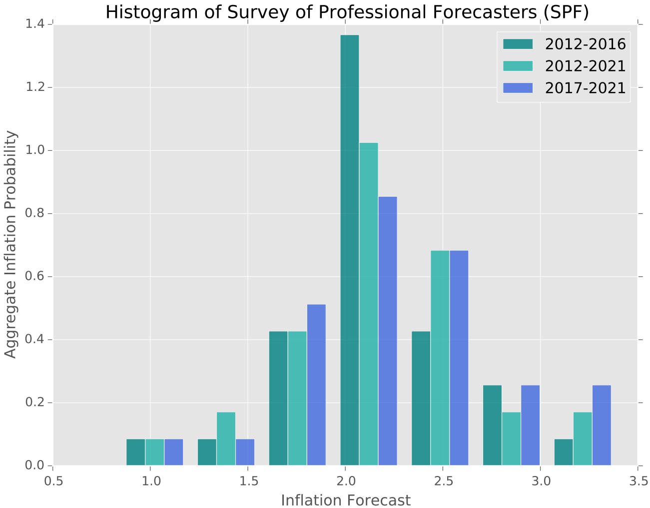
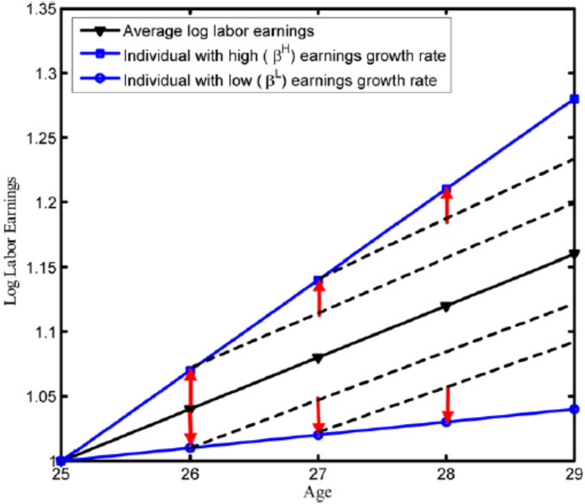
- Survey beats the model-based projection.(Ang, Bekaert and Wei 2007, JME)



Motivation

- Income process is individual specific (Guvenen 2008, RED). Cross-sectional dispersion in income growth has been rising since the 1970s. (Haider 2001, Guvenen and Kuruscu 2008)



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   - Ellsberg-style experiments. However, financial market is much more complicated than stylized Ellsberg urns.
   - Detection-error probability to measure relative entropy parameter. Hansen and Sargent (2002).

Contribution and finding

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- Findings
  1. Ambiguity aversion level is bounded around 1.5 with probability of 95%.
  2. Young investors should increase long-term bond portfolio and reduce stock exposure when inflation is high.

Literature Review

- Optimal asset allocation without labor income
  - Merton (1970, JET)
  - Brennan and Xia (2002, JF)
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  - Munk and Sørensen (2010, JFE)
- Robust Asset Allocation
  - Hansen and Sargent (2007, Book “Robustness”)
  - Maenhout (2004, RFS)
  - Branger Larsen and Munk (2013, JBF)

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Investment Opportunity

- Asset menu: stock, nominal 10-year bond, cash.

$$dW = \underbrace{\left(x_t^\top \left(\mu_t - \iota r_t\right) + r_t\right)}_{\text{expected portfolio return}} Wdt - Cdt + Ydt + Wx_t^\top \sigma^\top dZ$$

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- $dZ \in \mathbb{R}^4$ risk drivers
- W instantaneous financial wealth
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- r_t nominal risk free rate, affine function of latent factor X_t.
- Two-factor affine term structure model. (Sangvinatsos and Wachter 2005, JF)

Expected Inflation Uncertainty

- Reference model for inflation rate

$$\frac{d\Pi}{\Pi} = \pi_t dt + \sigma_\Pi^\top dZ$$

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- Perturbed inflation process:

$$\frac{d\Pi}{\Pi} = \pi_t dt + \sigma_\Pi^\top \left(dZ + \mathbf{e}\gamma_{1,t} dt\right)$$

  - Drift distortion γ_1 on the third Brownian motion process.
  - $\mathbf{e} = (0,0,1,0)^\top$ selects the expected inflation element in dZ

Income Growth Rate Uncertainty

- Reference model for income process

$$dY = (\mu_y)\, Ydt + \sigma_y Y \rho_y^\top dZ + \sigma_y Y \sqrt{1 - ||\rho_y||^2} dZ_Y$$

  - μ_y expected nominal income return is assumed equal to 0.1.
  - $\rho_y \in \mathbb{R}^4$ correlation vector be income shocks and financial market risks,

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- Drift distortions γ_1, γ_2 are penalized by **relative entropy**.

Preferences for a robust investor

- max-min preferences

$$\max_{\{C_t, x_t\}} \min_{\{\gamma_{1,t}, \gamma_{2,t}\}} \mathbb{E}_0 \left[\int_0^T \frac{\exp(-\beta t)}{1-\gamma} \left(\frac{C_t}{\Pi_t} \right)^{1-\gamma} dt + \frac{\varphi \exp(-\beta T)}{1-\gamma} \left(\frac{W_T}{\Pi_T} \right)^{1-\gamma} \right]$$

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- Indirect utility function $J(W, Y, \Pi, X, t)$

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- Model reduced to standard life-cycle model by setting $\gamma_{1,t} = \gamma_{2,t} = 0$.
- Indirect utility function $J(W, Y, \Pi, X, t)$
- Human capital: cumulative discounted remaining labor income stream under risk neutral measure

$$H_t = H(Y_t, X_t, t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T Y_u \exp\left(-\int_t^u r_s ds\right) du\right]$$

Solution Techniques

- First order condition implies, robust HJB equation, Anderson, Hansen and Sargent (2002)

$$0 = \max_{\{C,x\}} \min_{\{\gamma_1,\gamma_2\}} \left\{ \underbrace{\frac{1}{1-\gamma}\left(\frac{C}{\Pi}\right)^{1-\gamma} - \beta J + \mathcal{D}(C,x) J(W,Y,\Pi,X,t)}_{\text{refeece}-\text{model based HJB}} \right.$$

$$\left. + \underbrace{J_W W x^\top \sigma^\top \mathbf{e}\gamma_1 + J_Y Y(f(\gamma_1,\gamma_2)) + J_\Pi \Pi \sigma_\Pi^\top \mathbf{e}\gamma_1}_{\text{adjustment of Bellman equation due to drift distortions}} + \underbrace{\frac{1}{2\Psi}\left(\gamma_1^2+\gamma_2^2\right)}_{\text{penalty function}} \right\}$$

- $\Psi \to \infty$, no confidence over the underlying model.
- $\Psi \to 0$, extremely optimistic toward the reference model.

Solution Techniques

- Homothetic robustness, Maenhout (2004):

$$\boxed{\Psi(W,Y,\Pi,X,t)=\frac{\theta}{(1-\gamma)J(W,Y,\Pi,X,t)}}$$

- $\theta \geq 0$ represents investors' ambiguity aversion level. θ is a subjective number and is independent of time and state variables.
- ***What is a reasonable value of θ?***

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Robust Life-Cycle Portfolio

Proposition

Robust optimal portfolio with human capital and ambiguity protection is

$$x = \left(\gamma\sigma^\top\sigma + \theta\sigma^\top \mathbf{e}\mathbf{e}^\top\sigma\right)^{-1} \frac{W+H}{W}(\mu - \iota r)$$

$$+ \underbrace{\left(\gamma\sigma^\top\sigma + \theta\sigma^\top \mathbf{e}\mathbf{e}^\top\sigma\right)^{-1} \frac{W+H}{W}\left((\gamma-1)\,\sigma^\top\sigma_\Pi + \theta\sigma^\top \mathbf{e}\mathbf{e}^\top\sigma_\Pi\right)}_{\text{Inflation Replication Portfolio}}$$

$$+ \underbrace{\left(\gamma\sigma^\top\sigma + \theta\sigma^\top \mathbf{e}\mathbf{e}^\top\sigma\right)^{-1} \frac{W+H}{W}\sigma^\top\sigma_X \left[\Gamma_1(\tau)\,X_t + \Gamma_2(\tau)^\top\right]}_{\text{Long-run Risk Hedge}}$$

$$\underbrace{- \left(\gamma\sigma^\top\sigma + \theta\sigma^\top \mathbf{e}\mathbf{e}^\top\sigma\right)^{-1} \frac{H}{W}\left(\gamma\sigma^\top\rho_y\sigma_y + \theta\sigma^\top \mathbf{e}\mathbf{e}^\top\rho_y\sigma_y\right)}_{\text{Income Hedging Component}}$$

Nature's Optimal Decision

Proposition

Mother nature's optimal decision on the two drift distortions are

$$\gamma_1 = -\theta \left(\frac{W}{W+H} x^\top \sigma^\top \mathbf{e} + \frac{H}{W+H} \sigma_y \rho_y^\top \mathbf{e} - \sigma_\Pi^\top \mathbf{e} \right)$$

$$\gamma_2 = -\theta \frac{H}{W+H} \sigma_y \sqrt{1 - ||\rho_y||^2}$$

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Econometrics Property: Error of the Point Estimates

Parameter	True	Point Estimate
Inflation	$\sigma_\Pi^\top \mathbf{e}\gamma_1 + \hat{\pi}$	$\hat{\pi}$
Income Growth	$\hat{\mu}_y + \sigma_y \rho_y^\top \mathbf{e}\gamma_1 + \sigma_y \sqrt{1 - \lVert\rho_y\rVert^2}\gamma_2$	$\hat{\mu}_y$

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Proposition

Based on probability theory, drift distortions (γ_1, γ_2) satisfy following constraint set

$$\mathbf{S} = \{\gamma_1, \gamma_2 \mid \gamma_1^2 + \gamma_2^2 \leq \kappa^2\}$$

where $\kappa^2 = \frac{CV_{\alpha}}{N}$. *i.e.* $CV_{5\%} = 5.99$, $N = 632$ *(June 1961 - December 2013 Monthly data)*

Calibration of Ambiguity Aversion Parameter

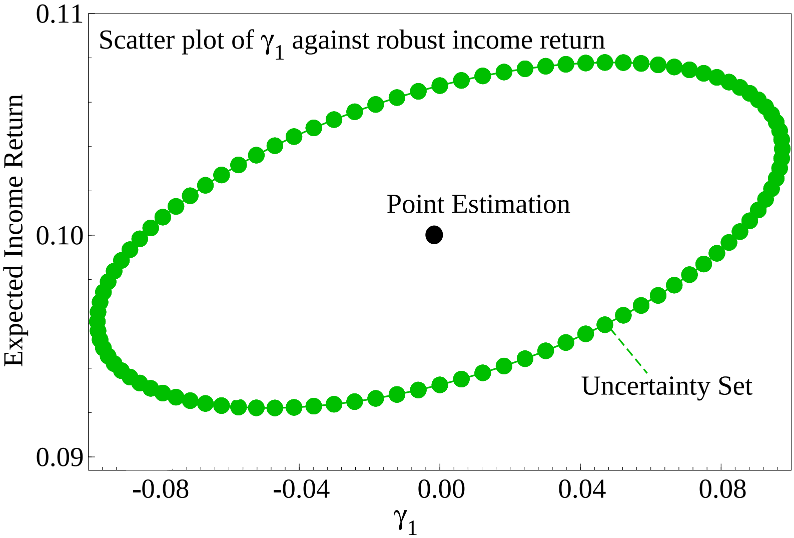

Figure: 95% Confidence Interval Uncertainty Set

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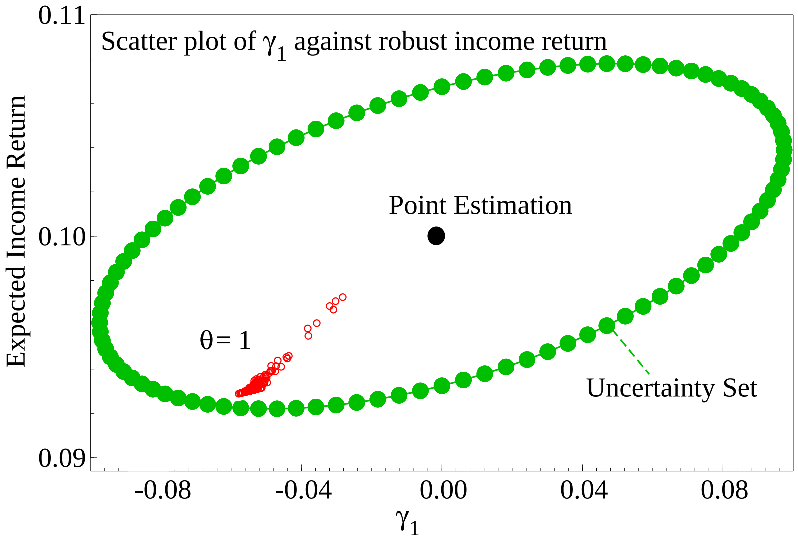

Figure: $\theta = 1$

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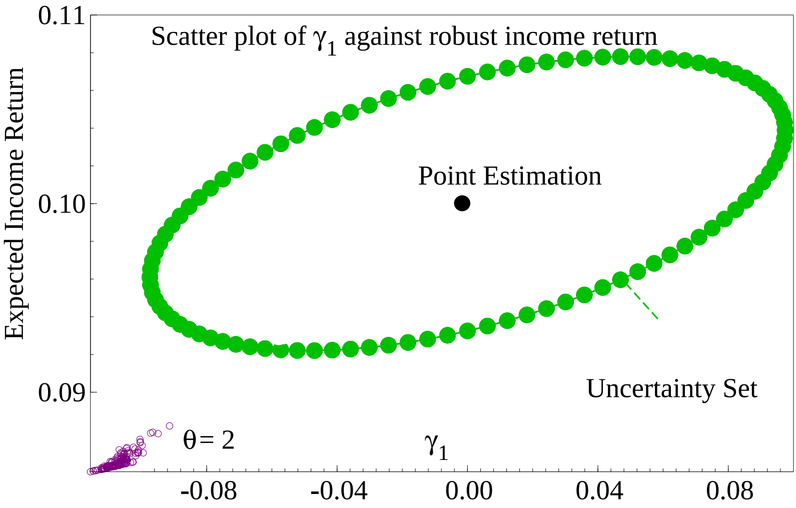
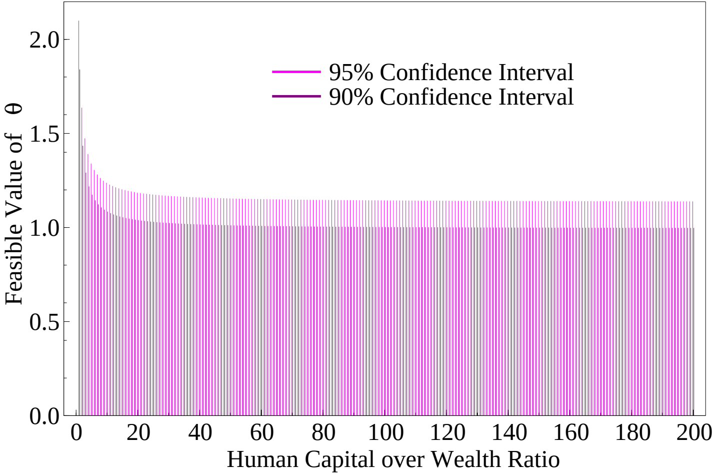


Figure: $\theta = 2$

Finding Feasible Value of θ for the Full Model



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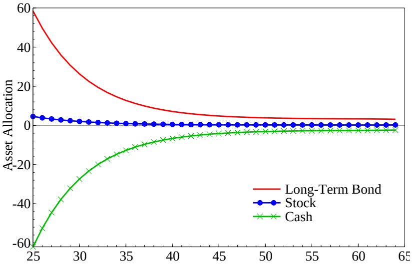
Optimal Consumption and Investment With Human Capital

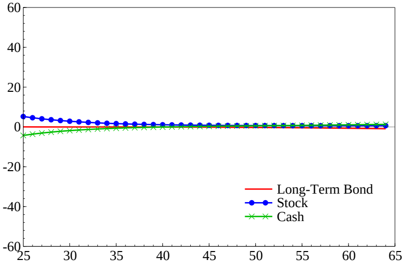
$\gamma = 5$	Myopic Demand		Interest Rate Hedge		Income Risk Hedge		Total Portfolio		$\frac{C}{W+H}$
Age	10Year	Stock	10Year	Stock	10Year	Stock	10Year	Stock	
30	75.23	88.30	63.42	-6.60	0.28	-1.79	138.94	79.91	37.39
40	52.71	61.87	74.81	-4.24	0.12	-0.75	127.64	56.88	30.66
50	41.37	48.55	65.41	-3.00	0.03	-0.22	106.81	45.34	31.57
60	37.33	43.81	34.00	-2.38	0.00	-0.03	71.34	41.41	41.45

- Strong horizon effect on each portfolio component due to human capital over wealth ratio $\frac{H}{W}$.
- Demand for long-term bond comes from both myopic portfolio and interest rate hedge.

Optimal Consumption and Investment With Human Capital



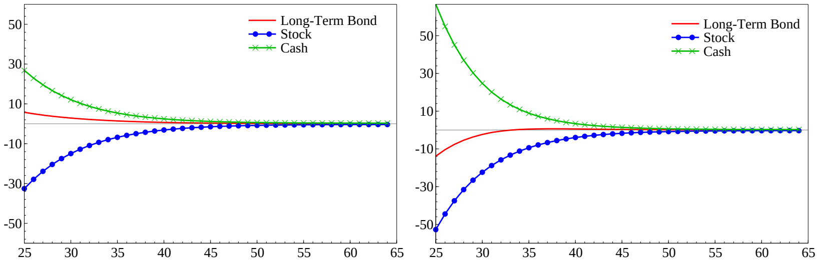
(a) Lower Interest Rate



(b) Higher Interest Rate

- $\gamma = 2, \theta = 0$.
- Low interest rate $\rightarrow$ high demand for long-term bond.

Risk Shifting from the Reference Model to the Robust Model



(c) Higher Real Rate (d) Lower Real Rate

- $\gamma = 2, \theta = 1.$
- Demand for long-term bond depends on state variables.
- Higher real rate results in extra demand for long-term bond.
- Strong horizon effect on risk shifting.

Conclusion

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    - New calibration approach: $\theta \leq 1$ at 95% confidence interval.

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    - Strong horizon effect on investment portfolio.