

How much should life-cycle investors adapt their behavior when confronted with model uncertainty? *

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Abstract

I investigate a dynamic life-cycle strategic asset allocation and consumption problem under model uncertainty, where both inflation rate and income growth rate are assumed to be estimated with errors. I present a feasible boundary for the uncertainty aversion parameter, which measures the investor's preference for robustness using econometric theory. I derive a closed-form solution for a robust investor characterized by min-max utility preference to insure against the worst case scenario. Robustness dramatically increases the demand for the long-term bonds when the instantaneous inflation rate is low.

JEL Codes: C61, D81, D91, G11

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1 Introduction

How to make a rational life-time investment decision in a world we do not understand? It is well-documented in the optimal life-cycle investment literature that labor income and inflation play crucial roles on optimal asset allocation of individual investors. Several examples, e.g.

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Early study by [Bodie et al. \(1992\)](#) incorporate nontradeable labor income into the standard [Merton \(1969\)](#)'s model and they find that an extra demand for risky assets is required when investors are employed. [Brennan and Xia \(2002\)](#) study the impact of stochastic inflation on optimal asset allocation and they observe a hypothetical indexed bond component in addition to the mean-variance tangency portfolio. In light of these studies, it is not surprising that investors have strong intention to protect themselves against income and inflation shocks. However, as these studies are cast in the setting where the underlying model parameters are based on point estimation, their optimal portfolio choice can be wrong if the parameters are misspecified.

In this paper, I extend the classic life-cycle problem by allowing for model misspecification. I provide a robust optimal asset allocation strategy as well as a robust optimal consumption ratio over life-time that can protect investors from model misidentification limited on the macroeconomic perspective. In particular, I assume investors are pessimistic towards the underlying labor income model and the inflation model and she considers these models as references. A robust investor concerns alternative models that perturb from the reference model. On the basis of [Anderson et al. \(2003\)](#) and [Maenhout \(2004\)](#) framework, an investor who is ambiguity aversion looks for a robust optimal investment strategy that maximize her working period utility function under some unexpected worse case scenarios.

How well can individuals estimate their income dynamics or the consumption price index level for a long run? Labor income process has been studied intensively, e.g. [Cocco et al. \(2005\)](#) and [Munk and Sørensen \(2010\)](#). Those studies employ complex stochastic models, while show that average individual household income volatility can reach as high as 30%. That means labor income dynamics can be even more volatile than the stock market, hence the forecasting result will lead a huge dispersion. In other words, labor income is difficult to estimate, even advanced models can make huge estimation error. Inflation rate is also difficult to estimate. Although the commodity price index (CPI) is relatively persistent compared with the stock price, in sample forecasting based on latent-factor models e.g [Wachter \(2010\)](#) can only capture around 35% of the variance of realized inflation. However, a small adjustment of inflation rate will results in a huge change on CPI level in the long-run. To conclude, macroeconomic factors play a unavoidable role in life-cycle problem and misspecified macro models can lead to suboptimal investment and consumption decisions. The empirical evidence however implies a poor estimation performance on these key factors. Therefore, it is important to take estimation errors into account when making life-time decisions.

In this paper, the financial reference model is closely related to [Kojien et al. \(2010\)](#) 's framework. I follow [Hamilton and Wu \(2012\)](#)'s approach to estimating the two-factor affine term structure model and use generalized method of moments GMM method to estimated the stock return process as well as the inflation dynamics. Both methods belongs to the school of minimum distance estimation. The reference financial model is based on monthly US return data from 1961 to 2013.

The perturbation between the unobservable true model and the reference model is captured

by a relative entropy parameter, which penalizes the failure of specification. The ambiguity aversion is limited at the first moment of return processes. In other words, the robust investor only worries about a misspecified expected inflation rate and / or a wrong expected income growth rate. Mother nature has a limited freedom to distort the two expected returns. The two drift distortions jointly, satisfying a chi-squared statistical constraint, reflect the investor's confidence towards the underlying model and can also indirectly help to derive the feasible values of relative entropy parameter.

I solved the min-max robust dynamic programming problem analytically and derive the robust optimal investment and consumption decisions in a closed form. To analyze the impact the macroeconomic uncertainty on optimal life-time investment decision, I proceed the investigation in two steps. First, I solve the robust life-cycle problem in absence of labor income, which means the robust investor confronts with inflation rate uncertainty only. The optimal portfolio is independent of instantaneous wealth while has a strong horizon effect. I find that the preference for robustness induces a larger demand for long-term bonds while interchanges a more conservative position on stocks. Further, nature's optimal decision on inflation rate distortion reflect a robust investor's fear for an underestimated inflation rate.

Second, I add labor income to the life-cycle problem and human capital is involved when considering optimal investment strategy. The labor income process is based on [Viceira \(2001\)](#)'s framework but in a continuous time setting. In addition, I allow for hedging opportunity for income risks by assuming that income shocks are partially correlated with the financial market. The remaining part of income risks are assumed idiosyncratic. I find that, first, the optimal hedging strategy is highly dependent on the human capital over wealth ratio. The ratio enlarge the horizon effects. Therefore, a larger risk exposure is expected for a young investor, because the human capital for a long-term investor is much larger than her financial wealth. Second I also find that in the presence of human capital, the risk shifting between difference asset classes induced by preference for robustness depends on state variables and the investor's risk aversion. A risk aversion investor is more in favour of long-term bond. Mother nature's distortions lead to a lower level of expected income growth rate while a higher inflation rate.

The contribution of this paper is twofold. First of all, I integrate and extend the literature on both life-cycle asset allocation problem and robust hedging problem. Most robust optimal portfolio studies as I mentioned above, that are based on [Hansen and Sargent \(2007\)](#) and [Anderson et al. \(1999\)](#) framework only consider one source of misspecification, namely the stock return ambiguity. None of these studies consider monetary policy uncertainty and/or income process misspecification. Why does the specification of the macroeconomic models matters? A robust investor can avoid stock return misspecification by reducing her equity exposure or simply stop participating the stock market for extreme case, but it is impossible for her to avoid inflation rate estimation error or unexpected income shocks. Further, I consider two sources of ambiguities jointly, which is also fresh among the existing literatures.

The second contribution of this paper is to construct a natural constraint for the relative

entropy parameter. Under the assumption that the estimation errors are asymptotically normal, together with the standard probability theory that a standardized normal distribution square becomes a Chi-squared distribution, one can derive a statistical constraint of the drift distortions at the second moment. The methodology resembles Bayesian paradigm, but in an opposite manner. For I do not consider the reference model obtained from the historical data and the true model. The existing literatures on robust hedging treats the relative entropy as a subjective parameter and the valuation is simple a role of thumb. However, some chosen level of entropies can be statistically impossible.

This paper is related to the optimal asset allocation problem for long-term investors. It is demonstrated in [Merton \(1971\)](#) that the time independent myopic demand is not sufficient to hedging the investment opportunity for a long-term investor when interest rate is stochastic. A large number of studies such as [Brennan and Xia \(2002\)](#), [Sørensen \(1999\)](#), [Wachter \(2003\)](#) and [Wachter \(2010\)](#) have argued that when investment opportunity is time-varying, the long-horizon investors intend to increase her risk exposure to the long-term bonds. Further, these studies also find a strong horizon effects. The longer the hedging horizon is, the bigger the long-term bonds are required. However, none of these studies take labor income into account, and none of these studies consider the problem of parameter uncertainty neither.

Pioneer work by [Bodie et al. \(1992\)](#) emphasis the importance of human capital on optimal asset allocation over life time and they argue that young should go short. More recent studies such as [Viceira \(2001\)](#) and [Cocco et al. \(2005\)](#) [Kojien et al. \(2010\)](#), [Benzoni et al. \(2007\)](#) and [Munk and Sørensen \(2010\)](#) consider the impact of stochastic income process on the optimal investment decision. However, none of these studies take income model uncertainty into consideration.

This paper is also related the studies of robust dynamic optimal portfolio choice. [Maenhout \(2004\)](#) extend [Merton \(1969\)](#)'s model by allowing for stock return misspecification, and he finds that ambiguity aversion reduce the investor's demand for risky asset. [Maenhout \(2006\)](#) considers again the expected equity return uncertainty on the basis of a mean-reversion market price of risk. [Branger et al. \(2013\)](#) extend [Maenhout \(2006\)](#)'s study in an incomplete equity market by introducing a non-tradable risk factor and they also allow for learning. They find that both ambiguity aversion and learning matter for a robust investor. Learning is not considered in this paper, as I assume that the data generation process do stay the same forever.

The rest of the paper proceeds as follows. In [Section 2](#) I introduce the reference financial and macroeconomic models first. Then I show the drift distortions as well as their influence on the underlying model. The robust optimization problem is also introduced in this section. In [Section 3](#), I derive the closed form solution for investors under different preferences. [Section 4](#) estimates the reference model using US data. [Section 5](#) presents the numerical solution for the robust life-time investor and [Section 6](#) concludes.

2 The Model

2.1 Financial Market

The financial model I propose is closely related to [Sangvinatsos and Wachter \(2005\)](#) and [Kojen et al. \(2010\)](#). I assume that a life-cycle investor can invest in a stock (index), a long-term nominal bond and a nominal money market account. The instantaneous nominal risk free rate, r_t is assumed to be affine in two state variables X_{1t} and X_{2t} .

$$r_t = \delta_0 + \delta^\top X_t, \quad \delta_0 > 0. \quad (1)$$

where δ_0 is scalar and $\delta \in \mathbb{R}^{2 \times 1}$. The latent factors $X_t = \begin{pmatrix} X_{1t} & X_{2t} \end{pmatrix}^\top$, governing yields are characterized by an Ornstein-Uhlenbeck process with mean-reverting around zero under the physical measure,

$$dX = -\kappa X dt + \sigma_X^\top dZ, \quad (2)$$

where $\kappa \in \mathbb{R}^{2 \times 2}$. Following [Dai and Singleton \(2000\)](#), I assume σ_X equal to an identity matrix with $\sigma_X = \begin{bmatrix} I_{2 \times 2} & 0_{2 \times 2} \end{bmatrix}^\top$. Let $dZ \in \mathbb{R}^{4 \times 1}$ represent a vector of independent risk drivers following Brownian motions. The nominal pricing kernel is given by

$$\frac{d\phi}{\phi} = -r_t dt - \Lambda_t^\top dZ \quad (3)$$

where market price of risk Λ_t is affine in X_t

$$\Lambda_t = \lambda_0 + \lambda_1 X_t, \quad (4)$$

with $\lambda_0 \in \mathbb{R}^{4 \times 1}$ and $\lambda_1 \in \mathbb{R}^{4 \times 2}$. Denote $P(t, T)$ as the nominal price at time t of a zero-coupon bond maturing at time $T = t + \tau$ with a nominal payoff of 1. Following [Duffie and Kan \(1996\)](#), I conjecture that bond price is exponential affine in the state variables,

$$P(t, T) = \exp(B(\tau)^\top X_t + A(\tau)), \quad (5)$$

where $B(\tau) \in \mathbb{R}^{2 \times 1}$ and $A(\tau)$ is a scalar. Therefore, the corresponding yield is

$$Y_t^\tau = a(\tau) + b(\tau)^\top X_t, \quad \text{with} \quad a(\tau) = -\frac{A(\tau)}{\tau}, \quad b(\tau) = -\frac{B(\tau)}{\tau}. \quad (6)$$

The expression of $B(\tau)$ and $A(\tau)$ can be solved in a closed form and is shown in [Appendix A](#). Applying Itô's lemma, bond price dynamics follow

$$\frac{dP}{P} = \left(r_t + B(\tau)^\top \sigma_X^\top \Lambda_t \right) dt + B(\tau)^\top \sigma_X^\top dZ \quad (7)$$

The commodity price level Π_t follows a diffusion process

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi^\top dZ, \quad \text{with} \quad \sigma_\Pi \in \mathbb{R}^{4 \times 1} \quad (8)$$

where the expected inflation, π_t is assumed to be affine in state variables as well

$$\pi_t = \xi_0 + \xi^\top X_t, \quad \text{with} \quad \xi \in \mathbb{R}^{2 \times 1}. \quad (9)$$

I assume the dynamics of stock price is given by

$$\frac{dS_t}{S_t} = (r_t + \eta_S) dt + \sigma_S^\top dZ \quad (10)$$

where $\sigma_S \in \mathbb{R}^{4 \times 1}$. The following constraint for expected excess stock return holds

$$\eta_S = \sigma_S^\top \Lambda_t \quad (11)$$

2.2 Investor's Labor Income

The nominal labor income model I propose is in line with [Viceira \(2001\)](#), while in addition, I assume income shocks are partially correlated with the financial market introduced in [Section 2.1](#).

$$dY = \left(g + \frac{\sigma_y^2}{2} \right) Y dt + \sigma_y Y \rho_y^\top dZ + \sigma_y Y \sqrt{1 - \|\rho_y\|^2} dZ_Y \quad (12)$$

where $\rho_y \in \mathbb{R}^{4 \times 1}$ and dZ_Y represents idiosyncratic risk.

2.3 Basic Optimal Portfolio Model

Turning now to the standard life cycle problem in the absence of model misspecification. Consider an investor who has constant relative risk aversion utility function (CRRA). Her preferences before retirement can be represented as followed

$$\max_{C_t, x_t} \mathbb{E}_0 \left[\int_0^T \frac{\exp(-\beta t)}{1-\gamma} \left(\frac{C_t}{\Pi_t} \right)^{1-\gamma} dt + \frac{\varphi \exp(-\beta T)}{1-\gamma} \left(\frac{W_T}{\Pi_T} \right)^{1-\gamma} \right] \quad (13)$$

where β is the subjective risk free rate with $\exp(-\beta) < 1$. The post-retirement consumption and investment decision is not included in agent's planning horizon. Instead, I introduce a bequest function scaled by φ to capture the retirement consumption. In particular, I assume that the value of bequest function is equivalent to the utility value of annuitization at retirement, as

such, the value of φ can be calibrated.¹

The asset menu includes one stock, one nominal long-term bond with maturity τ and a short term nominal government bond. Hence The nominal wealth follows the diffusion process

$$dW = (x_t^\top (\mu_t - \nu r_t) + r_t) W dt - C dt + Y dt + W x_t^\top \sigma^\top dZ \quad (14)$$

where

$$\mu_t = \begin{pmatrix} B(\tau)^\top \sigma_X^\top \Lambda \\ \sigma_s^\top \Lambda \end{pmatrix} + \nu r_t \quad (15)$$

$$\sigma^\top = \begin{pmatrix} B(\tau)^\top \sigma_X^\top \\ \sigma_s^\top \end{pmatrix} \quad (16)$$

with $\mu_t \in \mathbb{R}^{2 \times 1}$ and $\sigma \in \mathbb{R}^{4 \times 2}$.

Denote the value function at time t by $J(W, Y, \Pi, X, t)$, I omit the time subscripts for notation convenience. The Hamilton-Jacobi-Bellman HJB equation of the dynamic optimization problem (13) is given by

$$0 = \max_{C_t, x_t} \left[\frac{1}{1 - \gamma} \left(\frac{C}{\Pi} \right)^{1 - \gamma} - \beta J + \mathcal{D}(C_t, x_t) J(W, Y, \Pi, X, t) \right] \quad (17)$$

The full expression of HJB equation is explained in Appendix C.4.

2.4 Monetary Policy Uncertainty and Income Uncertainty

Suppose the expected inflation rate π_t and the expected income growth rate g are misspecified. A robust life cycle investor, who suspects the accuracy of her macroeconomic models, takes (8) and (12) as an approximation and considers alternative models. Adopting Anderson et al. (2003) framework, a perturbed model can be expressed with an added drift term on the Brownian motions $\mathbf{e}^\top dZ$ and dZ_Y of the reference model, with $\mathbf{e} = (0, 0, 1, 0)^\top$. Therefore an alternative inflation diffusion process is given by

$$\frac{d\Pi}{\Pi} = \pi_t dt + \sigma_\Pi^\top (dZ + \mathbf{e} \gamma_{1,t} dt) \quad (18)$$

¹ Along the line with Koijen et al. (2010)'s idea, the scaling factor φ is defined as

$$\varphi = A_T^{\gamma-1} \int_T^{85} \exp(-\beta t) dt$$

where A_T represents the annuity payment at retirement, and I assume a survival probability of 1 up to age of 85.

and a perturbed income diffusion process is read as

$$dY = \left(g + \frac{\sigma_y^2}{2}\right) Y dt + \sigma_y Y \rho_y^\top (dZ + \mathbf{e}\gamma_{1,t} dt) + \sigma_y Y \sqrt{1 - \|\rho_y\|^2} (dZ_Y + \gamma_{2,t} dt) \quad (19)$$

Each alternative macroeconomic model characterized by a joint stochastic process $(\gamma_{1,t}, \gamma_{2,t})$ is specified under a candidate measure which is equivalent to the reference measure \mathbb{P} .

Mother nature is entitled to decide the sign and the size of the drift distortions. Consider an investor with preferences (13), who can not observe the true model while is aware of the misspecification. The most aversive mother nature's decision to this investor, is when the reference model (9) underestimates the expected inflation while in contract, (12) overestimates the expected income return. Since both scenarios imply that investor's realized optimal preference is much lower than her expected level based on the reference model.

2.5 Robust Optimal Portfolio Model

The robust investor, who fears for some worst-case model, searches for a more resilient consumption and investment strategy to protect herself against model fragility. Her preferences can be read as

$$\min_{\{\gamma_{1,t}, \gamma_{2,t}\}} \max_{C_t, x_t} \mathbb{E}_0 \left[\int_0^T \frac{\exp(-\beta t)}{1-\gamma} \left(\frac{C_t}{\Pi_t}\right)^{1-\gamma} dt + \frac{\varphi \exp(-\beta T)}{1-\gamma} \left(\frac{W_T}{\Pi_T}\right)^{1-\gamma} \right] \quad (20)$$

The perturbation of inflation process 18 triggers the drift distortion of the nominal wealth dynamics

$$dW = (x_t^\top (\mu_t - r_t) + r_t) W dt - C dt + Y dt + W x_t^\top \sigma^\top (dZ + \mathbf{e}\gamma_1 dt) \quad (21)$$

Following Anderson et al. (2003), the robust HJB equation is given by

$$0 = \min_{\{\gamma_1, \gamma_2\}} \max_{C_t, x_t} \left\{ \frac{1}{1-\gamma} \left(\frac{C}{\Pi}\right)^{1-\gamma} - \beta J + \mathcal{D}(C_t, x_t) J(W, Y, \Pi, X, t) + J_W W x^\top \sigma^\top \mathbf{e}\gamma_1 \right. \\ \left. + J_Y Y \left(\sigma_y \rho_y^\top \mathbf{e}\gamma_1 + \sigma_y \sqrt{1 - \|\rho_y\|^2} \gamma_2 \right) + J_\Pi \Pi \sigma_\Pi^\top \mathbf{e}\gamma_1 + \frac{1}{2\Psi} (\gamma_1^2 + \gamma_2^2) \right\} \quad (22)$$

In addition to the reference-model based HJB equation (17), there are four additional terms. The first three additional terms reflect the adjustment of Bellman equation created from the additional drift components. The last term of (22) can be considered as a penalty function, to penalize those alternative models locating too far away from the reference model. Hence, mother nature's decision is constrained.

The penalty term is scaled by a non-negative function $\Psi = \Psi(W, Y, \Pi, X, t)$ which capture investor's preference for robustness. If $\Psi \rightarrow \infty$, the penalty term vanishes. As the result,

mother nature has tremendous freedom to choose drift distortion and the investor has almost no confidence over the underlying model. In contrast, when $\Psi \rightarrow 0$, the penalty function becomes infinitely large. Hence, it is too costly for mother nature to make any alternative decision deviating from the reference model. In this case, the investor is extremely optimistic toward the reference model.

In order to obtain an explicit solution, homotheticity of the HJB equation is usually required under CRRA preferences. Inspired by [Maenhout \(2004\)](#), I assume the penalty term Ψ is scaled function of the indirect utility $J(W, Y, \Pi, X, t)$, which takes the form:

$$\Psi(W, Y, \Pi, X, t) = \frac{\theta}{(1 - \gamma) J(W, Y, \Pi, X, t)} \quad (23)$$

where θ represents investor's ambiguity aversion level which is non-negative and is assume constant over time. The bigger the value of θ is, the more pessimistic the investor is towards the underlying model.

3 Optimal Portfolio Choice

Model 1: No Labor Income, No Model Misspecification I first consider a benchmark problem when the optimization problem (13) is absence of labor income $Y_t = 0$. Then it is a special case of the model of [Sangvinatsos and Wachter \(2005\)](#) and can be solved explicitly in two different ways. In this paper, I obtain the optimal investment and consumption by means of solving for the Bellman's equation. An alternative method is to use martingale technique of [Cox and Huang \(1989\)](#). Studies, such as [Brennan and Xia \(2002\)](#), [Wachter \(2002\)](#) and [Sangvinatsos and Wachter \(2005\)](#) use this method to solve their dynamic portfolio choice problem. However, the martingale technique is not applicable when the market price of risk is not unique, hence cannot be used to solve the robust optimization problem (20).

Proposition 3.1. *The indirect utility function for an investor with preference represented by (13) without labor income and without trading constraint is given by*

$$J^{M1}(W, \Pi, X, t) = \frac{1}{1 - \gamma} b^{M1}(X, t)^\gamma \left(\frac{W}{\Pi} \right)^{1-\gamma} \quad (24)$$

with

$$b^{M1}(X, t) = \exp \left\{ \frac{1}{\gamma} \left(\frac{1}{2} X^\top \Gamma_1^{M1}(\tau) X + \Gamma_2^{M1}(\tau) X + \Gamma_3^{M1}(\tau) \right) \right\} \quad (25)$$

where $\tau = T - t$ represents the remaining investment horizon before retirement and the deterministic functions $\Gamma_1^{M1} \in \mathbb{R}^{2 \times 2}$, $\Gamma_2^{M1} \in \mathbb{R}^{1 \times 2}$, and scalar Γ_3^{M1} solve a system of ordinary differential equations stated in [Appendix C.1](#).

The optimal consumption is $C^{M1} = \frac{W}{b^{M1}}$, and the optimal portfolio is given by

$$x^{M1} = \frac{1}{\gamma} (\sigma^\top \sigma)^{-1} (\mu - \iota r) + \frac{\gamma - 1}{\gamma} (\sigma^\top \sigma)^{-1} (\sigma^\top \sigma_\Pi) + (\sigma^\top \sigma)^{-1} (\sigma^\top \sigma_X) \frac{b_X^{M1}}{b^{M1}} \quad (26)$$

where $\frac{b_X^{M1}}{b^{M1}} = \frac{1}{\gamma} (\Gamma_1^{M1}(\tau) X + \Gamma_2^{M1}(\tau))$.

The optimal portfolio of **Model 1** consists of three components. The first component is simply Merton's solution that solves for the instantaneous mean-variance efficiency scaled by a risk aversion function. The second component means to replicate the unexpected inflation so as to adjust the nominal mean-variance portfolio to real terms. The first two terms together is also known as "myopic demand". Without labor income, the myopic demand is independent of both instantaneous wealth and hedging horizon. The last term represents the hedging demand for long-term nominal bond. It is also demonstrated in [Sangvinatsos and Wachter \(2005\)](#) that, a time-varying bond risk premia can generate hedging demand for long-term bond for those long-run investors. The third component depends on investor's remaining investment horizon and the value of state variables.

Model 2: No Labor Income, Under Model Misspecification Next, I investigate the optimal solution for a robust investor. Consider a special case of (20) subject to (18) and (21), disregarding the income effect $Y = 0$. The absence of labor income implicitly wipes away the effect of distortion drift term γ_2 . As the result, mother nature can only adjust the expected inflation rate to minimize investor's preferences. **Model 2** can also be understood as the robust solution of **Model 1**.

Proposition 3.2. *Under the assumptions stated above, the optimal consumption is $C^{M2} = \frac{W}{b^{M2}}$, where function b^{M2} takes the same structure as (25), while with the quadratic, linear and constant coefficients of the exponential function replaced by Γ_1^{M2} , Γ_2^{M2} and Γ_3^{M2} respectively.*

The robust optimal portfolio for an investor with preference (20) is given by

$$x^{M2} = (\gamma \sigma^\top \sigma + \theta \sigma^\top \mathbf{e} \mathbf{e}^\top \sigma)^{-1} \left\{ (\mu - \iota r) + (\gamma - 1) \sigma^\top \sigma_\Pi + \theta \sigma^\top \mathbf{e} \mathbf{e}^\top \sigma_\Pi + \sigma^\top \sigma_X \left[\Gamma_1^{M2}(\tau) X_t + \Gamma_2^{M2}(\tau) \right] \right\} \quad (27)$$

where Γ_1^{M2} and Γ_2^{M2} satisfying a system of ODE stated in Appendix C.2.

The worst case distortion is given by

$$\gamma_1^{M2} = -\theta (x^{M2\top} \sigma^\top \mathbf{e} - \sigma_\Pi^\top \mathbf{e}) \quad (28)$$

Without the preference for robustness $\theta = 0$, x^{M2} is equivalent to x^{M1} . The robust optimal portfolio of **Model 2** also contains three components. The first component

$$(\gamma \sigma^\top \sigma + \theta \sigma^\top \mathbf{e} \mathbf{e}^\top \sigma)^{-1} (\mu - \iota r)$$

modifies [Maenhout \(2004\)](#)'s solution in two aspects. First, instead of a constant investment opportunity, the investor faces a time-varying nominal myopic demand. Second, rather than fearing for a misspecified stock return, I assume the robust investor only worries about inflation rate misspecification, while the stock and bond return processes are assumed specified.

The adjustment term $\theta\sigma^\top \mathbf{e}\mathbf{e}^\top \sigma$, penalizes the misspecification of the inflation rate. A robust investor increases her risk-aversion level by a unit of θ limiting at the inflation risk dimension. The penalty parameter θ reduces the demand for nominal mean-variance hedging. The second component

$$(\gamma\sigma^\top \sigma + \theta\sigma^\top \mathbf{e}\mathbf{e}^\top \sigma)^{-1} ((\gamma - 1)\sigma^\top \sigma_\Pi + \theta\sigma^\top \mathbf{e}\mathbf{e}^\top \sigma_\Pi)$$

is an ambiguity-adjusted inflation replication portfolio. When the ambiguity parameter goes to infinity $\theta \rightarrow \infty$, this term converges to $(\sigma^\top \mathbf{e}\mathbf{e}^\top \sigma)^{-1} \sigma^\top \mathbf{e}\mathbf{e}^\top \sigma_\Pi$ and the first component vanishes. It indicates that when inflation rate is extremely unpredictable, the most robust investment strategy to stay inflation risk neutral (in the absence of time-varying bond premia). As the penalty function occurs on both numerator and denominator of the fraction, the joint effect on inflation adjustment is unclear. The last part

$$(\gamma\sigma^\top \sigma + \theta\sigma^\top \mathbf{e}\mathbf{e}^\top \sigma)^{-1} \sigma^\top \sigma_X \left[\Gamma_1^{M2}(\tau) X_t + \Gamma_2^{M2}(\tau)^\top \right]$$

shows the adjustment of hedging demand on long-term bond. As the penalty parameter θ also plays a role in function Γ_1^{M2} and Γ_2^{M2} , the quantitative influence on hedging demand of the long-term interest rate risk is not clear at this stage.

It is also interesting to notice that, the optimal nature's decision γ_1^{M2} depends on investor's investment decision. As the impact of ambiguity aversion on the portfolio choice is not obvious, it is hard to detect mother nature's intention.

Model 3: With Labor Income, No Model Misspecification Classic optimal life-cycle investment studies, such as [Merton \(1969\)](#) and [Samuelson \(1969\)](#) show that the optimal asset allocation is independent of age. However, pioneer work by [Bodie et al. \(1992\)](#) show that young should go short and investor's life-time risk exposure is a diminishing function of age if the non-tradeable human capital is taken into account.

In **Model 3**, I assume investor has preferences (13) subject to both financial wealth constraint (14) and human capital constraint (12). This problem is comparable to [Munk and Sørensen \(2010\)](#)'s work but with two small variations. First, the inflation risk is not considered in their model and the investor's asset menu is assumed in real terms. Second, they employ one-factor [Vasicek \(1977\)](#) model instead of a multi-factor affine model.

Human capital measures the cumulative discounted remaining labor income stream under risk neutral measure. Assume that the income stream is locally risk free, $\sigma_y \equiv 0$,² market

²This assumption is crucial and is also used in [Munk and Sørensen \(2010\)](#). If σ_y is not spanned, then the

value of the income stream at time t over the working period $[t, T]$ is defined as

$$H_t = H(Y_t, X_t, t) = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^T Y_u \exp \left(- \int_t^u r_s ds \right) du \right] \quad (29)$$

The explicit expression of H_t is shown in the following proposition. The proof is given in Appendix C.3.

Proposition 3.3. *Under the assumption that the idiosyncratic risk is vanished, the nominal human capital is given by*

$$H(Y, X, t) = Y_t M(X, t) = Y_t \int_t^T \exp(M_1(u-t) + M_2(u-t)X_t) du \quad (30)$$

The expression of $M(X, t)$ is derived in Appendix C.3. Proposition 3.3 indicates that human capital is a linear function of instantaneous income Y_t . The value of human capital depends also on age. As the younger the investor is, the longer her future expected labor income stream will be.

Proposition 3.4. *The indirect utility function for an ambiguity-neutral investor with preference (13) subject to (12) and (14) is given by*

$$J^{M3}(W, Y, \Pi, X, t) = \frac{1}{1-\gamma} b^{M3}(X, t)^\gamma \left(\frac{W+H}{\Pi} \right)^{1-\gamma} \quad (31)$$

where the human capital H follows Proposition C.3, under a special case when state variables take values of their unconditional means.³ Function $b^{M3}(X, t)$ takes the same form as (25), but with the coefficients of the polynomial function replaced by $\Gamma_1^{M3}(\tau)$, $\Gamma_2^{M3}(\tau)$, and $\Gamma_3^{M3}(\tau)$. See Appendix C.4 for full expression.

For an ambiguity neutral investor, the optimal portfolio weight is

$$\begin{aligned} x^{M3} = & \frac{W+H}{\gamma W} (\sigma^\top \sigma)^{-1} (\mu - \iota r) - \frac{H}{W} (\sigma^\top \sigma)^{-1} \sigma^\top \rho_y \sigma_y \\ & + \frac{\gamma-1}{\gamma} \frac{W+H}{W} (\sigma^\top \sigma)^{-1} (\sigma^\top \sigma_\Pi) + \frac{W+H}{W} (\sigma^\top \sigma)^{-1} (\sigma^\top \sigma_X) \frac{b_X^{M3}}{b^{M3}} \end{aligned} \quad (32)$$

where $\frac{b_X^{M3}}{b^{M3}} = \frac{1}{\gamma} (\Gamma_1^{M3}(\tau) X + \Gamma_2^{M3}(\tau))$. The remaining part of wealth, $1 - x^\top \iota$ is invested in nominal short-term bond. The optimal nominal consumption equals to

$$C^{M3} = \frac{W+H}{b^{M3}}.$$

market is incomplete, due to the non-traded risk driver dZ_Y . As the result, the risk neutral measure is no longer unique and the expected value of human capital cannot be priced.

³This assumption implicitly assumes human capital function is insensitive to the state variables, but is only a function of age.

The human-capital adjusted optimal portfolio (32) differs from the benchmark case (**Model 1**) in two aspects. First, the solution is no longer instantaneous wealth neutral, but instead, each of the three hedging components discussed in **Model 1** is scaled by a total wealth over financial wealth ratio, where total wealth is the sum of financial wealth W and human capital H . It is demonstrated in [Bovenberg et al. \(2007\)](#) that human capital plays a crucial role in life-cycle financial planning. The ratio $\frac{W+H}{W}$ is downward sloping as a function of age. For the human capital dominates the total wealth and it is a decreasing function over age. As the result, the optimal portfolio x^{M3} also decreases over age.

The second difference comes from the additional income hedging component

$$-\frac{H}{W} (\sigma^\top \sigma)^{-1} \sigma^\top \rho_y \sigma_y.$$

As income risk dZ_Y is assumed correlated to the financial shocks, this hedging portfolio, scaled by human capital over wealth ratio, reflects the hedging demand against non-idiosyncratic income risks.

Model 4: With Labor Income and Model Misspecification

Proposition 3.5. *Under the assumption that the human capital H is ambiguity neutral and is state variable insensitive, the indirect utility function of a robust investor with preferences represented by (20) subject to (19), (18) and (21) is given by*

$$J^{M4}(W, Y, \Pi, X, t) = \frac{1}{1-\gamma} b^{M4}(X, t)^\gamma \left(\frac{W+H}{\Pi} \right)^{1-\gamma} \quad (33)$$

where b^{M4} is an exponential affine function of X , taking the same form as (25) while with coefficients replaced by $\Gamma_1^{M4}(\tau)$, $\Gamma_2^{M4}(\tau)$ and $\Gamma_3^{M4}(\tau)$ respectively. The robust optimal portfolio is given by

$$x^{M4} = \left(\frac{\gamma W}{W+H} \sigma^\top \sigma + \frac{\theta W}{W+H} \sigma^\top \mathbf{e} \mathbf{e}^\top \sigma \right)^{-1} \left\{ (\mu - \iota r) + ((\gamma - 1) \sigma^\top \sigma_\Pi + \theta \sigma^\top \mathbf{e} \mathbf{e}^\top \sigma_\Pi) \right. \\ \left. - \left(\frac{\gamma H}{W+H} \sigma^\top \rho_y \sigma_y + \frac{\theta H}{W+H} \sigma^\top \mathbf{e} \mathbf{e}^\top \rho_y \sigma_y \right) + \gamma \sigma^\top \sigma_X \frac{b_X^{M4}}{b^{M4}} \right\} \quad (34)$$

where $\frac{b_X^{M4}}{b^{M4}} = \frac{1}{\gamma} (\Gamma_1^{M4}(\tau) X + \Gamma_2^{M4}(\tau))$. The optimal consumption takes the same form as **Model 3** while applying indirect utility function J^{M4} . Mother nature's optimal decision on the two drift distortions are

$$\gamma_1^{M4} = -\theta \left(\frac{W}{W+H} x^{M4 \top} \sigma^\top \mathbf{e} + \frac{H}{W+H} \sigma_y \rho_y^\top \mathbf{e} - \sigma_\Pi^\top \mathbf{e} \right) \quad (35)$$

$$\gamma_2^{M4} = -\theta \frac{H}{W+H} \sigma_y \sqrt{1 - \|\rho_y\|^2} \quad (36)$$

Appendix C.5 provides the proof.

The robust optimal portfolio contains four hedging components. The first two components of x^{M4} are equivalent to the robust myopic demand of **Model 2** but scaled by the labor-income adjusted scale $\frac{W+H}{W}$. Therefore, instead of age independent portfolios, these two hedging portfolios are aggregately decreasing with age. With age given, the presence of preference for robustness reduced the hedging demand for nominal mean-variance portfolio, however, the impact of inflation hedging portfolio is unclear but depends on the input parameters.

The simplified third component is

$$-\frac{H}{W} (\gamma\sigma^\top\sigma + \theta\sigma^\top\mathbf{e}\mathbf{e}^\top\sigma)^{-1} (\gamma\sigma^\top\rho_y\sigma_y + \theta\sigma^\top\mathbf{e}\mathbf{e}^\top\rho_y\sigma_y)$$

It captures the hedging demand for tradeable income risk in the present of model misspecification. As the entropy parameter θ brings a structure change on both sides of the income-hedge portfolio fraction, the hedging demand for this term is not clear.

Mother nature's decision on inflation rate drift distortion contains three mixed-sign components and is also portfolio decision dependent. Hence it is difficult to foretell the sign of γ_1^{M4} . However, the sign of the other drift distortion term γ_2^{M4} is obvious. Mother nature would always prefer a non-positive drift distortion on income expected return. In other words, the robust investor is panic about an overestimated expected income growth rate.

4 Model Calibration

4.1 Data

I use monthly US data from June 1961 to December 2013. The US government yield data are taken from [Gürkaynak et al. \(2007\)](#)⁴, where data is only available on daily base. I collect the last business day of the month to obtain monthly data. I use three yields with maturities of 1, 3 and 6 years to estimate the two-factor affine term structure model. Data on price index and stock market are in line with [Kojien et al. \(2010\)](#) and [Sangvinatsos and Wachter \(2005\)](#). Consumption price index data is obtained from the Bureau of Labor Statistics.⁵ The stock return data I use is from CRSP value-weighted NYSE/Ames/Nasdaq index. I use monthly US data from June 1961 to December 2013. The US government yield data are taken from [Gürkaynak et al. \(2007\)](#)⁶, where data is available on daily base. I collect the last business day of the month to obtain monthly data. I use three yields with maturities of 1, 3 and 6 years to estimate the two-factor affine term structure model. Data on price index and stock market are in line with [Kojien et al. \(2010\)](#) and [Sangvinatsos and Wachter \(2005\)](#). Consumption price

⁴See <http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html>

⁵See <http://www.bls.gov/> for details.

⁶See <http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html>

index data is obtained from the Bureau of Labor Statistics.⁷ The stock return data I use is from CRSP value-weighted NYSE/Ames/Nasdaq index.⁸

4.2 Estimation

The financial model introduced in Section 2.1 consists of four independent risk driver. I assume that the volatility matrix stacking σ_X , σ_Π and σ_S is lower triangular.

$$\begin{pmatrix} \sigma_X^\top \\ \sigma_\Pi^\top \\ \sigma_S^\top \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \sigma_{\Pi_1} & \sigma_{\Pi_2} & \sigma_{\Pi_3} & 0 \\ \sigma_{S_1} & \sigma_{S_2} & \sigma_{S_3} & \sigma_{S_4} \end{pmatrix} \quad (37)$$

Market price of risk (4) can take the following form to satisfy no arbitrage assumption

$$\Lambda_t = \begin{pmatrix} \lambda_{01} \\ \lambda_{02} \\ 0 \\ \star \end{pmatrix} + \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \\ 0 & 0 \\ \star & \star \end{pmatrix} X_t \quad (38)$$

where the fourth row of Λ_t satisfies the restriction that $\sigma_S^\top \lambda_0 = \eta_S$ and $\sigma_S^\top \lambda_1 = 0$. Under the assumption that price of inflation rate risk is only driven by the two state variables, X_1 and X_2 , the third row of Λ_0 turns to zero.

4.3 Hybrid Estimation Method

Quasi maximum likelihood estimation (QMLE) is considered a standard method to calibrate the affine term structure models, e.g. Duffee (2002) and Sangvinatsos and Wachter (2005). However, practical experience has seen tremendous estimation difficulties. The numerical challenges come from two aspects. First, these latent factor models are characterized by nonlinear and complex likelihood surface. Due to a large number of parameters to be estimated, the nonlinear optimization problem can hardly converge. Second, it is hard to find a good starting values to achieve convergence due to high degrees of freedom, argued by Ang and Piazzesi (2003).

I propose a hybrid estimation method that combines in essence the method of Hamilton and Wu (2012) and Generalized method of moments (GMM) method introduced by Hansen and Singleton (1982). In particular, I use minimum chi-square estimation (MCSE) as an alternative to QMLE to estimate the affine term structure model. Estimation of stock and inflation diffusion process relies on GMM method. More generally, both methods could be viewed as spe-

⁷See <http://www.bls.gov/> for details.

⁸See <http://www.crsp.com/products/research-products/crsp-us-stock-databases> for further details.

cially cases of minimum distance estimation (MDE), in which one minimizes a distance square between restricted and unrestricted statistics. Estimation details are described in Appendix B.

Table 1 reports the parameter estimation results followed by their estimation errors. The results are displayed in annual terms. First, I summarize some properties of the short rate process and the inflation rate. The constant terms δ_0 and ξ_0 are both comparable to [Kojien et al. \(2010\)](#) and [Sangvinatsos and Wachter \(2005\)](#). The short rate is increasing with the two latent factors and is more sensitive to X_2 . One unit of increase on X_2 with the other factor X_1 fixed, results in a 188 basis point of increase on the nominal interest rate. However, the two factors X_1 and X_2 have an opposite impact on the value of inflation rate.

Next I turn to the mean reversion parameters. The risk neutral mean-reversion matrix κ^Q is a lower triangular matrix. The bond market price of risk excluding the unconditional part $\sigma_X^\top \lambda_1$ fills the difference between κ^Q and κ . The eigenvalues of the mean-reversion matrix κ are 0.57 and 0.10. It corresponds to a half-life innovation is 1.2 years for X_1 and is approximately seven years for X_2 . Therefore, X_2 is estimated to be more persistent than X_1 . The half-life innovation for nominal interest rate must be also 1.2 year, the same as innovation speed of X_1 .

9

Last, I highlight some properties of the stock return parameters from the penultimate panel of Table 1. First, the value of η_S represents the expected excess return. The standard deviation of stock return parameter is relatively higher than the volatility of other (unconditional) expected return parameters, such as δ_0 and ξ_0 . This result confirms that the expected stock return is harder to estimate than the other return parameters in the model. Second, negative values of σ_{S_1} and σ_{S_2} indicates that stock and bond returns must be positively correlated, since the first two rows of bond volatility vector elements $\sigma_X B(\tau)$ are also negative.

Table 2 provides some insights in the performance of the estimation results. I compare the summary statistics of the raw sample series with the estimated series from the discretized model using the estimation results from Table 1. Table 2 provides an overview of the first two moments of stock returns, inflation and the three yields on a monthly basis. The estimated latent factors X_t are linear combinations of the one-year and six-year yields. Hence, by construction, the estimated sample moments of the one-year and six-year yields match the distribution of raw sample precisely. The remaining three-year yield, stock return and inflation process all fit the raw sample distribution reasonably well. The model underestimates the expected monthly stock return by approximately 10 basis points, but the high volatility of the stock return series make the difference between the two statistically insignificant. To conclude, Table 2 implied that the long-term bond yields can capture the dynamics of inflation and stock returns substantially.

Figure 1 shows that the expected inflation rate implied from the model does not always manage to forecast the realized inflation rate despite the highly fitted moments shown in Table 2. For instance, in 1973 during the oil crisis, the model underestimates the inflation rate. In

⁹To calculate half-life of nominal rate under the multi factors affine model, one has to use the largest eigenvalue of mean reversion matrix κ as a proxy of mean reversion parameter.

Table 1: **Estimation of Model Parameters.**

Estimation result of the two-factor financial model described in Section 2.1. The financial model is estimated using monthly U.S. data on three bond yields, inflation and stock return over the period from June 1961 to December 2013. The bond maturities used in estimation are one-year, three-year and six-year. The three-year bond yield is assumed to be measured with error.

Parameter	Estimate	Standard error
Nominal Interest Rate $r_t = \delta_0 + \delta^\top X_t$		
δ_0	4.62%	2.11%
δ_1	0.31%	0.10%
δ_2	1.88%	0.05%
Expected inflation rate $\pi_t = \xi_0 + \xi^\top X_t$		
ξ_0	3.67%	0.26%
ξ_1	-0.14%	0.18%
ξ_2	1.43%	0.30%
Market price of risk $\Lambda_t = \lambda_0 + \lambda_1 X_t$		
λ_{01}	-0.1083	0.1313
λ_{02}	-0.3451	0.1313
λ_{11}	-0.2551	0.1023
λ_{12}	0.4189	0.1316
λ_{21}	-0.1883	0.1011
λ_{22}	0.0914	0.1316
Latent factors $dX_t = -\kappa X_t dt + \sigma_X^\top dZ_t$		
κ_{11}	0.3473	0.1025
κ_{12}	-0.4189	0.1316
κ_{21}	-0.1310	0.1029
κ_{22}	0.3192	0.1341
Mean reversion under risk neutral measure $\kappa^Q = \kappa + \sigma_X^\top \lambda_1$		
κ_{11}^Q	0.0922	0.0076
κ_{21}^Q	-0.3193	0.0194
κ_{22}^Q	0.4107	0.0260
Realized inflation process $\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi^\top dZ_t$,		
σ_{Π_1}	0.16%	0.06%
σ_{Π_2}	0.10%	0.04%
σ_{Π_3}	-1.09%	0.08%
Stock return process $\frac{dS_t}{S_t} = (r_t + \eta_S) dt + \sigma_S^\top dZ_t$		
η_S	5.59%	2.25%
σ_{S_1}	-0.84%	0.86%
σ_{S_2}	-2.26%	0.96%
σ_{S_3}	1.01%	0.79%
σ_{S_4}	15.31%	0.88%
Standard errors of yield measurement error		
Σ_e	0.48%	0.11bp

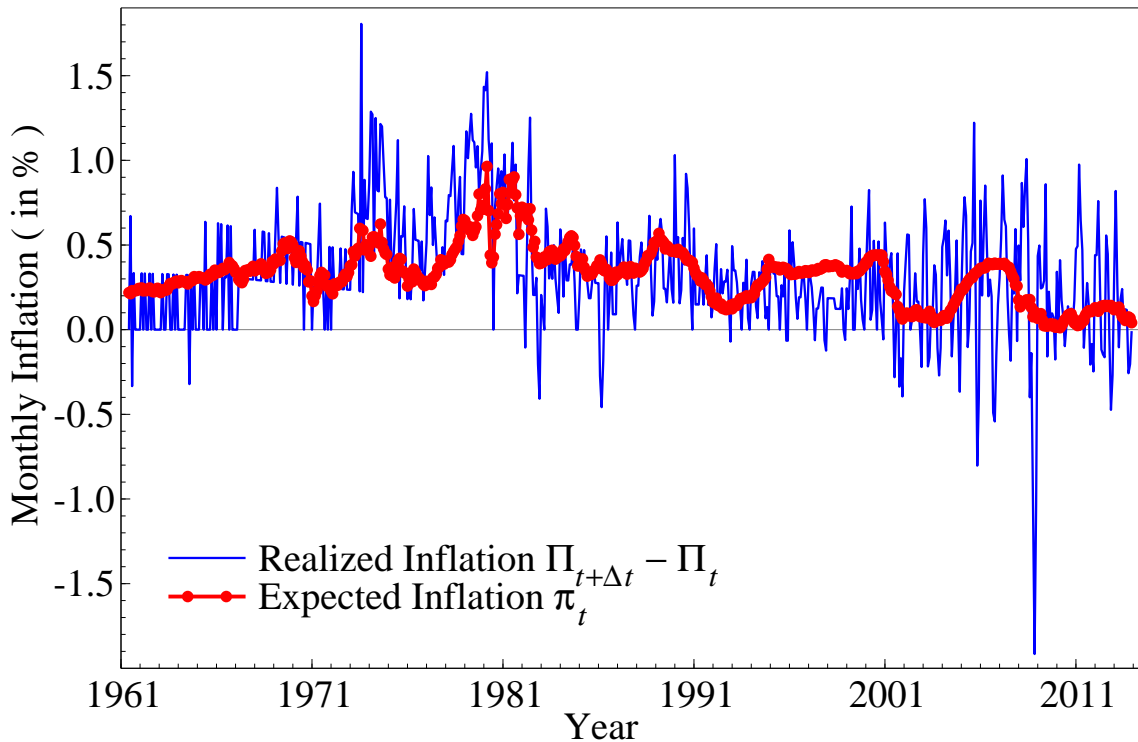
Table 2: **Comparison of Sample Moments with Model Implied Distribution.**

Compare mean and standard deviation of inflation rates, stock returns and yields of three maturities that follow from the data and from the model using the estimated parameters reported from Table 1. The two estimated latent factors are estimated based on one-year and six-year yields. The distribution from both historical data and model estimation are on a monthly basis.

	Average		Volatility	
	Data	Model	Data	Model
Stock Return	0.89%	0.78%	4.47%	4.48%
Inflation	0.33%	0.33%	0.36%	0.36%
1 Year Yield	5.45%	5.45%	3.17%	3.17%
3 Year Yield	5.85%	5.85%	3.01%	3.00%
6 Year Yield	6.24%	6.24%	2.78%	2.78%

contract, the inflation rate is over estimated during the dot-com bubble, around the year of 2001.

Figure 1: **Realized vs expected inflation rate.** Lighter solid curve plots the rate of monthly change in consumer price index over the sample period (from data) . The dotted curve presents the expected inflation $\pi_t = \xi_0 + \xi^\top X_t$ implied by estimation results shown in Table 1.



Next, I elaborate some properties of bond market price of risk parameters. Table 3 reports the unconditional bond premia $B(\tau)^\top \sigma_X^\top \lambda_0$ as well as their volatilities $\sqrt{B(\tau)^\top \sigma_X^\top \sigma_X B(\tau)}$ under various maturities with latent factors equal to their long-term means of zero. It is reported in Table 1 that both unconditional bond market price of risk factors λ_{01} and λ_{02} are negative. Since $B(\tau)$ is a negative vector, the unconditional bond premia $B(\tau)^\top \sigma_X \lambda_0$ must

Table 3: **Bond Risk Premia and Volatilities.**

This table displays the implied bond risk premia and return volatility with various maturities using the estimation results of Table 1. The market price of risk Λ_t is at the unconditional expectation with $X_t = 0_{2 \times 1}$.

Bond Maturity	1 year	3 year	6 year	10 year
Bond Premia $B(\tau)^\top \sigma_X^\top \lambda_0$	0.59%	1.39%	2.09%	2.62%
Volatility $\sqrt{B(\tau)^\top \sigma_X^\top \sigma_X B(\tau)}$	1.64%	4.10%	7.26%	10.82%

Table 4: **Correlation between returns on the stock, CPI and bonds with maturities of 1, 3, 6 and 10 years..** The table depicts the correlation between stock return, inflation rate and returns on nominal bonds with maturities 1 year, 3 year, 6 year and 10 year on the basis of estimation results shown in Table 1. For example, the correlation between ten - year bond return and inflation rate is equal to $B(10)^\top \sigma_X^\top \sigma_\Pi \left(B(10)^\top \sigma_X^\top \sigma_X B(10) \right)^{-1/2} \left(\sigma_\Pi^\top \sigma_\Pi \right)^{-1/2}$.

	Stock	Inflation	1 Year	3 Year	6 Year	10 Year
Stock return	1					
Inflation	-0.0930	1				
1 Year bond return	0.1574	-0.1330	1			
3 Year bond return	0.1561	-0.1582	0.9511	1		
6 Year bond return	0.1407	-0.1678	0.8184	0.9559	1	
10 Year bond return	0.1246	-0.1663	0.6975	0.8848	0.9826	1

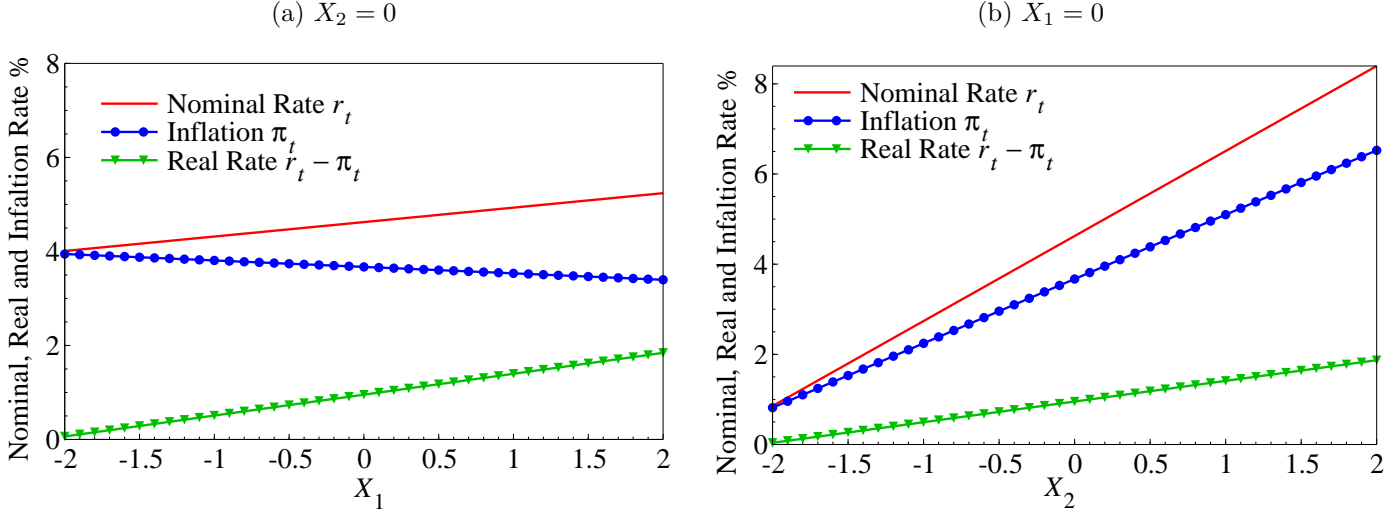
be strictly positive and are increasing with bond maturity. Further, Table 3 also implies that both bond risk premia and volatilities are positively correlated with their maturity τ , however the Sharpe ratio¹⁰ is negatively correlated with the bond maturity. Therefore, the one-year nominal bond must have a higher Sharpe ratio than the ten-year nominal bond (0.36 versus 0.18).

Table 4 reports the correlations between the returns of different assets included in the asset menu and the inflation rate. These correlations can influence the hedging demand to hedge future investment opportunities, hence are important to investors. I first summarize the property of the correlation between inflation rate and other asset returns. The value of σ_Π shown in Table 1 implies that inflation rate risk is negatively correlated with the bond return risk. Indeed, as Table 4 shows that correlation is negative. Further, Table 4 also reports a weak negative correlation between stock return and inflation rate, since the third elements of σ_Π and σ_S have opposite sign. Second, bond returns are always positively correlated with the stock returns, but the correlation value is negatively correlated with the bond maturity. This property is consistent with Koijen et al. (2010).

Figure 7 present the conditional nominal, real and inflation rate for a reasonable range of

¹⁰Sharp ratio is a fraction of a bond risk premium and its corresponding volatility.

Figure 2: **State-Variable Dependent Interest Rate and Inflation** The figure presents the interest rate in both nominal and real terms and inflation rate (in percentage) as a function of the two state variables.



X_1 and X_2 .¹¹ Panel (a) fixes the latent factor X_2 at zero and X_1 varies between -2 to 2. Panel (b) is the other way around. Figure 7 shows that both nominal and real interest rates are positively correlated to the two state variables. While inflation rate is negatively correlated with X_1 , for ξ_1 (see Table 1) is negative. Further, the three rates are much more sensitive to X_2 but is relatively persistent to X_1 . The time-variation of bond premia is governed of λ_1 . Figure 3 demonstrates that bond premia are increasing with X_1 but is decreasing with X_2 . Also, the figure shows that the long-term bond premia are more sensitive to the movement of both state variables than the short term bond premia.

5 Numerical Solution

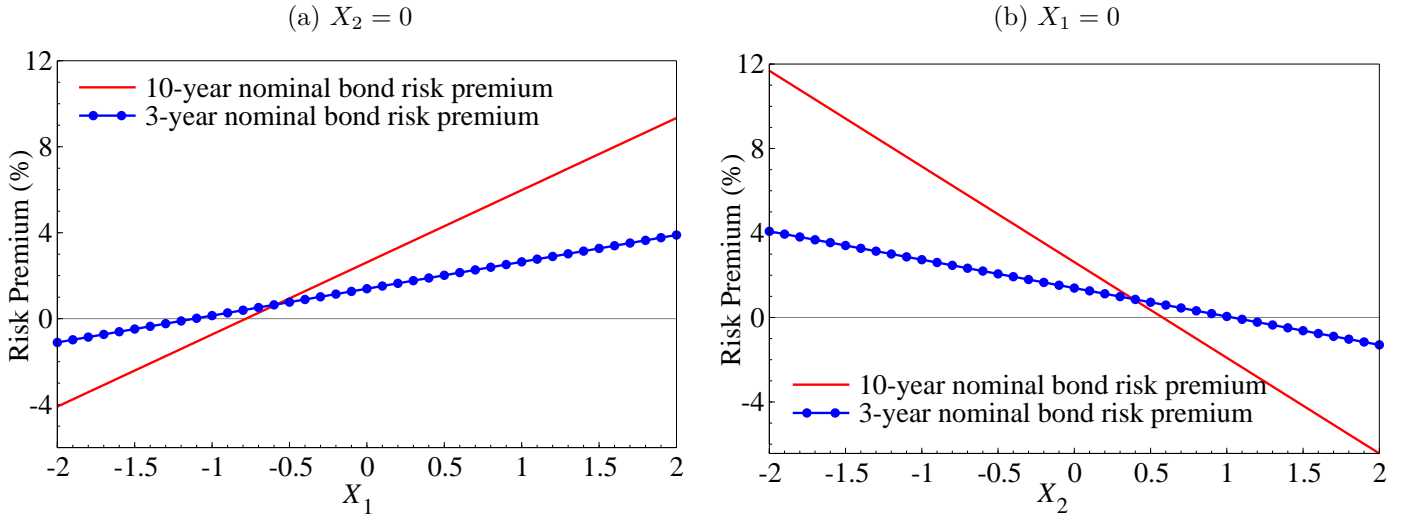
This section aims to discover the quantitative impact of parameter uncertainty on the optimal life-time asset allocation. I first determine the feasible region of the penalty parameter θ in Section 5.1. In Section 5.2 I demonstrate the optimal risk taking under various preferences for different age cohorts.

¹¹The latent factors follow OrnsteinUhlenbeck process 2. The explicit solution of 2 is

$$X_T = \exp(-\kappa(T-t)) X_t + \int_t^T \exp(-\kappa(T-s)) \sigma_X^\top dZ_s$$

therefore X_t is a Gaussian with mean $\mathbb{E}[X_T | \mathcal{F}_t] = \exp(-\kappa(T-t)) X_t$ and variance $\text{Var}[X_T | \mathcal{F}_t] = \int_t^T \exp(-\kappa(T-s)) \sigma_X^\top \sigma_X \exp(-\kappa(T-s))^\top ds$. The volatilities for X_1 and X_2 over one year is 0.57 and 0.91 respectively based on the estimation results 1. Assume that the long-term mean of X_1 and X_2 are close to zero, the 99% confidence interval for X_1 approximately between -1.5 to 1.5 and for X_2 is between -2.4 to 2.4.

Figure 3: **State-Variable Dependent Bond Premia** The figure plots the 3-year and 10-year nominal bond premia as functions of state variable X_1 (Panel (a)) and state variable X_2 (Panel (b)).



5.1 Uncertainty Set and Penalty Valuation

In most robust asset allocation studies, such as Maenhout (2004) and Branger et al. (2013), the preference for robustness parameter θ is taken subjectively. In this paper, I propose a statistically approach to determine the feasible value for θ .

Proposition 5.1. *Under the assumption that the estimation for parameter $\psi = \begin{pmatrix} \pi_t \\ g + \frac{\sigma_y^2}{2} \end{pmatrix}$ is misspecified and the perturbed point estimate $\hat{\psi}$ is asymptotically normal. The perturbation parameters (γ_1, γ_2) satisfy following constraint set*

$$\mathbf{S} = \{\gamma_1, \gamma_2 \mid \gamma_1^2 + \gamma_2^2 \leq \kappa^2\} \quad (39)$$

where κ^2 represents the ratio between the critical value of chi-squared statistics with two degrees of freedom $\chi^2(2)_\alpha$ and the number of observations.

See Appendix C.6 for the derivation of Proposition 5.1. The uncertainty set \mathbf{S} transfers mother nature's freedom to a confidence interval of the estimation parameters. The small significance level triggers a bigger set \mathbf{S} , hence will result in a larger perturbation.

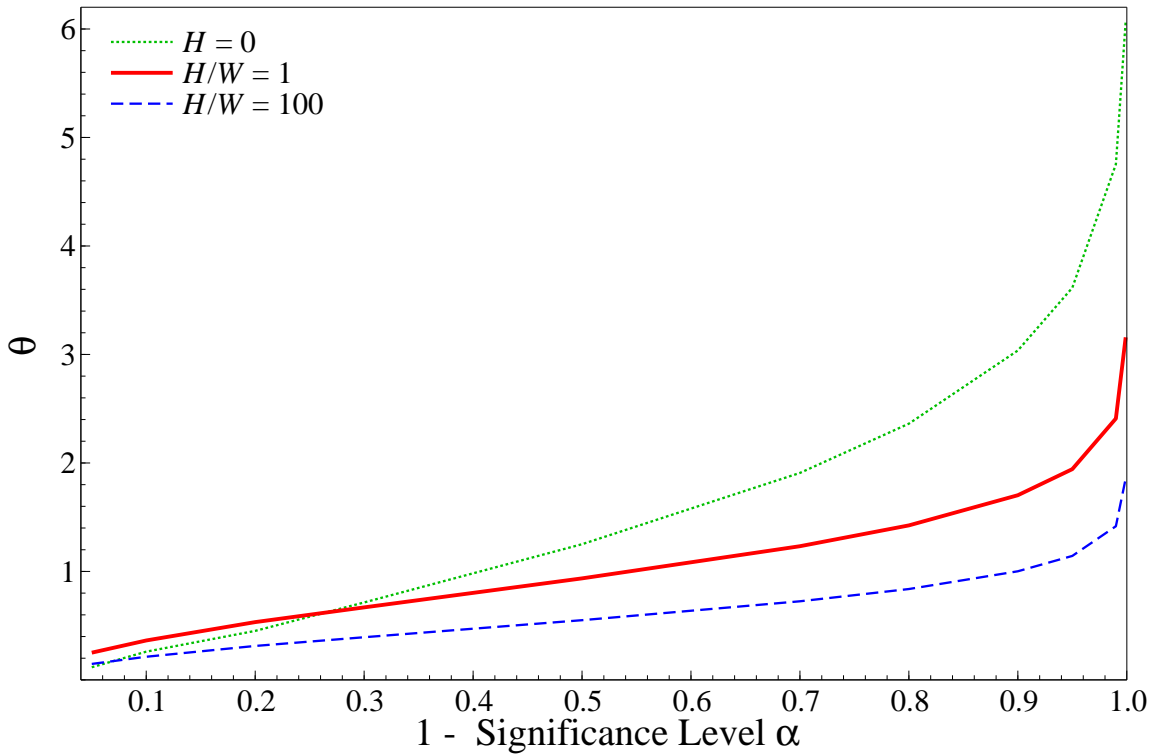
As demonstrated in Proposition 5.1 that, the sum squared drift distortions are bounded. By replacing the optimal distortions obtained in Proposition 3.5, one can obtain the constraint of θ as a function of κ^2 .

Figure 4 plots the maximum value of penalty θ under different statistical significance level α . The three curves in the figure refer to the movement of θ under three different $\frac{H}{W}$ ratios. Younger cohorts, usually have a high human capital over wealth ratio and is less ambiguity

aversion than the middle-age cohorts. Remark, when $H = 0$, I use $\chi^2(1)_\alpha$ to calculate the constraint κ^2 , for the second distortion parameter γ_2 vanishes.

To conclude, the value of θ can be chosen between 0 to 6 for **Model 2**. For **Model 4**, however, a feasible region of θ depends on the $\frac{H}{W}$ ratio. For example, if an investor has a human capital over wealth ratio equal to 100, then her maximum preference for robustness even under the lowest significance level is barely above 1.

Figure 4: Penalty parameter as a function of statistical significance level under different human capital and financial wealth ratios. The dotted curve refers to the feasible value of θ for **Model 2**, when human capital is not considered. The dashed curve and the solid curve show feasible θ of **Model 4** under two scenarios. The high $\frac{H}{W}$ ratio curve refers the anti-fragility preference for young cohorts, and the other one represents the preference for the older cohorts.



5.2 Robust Optimal Life-Cycle Decision without Labor Income

Table 5 reports the optimal strategies of **Model 1** and **Model 2** for different age cohorts without considering labor income. The risk aversion level is chosen at $\gamma = 5$ and the ambiguity aversion level for **Model 2** is $\theta = 4$, which is a reasonable value according to Proposition 5.1. Investor's age indirectly tells the remaining hedging horizon, as I assume a fixed retirement age. Therefore, the younger an investor is, the longer her hedging horizon will be.

For both models, the myopic demand contains a long position in 10-year nominal bonds, a long position in stocks. The myopic demand depends on the current value of the state variable, but is independent of hedging horizon. Therefore, if X is fixed, the myopic demand also stays the same for different age cohorts. When both state variables take values of their long-run

mean of zero, the risk premium for 10-year bond is 2.6% (see Table 3), which is higher than the stock premium. Not surprisingly, the myopic demand for long-term bond is higher than stock. Because the 10-year bond enjoys a lower level of volatility hence has a higher marker price of risk compared with the stock market.

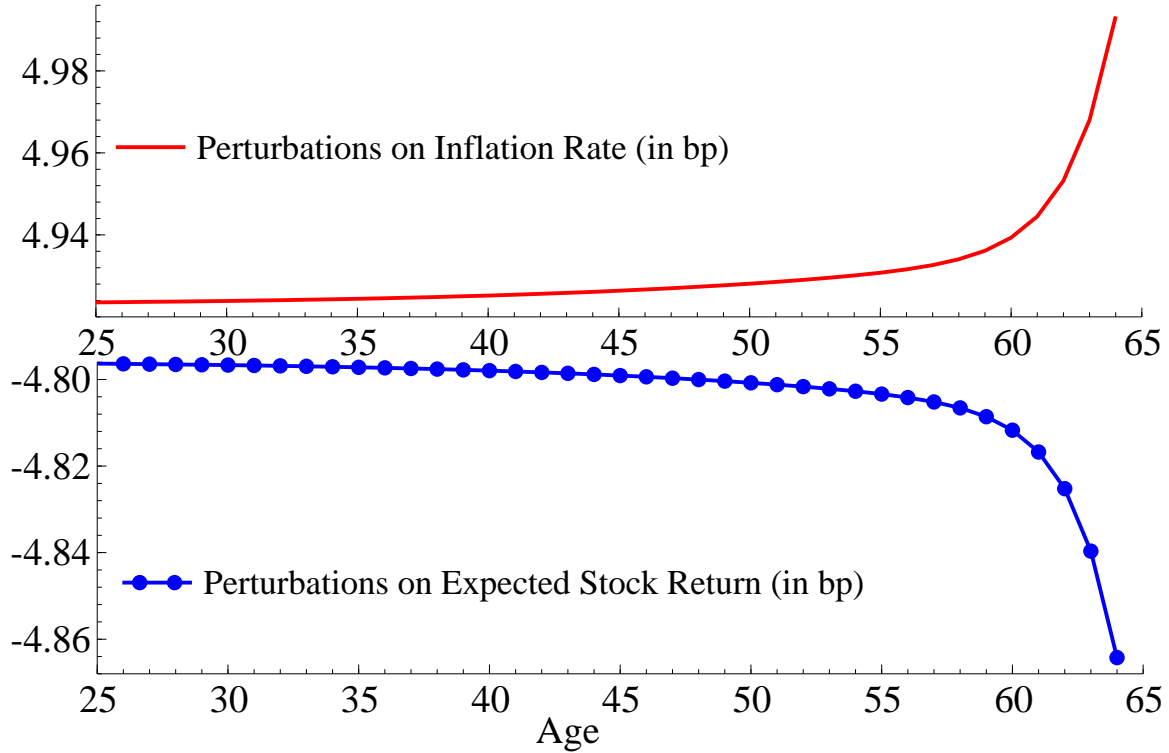
The preference for robustness creates a small myopic demand for long-term bond while in contract, reduced the demand for stocks. What drifts the risk shifting effect? On one hand, the inflation penalty term $\theta\sigma^\top \mathbf{e}\mathbf{e}^\top \sigma_\Pi$ shown in (27) brings a negative shock on the expected stock return. However, it will not influence the expected bond premia. On the other hand, the additional volatility penalty term $\theta\sigma^\top \mathbf{e}\mathbf{e}^\top \sigma$ will raise the volatility effect on the stock dimension while by construction will keep the bond volatility part unchanged. Because of the positive correlation between returns on the stock and the 10-year bond, a decrease of price of risk for the stock leads to a higher allocation on the long-term bond.

The interest rate hedging demand contains a long position in nominal bonds and a short position in stocks. There are strong horizon effects on the interest rate hedge, because functions $\Gamma_1^{Mi}(\tau)$, $\Gamma_1^{Mi}(\tau)$ with $i = 1, 2$ are increasing with the remaining hedging horizon τ . The horizon effects on long-term bond is much more dramatic. For the correlation between nominal interest rate and long-term bond return is much higher than the correlation between the nominal rate and the stock return. Ditto, the penalty term also brings an extra hedging demand for long-term bonds. However it has a very subtle effect on the allocation of stocks.

A large number of studies (e.g Wachter (2010), Brennan and Xia (2002), Kojien et al. (2010) and Wachter (2003)) discover that a time-varying short rate can generate a large amount of hedging demand for long-term bonds, The closed-form solutions Proposition 3.1 and 3.2 implies that the hedging opportunity for bond portfolio comes from two sources. One is from the time-varying risk bond premia. and the other one comes from the investor's desire for real interest rate hedge $r - \pi$ which can be found is PDE 57. The second source creates the well-known horizon effects for long-term bonds.

Mother nature's decision influences the reference model in two aspects. On one hand, nature's decision obviously would perturb the expected inflation rate to a different level. Figure 5 shows that a positive inflation shock is considered as the worst scenario, which also means that mother nature much be choosing a negative drift distortion term $\gamma_1 < 0$ for different age cohorts. Hence, a robust investor must be panic about an under estimation of the inflation level. On the other hand, a negative drift distortion would also brings a downside shock to the expected stock return. It also explained why a robust investor wants to reduce her risk exposure on the stock market shown in Table 5. There also are horizon effects of the optimal perturbations, as the older investors are more panic about the inflation rate misspecification.

Figure 5: Distortion (in *bp*) on inflation rate and expected stock return. The penalty parameter θ is assumed equal to 4. State variables are assumed equal to their long-run means zero.



In order to understand how the robust decision varies under different state variables and different risk preference, I compare the changes of the optimal hedge at each asset class from **Model 1** to **Model 2** for different scenarios of state variables as well for difference risk aversion level. Figure 6 demonstrates the risk shifting effects when an ambiguity neutral investor is aware of the model misspecification and becomes ambiguity aversion. First, with the preference for robustness, a robust investor reduces the hedge demand for stocks. for the inflation drift distortion would reduce the expected stock return. In other words, the realized hedging opportunity for stocks is over estimated (see also Figure 5). Secondly, Figure 6 also shows that a robust investor always required more hedging demand for long-term bond and this demand increases with X_2 while decreases with age. As shown in Figure 7 that the real interest rate is increasing with X_2 , and a higher real interest rate triggers a larger demand for long-term bond. Not surprisingly, risk takers (e.g $\gamma = 2$) enjoy a more aggressive risk shifting movement than the high risk aversion investors. For the risk takers are willing to put a higher weight on each investment opportunity.

5.3 Robust Optimal Life-Cycle Decision with Labor Income

Differ from the first two models, when labor income is included and is assumed partially correlated with the financial market, an additional investment opportunity is created. Therefore, in addition to myopic demand and interest rate hedging components, there is a new hedging

Table 5: **Optimal Portfolio Choice and Consumption without Labor Income.** The optimal portfolio choice between 10-year nominal bonds, stock market and a nominal money market (in percentage of total wealth) as well as the optimal consumption over wealth ratio for investors at age 30, 40, 50 and 60. The state variables X_1 and X_2 are set equal to their unconditional mean, namely zero. Panel A is on the basis of Proposition 3.1. Panel B is based on **Model 2** with the portfolio composition shown in Proposition 3.2. The penalty term θ is set equal to 4, which is approximately equivalent to 5% significance level of the perturbation distribution discussed in Figure 4.

$\gamma = 5$	Myopic Demand		Interest Rate Hedge		Total Portfolio		$\frac{C}{W}$
Age	10Year	Stock	10Year	Stock	10Year	Stock	
Panel A: Model 1							
30	40.46	17.38	84.05	-2.60	124.51	14.79	40.41
40	40.46	17.38	78.14	-2.57	118.60	14.82	40.31
50	40.46	17.38	64.46	-2.50	104.92	14.88	40.30
60	40.46	17.38	34.61	-2.30	75.07	15.08	43.14
Panel B: Model 2 with $\theta = 4$							
30	40.54	16.94	84.18	-2.59	124.71	14.35	40.35
40	40.54	16.94	78.24	-2.56	118.77	14.38	40.26
50	40.54	16.94	64.52	-2.50	105.06	14.44	40.25
60	40.54	16.94	34.63	-2.30	75.17	14.64	43.11

component occurs in **Model 3** and **Model 4** named income risk hedging component. Table 6 displays the hedging components for **Model 3** and **Model 4** under various age cohorts. Remark, when labor income is introduced, the nature has a freedom to control two drift distortions. As the result, the impact of parameter uncertainty becomes much more complicated than **Model 2**.

The optimal portfolio choices are no longer wealth independent. However, as shown in (32) and (34) that the human capital over wealth ratio $\frac{H}{W}$ plays a crucial rule in life-time optimal asset allocation. Munk and Sørensen (2010) also emphasize the importance of $\frac{H}{W}$ ratio but not the two separately to the investment decision making. According to Proposition 3.3, human capital, the accumulative expected future income, is a function of the remaining working time horizon and the income growth rate. Figure C.3 indicates that the human capital over wealth ratio, $\frac{H}{W}$ is a decreasing function of age, for human capital diminishing over time, while in the meantime financial wealth is growing over employment years.

Two important messages can be refined from Table 6. First, the impact of human capital parameter on the optimal decision is dramatic for the young cohort. As demonstrated in Proposition 3.4 that, both myopic demand and the interest ratio hedging demand are scaled by the total wealth over fiance wealth ratio $\frac{W+H}{W}$. The scaling effect influence both hedging components but in different ways. For myopic demand, it is not surprising that the youth should enlarge their risk exposure on both long-term bonds and stocks by a noticeable amount, since for instance the mean-variance investment opportunity is enlarged for more than six times

for investors younger than 30.

However, it is interesting to notice the interest rate hedging component suggests to short the long-term bonds for young investors. What drives such a theatrical change? It is demonstrated in the previous section that real interest rate $r - \pi$ also generates hedge demand for long-term bond and creates horizon effect in the first two models. Mathematically, when $X = 0$, the interest rate hedging component depends on Γ_2^{Mi} for $i = 3, 4$ with other parameters given. The full expression of Γ_2^{Mi} are shown in Appendix C.4 and C.5. It shows in function (68) that the last two terms $\frac{W}{W+H}\delta^\top - \xi^\top$ which represents the desire for interest rate change make the real rate hedging much smaller or even negative when H is high compared with the previous two model. However, when an investor is about to retire, her human capital vanishes and her preferences converges to the cases without human capital influences. That also explains the similarity between the four preferences under the older cohorts for each hedging component.

Table 6: Optimal Portfolio Choice and Consumption with Labor Income. The optimal portfolio choice (in percentage) on the basis of **Model 3** and **Model 4** for investors at age 30, 40, 50 and 60. The risk aversion level is set at $\gamma = 5$. The state variables X_1 and X_2 are set equal to zero, namely zero. The penalty term θ is set equal to 1, which is approximately equivalent to 10% significance level of the perturbation distribution discussed in Figure 4. The first six columns show each the demand for each hedging components which contains myopic demand, interest hedging ratio and income risk hedging demand. The next two columns aggregate the total hedging demand for the long-term bonds and the stocks. The last column gives the optimal consumption over total wealth ratio, which is $\frac{1}{b^{M3}}$ for **Model 3** and is $\frac{1}{b^{M4}}$ for **Model 4**.

$\gamma = 5$	Myopic Demand		Interest Rate Hedge		Income Risk Hedge		Total Portfolio		$\frac{C}{W+H}$
Age	10Year	Stock	10Year	Stock	10Year	Stock	10Year	Stock	
Panel A: Model 3									
30	344.03	147.82	-111.53	-29.96	2.49	-14.43	235.00	103.43	25.87
40	98.87	42.483	52.19	-7.83	0.48	-2.78	151.53	31.88	11.80
50	50.05	21.506	66.13	-3.35	0.08	-0.46	116.25	17.70	20.85
60	41.44	17.807	35.01	-2.38	0.81bp	-0.05	76.46	15.38	39.70
Panel B: Model 4 with $\theta = 1$									
30	344.21	146.79	-112.68	-29.96	4.98	-28.83	236.51	87.99	24.81
40	98.92	42.19	52.234	-7.83	0.96	-5.55	152.11	28.81	11.51
50	50.08	21.36	66.145	-3.35	0.16	-0.91	116.38	17.09	20.79
60	41.47	17.68	35.014	-2.38	0.02	-0.09	76.49	15.21	39.69

Figure 8 shows more details about the asset allocation changes when human capital parameter is involved. The allocation to each asset class depends on both investors' risk aversion level and the state variables. First, the demand for long-term bond is decreasing with X_2 , because of the negative relationship between bond premia and X_2 , shown in Figure 3. Therefore, a lower X_2 with X_1 given implies a higher bond premia hence leads to a bigger risk exposure. Second, the optimal long-term bond allocation is very sensitive to the risk aversion level. An extreme

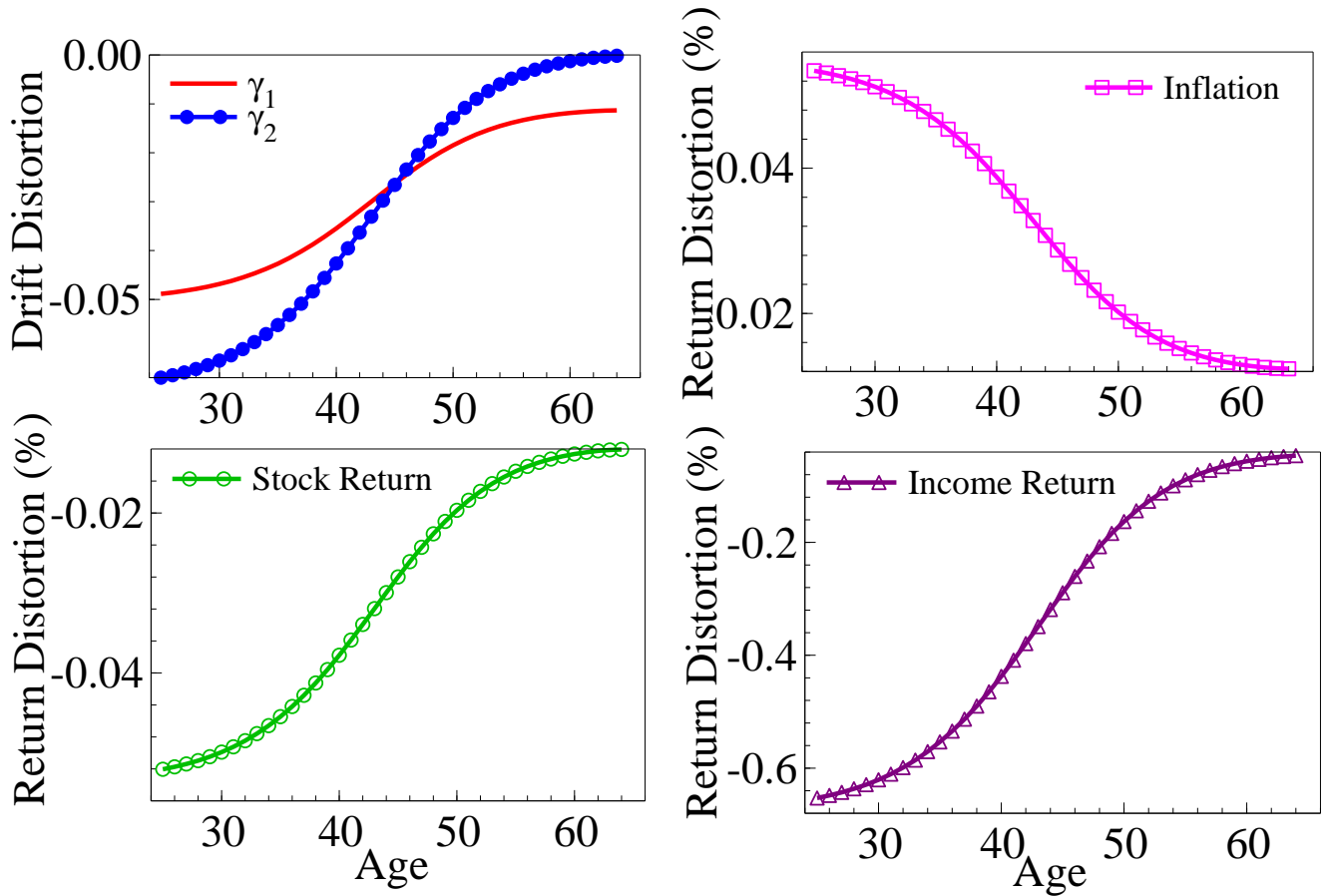
long-position on long-term bond is suggested for a risk takers. However, a short position is required from risk aversion investor. What makes the variation so dramatic? It is notable from Table 6 that both myopic demand and interest rate hedging component dominate the demand for long-term bond and have two opposite signs for the young cohorts. A high risk aversion level, say $\gamma = 10$ brings a bigger negative effect on function $\frac{b^M X}{b^M A}$ and also reduce the myopic demand although the myopic demand always stays positive. Therefore, the joint effect leads to a short position on the long-term bond.

The second message from Table 6 is that the perturbations bring a much bigger risk shifting effect especially for the young cohorts compared with **Model 2**. Because the investor worries about ambiguity exposure of total wealth to the drift distortions but not the financial wealth only. 9 provides more details about the risk shifting effects over as functions of age. It is important to notice that the risk shifting unit used in Figure XX is percentage, while in Figure 6, the risk shirking amount is in unit of basis point. I only focus on the dynamics for the young cohorts, because the human capital parameter can only influence the young cohorts' investment behavior.

The asset allocation shifting is caused by the misspecification of expected returns. On the basis of min-max optimization framework, nature's decision can represents an robust investor's fear for the worst case scenario. It is not difficult to imagine that, a robust investor can be worried about an overestimated equity return, an under estimated inflation rate. Further, she may also afraid that the true income growth rate is lower than the reference model expected. Therefore, to sum up, as shown in Figure 10 that the perturbation will reduce expected equity return as well as the expected income growth rate, while in contract will increase the inflation rate, since all those three scenarios can make the value of investor's utility lower than expected. The fear for the an overestimated equity return leads to a more conservative decision on the stock market, therefore in Figure 9, the shifting of stock allocation is always negative.

What makes the robust demand for long-term bond so sensitive to the state variables as well as the risk aversion level? The extra demand for long-term bond comes from the three hedging component. The myopic demand and income risk hedging demand both requires a longer position on the long-term bond when an investor is ambiguity averse. Therefore, it is the interest rate hedging component that controls the sign of the the demand change.

Figure 10: **Mother Nature’s Perturbation Effect.** The figure presents mother nature’s optimal decision on the basis on **Model 4**, as well as the perturbation impact on equity and income returns.



6 Conclusion

In this paper, I show the importance of being aware of macro-model uncertainty for a life-time investor. The impact of parameter uncertainty on the optimal asset allocation is heterogeneous. It depends on an investor’s age, risk aversion level, income stability as well as the instantaneous state variables.

Throughout the paper, I ignore shorting selling constraint and allow for borrowing from the future income. Of course, these assumptions are necessary in order to obtain closed-form solution but they can be unrealistic in reality. Therefore, a numerical solution with borrowing constraint bearing in mind is an useful extending of this study.

A Nominal bond pricing

In line with [Cox et al. \(1985\)](#), [Sangvinatsos and Wachter \(2005\)](#) and many other term structure studies, I assume that bond prices are continuous functions of state variables X and of time t .

I omit time subscription for notation convenience. No arbitrage condition implies that $P(t, T)$ satisfies

$$-\kappa X P_X + \frac{1}{2} \text{tr} (P_{XX} \sigma_X^\top \sigma_X) + P_t - rP = P_X \sigma_X^\top \Lambda \quad (40)$$

Assume that bond price is exponential affine in the X , (see (5)) and substitute this conjecture bond price function to (40),

$$-PB(\tau)^\top \kappa X + \frac{1}{2} PB(\tau)^\top \sigma_X^\top \sigma_X B(\tau) + P(-B'(\tau)X - A') - rP = PB(\tau)^\top \sigma_X^\top \Lambda$$

and matching coefficients on X and the constants produce the following system of ordinary differential equation for row vector $B(\tau)$, $1 \times m$ and the scalar $A(\tau)$

$$B'(\tau) = -(\kappa^\top + \lambda_1^\top \sigma_X) B(\tau) - \delta \quad (41)$$

$$A'(\tau) = \frac{1}{2} B(\tau)^\top \sigma_X^\top \sigma_X B(\tau) - \delta_0 - B(\tau)^\top \sigma_X^\top \lambda_0 \quad (42)$$

The boundary condition is that $B(0) = 0$ and $A(0) = 0$. The ODE can be solved analytically,

$$B(\tau) = (\kappa^\top + \lambda_1^\top \sigma_X^\top)^{-1} [\exp(-(\kappa^\top + \lambda_1^\top \sigma_X^\top) \tau) - I_{2 \times 2}] \delta_1 \quad (43)$$

$$A(\tau) = \int_0^\tau A'(s) ds \quad (44)$$

I define $\kappa^Q = \kappa + \sigma_X^\top \lambda_1$, where κ^Q is a lower triangular matrix.

B Estimation Procedure

The estimation procedure is as follows. First, I estimation the term structure model using MCSE method. Second, I use GMM method to estimation inflation and stock return processes. (*Peter mentioned to combine the variance covariance matrix of the two parts together, I have not figured out how to do that.*)

The term structure model estimation method I use combines in essence the method of [Hamilton and Wu \(2012\)](#). I use three different yields to estimate the term structure, namely $\tau_1 = 1$ year, $\tau_2 = 6$ year and $\tau_3 = 3$ year maturities. In line with [Duffee \(2002\)](#), I assume that 1 year and 10 year yield bonds are measured without error, and the other 3-year to maturity bond is assumed to be measured with serially uncorrelated, mean-zero measurement error. Yield function with maturity τ_i , with $i = 1, 2, 3$ is given by

$$Y_t^{\tau_i} = a(\tau_i) + b(\tau_i)^\top X_t$$

Let $Y_{1,t}$ denote the vector of perfectly observed yield at time t and $Y_{2,t}$ denotes the yield which

is observed imperfectly. More specifically

$$\begin{pmatrix} Y_{1,t} \\ (2 \times 1) \\ Y_{2,t} \\ (1 \times 1) \end{pmatrix} = \begin{pmatrix} A_1 \\ (2 \times 1) \\ A_2 \\ (1 \times 1) \end{pmatrix} + \begin{pmatrix} B_1 \\ 2 \times 2 \\ B_2 \\ (1 \times 1) \end{pmatrix} X_t + \begin{pmatrix} 0 \\ (2 \times 1) \\ \Sigma_e \\ (1 \times 1) \end{pmatrix} u_t.$$

where Σ_e represents the variance of estimation error with $u_t \sim N(0, \Delta t)$. There are 13 unknown parameters involved in the model, namely 1 in δ_0 , 2 in δ , 4 in κ , 3 in κ^Q , 2 in λ_0 and 1 in Σ_e .

MCSE algorithm consists of three steps. First, I reconstruct the affine term structure model to a vector autoregression (VAR) model of Y_t^r , then estimate VAR model using OLS. The coefficients obtained from VAR is called reduced-form parameters. Second step is to map between structural parameters and reduced-form parameters. Last step is solve minimum chi-square criterion to obtain the structural parameters.

Step 1 I estimate parameters of continuous time model using a discrete time econometrics specification. The implied state variable can be derived from (6)

$$X_t = B_1^{-1}(Y_{1,t} - A_1) \quad (45)$$

A discretization of (2) yields the discrete time model of state variables

$$X_{t+\Delta t} = \mu_X X_t + \sqrt{\Delta t} z_{t+\Delta t} \quad \text{with} \quad \mu_X = \exp(-\kappa \Delta t)$$

where $\Delta t = \frac{1}{12}$ and $z_{t+\Delta t}$ represents bivariate standard normal distribution. Both sides of equation above multiply by B_1 and plus A_1 , I get

$$A_1 + B_1 X_{t+\Delta t} = A_1 + B_1 \mu_X B_1^{-1} B_1 X_t + B_1 \sqrt{\Delta t} z_{t+\Delta t}$$

which can be rewritten as a VAR model of $Y_{1,t}$

$$\begin{aligned} Y_{1,t+\Delta t} &= A_1 - B_1 \mu_X B_1^{-1} A_1 + B_1 \mu_X B_1^{-1} Y_{1,t} + B_1 \sqrt{\Delta t} z_{t+\Delta t} \\ &= A_1^* + \phi_{11}^* Y_{1,t} + u_{1,t+\Delta t}^* \end{aligned} \quad (46)$$

where $A_1^* = A_1 - B_1 \mu_X B_1^{-1} A_1$, $\phi_{11}^* = B_1 \mu_X B_1^{-1}$ and $\Omega_1^* = B_1 B_1^\top \Delta t = u_{1,t+\Delta t}^* u_{1,t+\Delta t}^*$. Similarly

$$Y_{2,t} = A_2^* + \phi_{21}^* + u_{2,t}^* \quad (47)$$

with $A_2^* = A_2 - B_2 B_1^{-1} A_1$, $\phi_{21}^* = B_2 B_1^{-1}$ and $\Omega_2^* = \Sigma_e \Sigma_e \Delta t = u_{2,t}^* u_{2,t}^*$. Hence the reduced-form parameter vector is

$$\pi = \{ \text{vec} [(A_1^*, \phi_{11}^*)'], \text{vech}(\Omega_1^*), \text{vec} [(A_2^*, \phi_{21}^*)'], \Omega_2^* \}$$

where $\text{vec}(X)$ collects the columns of the matrix X into a vector. If X is symmetric matrix, $\text{vech}(X)$ does the same but only use the elements below and including diagonal.

The reduced-form parameters can easily be obtained via OLS estimation with Ω_1^* and Ω_2^* products of those OLS residuals:

$$\begin{aligned}\Omega_1^* &= N^{-1} \sum_{t=\Delta t}^{N\Delta t} (Y_{1,t+\Delta t} - A_1^* - \phi_{11}^* Y_{1,t}) (Y_{1,t+\Delta t} - A_1^* - \phi_{11}^* Y_{1,t})^\top \\ \Omega_2^* &= N^{-1} \sum_{t=\Delta t}^{N\Delta t} (Y_{2,t} - A_2^* - \phi_{21}^* Y_{1,t})^2\end{aligned}$$

where T represents to the number of observations from the historical sample data. Next I will map the structural parameters to the

Step 2 Mapping between structural and reduced-form parameters can be done in four steps:

1. Estimation of Σ_e is obtained analytically, $\Omega_2^* = \Sigma_e^2 \Delta t$.
2. Estimates of the 5 unknowns in κ^Q and δ , are found by numerically solving 5 equations from following relation

$$\begin{aligned}B_1 B_1^\top \Delta &= \Omega_1^* \\ B_2 B_1^\top \Delta &= \phi_{21}^* \Omega_1^*\end{aligned}$$

3. Estimate of 4 unknowns in κ can be obtained analytically via

$$\mu_X = B_1^{-1} \phi_{11}^* B_1, \quad \text{with} \quad \mu_X = \exp(-\kappa \Delta t) = N \exp(-D \Delta t) N^{-1}$$

under the assumption that κ can be diagonalized and $\kappa = N D N^{-1}$.

4. Numerically solve the remaining 3 unknowns δ_0 , λ_0 from equation A_1^* , A_2^* .

$$\begin{aligned}A_1 - B_1 \mu_X B_1^{-1} A_1 &= A_1^* \\ A_2 - B_2 B_1^{-1} A_1 &= A_2^*\end{aligned}$$

Step 3 Let θ_B stack all 13 structure parameters of the term structure model into a vector. Next one can test the hypothesis that $\pi = g(\theta_B)$ using Wald statistics $T[\hat{\pi} - g(\theta_B)]' \hat{R}[\hat{\pi} - g(\theta_B)]$ which would have an asymptotic $\chi^2(13)$ distribution, in which \hat{R} is a consistent estimate of the information matrix $R = -\frac{1}{T} \mathbb{E} \left[\frac{\partial^2 \mathbf{L}(\pi; X)}{\partial \pi \partial \pi'} \right]$ with $\mathbf{L}(\pi; X)$ the log likelihood for the entire sample.

The reduced-form equation (46) and (47) form 2 independent blocks. For $i = 1, 2$

$$Y_{i,t} = \Pi_i' x_{it} + u_{i,t}^* \tag{48}$$

with $u_{i,t}^* \sim N(0, \Omega_i^*)$. The information matrix R for the full system of reduced-form parameter is given by

$$\hat{R} = \begin{pmatrix} \hat{R}_1 & 0 \\ 0 & \hat{R}_2 \end{pmatrix} \quad (49)$$

where as shown in [Neudecker and Magnus \(1988\)](#),

$$\hat{R}_i = \begin{pmatrix} (\Omega_i^*)^{-1} \otimes \sum x_{it} x_{it}' & \\ & \frac{1}{2} D'_{q_i} ((\Omega_i^*)^{-1} \otimes (\Omega_i^*)^{-1}) D_{q_i} \end{pmatrix} \quad (50)$$

for D_N the $N^2 \times N(N+1)/2$ duplication matrix satisfying $D_N \text{vech}(\Omega) = \text{vec}(\Omega)$, and q_i represents the dimension of Ω matrix.

Following [Rothenberg \(1973\)](#), the minimum-chi-square estimator $\hat{\theta}_B$ is the solution of

$$\min_{\theta_B} T [\hat{\pi} - g(\theta_B)]' R [\hat{\pi} - g(\theta_B)] \quad (51)$$

According to [Hamilton and Wu \(2012\)](#), the asymptotic distribution of θ_B is

$$\sqrt{T} (\hat{\theta}_B - \theta_B) \rightarrow N \left(0, [\Gamma' R \Gamma]^{-1} \right) \quad (52)$$

where $\Gamma = \frac{\partial g(\theta_B)}{\partial \theta_B}$ is the Jacobian matrix of $g(\theta_B)$ as a function of θ_B .

Next, I estimation inflation and stock return process using GMM method. The discrete time inflation return process is as follows

$$\Delta\pi_{t+\Delta t} = (\xi_0 + \xi^\top X_t) \Delta t + \sqrt{\Delta t} \sigma_\pi^\top z_{t+\Delta t}$$

where $\Delta\pi_{t+\Delta t} = \frac{\Delta\Pi_{t+\Delta t}}{\Pi_t}$. Hence, there are 6 unknown parameters, 1 in ξ_0 , 2 in ξ , 3 in σ_π . I can generate six moment conditions based on inflation dynamics and state variable diffusion process, namely

$$\begin{aligned} \mathbb{E} [\Delta\pi_{t+\Delta t} - (\xi_0 + \xi^\top X_t) \Delta t] &= 0 \\ \mathbb{E} [X_t (\Delta\pi_{t+\Delta t} - (\xi_0 + \xi^\top X_t) \Delta t)] &= 0 \\ \mathbb{E} [(\Delta\pi_{t+\Delta t} - (\xi_0 + \xi^\top X_t) \Delta t)^2 - (\sigma_\pi^\top \sigma_\pi) \Delta t] &= 0 \\ \mathbb{E} [(\Delta X_{t+\Delta t} - (\mu_X - 1) X_t) (\Delta\pi_{t+\Delta t} - (\xi_0 + \xi^\top X_t) \Delta t)] - \sigma_X^\top \sigma_\pi \Delta t &= 0 \end{aligned}$$

Similarly, one generate five moment conditions based on the discrete time stock return model

$$r_{s,t+\Delta t} = (\delta_0 + \delta^\top X_t + \eta_s) \Delta t + \sqrt{\Delta t} \sigma_S^\top z_{t+\Delta t}$$

Let $r_{s,t+\Delta t} = \frac{\Delta S_{t+\Delta t}}{S_t}$. Five moment conditions are

$$\begin{aligned} \mathbb{E} [r_{s,t+\Delta t} - (\delta_0 + \delta^\top X_t + \eta_s) \Delta t] &= 0 \\ \mathbb{E} [(r_{s,t+\Delta t} - (\delta_0 + \delta^\top X_t + \eta_s) \Delta t) (\Delta X_{t+\Delta t} - (\mu_X - 1)X_t) - \Delta t \sigma_X^\top \sigma_S] &= 0 \\ \mathbb{E} [(r_{s,t+\Delta t} - (\delta_0 + \delta^\top X_t + \eta_s) \Delta t)^2 - \Delta t \sigma_S^\top \sigma_S] &= 0 \\ \mathbb{E} [(r_{s,t+\Delta t} - (\delta_0 + \delta^\top X_t + \eta_s) \Delta t) (\Delta \pi_{t+\Delta t} - (\xi_0 + \xi^\top X_t) \Delta t) - \Delta t \sigma_S^\top \sigma_\pi] &= 0 \end{aligned}$$

Denote θ_π a vector of inflation process parameters and θ_S a vector of stock process parameters. The GMM estimates $\hat{\theta}_{GMM} = (\hat{\theta}_\pi, \hat{\theta}_S)^\top$ aim to minimize a quadratic form of sample mean of the moment conditions $g(\theta_{GMM}) = \frac{1}{T} \sum f_t(\theta_{GMM})$, where $f_t(\theta_{GMM})$ denotes a vector of errors of moment conditions at time t . The long run covariance matrix is defined by

$$\mathbf{S} = \sum_{j=-\infty}^{\infty} \mathbb{E}[f_t(\theta_{GMM}) f_{t-j}'(\theta_{GMM})] \quad (53)$$

I use Newey West method to estimate \mathbf{S} with $j = 12$. Hence under the efficient GMM, the estimate θ is asymptotically normal

$$\sqrt{T}(\hat{\theta}_{GMM} - \theta_{GMM}) \sim N(0, (d' \mathbf{S}^{-1} d)^{-1}) \quad (54)$$

where d is the Jacobian matrix of the population moments vector estimated by $d(\theta_{GMM}) = \frac{\partial g(\theta_{GMM})}{\partial(\theta_{GMM})}$.

C Proofs

C.1 Proof of Proposition 3.1

Model 1 is a standard life-cycle problem with consumption. Wachter (2010) provides a systematic overview of modern techniques on solving asset allocation problems. In this paper, I use standard dynamic programming method by means of solving a HJB equation to obtain the optimal decision. Define value function of **Model 1** at time t as $J^{M1}(W, \Pi, X, t)$. The HJB equation of **Model 1** reads as

$$\begin{aligned} \beta J^{M1} = \max_{C_t, x_t} \{ & \frac{1}{1-\gamma} \left(\frac{C}{\Pi} \right)^{1-\gamma} + J_t^{M1} + J_W^{M1} [(x^\top (\mu - r) + r) W - C] \\ & + \frac{1}{2} J_{WW}^{M1} W^2 x^\top \sigma^\top \sigma x + J_\Pi^{M1} \Pi \pi_t + \frac{1}{2} J_{\Pi\Pi}^{M1} \Pi^2 \sigma_\pi^\top \sigma_\pi - J_X^\top \kappa X + \frac{1}{2} \text{tr} (J_{XX}^{M1} \sigma_X^\top \sigma_X) \\ & \left. J_{W\Pi}^{M1} W \Pi x^\top \sigma^\top \sigma_\Pi + W x^\top \sigma^\top \sigma_X J_{WX}^{M1} + \Pi \sigma_\Pi^\top \sigma_X J_{\Pi X}^{M1} \right\} \quad (55) \end{aligned}$$

The first order condition for consumption equals to

$$C^{M1} = (J_W^{M1} \Pi^{1-\gamma})^{-\frac{1}{\gamma}} \quad (56)$$

Then, substitute the conjecture function (24) as well the the optimal consumption (56) back to the equation. Function b^{M1} satisfies the following partial differential equation (PDE)

$$\begin{aligned} 0 = & \left(-\frac{\beta}{1-\gamma} + x^\top (\mu - \iota r) + (r - \pi) - \frac{1}{2} \gamma x^\top \sigma^\top \sigma x - \frac{\gamma-2}{2} \sigma_\Pi^\top \sigma_\Pi + (\gamma-1) x^\top \sigma^\top \sigma_\Pi \right) \\ & + \frac{1}{b^{M1}} \frac{\gamma}{1-\gamma} + \frac{b_t^{M1}}{b^{M1}} - \frac{b_X^{M1}}{b^{M1}} \left(\frac{\gamma}{1-\gamma} \kappa X - \gamma x^\top \sigma^\top \sigma_X + \gamma \sigma_\Pi^\top \sigma_X \right) \\ & - \frac{1}{2} \text{tr} \left\{ \left(-\gamma \frac{b_X^{M1} b_X^{M1}}{(b^{M1})^2} + \frac{\gamma}{1-\gamma} \frac{b_{XX}^{M1}}{b^{M1}} \right) \sigma_X^\top \sigma_X \right\} \end{aligned} \quad (57)$$

The PDE above is comparable to equation (36) of [Sangvinatsos and Wachter \(2005\)](#) expect the additional term $\frac{1}{b^{M1}} \frac{\gamma}{1-\gamma}$ which is generated from the optimal consumption decision C^{M1} . Their work shows that as long as the optimal portfolio x^{M1} is affine in state variable X , the PDE above can be solve explicitly in the absence of consumption, and the trial function is exponentially quadratic in the state variable.

Conjecture the form of b^{M1} as (25). Differ from [Sangvinatsos and Wachter \(2005\)](#) case, however, when consumption is considered, the PDE above is no longer homothetic hence cannot be solved explicitly by coefficients matching. Inspired by [Wachter \(2010\)](#) equation (49), the inverse of trial function is approximately affine in $\log(b^{M1})$

$$\frac{1}{b^{M1}} = b_0^{M1} - b_1^{M1} \log(b^{M1}) \quad (58)$$

where $b_1^{M1} = \exp(\mathbb{E}[-\log(b^{M1})])$ and $b_0^{M1} = b_1^{M1} (1 - \log b_1^{M1})$. This approximation technique retrieves the PDE to a homothetic environment. The optimal portfolio obtained from the first order condition on HJB equation can be expressed as

$$x^{M1} = a_0^{M1}(\tau) + a_1^{M1}(\tau) X \quad (59)$$

where

$$\begin{aligned} a_0^{M1}(\tau) &= \frac{1}{\gamma} (\sigma^\top \sigma)^{-1} \sigma^\top \lambda_0 + \frac{\gamma-1}{\gamma} (\sigma^\top \sigma)^{-1} \sigma^\top \sigma_\Pi + \frac{1}{\gamma} (\sigma^\top \sigma)^{-1} \sigma^\top \sigma_X \Gamma_2^{M1}(\tau)^\top \\ a_1^{M1}(\tau) &= \frac{1}{\gamma} (\sigma^\top \sigma)^{-1} \sigma^\top \lambda_1 + \frac{1}{2\gamma} (\sigma^\top \sigma)^{-1} \sigma^\top \sigma_X \left(\Gamma_1^{M1}(\tau) + \Gamma_1^{M1}(\tau)^\top \right) \end{aligned}$$

Substituting the trial solution (25) to the PDE (57) and match the coefficients on $X^\top (\cdot) X$, X

and the constant term, the parameters Γ_1^{M1} and Γ_2^{M1} satisfy the following system of ODE

$$\begin{aligned}\Gamma_1^{M1}(\tau)' &= \left(\Gamma_1^{M1}(\tau) + \Gamma_1^{M1}(\tau)^\top \right) (-\kappa + (1-\gamma) \sigma^\top \sigma_X a_1^{M1}) \\ &\quad + \frac{\Gamma_1^{M1}(\tau) + \Gamma_1^{M1}(\tau)^\top}{2} \sigma_X^\top \sigma_X \frac{\Gamma_1^{M1}(\tau) + \Gamma_1^{M1}(\tau)^\top}{2} + 2(1-\gamma) a_1^{M1\top} \sigma^\top \lambda_1 \\ &\quad - \gamma(1-\gamma) a_1^{M1\top} \sigma^\top \sigma a_1^{M1} - b_1^{M1} \Gamma_1^{M1}(\tau)\end{aligned}\quad (60)$$

$$\begin{aligned}\Gamma_2^{M1}(\tau)' &= \Gamma_2^{M1}(\tau) \left[-b_1^{M1} - \kappa + (1-\gamma) \sigma^\top \sigma_X a_1^{M1} + \frac{\Gamma_1^{M1}(\tau) + \Gamma_1^{M1}(\tau)^\top}{2} \sigma_X^\top \sigma_X \right] \\ &\quad + [(1-\gamma) a_0^{M1\top} \sigma^\top \sigma_X - (1-\gamma) \sigma_\Pi^\top \sigma_X] \frac{\Gamma_1^{M1}(\tau) + \Gamma_1^{M1}(\tau)^\top}{2} - (1-\gamma)^2 \sigma_\Pi^\top \sigma a_1^{M1} \\ &\quad - \gamma(1-\gamma) a_0^{M1\top} \sigma^\top \sigma a_1^{M1} + (1-\gamma) [a_0^{M1\top} \sigma^\top \lambda_1 + \lambda_0^\top \sigma a_1^{M1} + \delta^\top - \xi^\top]\end{aligned}\quad (61)$$

with the boundary condition that $\Gamma_1^{M1}(0) = 0$, $\Gamma_2^{M1}(0) = 0$ and $\Gamma_3^{M1}(0) = 0$. The expression for Γ_3^{M1} is not required for the optimal portfolio and hence I do not provide.

C.2 Proof of Proposition 3.2

Denote $J^{M2}(W, \Pi, X, t)$ an indirect utility function of **Model 2**. The robust HJB equation with only one source of parameter uncertainty is a special case of (22) stated in Section 2.5, with $Y = 0$ and $\gamma_2 = 0$. The solution technique is very similar to **Model 1**, expect that robust HJB equation has an additional penalty function. Thanks to Maenhout (2004)'s trick (23), the robust HJB equation still stays homothetic hence is friendly to the closed-form solution.

The robust solution can be obtained in three steps.

- Step 1. Replace the optimal consumption, the optimal affine portfolio choice and the optimal distortion, obtained from the first order conditions respectively, back to the robust HJB equation. The optimal portfolio takes the form

$$x^{M2} = a_0^{M2} + a_1^{M2} X, \text{ with } a_0^{M2} \in \mathbb{R}^{2 \times 1}, \quad a_1^{M2} \in \mathbb{R}^{2 \times 2}$$

- Step 2. Substitute the conjecture for J^{M2} back to the PDE derived from Step 1. The trail function of J^{M2} takes the same form as (24)

$$J^{M2} = \frac{1}{\gamma} \left(\frac{W}{\Pi} \right)^{1-\gamma} \exp \left\{ \frac{1}{\gamma} \left(\frac{1}{2} X^\top \Gamma_1^{M2}(\tau) X + \Gamma_2^{M2}(\tau) X + \Gamma_3^{M2}(\tau) \right) \right\} \quad (62)$$

- Step 3. Eventually, the simplified PDE is an affine function of $X^\top (\cdot) X$ and X . The last step is to match the coefficients of each polynomial order of the state variable.

$$\begin{aligned}\Gamma_1^{M2}(\tau)' &= \Gamma_1^{M1}(\tau)' (a_0^{M2}, a_1^{M2}) - (1-\gamma) \theta a_1^{M2\top} \sigma^\top \mathbf{e} \mathbf{e}^\top \sigma a_1^{M2} \\ \Gamma_2^{M2}(\tau)' &= \Gamma_2^{M1}(\tau)' (a_0^{M2}, a_1^{M2}) + (1-\gamma) \theta [\sigma_\Pi^\top \mathbf{e} \mathbf{e}^\top \sigma a_1^{M2} - a_0^{M2} \sigma^\top \mathbf{e} \mathbf{e}^\top \sigma a_1^{M2}]\end{aligned}$$

The first part of ODE system repeats the corresponding ODE of **Model 1** while replacing the deterministic functions function a_0^{M1} and a_1^{M1} to a_0^{M2} and a_1^{M2} respectively.

Then substituting the explicit solution of J^{M2} back to the first-order conditions, one can arrive at Proposition 3.2. The expression of a_0^{M2} and a_1^{M2} can easily be obtained from (27) via coefficients matching.

C.3 Proof of Proposition 3.3

The state variable under \mathbb{Q} measure follows the diffusion process

$$dX_t = (-\kappa^{\mathbb{Q}} X_t - \sigma_X^\top \lambda_0) dt + \sigma_X^\top dZ_t^{\mathbb{Q}} \quad (63)$$

where $\kappa^{\mathbb{Q}} = \kappa + \sigma_X^\top \lambda_1$. Assume that $\kappa^{\mathbb{Q}}$ is diagonalizable, then there exists a diagonal matrix D and a matrix N such that

$$\kappa = NDN^{-1}$$

Hence the explicit solution of X_s , $s \geq t$ is given by

$$\begin{aligned} X_s &= N \exp(-D(s-t)) N^{-1} X_t - \int_t^s N \exp(-D(s-v)) N^{-1} \sigma_X^\top \lambda_0 dv \\ &\quad + \int_t^s N \exp(-D(s-v)) N^{-1} \sigma_X^\top dZ_v^{\mathbb{Q}} \end{aligned}$$

hence

$$\begin{aligned} \int_t^u X_s ds &= \int_t^u N \exp(-D(s-t)) N^{-1} X_t ds - \int_t^u \int_t^s N \exp(-D(s-v)) N^{-1} \sigma_X^\top \lambda_0 dv ds \\ &\quad + \int_t^u \int_t^s N \exp(-D(s-v)) N^{-1} \sigma_X^\top dZ_v^{\mathbb{Q}} ds \end{aligned}$$

The integration above contains three parts. The first two parts can be solve analytically. One can reduce one level of integration of the third part by applying the Fubini rule. Then I get

$$\begin{aligned} \int_t^u X_s ds &= N f(d_i, u-t)_i N^{-1} X_t - N \left(\frac{1}{d_i} (u-t) - \frac{1}{d_i} f(d_i, u-t)_i \right) N^{-1} \sigma_X^\top \lambda_0 \\ &\quad + \int_t^u f(d_i, u-s)_i \sigma_X^\top dZ_s^{\mathbb{Q}} \end{aligned} \quad (64)$$

where d_i represents the diagonal element of matrix D with $\{i = 1, 2\}$, and $f(d_i, \tau) = -\frac{1}{d_i} [\exp(-d_i(u-t)) - 1]$. The nominal income process under \mathbb{Q} measure is given by

$$dY_t = \left(g + \frac{\sigma_y^2}{2} - \sigma_y \rho_y^\top \lambda_0 - \sigma_y \rho_y^\top \lambda_1 X_t \right) Y_t dt + Y_t \sigma_y \rho_y^\top dZ_t^{\mathbb{Q}} \quad (65)$$

Hence the explicit solution of nominal income stream at time u is given by

$$Y_u = Y_t \exp \left[\int_t^u \left(g + \frac{\sigma_y^2}{2} - \sigma_y \rho_y^\top \lambda_0 - \frac{1}{2} \sigma_y^2 \rho_y^\top \rho_y \right) ds - \int_t^u \sigma_y \rho_y^\top \lambda_1 X_s ds + \int_t^u \sigma_y \rho_y^\top dZ_s^Q \right]$$

The explicit expression of $Y_u \exp \left(- \int_t^u r_s ds \right)$ is log normally distributed

$$Y_u \exp \left(- \int_t^u r_s ds \right) = Y_t \exp \left\{ \int_t^u \left(g + \frac{\sigma_y^2}{2} - \sigma_y \rho_y^\top \lambda_0 - \frac{1}{2} \sigma_y^2 \rho_y^\top \rho_y - \delta_0 \right) ds - (\sigma_y \rho_y^\top \lambda_1 + \delta^\top) \left[N f(d_i, u-t)_i N^{-1} X_t - N \left(\frac{1}{d_i} (u-t) - \frac{1}{d_i} f(d_i, u-t)_i \right) N^{-1} \sigma_X^\top \lambda_0 + \int_t^u f(d_i, u-s)_i \sigma_X^\top dZ_s^Q \right] + \int_t^u \sigma_y \rho_y^\top dZ_s^Q \right\} \quad (66)$$

Taking the expectation of exponential of normal random variable

$$\begin{aligned} & \mathbb{E}_t^Q \left[Y_u \exp \left(- \int_t^u r_s ds \right) \right] \\ &= Y_t \exp \left\{ \mu_{H_1} (u-t) + \mu_{H_2} N \left(\frac{1}{d_i} (u-t) - \frac{1}{d_i} f(d_i, u-t)_i \right) N^{-1} \sigma_X^\top \lambda_0 \right. \\ & \quad \left. + \frac{1}{2} \int_t^u \Sigma_H(s) \Sigma_H(s)^\top ds + \mu_{H_2} N f(d_i, u-t)_i N^{-1} X_t \right\} \\ &= Y_t \exp (M_1 (u-t) + M_2 (u-t) X_t) \end{aligned}$$

where

$$\begin{aligned} \mu_{H_1} &= g + \frac{\sigma_y^2}{2} - \sigma_y \rho_y^\top \lambda_0 - \frac{1}{2} \sigma_y^2 \rho_y^\top \rho_y - \delta_0 \\ \mu_{H_2} &= \sigma_y \rho_y^\top \lambda_1 + \delta^\top \\ \Sigma_H(s) &= f(d_i, u-s) \sigma_X^\top + \sigma_y \rho_y^\top \end{aligned}$$

The exponential term consists of four parts and are separated into two groups. The first three parts together named M_1 is a scalar. The remaining part is state variable dependent. Integrating the expectation above over u arrives Proposition 3.3.

C.4 Proof of Proposition 3.4

The complete version of the Hamilton-Jacobi-Bellman (HJB) equation (17) of the dynamic optimization problem (13) is given by

$$\begin{aligned}
\beta J = \max_{C_t, x_t} \{ & \frac{1}{1-\gamma} \left(\frac{C}{\Pi} \right)^{1-\gamma} + J_t + J_W [(x^\top (\mu - r) + r) W - C + Y] \\
& + \frac{1}{2} J_{WW} W^2 x^\top \sigma^\top \sigma x + J_Y Y \left(g + \frac{\sigma_y^2}{2} \right) + \frac{1}{2} J_{YY} Y^2 \sigma_y^2 \\
& + J_\Pi \Pi \pi_t + \frac{1}{2} J_{\Pi\Pi} \Pi^2 \sigma_\pi^\top \sigma_\pi - J_X^\top \kappa X + \frac{1}{2} \text{tr} (J_{XX} \sigma_X^\top \sigma_X) \\
& J_{WY} W Y x^\top \sigma^\top \rho \sigma_y + J_{W\Pi} W \Pi x^\top \sigma^\top \sigma_\Pi + W x^\top \sigma^\top \sigma_X J_{WX} \\
& \left. + Y \Pi \sigma_y \rho_y^\top \sigma_\Pi J_{Y\Pi} + Y \sigma_y \rho_y^\top \sigma_X J_{YX} + \Pi \sigma_\Pi^\top \sigma_X J_{\Pi X} \right\} \quad (67)
\end{aligned}$$

Hence the right side of (67) except the first term represents the term $\mathcal{D}(C_t, x_t) J(W, Y, \Pi, X, t)$ of (17). The HJB equation of **Model 1** is a special case of (67), under the condition that $Y = 0$. Solution technique is the same as previous two models, expected the form of indirect utility function. Under the assumption of **Model 3**, the indirect utility function denoting J^{M3} must be a function of labor income.

$$J(W, Y, \Pi, X, t) = J^{M3}(W + H(Y, X, t), \Pi, X, t)$$

where H is the value of human capital for an investor at age t (See Proposition 3.3 for details) under the condition of $X = \bar{X} = 0$.¹² Then I borrow the idea of Bodie et al. (1992) by assuming a trial function of (31). The rest of the work repeats the solution procedure of **Model 1**.

The coefficients of an exponential affine function b^{M3} (see (31)) has to satisfy following ODE

¹²The assumption that $\frac{\partial H}{\partial X} = 0$ is crucial for the closed form solution. It is stated in Munk and Sørensen (2010) that, human capital is relatively insensitive to the latent factors, hence it is valid to assume that $H(Y, X, t) = H(Y, \bar{X}, t)$. Without this assumption, the state variable X appears in both function H and b^{M3} . If b^{M3} is still assumed exponential, the HJB equation of **Model 3** will not be homothetic hence cannot be solved analytically.

system

$$\begin{aligned}
\Gamma_1^{M3}(\tau)' &= \left(\Gamma_1^{M3}(\tau) + \Gamma_1^{M3}(\tau)^\top \right) \left(-\kappa + (1-\gamma) \frac{W}{W+H} \sigma^\top \sigma_X a_1^{M3} \right) \\
&+ \frac{\Gamma_1^{M3}(\tau) + \Gamma_1^{M3}(\tau)^\top}{2} \sigma_X^\top \sigma_X \frac{\Gamma_1^{M3}(\tau) + \Gamma_1^{M3}(\tau)^\top}{2} + 2(1-\gamma) \frac{W}{W+H} a_1^{M3\top} \sigma^\top \lambda_1 \\
&- \gamma(1-\gamma) \frac{W^2}{(W+H)^2} a_1^{M3\top} \sigma^\top \sigma a_1^{M3} - b_1^{M3} \Gamma_1^{M3}(\tau) \tag{68}
\end{aligned}$$

$$\begin{aligned}
\Gamma_2^{M3}(\tau)' &= \Gamma_2^{M3}(\tau) \left[-b_1^{M3} - \kappa + (1-\gamma) \frac{W}{W+H} \sigma^\top \sigma_X a_1^{M3} + \frac{\Gamma_1^{M3}(\tau) + \Gamma_1^{M3}(\tau)^\top}{2} \sigma_X^\top \sigma_X \right] \\
&+ \left[(1-\gamma) \frac{W}{W+H} a_0^{M3\top} \sigma^\top \sigma_X + (1-\gamma) \frac{H}{W+H} \sigma_y \rho_y^\top \sigma_X \right. \\
&- \left. (1-\gamma) \sigma_\Pi^\top \sigma_X \right] \frac{\Gamma_1^{M3}(\tau) + \Gamma_1^{M3}(\tau)^\top}{2} - (1-\gamma)^2 \frac{W}{W+H} \sigma_\Pi^\top \sigma a_1^{M3} \\
&- \gamma(1-\gamma) \frac{W^2}{(W+H)^2} a_0^{M3\top} \sigma^\top \sigma a_1^{M3} - \gamma(1-\gamma) \frac{HW}{(W+H)^2} a_1^{M3\top} \sigma^\top \rho_y \sigma_y \\
&+ (1-\gamma) \frac{W}{W+H} \left[a_0^{M3\top} \sigma^\top \lambda_1 + \lambda_0^\top \sigma a_1^{M3} + \frac{W}{W+H} \delta^\top - \xi^\top \right] \tag{69}
\end{aligned}$$

where a_0^{M3} and a_1^{M3} represents the coefficients of affine optimal portfolio x^{M3} , and can easily obtained from (32).

C.5 Proof of Proposition 3.5

The solution procedure of **Model 4** follows the same steps as solving for **Model 2**. Hence, I show directly the explicit expression of the deterministic coefficients appeared in the indirect utility function (33).

$$\Gamma_1^{M4}(\tau)' = \Gamma_1^{M3}(\tau)'(a_0^{M4}, a_1^{M4}) - \frac{\theta(1-\gamma)W^2}{(W+H)^2} a_1^{M4\top} \sigma^\top \mathbf{e} \mathbf{e}^\top \sigma a_1^{M4} \tag{70}$$

$$\begin{aligned}
\Gamma_2^{M4}(\tau)' &= \Gamma_2^{M3}(\tau)'(a_0^{M4}, a_1^{M4}) - \frac{\theta(1-\gamma)W^2}{(W+H)^2} a_0^{M4\top} \sigma^\top \mathbf{e} \mathbf{e}^\top \sigma a_1^{M4} \\
&- \frac{\theta(1-\gamma)W}{(W+H)^2} \left[\mathbf{e}^\top \rho_y \sigma_y H - \mathbf{e}^\top \sigma_\Pi (W+H) \right] \mathbf{e}^\top \sigma a_1^{M4} \tag{71}
\end{aligned}$$

The expressions of each coefficient is comparable to the corresponding parameters in **Model 3**, but under a different affine function optimal portfolio. I ignore the scalar term Γ_3^{M4} , as it is not required in the optimal function function x^{M4} .

C.6 Proof of Proposition 5.1

Define $\psi = \begin{pmatrix} \pi_t \\ g + \frac{\sigma_y^2}{2} \end{pmatrix}$. I write $\hat{\psi}$ to denote an estimate. Hence the estimation error is defined as

$$\hat{\psi} - \psi = \begin{pmatrix} \sigma_{\Pi}^{\top} \mathbf{e} \gamma_1 \\ \sigma_y \rho_y^{\top} \mathbf{e} \gamma_1 + \sigma_y \sqrt{1 - \|\rho_y\|^2} \gamma_2 \end{pmatrix} = \Sigma \Gamma' \quad (72)$$

where $\Gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \sigma_{\Pi}^{\top} \mathbf{e} & 0 \\ \sigma_y \rho_y^{\top} \mathbf{e} & \sigma_y \sqrt{1 - \|\rho_y\|^2} \end{pmatrix}$. Assume the perturbation vector is jointly normal with mean zero and volatility $\Sigma \Sigma'$

$$\sqrt{T} (\hat{\psi} - \psi) \sim N(0, \Sigma \Sigma') \quad (73)$$

In other words, standardized error term square is chi-square distributed

$$T (\hat{\psi} - \psi)' (\Sigma \Sigma')^{-1} (\hat{\psi} - \psi) \sim \chi^2(2) \quad (74)$$

Suppose the critical value at α significance level as CV_{α} , then I can derive that

$$(\hat{\psi} - \psi)' (\Sigma \Sigma')^{-1} (\hat{\psi} - \psi) \leq \frac{CV_{\alpha}}{T} = \kappa^2$$

The simplification of the constraint reaches Proposition 5.1.

References

- Anderson, E., Hansen, L., and Sargent, T. (1999). Robustness, detection and the price of risk. *University of Chicago*. 3
- Anderson, E., Hansen, L., and Sargent, T. (2003). A quartet of semigroups for model specification, robustness, prices of risk, and model detection. *Journal of the European Economic Association*, 1(1):68–123. 2, 7, 8
- Ang, A. and Piazzesi, M. (2003). A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary economics*, 50(4):745–787. 15
- Benzoni, L., Collin-Dufresne, P., and Goldstein, R. S. (2007). Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. *The Journal of Finance*, 62(5):2123–2167. 4
- Bodie, Z., Merton, R. C., and Samuelson, W. F. (1992). Labor supply flexibility and portfolio choice in a life cycle model. *Journal of economic dynamics and control*, 16(3):427–449. 2, 4, 11, 38
- Bovenberg, L., Koijen, R., Nijman, T., and Teulings, C. (2007). Saving and investing over the life cycle and the role of collective pension funds. *De Economist*, 155(4):347–415. 13
- Branger, N., Larsen, L. S., and Munk, C. (2013). Robust portfolio choice with ambiguity and learning about return predictability. *Journal of Banking & Finance*, 37(5):1397–1411. 4, 21
- Brennan, M. J. and Xia, Y. (2002). Dynamic asset allocation under inflation. *The journal of finance*, 57(3):1201–1238. 2, 4, 9, 23
- Cocco, J. F., Gomes, F. J., and Maenhout, P. J. (2005). Consumption and portfolio choice over the life cycle. *Review of financial Studies*, 18(2):491–533. 2, 4
- Cox, J. and Huang, C. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of economic theory*, 49(1):33–83. 9
- Cox, J. C., Ingersoll Jr, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica: Journal of the Econometric Society*, pages 385–407. 28
- Dai, Q. and Singleton, K. J. (2000). Specification analysis of affine term structure models. *The Journal of Finance*, 55(5):1943–1978. 5
- Duffee, G. R. (2002). Term premia and interest rate forecasts in affine models. *The Journal of Finance*, 57(1):405–443. 15, 29

- Duffie, D. and Kan, R. (1996). A yield-factor model of interest rates. *Mathematical finance*, 6(4):379–406. 5
- Gürkaynak, R. S., Sack, B., and Wright, J. H. (2007). The us treasury yield curve: 1961 to the present. *Journal of Monetary Economics*, 54(8):2291–2304. 14
- Hamilton, J. D. and Wu, J. C. (2012). Identification and estimation of gaussian affine term structure models. *Journal of Econometrics*, 168(2):315–331. 2, 15, 29, 32
- Hansen, L. and Sargent, T. (2007). *Robustness*. Princeton University Press Princeton, NJ. 3
- Hansen, L. P. and Singleton, K. J. (1982). Generalized instrumental variables estimation of nonlinear rational expectations models. *Econometrica: Journal of the Econometric Society*, pages 1269–1286. 15
- Koijen, R. S., Nijman, T. E., and Werker, B. J. (2010). When can life cycle investors benefit from time-varying bond risk premia? *Review of Financial Studies*, 23(2):741–780. 2, 4, 5, 7, 14, 16, 19, 23
- Maenhout, P. (2004). Robust portfolio rules and asset pricing. *Review of Financial Studies*, 17(4):951. 2, 4, 9, 11, 21, 35
- Maenhout, P. J. (2006). Robust portfolio rules and detection-error probabilities for a mean-reverting risk premium. *Journal of Economic Theory*, 128(1):136–163. 4
- Merton, R. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, 51(3):247–257. 2, 4, 11
- Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory*, 3(4):373–413. 4
- Munk, C. and Sørensen, C. (2010). Dynamic asset allocation with stochastic income and interest rates. *Journal of Financial Economics*, 96(3):433–462. 2, 4, 11, 25, 38
- Neudecker, H. and Magnus, J. R. (1988). Matrix differential calculus with applications in statistics and econometrics. 32
- Rothenberg, T. J. (1973). *Efficient estimation with a priori information*, volume 23. Yale University Press New Haven. 32
- Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *The review of economics and statistics*, pages 239–246. 11
- Sangvinatsos, A. and Wachter, J. (2005). Does the failure of the expectations hypothesis matter for long-term investors? *The Journal of Finance*, 60(1):179–230. 5, 9, 10, 14, 15, 16, 28, 34

- Sørensen, C. (1999). Dynamic asset allocation and fixed income management. *Journal of financial and quantitative analysis*, 34(04):513–531. [4](#)
- Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of financial economics*, 5(2):177–188. [11](#)
- Viceira, L. M. (2001). Optimal portfolio choice for long-horizon investors with nontradable labor income. *The Journal of Finance*, 56(2):433–470. [3](#), [4](#), [6](#)
- Wachter, J. (2010). Asset allocation. Technical report, National Bureau of Economic Research. [2](#), [4](#), [23](#), [33](#), [34](#)
- Wachter, J. A. (2002). Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets. *Journal of financial and quantitative analysis*, 37(01):63–91. [9](#)
- Wachter, J. A. (2003). Risk aversion and allocation to long-term bonds. *Journal of Economic Theory*, 112(2):325–333. [4](#), [23](#)

Figure 6: **Risk Shifting** $x^{M2} - x^{M1}$. The figure shows the change of asset allocation (in basis point) among different asset classes from an ambiguity neutral investor to an ambiguity aversion investor under different risk aversion and different state variables scenarios. The difference between the two optimal strategies (x^{M1} and x^{M2}) is named risk shifting. The figure shows the risk shifting effect as a function of age. The state variable X_1 is assumed equal to zero. The other state variable X_2 is zero at the first column and is set equal to 1 and -1 in the other two column. The relation between state variables and interest rate or inflation rate is described in Figure 7.

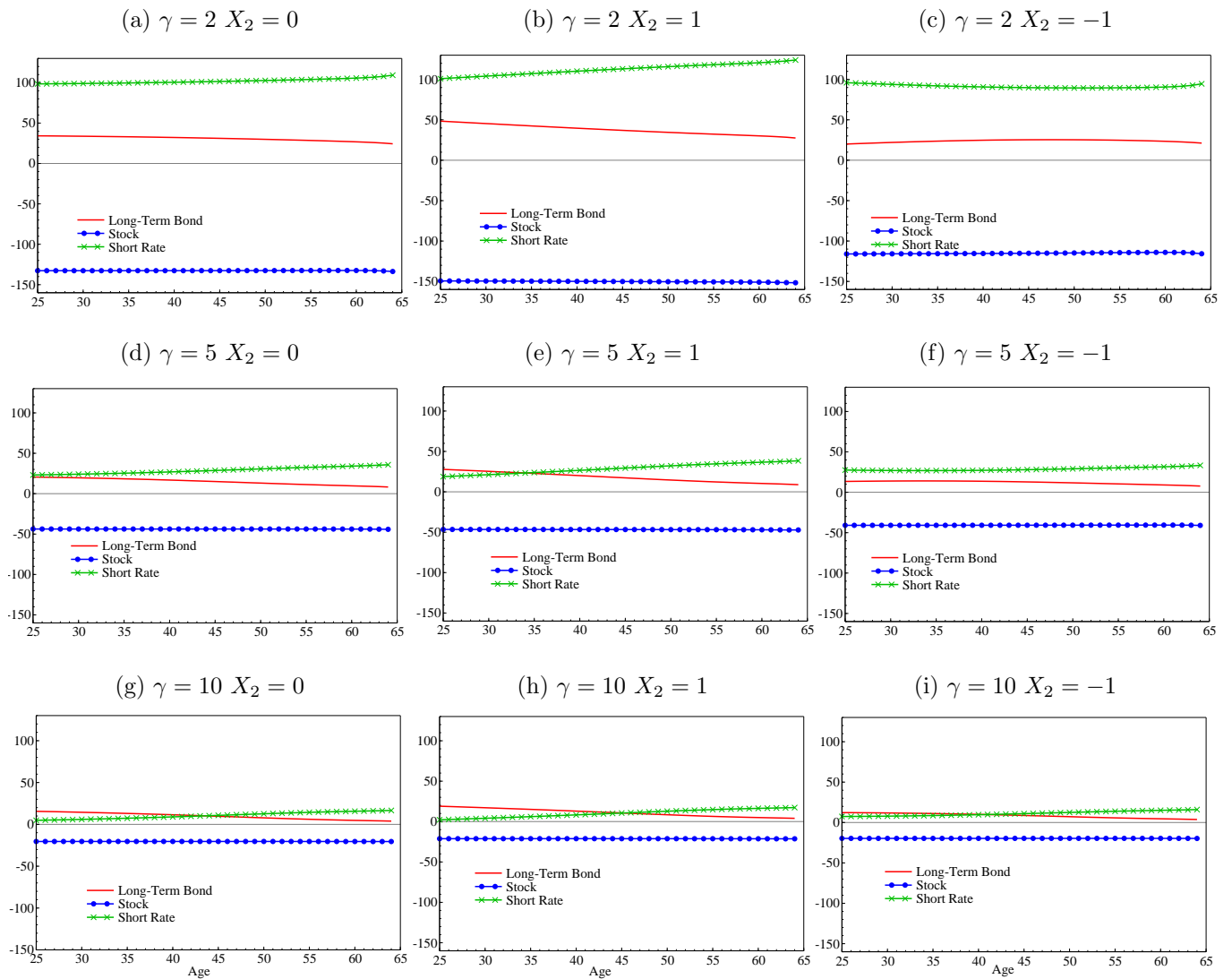


Figure 7: **Human capital, financial wealth and their ratio.** Panel (a) plots the value human capital, financial wealth and total wealth as a function of age. Human capital is calculated on the basis of (29) with $X_1 = X_2 = 0$ and the initial nominal income $Y_0 = 1$. Income volatility is set equal to $\sigma_y = 0.08$ and the expected return is assumed equal to $g + \frac{\sigma_y^2}{2} = 0.1$. Initial wealth is set to $W_0 = 10$ and is increasing with accumulative income over life cycle. Panel (b) shows the human capital over financial wealth under different age cohorts.

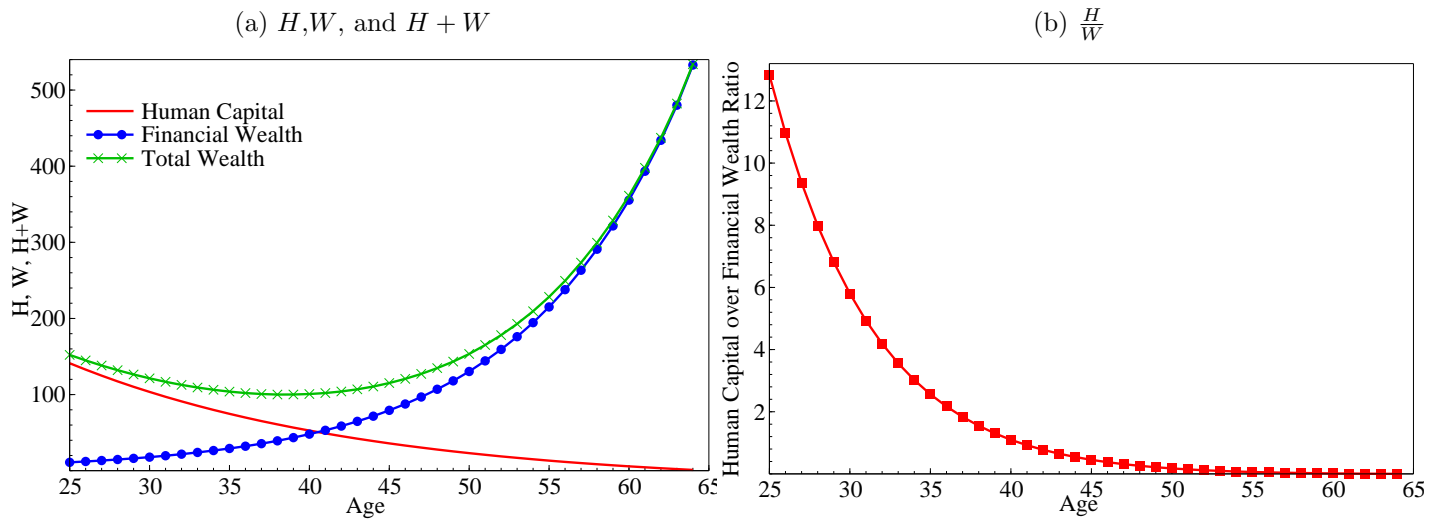


Figure 8: **Robust Optimal Portfolio with Human Capital** The figure presents the optimal asset allocation on different classes for investors with preferences of **Model** when both labor and parameter uncertainty are concerned. State variable $X_1 = 0$ and state variable $X_2 = -1$ for the first column and is 1 for the second column. The penalty parameter is assumed equal to $\theta = 1$.

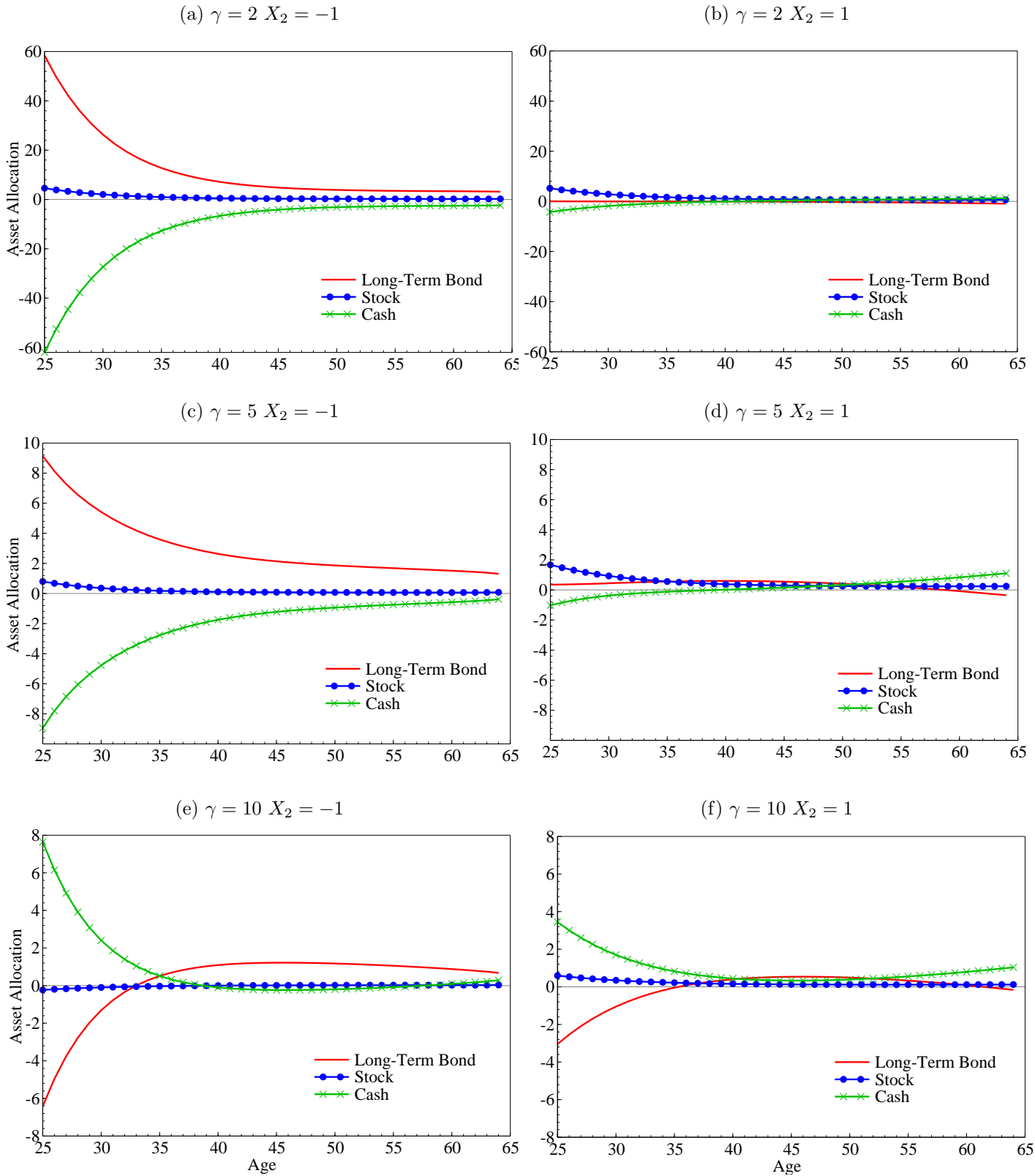


Figure 9: **Risk Shifting** $x^{M4} - x^{M3}$. The figure presents the risk shifting (in percentage) effect for an investor with preferences moving from **Model 3** to **Model 4**. The state variable X_1 is assumed equal to zero. The other state variable X_2 is zero at the first column and is set equal to 1 and -1 in the other two column.

