Housing Habits and Their Implications for Life-Cycle Consumption and Investment

Holger Kraft[a] Claus Munk[b] Sebastian Wagner[c]

December 3, 2015

Abstract: We solve a rich life-cycle model of household decisions involving consumption of perishable goods and housing services, habit formation for housing consumption, stochastic labor income, stochastic house prices, home renting and owning, stock investments, and portfolio constraints. In line with empirical observations, the optimal decisions involve (i) stock investments that are low or zero for many young agents and then gradually increasing over life, (ii) an age- and wealth-dependent housing expenditure share, (iii) non-housing consumption being significantly more sensitive to wealth and income shocks than housing consumption, and (iv) non-housing consumption being hump-shaped over life.

Keywords: Habit formation, life-cycle household decisions, housing expenditure share, consumption hump, stock market participation, human capital

JEL subject codes: G10, D14, D91, E21, R21

[a] Department of Finance, Goethe University Frankfurt am Main, Faculty of Economics and Business Administration, Germany. E-mail: `holgerkraft@finance.uni-frankfurt.de`

[b] Department of Finance, Copenhagen Business School, Denmark. E-mail: `cm.fi@cbs.dk`

[c] Department of Finance, Goethe University Frankfurt am Main, Faculty of Economics and Business Administration, Germany. E-mail: `Sebastian.Wagner@hof.uni-frankfurt.de`

We appreciate comments from Martin Brown, Thomas Dangl, Roland Füss, Peter N. Smith, Peter Spencer, Alex Weissensteiner, and participants at presentations at the Gutmann Center Symposium 2015 in Vienna, Lund University, the Swiss Institute of Banking and Finance in St. Gallen, the Technical University of Kaiserslautern, the University of Bonn, the University of Duisburg-Essen, the University of York, and the Copenhagen Business School. We are grateful to Fang Yang for sharing her data with us. Holger Kraft gratefully acknowledges financial and research support from the Center of Excellence SAFE, funded by the State of Hessen research initiative LOEWE, and both Holger Kraft and Sebastian Wagner gratefully acknowledge financial support by Deutsche Forschungsgemeinschaft (DFG).

Housing Habits and Their Implications for Life-Cycle Consumption and Investment

Abstract: We solve a rich life-cycle model of household decisions involving consumption of perishable goods and housing services, habit formation for housing consumption, stochastic labor income, stochastic house prices, home renting and owning, stock investments, and portfolio constraints. In line with empirical observations, the optimal decisions involve (i) stock investments that are low or zero for many young agents and then gradually increasing over life, (ii) an age- and wealth-dependent housing expenditure share, (iii) non-housing consumption being significantly more sensitive to wealth and income shocks than housing consumption, and (iv) non-housing consumption being hump-shaped over life.

Keywords: Habit formation, life-cycle household decisions, housing expenditure share, consumption hump, stock market participation, human capital

JEL subject codes: G10, D14, D91, E21, R21

1 Introduction

Residential real estate is important for households both as a consumption good, an investment asset, and as collateral facilitating borrowing.[1] Therefore housing consumption and investment should be included in life-cycle models of household decisions but, due to modeling complexity, only few existing papers do so. We solve for the optimal life-cycle decisions in a rich model featuring consumption of perishable goods and housing services, stochastic labor income, stochastic house prices, renting/owning decisions, stock investments, and portfolio constraints. An innovative model feature is habit formation in the preferences for housing consumption. Calibrated to U.S. data, our model is the first to jointly explain why households hold no or few stocks early in life, choose a hump-shaped non-housing consumption pattern and an increasing housing expenditure share over life, and adjust non-housing consumption much more than housing consumption in response to wealth and income shocks. Our results highlight the importance of collateralized borrowing for the optimal investment and consumption decisions over the life cycle.

Habit formation finds empirical support (Browning and Collado 2007; Ravina 2007), and the consequences of perishable consumption habits for consumption and investment decisions are well-studied (Ingersoll 1992; Munk 2008). Unexplored until now, housing habits are highly plausible because homes serve as very visible status symbols and because individuals build up affection for their home and neighborhood as well as social ties within the community.[2] Moreover, explicit and implicit costs of home transactions are likely to induce consumption and trading patterns similar to those caused by habit persistence. We assume an internal, additive, multi-period habit meaning that the agent's utility depends on the difference between her current housing consumption and a weighted average of her own past housing consumption.[3] In the following we explain how the housing habit affects

[1]Housing services (shelter) weigh 31.9% in the November 2013 U.S. Consumer Price Index for All Urban Consumers, cf. `http://www.bls.gov/cpi/`. In the 2010 U.S. Survey of Consumer Finances, residential property constituted 36% of total household wealth, see Tables 8 and 9.1 in Bricker, Kennickell, Moore, and Sabelhaus (2012); similar numbers are reported by Campbell (2006) and Guiso and Sodini (2013). The home ownership rate was 65-70% in the U.S. in the period 1996-2013 and also well above 50% in most other developed countries, see the Census Bureau at `http://www.census.gov/housing/hvs/` for the U.S. and Andrews and Sánchez (2011) for selected OECD countries. Moreover, 27% of U.S. home sales in 2011 were purely for investment purposes, cf. Choi, Hong, Kubik, and Thompson (2013).

[2]Solnick and Hemenway (2005) report empirical evidence supporting status concerns regarding housing.

[3]Aydilek (2013) considers a one-period multiplicative housing habit in a two-good, discrete-time, life-cycle model that disregards stock investments and focuses on housing late in life. The habit has stronger implications in our setting because of both the multi-period and the additive specification of the habit, and we also consider the interaction between housing decisions and stock investments.

the agent's optimal life-cycle decisions.

Stock investments. In our baseline parametrization the optimal stock investment is zero in the early years and then gradually increasing until retirement. Typical models ignoring housing (e.g., Cocco, Gomes, and Maenhout 2005) conclude that a borrowing-constrained agent early in life invests 100% of financial wealth in stocks and then gradually less towards retirement, unless the agent is extremely risk-averse or has a labor income highly correlated (or even co-integrated, cf. Benzoni, Collin-Dufresne, and Goldstein 2007) with the stock. This is at odds with the low stock market participation and the life-cycle variations in the stock weight observed empirically (Guiso and Sodini 2013).[4] Adding housing to the model, Cocco (2005) concludes that house price risk crowds out stock holdings which, together with a sizable stock market entry cost, can explain the limited stock market participation. His model ignores the renting/owning decision and assumes a perfect correlation between house prices and aggregate income shocks. We obtain early-life non-participation in a more general model even for low risk aversion and without imposing entry costs. The housing habit induces a minimum future housing consumption which the investor finances by maintaining a wealth buffer invested in the housing market. Hence, the habit reduces the wealth disposable for speculative investments and further postpones stock market entry.[5] The stock's share in the financial portfolio is hump-shaped over life, starting at zero, but eventually increasing and staying in the range 50-80%, and always significantly lower than in the corresponding no-habit case.

Also consistent with data, but unlike standard models, the stock weight goes up when increasing wealth since the habit-induced wealth buffer in our model then seizes a smaller share of wealth. Wachter and Yogo (2010) show that an "addilog" utility of basic consumption and luxury consumption can explain why the stock portfolio weight increases in wealth, but their model does not explain a zero or low stock weight of young investors.

Adding housing as a second investment asset reduces the speculative stock demand because of diversification and because housing investments provide access to collateralized borrowing, which stocks generally do not. We show that when the maximum house loan-to-value ratio is reduced, the optimal stock [house] investment increases [decreases].

[4] Cocco, Gomes, and Maenhout (2005) and Davis, Kubler, and Willen (2006) show that young investors with access to unsecured borrowing at a high interest rate may optimally refrain from investing in stocks. In contrast, our model allows for fully collateralized borrowing at the risk-free rate.

[5] Habits for perishable consumption reduce stock investments somewhat, but cannot alone justify zero stock investments, cf. Gomes and Michaelides (2003), Polkovnichenko (2007), and Munk (2008).

Furthermore, we find that optimal stock investments are increasing in risk aversion (in some range) as stocks are a less risky alternative to leveraged house investments.

Perishable consumption. Empirical studies find that perishable consumption over life is hump shaped (e.g., Thurow 1969), but frictionless consumption-savings models with one good lead to optimal consumption being either increasing, decreasing, or flat. The predominant explanation is that borrowing constraints or non-hedgeable risks cause the hump (Nagatani 1972, Carroll 1997, Gourinchas and Parker 2002). These models prohibit the agent from any borrowing, although homeowners in reality have access to typically cheap borrowing at a large scale. Relaxing the borrowing constraint postpones the consumption hump, but we show that housing habits can restore an early consumption hump. More precisely, the housing habit induces a hump in perishable consumption without constraints or risks, and it moves the hump to an earlier age when reasonable constraints and risks are included. Housing consumption early in life is expensive since the individual then commits to a certain level of housing consumption in the remaining lifetime, which necessitates the above-mentioned wealth buffer. As the individual's horizon shrinks, the buffer decreases, and housing consumption effectively becomes less expensive. Early in life the individual thus tilts the consumption bundle towards perishable consumption, and less so as time passes. By embedding this mechanism in a setting in which overall consumption tends to increase over life, the hump-shaped perishable consumption emerges.

Housing expenditure share. The Cobb-Douglas utility assumed in standard two-good models implies a constant relation between expenditures on housing and expenditures on perishable goods, i.e., a constant housing expenditure share. This contradicts empirical observations. First, the housing expenditure share varies counter cyclically. For example, Figure 1 shows the 1980-2013 annual U.S. GDP growth rates and percentage changes in the housing expenditure share of U.S. households reported in the National Income and Product Accounts (NIPA).[6] The correlation over the period shown is -0.51 (even -0.59 in 1990-2013), and the housing expenditure ratio has varied between 21.0% (in 2012) and 23.1% (1986) with an average of 22.0%. The Consumer Expenditure Survey (CEX) confirms this variation: in 1993-2013 the housing expenditure share for all age groups averaged 32.9%

[6]See NIPA Table 2.3.5 published by the Bureau of Economic Analysis, the U.S. Department of Commerce, at http://www.bea.gov/itable/. Following Piazzesi, Schneider, and Tuzel (2007), the housing expenditure share is the ratio of housing consumption expenditures to the sum of housing and non-housing consumption expenditures. Housing consumption expenditures are represented by the item 'housing and utilities' and non-housing consumption is the sum of 'nondurable goods' (less 'clothing and footwear') and 'services' (less 'housing and utilities'). While NIPA includes a quantity index for housing consumption, Piazzesi et al. (2007) argue that this measure is imprecise so, following their lead, we do not use it.

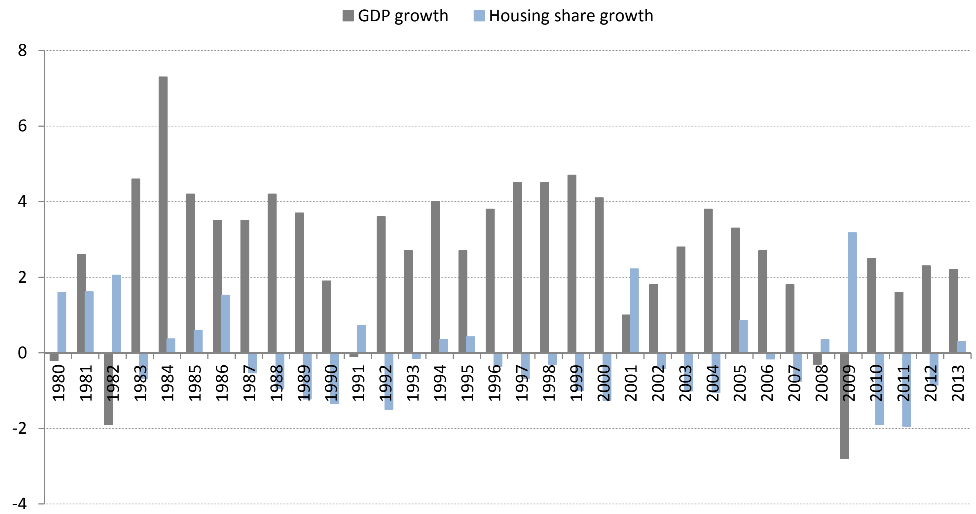

Figure 1: Growth rates of GDP and the housing expenditure share. The data is taken from the U.S. National Income and Product Accounts published by the Bureau of Economic Analysis. GDP growth rates are in Table 1.1.1 (dated January 30, 2015), and the housing expenditure share is in Table 2.3.5 (dated October 30, 2014).

and varied between 31.4% (1993) and 34.4% (2009, 2010).[7] Across households, the housing expenditure share decreases in income. For example, in the 2013 CEX, the share in the five income quintiles (from lowest to highest) is 40.0%, 36.8%, 34.8%, 32.4%, and 31.1%. On a micro level, Chetty and Szeidl (2007) report from Panel Study of Income Dynamics (PSID) data that in the year following a job loss, home owners on average reduce food consumption much more than housing consumption. Our model with habit formation in housing explains these empirical findings: in times of declining [increasing] wealth or income, housing expenditures are reduced [increased] less than non-housing expenditures.

Secondly, the housing expenditure share varies with age. By analyzing CEX consumption data and housing ownership data from the Survey of Consumer Finances (SCF), Yang (2009) finds that non-housing consumption is hump shaped, whereas housing consumption per adult-equivalent increases throughout life, quite steeply early in life and then flattening out. Based on her data, Figure 2 shows the housing expenditure share over the life cycle for renters and owners as well as the average of the two.[8] The housing expenditure

[7]The CEX data was downloaded from the Bureau of Labor Statistics, the U.S. Department of Labor, at `http://www.bls.gov/cex/csxshare.htm`. We use the expenditure share for the item 'housing'. We obtain very similar results by using 'shelter' that subtracts various expenditures from 'housing.'

[8]We are grateful to Fang Yang for sharing the data with us. The data includes the housing stock of owners, and we assume that housing consumption is a constant fraction of the stock. The fraction is set to 8.22% since then 20-year old owners have the same expenditure share as 20-year old renters, but the life-cycle profile of the expenditure share is the same for a wide range of values for this constant. We use the mean housing and non-housing consumption for each age to compute the expenditure share.

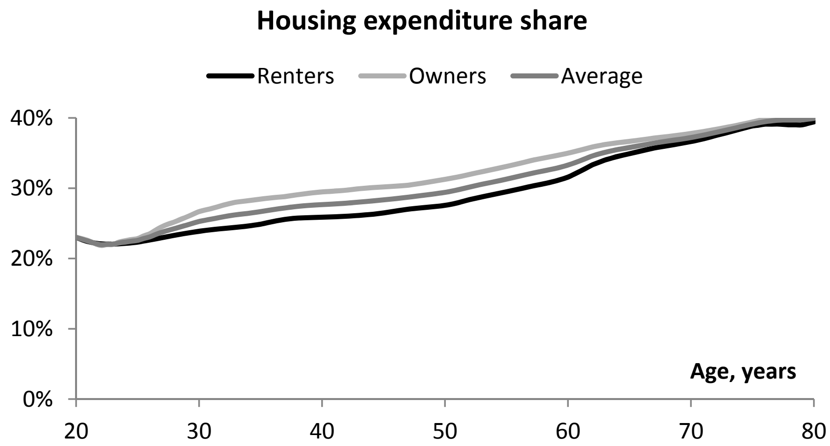

Figure 2: The housing expenditure share over the life cycle. The figure shows housing expenditure shares of renters and owners, as well as the average of the two, over the life cycle. The housing expenditure share is the ratio of housing consumption to the sum of housing and non-housing consumption. The data is based on CEX and SCF, and processed and supplied by Yang (2009).

share increases with age from around 22% to 40%, with the steepest increases at age 25-35 and age 60-70. Furthermore, the correlation between the housing expenditure share and GDP growth is more negative for young households than for old. For the 25-34 year olds the correlation is -0.22, whereas for households above 75 years the correlation is -0.03. Also these features are consistent with habit formation. Current housing consumption decisions are influenced by their impact on the housing habit in the remaining life, which obviously is less important for households with a shorter expected remaining lifetime.

The housing expenditure share in our model depends on the agent's age and the housing consumption relative to the habit level, which again depends on wealth and the house price. With CES (Constant Elasticity of Substitution) utility the share depends only on the house price. CES utility was assumed in the housing-extended Consumption-CAPM of Piazzesi, Schneider, and Tuzel (2007). Their key mechanism is that agents are especially concerned about states with low overall consumption when the share of housing consumption is low relative to perishable consumption. This holds for CES utility if the intratemporal elasticity of substitution between housing and non-housing consumption exceeds one. In our model low housing consumption is particularly bad due to the habit. The addi-log specification used by Wachter and Yogo (2010) for the utility of basic and luxury goods implies an expenditure share depending on good prices and the level of consumption, but not directly on the agent's age as in our model and as observed in the data.

Housing consumption vs. investment. The housing investment consists of a speculative position, a hedge against increases in future housing consumption costs, the above-mentioned habit-induced buffer, and a term adjusting for the extent to which human capital replaces a housing investment. With our baseline parameters, the resulting housing investment position is positive from the beginning, increasing to around retirement, after which it drops as wealth is reduced to finance retirement consumption. Both in the early and the late years of adult life, the optimal housing consumption exceeds the optimal housing investment, which can be implemented by renting the home and having a limited investment in the housing market, maybe through REITs (Real Estate Investment Trusts). In the intermediate life phase, the optimal housing investment exceeds consumption, which can be implemented by owning the home and investing additionally in the housing market. This renting-owning-renting life-cycle pattern is often seen in real life.

Our paper is related to several recent papers on housing decisions. Campbell and Cocco (2003) and van Hemert (2010) allow for stochastic interest rates and focus on adjustable- vs. fixed-rate mortgage decisions of home owners. Adding stochastic interest to our framework would allow us to recover their findings, without changing our results concerning consumption decisions. Corradin, Fillat, and Vergara-Alert (2014) show that predictability in house prices causes house [stock] investments to be increasing [decreasing] in the current expected house price growth. Even with a relatively high risk aversion, they report significant stock investments. By abstracting from the renting-owning decision these papers force housing investment and housing consumption to be identical. Yao and Zhang (2005) find that home owners invest less in stocks than renters, supporting that housing risk crowds out stock risk, but stock investments are large both for owners and renters, in particular early in life. Chetty and Szeidl (2014) show empirically and in a very stylized model that mortgage debt generally reduces stockholding while home equity wealth increases it. In contrast to our paper, these papers do not address the life-cycle consumption patterns. In a life-cycle setting with renting-owning decisions, Fischer and Stamos (2013) find that also home ownership rates and mortgage debt comove with the expected house price growth rate. In their baseline parametrization, zero stock investments are optimal only when expected house price growth is very high. The life-cycle consumption profile they report is steeply increasing until retirement where it flattens out, and they do not address the housing expenditure share. Our model disregards house price predictability, but adding it is unlikely to qualitatively change the consumption patterns, while it would make stock investments relative less attractive compared to housing investments and thus further postpone optimal stock market entry.

For tractability we abstract from transaction costs which most above-mentioned papers explicitly model by assuming that a household can only adjust its wealth exposure to the housing market by selling the existing house (with proportional costs) and purchasing a new house of the desired size. The results of Kraft and Munk (2011) indicate that household welfare is reduced very little if housing consumption and investments are only infrequently adjusted suggesting that, without exogenously triggered forced moves, transaction costs have modest effects on the overall housing position. Moreover, in contrast to our paper, the above-mentioned papers (except for Kraft and Munk 2011) ignore that households can disentangle housing consumption from housing investments by investing in REITs or other housing-related assets, by renting out part of the home they own, or by simultaneously renting and owning (e.g., renting their primary residence and owning a secondary residence or vice versa). Moreover, households can implement some variation in the desired housing investment position without sizeable costs by remodeling or adjusting home maintenance.

As our full model features portfolio constraints and unspanned labor income risk, the optimal consumption and investment decisions are determined by a numerical method. To understand the economic forces at play, we first solve two simpler versions of the model in closed form. The first model is designed to demonstrate the impact of the housing habit on the optimal perishable and housing consumption. The second model focuses on the impact of housing and the housing habit on portfolio decisions, but has to abstract from portfolio constraints and unspanned income risk to facilitate a closed-form solution.

To solve the full model, mainstream numerical methods are computationally infeasible due to the high number of state variables (time, wealth, income, house price, habit level). Instead, we extend and adapt the SAMS (Simulation of Artificial Market Strategies) method introduced by Bick, Kraft, and Munk (2013) to habit formation, two consumption goods, and two risky assets. The method optimizes over a family of consumption and investment strategies parameterized by a low number of constants. Each strategy is a minor transformation of the optimal strategy in a closely related unconstrained, complete market, and we derive this optimal strategy in closed form. The expected utility in the true market of each strategy is evaluated by Monte Carlo simulations and, by embedding that in a standard numerical optimization over the parameters, we determine the best of these strategies. In our baseline parametrization this strategy deviates from the unknown, truly optimal strategy by at most 1.1% in terms of the certainty equivalent of wealth.[9]

The paper is organized as follows. Section 2 introduces the ingredients of our model. Section 3 derives and illustrates the optimal decisions in the two simple models. The full model and the numerical solution technique are presented in Section 4. Results for a baseline parametrization of the model are discussed in Section 5, whereas Section 6 considers alternative parametrizations. Section 7 concludes.

2 Model ingredients

2.1 Consumption goods

Our economy features two consumption goods: perishable (or non-housing) consumption and housing consumption. The housing good is measured in a number of "units" reflecting size, quality, and location of the residential property. For concreteness, think of one unit as one square foot residence of average quality and location. For convenience we refer to this good as houses. The agent consumes perishable consumption at the rate c_t and units of housing consumption at the rate q_t. We take the perishable good to be the numeraire. The time t unit price of the housing good is denoted by H_t, which varies over time as explained in the specific models in the following sections. The agent can rent housing units at a rent proportional to the price of the rented property; renting q units over the time interval $[t, t+dt]$ costs $\chi q H_t \, dt$, and we refer to $\chi \geq 0$ as the rental rate.

2.2 Preferences

The agent develops habits for housing consumption; the habit level $\bar{q}_t$ satisfies

$$\bar{q}_t = \bar{q}_0 e^{-\varepsilon t} + \alpha \int_0^t e^{-\varepsilon(t-s)} q_s \, ds \quad \Rightarrow \quad d\bar{q}_t = (\alpha q_t - \varepsilon \bar{q}_t) \, dt, \tag{1}$$

where the initial habit level $\bar{q}_0$, the persistence parameter ε, and the scaling parameter α are non-negative constants. The agent derives utility from the consumption at any given time according to the habit-extended Cobb-Douglas function

$$U(c, q, \bar{q}) = \frac{1}{1-\gamma} \left[c^b \left(q - \bar{q}\right)^{1-b} \right]^{1-\gamma},$$

[9] This is an upper bound. The losses are typically much lower.

where $\gamma > 1$ is a risk aversion parameter, and $b \in (0, 1)$ is a weighting parameter. Note that $\gamma > 1$ implies that perishable and housing consumption are substitutes in the sense that $U_{cq} < 0$. The habit level $\bar{q}$ represents an endogenously determined subsistence level of housing consumption. If, from time t on, the agent's housing consumption is exactly at the minimum, $q_s = \bar{q}_s$ for $s \geq t$, the future habit level is $\bar{q}_u = \bar{q}_t e^{-(\varepsilon-\alpha)(u-t)}$ housing units. The difference $\varepsilon - \alpha$ indicates the strength of the habit by determining how much the current habit level restricts future decisions. For later use, we define

$$k = \frac{(1-b)(\gamma-1)}{\gamma}, \qquad \hat{b} = b^{-kb/(1-b)}(1-b)^{-k}.$$

The agent lives until time T. The agent's objective at any time $t < T$ is to maximize $\mathrm{E}_t[\int_t^T e^{-\delta(s-t)} U(c_s, q_s, \bar{q}_s)\, ds]$ over the feasible consumption and investment strategies as will be specified in more detail. Here, δ is the agent's subjective time preference rate.

2.3 Labor income

Throughout life the agent receives an income stream from non-financial sources at a rate of Y_t. The dynamics of Y_t are specified in the concrete models considered in the following sections. The individual retires at a predetermined time $\tilde{T} \leq T$. At retirement the income drops to a known fraction Υ of the income immediately before,

$$Y_{\tilde{T}+} = \Upsilon Y_{\tilde{T}-}. \tag{2}$$

This reflects the wide-spread final-salary pension schemes and is a common assumption in the literature (e.g., Cocco, Gomes, and Maenhout 2005; Lynch and Tan 2011).

2.4 Investments, wealth, and potential constraints

In all the models we study, the agent can invest in a risk-free asset offering a continuously compounded rate of return r. We assume a constant r throughout to focus on the impact of housing decisions and habits on consumption and investment, and the effects of stochastic interest rates on life-cycle decisions are already well-studied.[10] In our simplest model the risk-free asset is the only investment object. In the other models the agent can also invest in a single stock (the market index), and in housing units in order to capture the dual role

[10] See, e.g., van Hemert (2010), Koijen, Nijman, and Werker (2010), Munk and Sørensen (2010), and Kraft and Munk (2011).

of housing as both a consumption good and an investment object. The agent's *tangible wealth* at a given date is the sum of the values of her investments in the available assets. In addition, she has human wealth in terms of the present value of her future labor income.

The first two models we consider disregard constraints on investments, and any uncertainty (also about future labor income) is assumed to be spanned by traded assets, so that markets are complete. In particular, the labor income stream can then be valued as a dividend stream and is modeled so that the *human wealth* at any time t is of the form $Y_t F(t)$ for some deterministic function F. The total wealth of the agent is the sum of the tangible wealth X_t and the human wealth $Y_t F(t)$. The agent has to make sure that she can meet the minimum housing consumption defined by the habit level also in the future. As explained earlier, if the agent from time t on keeps housing consumption exactly at the minimum, the future habit level is $\bar{q}_u = \bar{q}_t e^{-(\varepsilon-\alpha)(u-t)}$ housing units with a total cost rate of $\bar{q}_u \chi H_u$. The present value of these costs, which we refer to as the *housing habit buffer*, constitutes the amount the agent must set aside at time t to cover minimum future housing consumption. In the models we consider, the housing habit buffer is of the form $\bar{q}_t \chi H_t B(t)$ for some deterministic function B. The *disposable wealth* of the agent at time t is thus the tangible wealth plus the human wealth less the housing habit buffer:

$$\widehat{X}_t = X_t + Y_t F(t) - \bar{q}_t \chi H_t B(t). \tag{3}$$

Our main model features unspanned risks and investment constraints that are binding in some states, and then the above conclusions are invalid. But, as explained in the Introduction, we solve that decision problem by embedding it in certain unconstrained complete markets where the above considerations hold.

3 Two instructive models with closed-form solutions

Our full-blown model involves unspanned risks and borrowing and short-selling constraints. Consequently, we cannot solve it in closed form. To build intuition for the economic forces at play, this section considers two simpler models with closed-form solutions.

3.1 Model with full certainty

Here we set up and solve a simple model disregarding uncertainty and frictions. The unit house price H_t and labor income rate develop as

$$dH_t = H_t\,(r + \mu_H)\,dt, \qquad dY_t = Y_t \mu_Y(t)\,dt,$$

so that house prices grow at a constant and labor income at a deterministic rate. The agent can only invest in the risk-free asset so her wealth X_t evolves as

$$dX_t = rX_t\,dt + Y_t\,dt - c_t\,dt - q_t H_t \chi\,dt.$$

The value function (aka. the indirect utility function) of the agent is

$$J(t, x, h, y, \bar{q}) = \sup_{c,q} \int_t^T e^{-\delta(s-t)} U(c_s, q_s, \bar{q}_s)\,ds,$$

where $x, h, y, \bar{q}$ denote the time t values of the wealth, the house price, the labor income, and the housing habit. By design, this model focuses on the effects of housing habits on consumption. In the more realistic settings considered subsequently, the value function and the optimal consumption policy have the same structure as in this simple case.

Theorem 1 *The value function is*

$$J(t, x, h, y, \bar{q}) = \frac{1}{1-\gamma} (\chi h)^{\gamma k} G(t)^{\gamma} \left(x + yF(t) - \bar{q}\chi h B(t)\right)^{1-\gamma}, \tag{4}$$

where

$$B(t) = \frac{1}{r_B}\left(1 - e^{-r_B(T-t)}\right), \tag{5}$$

$$G(t) = \hat{b} \int_t^T e^{-r_G(s-t)} \left(1 + \alpha B(s)\right)^k\,ds, \tag{6}$$

$$F(t) = \begin{cases} \int_t^T e^{-\int_t^u r_F(s)\,ds}\,du, & t \in [\tilde{T}, T], \\ \int_t^{\tilde{T}} e^{-\int_t^u r_F(s)\,ds}\,du + \Upsilon \int_{\tilde{T}}^T e^{-\int_t^u r_F(s)\,ds}\,du, & t < \tilde{T}, \end{cases} \tag{7}$$

$$r_B = \varepsilon - \alpha - \mu_H, \qquad r_G = \frac{\delta}{\gamma} + \frac{\gamma - 1}{\gamma} r - (r + \mu_H)k, \qquad r_F(t) = r - \mu_Y(t). \tag{8}$$

With $\widehat{X}_t$ defined in (3), the optimal perishable and housing consumption rates are

$$c_t = b\hat{b}\frac{(1+\alpha B(t))^k}{G(t)}\widehat{X}_t, \qquad q_t = \bar{q}_t + (1-b)\hat{b}\,\frac{(1+\alpha B(t))^{k-1}}{\chi H_t G(t)}\widehat{X}_t. \tag{9}$$

The life-cycle profile of perishable consumption is hump shaped if $r > \delta$, $\mu_H < \varepsilon$, and

$$k\left[\frac{\alpha r_B}{(\alpha + r_B)e^{r_B T} - \alpha} - (r + \mu_H)\right] \leq \frac{r-\delta}{\gamma} \leq k(\alpha - [r + \mu_H]).$$

See Appendix A for a proof. Optimal perishable and housing consumption satisfy

$$c_t = \frac{b}{1-b}(q_t - \bar{q}_t)\chi H_t\left(1 + \alpha B(t)\right), \tag{10}$$

and the housing expenditure share is

$$\frac{\chi q_t H_t}{c_t + \chi q_t H_t} = \frac{1-b}{1 - b + b\left(1 - \frac{\bar{q}_t}{q_t}\right)(1 + \alpha B(t))}. \tag{11}$$

Without habit formation ($\bar{q}_t = 0, \alpha = 0$), perishable consumption is a constant multiple of housing consumption expenditures, and the housing expenditure share is the constant $1-b$. Furthermore, the growth rate in perishable consumption is the constant $[r - \delta + (1-b)(\gamma-1)(r+\mu_H)]/\gamma$. With $r > \delta$, $\gamma > 1$, and a non-negative house price growth rate $r + \mu_H$, perishable consumption is increasing throughout life.

The housing habit implies a time-dependent perishable-housing consumption ratio. Eq. (10) shows that $\chi H_t\left(1 + \alpha B(t)\right)$ is the effective time t unit price of housing consumption. Since B is decreasing over time, so is the effective price of housing, other things equal. Housing consumption is relatively more expensive for young agents: through the housing habit, the consumption of one extra unit of housing costs not only the current rent but also future required rents due to the increase in the habit level. The present value of the habit-induced required future rents is larger for young than for old agents. Since perishable goods and housing services are substitutes, the optimal composition of consumption is tilted towards perishable goods when young and towards housing when old. Embedding this mechanism in a setting where overall consumption is optimally increasing over life can lead to a hump-shaped life-cycle pattern in perishable consumption as seen in the data. With housing habits, the housing expenditure share is time dependent, and it decreases with q_t and thus with disposable wealth $\widehat{X}_t$. The housing expenditure share

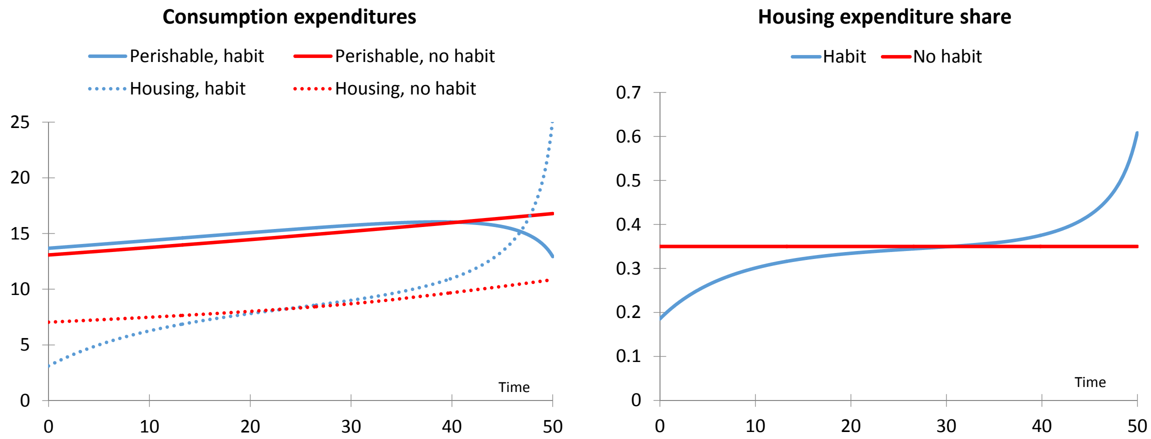

Figure 3: Consumption in the deterministic model. The left panel shows expenditures on perishable goods and housing and the right panel shows the housing expenditure share, with and without a housing habit. The graphs are based on the model of Section 3.1. The parameter values are: $\delta = 0.02$, $\gamma = 2$, $t = 0$, $\tilde{T} = 35$, $T = 50$, $X_0 = 20$, $Y_0 = 20$, $\mu_Y = 0.01$ (zero in retirement), $\Upsilon = 0.6$, $r = 0.03$, $\chi = 0.06$, $H_0 = 0.25$, $\mu_H = -0.03$. Without habit formation $b = 0.65$. With habit formation $\bar{q}_0 = 150$, $\alpha = 0.8$, $\varepsilon = 0.9$, and $b = 0.69$ (to obtain the same average housing expenditure share as in the no-habit case).

is therefore higher in hard times than in good times, as seen in the data.

Figure 3 shows optimal consumption over a 50-year period, representing the situation of a 30-year old, retiring at 65, and living on until age 80. Initial wealth and (after-tax) income are set to 20—representing \$20,000—in line with the median net worth and before-tax income statistics for young individuals according to the 2013 SCF.[11] The left panel shows expenditures (in \$1,000, per year) with a clear hump in perishable consumption with, but not without, housing habits. The assumed habit is relatively weak in the sense that $\varepsilon - \alpha = 0.1$ so as long as the agent consumes housing at the minimum, this minimum decreases by 10% per year. Optimal housing expenditures are increasing early in life, then quite flat, and then increasing again in the final years. The housing expenditure share shown in the right panel exhibits the shape seen in the data, cf. Figure 2.

Mortality risk is known to affect the life-cycle profile of consumption and can generate a hump. Appendix B extends the above model to mortality risk and bequest. The bequest utility is $w^{\gamma} \frac{1}{1-\gamma} X_{\tau}^{1-\gamma}$, where τ is the time of death, X_{τ} is the bequeathed wealth, and w is the preference weight of bequest. We use a deterministic mortality intensity $\zeta(t)$ derived

[11]See `http://www.federalreserve.gov/econresdata/scf/scfindex.htm`. Summary results are presented in "Changes in U.S. Family Finances from 2010 to 2013: Evidence from the Survey of Consumer Finances" published in the Federal Reserve Bulletin (Sep. 2014) and available on the above homepage. According to Table 1 of this document, the *before*-tax median *family* income was \$35.300 for age (of family head) less than 35 and \$60.900 for age 35-44. From Table 2, the median net worth per family was \$10.400 for age less than 35 and \$46.700 for age 35-44. Our numbers reflect after-tax income and wealth per adult.

from the life tables for the total U.S. population as of 2009 with an imposed maximum age of 100.[12] Following Richard (1975), Blanchard (1985), and others, we assume the agent has access to a life insurance contract that pays a continuous flow at the rate of $\Gamma\zeta(t)N_t$ per year to the agent. Upon death the agent pays the amount $N_{\tau-}$ to the contract issuer (formally the bequest is $X_\tau = X_{\tau-} - N_{\tau-}$). Here N_t is chosen by the agent. With a moderate or high bequest motive, the optimal N_t is typically negative. With a low bequest motive, the optimal N_t tends to be positive except when the agent is relatively young or very close to 100 years. We let $\Gamma = 0.8$, corresponding to a 20% margin to the insurance company. The agent's optimal strategy is similar to the no-mortality case but with the (deterministically increasing) mortality intensity $\zeta(t)$ added to the constant time preference δ, so that the agent effectively becomes more impatient with age.

Figure 4 depicts perishable consumption over the life cycle with mortality risk. The agent is initially 30 years and retires at 65. The panels differ in bequest preference with $w = 1$ in the left and $w = 10$ in the right panel; if surviving until 100, an agent with a weak habit, $\varepsilon - \alpha = 0.1$, leaves a wealth of 25.9 if $w = 1$ and 233.8 if $w = 10$. Each panel shows the consumption profile for the no-habit case, a weak habit, and a strong habit ($\varepsilon - \alpha = 0.02$). The diamonds indicate the peaks. Without habit formation mortality risk induces a consumption hump rather late in life, but habit formation produces an earlier hump more in line with the observed hump. The bequest weight does not significantly influence the shape or peak age of the consumption pattern. With a high bequest weight consumption is scaled down throughout life to build up more wealth to bequeath.

Habits also impact the sensitivity of consumption with respect to changes in wealth or income. The ratio $\frac{\partial c_t}{\partial \widehat{X}_t} \big/ \frac{\partial(\chi q_t H_t)}{\partial \widehat{X}_t}$ of the marginal propensities to consume out of disposable wealth equals $(1 + \alpha B(t))b/(1 - b)$, which is constant in the no-habit case, but bigger and time-dependent with habits. With the parameters used to generate Figure 3, the ratio is 1.9 in the no-habit case, whereas it declines from 15.9 initially to 2.3 at the end of life with the weak habit and from 38.2 to 2.3 with the strong habit. In the habit model, a young agent being hit by a negative wealth or income shock reduces perishable consumption expenditures much more than housing expenditures, as seen in the data.

[12] Published at the Centers for Disease Control and Prevention under the U.S. Department of Health and Human Services, see http://www.cdc.gov/nchs/data/nvsr/nvsr62/nvsr62_07.pdf.

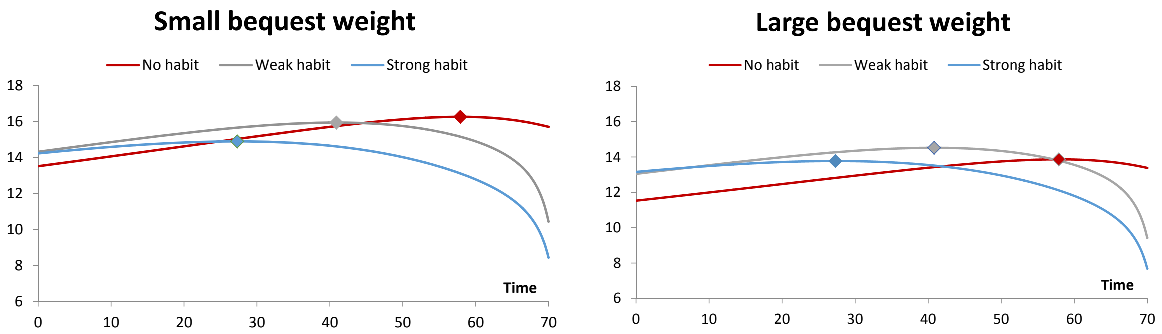

Figure 4: Perishable consumption with mortality risk and bequest. Each panel shows perishable consumption over the life cycle without habit, with a weak habit ($\alpha = 0.8, \varepsilon = 0.9$), and with a strong habit ($\alpha = 0.88, \varepsilon = 0.9$). The graphs are generated using the mortality and bequest extension of the deterministic model. The left (right) panel assumes a bequest utility parameter of $w = 1$ ($w = 10$). For both panels the following parameter values are used: $\delta = 0.01$, $\gamma = 5$, $t = 0$, $\tilde{T} = 35$, $T = 70$, $X_0 = 20$, $Y_0 = 20$, $\mu_Y = 0.01$ (zero in retirement), $\Upsilon = 0.6$, $r = 0.03$, $\chi = 0.06$, $H_0 = 0.25$, $\mu_H = -0.03$, $\Gamma = 0.8$. Without habit formation $b = 0.65$. With habit formation $b = 0.69$ and $\bar{q}_0 = 150$.

3.2 A model with spanned risks and no frictions

Next we add the stock index and uncertainty about the house price, the income, and the stock price. The agent can own any non-negative number of housing units and can simultaneously rent out some of the owned housing units or rent additional housing units. Hence, housing consumption can be disentangled from the housing investment position.

To obtain a closed-form solution, we let the agent continuously adjust the number of units rented or owned without transaction costs. While changes in the physical ownership of homes seem rare and costly, the remodeling or the extension of a house also constitutes an increase in the number of housing units owned due to the higher quality or increased space. In addition, individuals can indirectly invest in housing units by purchasing shares in residential REITs, exchange-traded funds emulating the REIT market, or other financial assets linked to house prices such as the Case-Shiller derivatives.[13] Home owners can implement minor variations in the desired housing investment position through remodeling or REITs, whereas they can implement larger changes in both desired housing consumption and investment through infrequent physical house transactions. A pure housing investment

[13] Well-developed REIT markets exist in many countries. Cotter and Roll (2015) study the risk and return characteristics of U.S. REITs. As explained by Ang (2014, Ch. 11), REIT returns exhibit only a low short-run correlation with returns on directly owned real estate (and a higher correlation with common stocks), but longer-term correlations are significantly higher, cf., e.g., Hoesli and Oikarinen (2012). Pagliari, Scherer, and Monopoli (2005) argue that after various relevant adjustments REIT returns and direct real estate returns are much more highly correlated even in the short run, and Lee, Lee, and Chiang (2008) and others report that REITs behave more and more like real estate and less and less like ordinary stocks.

position is obtained by owning housing units and renting them out either directly or through REITs. Ownership entails maintenance costs (including property taxes) equal to a constant fraction $m \geq 0$ of the property value. Hence the return on a pure investment in a housing unit is $dH_t + (\chi - m)H_t\,dt$ over the dt-interval following time t.

In this section we deliberately abstract from any portfolio constraints and assume that all risks are spanned by traded assets.[14] More precisely, the stock price S_t (including reinvested dividends), the unit house price H_t, and the labor income rate Y_t satisfy

$$dS_t = S_t\left[(r+\mu_S)\,dt + \sigma_S\,dW_{St}\right],$$

$$dH_t = H_t\left[(r+\mu_H)\,dt + \sigma_H\left(\rho_{HS}\,dW_{St} + \sqrt{1-\rho_{HS}^2}\,dW_{Ht}\right)\right], \tag{12}$$

$$dY_t = Y_t\left[\mu_Y(t)\,dt + \sigma_Y(t)\left(\rho_{YS}\,dW_{St} + \sqrt{1-\rho_{YS}^2}\,dW_{Ht}\right)\right], \tag{13}$$

where W_S and W_H are independent standard Brownian motions. We define $\tilde{\rho}_{HY} = \sqrt{1-\rho_{YS}^2}/\sqrt{1-\rho_{HS}^2}$ and note that the house-income correlation is $\rho_{HY} = \rho_{HS}\rho_{YS} + \tilde{\rho}_{HY}(1-\rho_{HS}^2)$. As there is no income-specific shock term, the agent can adjust the exposure to both stock price, house price, and income shocks through her positions in the stock and the housing units. The excess expected return μ_S and the volatility σ_S of the stock are constant. Similarly, the excess expected house price growth rate μ_H and volatility σ_H are constants, as are the stock-house price correlation ρ_{HS} and the stock-income correlation ρ_{YS}. The instantaneous Sharpe ratios on investments in stocks and houses are then

$$\lambda_S = \frac{\mu_S}{\sigma_S}, \qquad \lambda_H = \frac{\mu_H + \chi - m}{\sigma_H}.$$

Both the expected growth rate and volatility of income may depend on the individual's age. Where most life-cycle papers assume a constant retirement income, we allow for risk. This is motivated by (i) some retirees continue to earn income from proprietary businesses or other non-traded assets; (ii) uncertainty about medical expenses implies that the disposable income is risky (see De Nardi, French, and Jones 2010); (iii) because of mortality risk, the individual may miss retirement payments and, when we do not model mortality formally, retirement income risk captures this effect parsimoniously.

Let Π_{St} be the fraction of tangible wealth invested in the stock, and let ϕ_{ot} and ϕ_{rt} denote the housing units owned (physically or through REITs) and rented. Then the

[14] This model extends Kraft and Munk (2011) to housing habits (they allow stochastic interest rates).

units of housing consumption and the fraction of tangible wealth invested in housing are $q_t \equiv \phi_{ot} + \phi_{rt}, \Pi_{Ht} \equiv \frac{\phi_{ot} H_t}{X_t}$, respectively. The wealth invested in the risk-free asset is residually determined as $M_t = X_t(1 - \Pi_{St} - \Pi_{Ht})$. The dynamics of tangible wealth are

$$\begin{aligned} dX_t &= \Pi_{St} X_t \frac{dS_t}{S_t} + M_t r\, dt + \phi_{ot}\left(dH_t - mH_t\, dt\right) - \phi_{rt}\chi H_t\, dt - c_t\, dt + Y_t\, dt \\ &= \left(X_t\left[r + \Pi_{St}\mu_S + \Pi_{Ht}(\mu_H + \chi - m)\right] + Y_t - c_t - q_t\chi H_t\right) dt \\ &\quad + X_t\left(\Pi_{St}\sigma_S + \Pi_{Ht}\sigma_H\rho_{HS}\right) dW_{St} + X_t \Pi_{Ht}\sigma_H\sqrt{1-\rho_{HS}^2}\, dW_{Ht}. \end{aligned} \tag{14}$$

The value function is now defined as

$$J(t, x, h, y, \bar{q}) = \sup_{c,q,\Pi_S,\Pi_H} \mathrm{E}_t\left[\int_t^T e^{-\delta(s-t)} U(c_s, q_s, \bar{q}_s)\, ds\right].$$

Theorem 2 *The value function is given by (4)–(7) with*

$$r_G = \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma} r + \frac{\gamma-1}{2\gamma^2}\Lambda^2 - k\left(r + \frac{\mu_H}{\gamma} - \frac{\gamma-1}{\gamma}(\chi - m) + \frac{1}{2}(k-1)\sigma_H^2\right), \tag{15}$$

$$r_F(t) = r - \mu_Y(t) + \sigma_Y(t)\left(\lambda_S\rho_{YS} + \tilde{\rho}_{HY}\left[\lambda_H - \rho_{HS}\lambda_S\right]\right), \tag{16}$$

$$r_B = \varepsilon - \alpha + \chi - m, \tag{17}$$

$$\Lambda^2 = \lambda_S^2 + \frac{1}{1-\rho_{HS}^2}\left(\lambda_H - \rho_{HS}\lambda_S\right)^2.$$

With disposable wealth $\widehat{X}_t$ defined in (3), the optimal consumption decisions are still given by (9), and the optimal investment strategy is

$$\Pi_{St} = \frac{1}{\gamma}\frac{\lambda_S - \rho_{HS}\lambda_H}{\sigma_S(1-\rho_{HS}^2)}\frac{\widehat{X}_t}{X_t} - \frac{\sigma_Y(t)}{\sigma_S}\left(\rho_{YS} - \rho_{HS}\tilde{\rho}_{HY}\right)\frac{Y_t F(t)}{X_t}, \tag{18}$$

$$\Pi_{Ht} = \frac{1}{\gamma}\frac{\lambda_H - \rho_{HS}\lambda_S}{\sigma_H(1-\rho_{HS}^2)}\frac{\widehat{X}_t}{X_t} - \frac{\sigma_Y(t)}{\sigma_H}\tilde{\rho}_{HY}\frac{Y_t F(t)}{X_t} + k\frac{\widehat{X}_t}{X_t} + \bar{q}_t\chi B(t)\frac{H_t}{X_t}. \tag{19}$$

The expected consumption $\mathrm{E}[c_t]$ is a hump-shaped function of time if $\varepsilon + \chi - m > 0$ and

$$\frac{k\alpha r_B}{(\alpha + r_B)e^{r_B T} - \alpha} < \frac{r-\delta}{\gamma} + \frac{\gamma+1}{2\gamma^2}\Lambda^2 + k\left(r + \mu_H + \frac{\mu_H + \chi - m}{\gamma} + \frac{k-1}{2}\sigma_H^2\right) < k\alpha. \tag{20}$$

The value function and the consumption policy are as in the full-certainty model, only the "discount rates" r_F and r_G are adjusted to account for risk premia. If we let $W_t = X_t + Y_t F(t)$ denote total wealth, the relative risk aversion derived from the value function is

$$\text{RRA} = -\frac{WJ_{WW}}{J_W} = \frac{\gamma W}{W - \bar{q}\chi h B(t)}, \tag{21}$$

in line with other habit settings, e.g., Campbell and Cochrane (1999). Early in life, human capital and therefore total wealth are typically large relative to the habit buffer, so the RRA is only marginally larger than γ.

The housing habit can generate a hump in expected perishable consumption. The optimal stock investment in (18) is the sum of a speculative demand determined by the current risk-return tradeoff and an income-adjustment term. The first two terms of the optimal housing investment have a similar interpretation. The third term in (19) hedges against increases in the price of housing consumption, which is accomplished by having a higher wealth exposure to house prices, cf. Sinai and Souleles (2005). The final term ensures that the agent can reach at least the minimum housing consumption in the remaining life time. As the costs of ensuring that the minimum is achieved vary with house prices, this calls for an increased wealth exposure to house prices. Both the hedge term and the habit-insurance term are thus positive. The income-adjustment terms undo the extent to which the human capital resembles an investment in the stock or housing units.

Habit formation lowers the disposable wealth $\widehat{X}_t$ through the buffer $\bar{q}_t \chi H_t B(t)$, which is typically slightly increasing early in life (since $\bar{q}_t$ increases) and then decreasing as the remaining life time shrinks ($B(t)$ is decreasing). Therefore, the habit reduces the optimal stock investment, in particular for young investors. The habit also reduces the speculative housing demand and the house price hedge demand but, on the other hand, induces a positive habit-insurance term so that the net effect is parameter dependent.

For a numerical example we fix most parameters values at the empirical estimates explained in Section 5, including the expected returns and variances of the assets, and the house-stock correlation of $\rho_{HS} = 0.25$. But, to ensure that the income risk is spanned, values of the income-asset correlations different from the empirical estimates have to be used, which primarily affects the income-adjustment component of the portfolio. We let $\rho_{YS} = \rho_{HY} = 0.7906$ so that stocks and houses are equally "income-like." This generates a large negative adjustment in the portfolio weights, which is not expected to be seen in a realistic setting with unspanned income risk. The speculative term is unaffected by the income-asset correlations except for a small effect on the human capital through r_F.

The left panel of Figure 5 illustrates the life-cycle variations in the optimal fraction of tangible wealth invested in the risk-free asset and the stock, with the latter decomposed

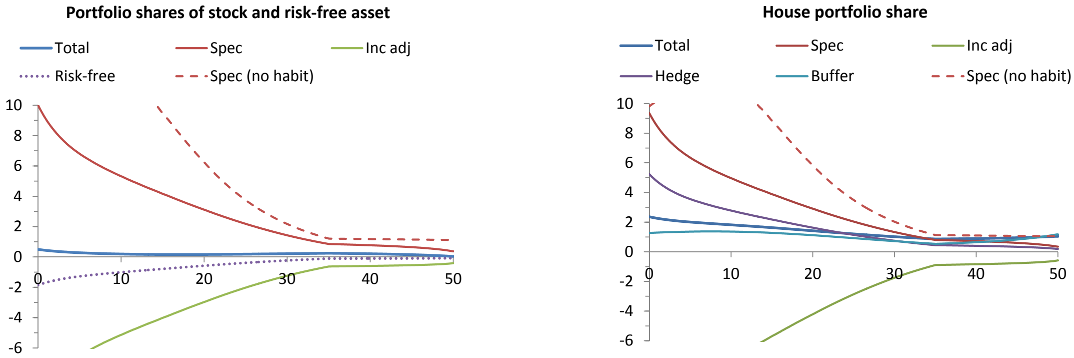

Figure 5: Optimal portfolio shares over life when income risk is spanned. The left panel shows the life-cycle variations in the portfolio weights of the risk-free asset (dotted curve) and the stock (blue), with the latter decomposed into the speculative term (red) and the income-adjustment term (green). The right panel shows the life-cycle variations in the portfolio share of housing (dark blue) decomposed into the speculative term (red), the income-adjustment term (green), the house price hedge term (purple), and the habit buffer term (light blue). In both panels, the dashed red curve shows the speculative term in the absence of habit formation. The parameter values listed in Table 1 are used except that $\rho_{YS} = \rho_{HY} = 0.7906$ to ensure spanned income risk.

into the speculative and the income-adjustment part. The right panel shows the housing portfolio share and its four components. We consider a modestly risk-averse agent ($\gamma = 3$, RRA from (21) is 3.01) with a 35-year working period and a subsequent 15-year retirement. The results presented are averages over 10,000 simulated paths with strategies updated 50 times per year.

The optimal portfolio early in life is extreme. The human capital dominates the financial wealth so the ratio $\widehat{X}_t/X_t$ is much bigger than one. Consequently, unless the young agent is highly risk averse, the speculative demand exceeds 100% of financial wealth as seen in the figure both for stock and house investments, even though the habit dampens the speculative term substantially. Imposing more realistic lower income-adjustment terms, the overall portfolio weights of both stocks and housing units would be very large early in life, which calls for significant borrowing. The main source of life-cycle variations is the human capital, which is typically decreasing over life as the remaining working period shrinks. This tends to reduce the ratios $\hat{X}_t/X_t$ and $Y_t F(t)/X_t$ and thus lead to the speculative terms, income-adjustment terms, and a house price hedge term that decrease in magnitude over life.[15] Note that the habit-buffer component and, especially, the hedge component of the housing share are sizable and thus important for the optimal investment

[15]With a sufficiently high income growth rate, the human wealth can be locally increasing with age, but eventually it decreases. Also note that the denominator X_t, the tangible wealth, is generally hump-shaped over life with net savings during the working phase and consumption out of savings in retirement.

strategy. Next, we turn to the full-fledged model with short-selling constraints, limited borrowing, and unspanned income risk.

4 The full-fledged model

4.1 Model specification

Our main model differs from the model considered in Section 3.2 by having unspanned labor income risk and investment constraints. Instead of (13), the income rate follows

$$dY_t = Y_t \left[\mu_Y(t)\, dt + \sigma_Y(t) \left(\rho_{YS}\, dW_{St} + \hat{\rho}_{HY}\, dW_{Ht} + \hat{\rho}_Y\, dW_{Yt} \right) \right], \tag{22}$$

where $W_Y = (W_{Yt})$ is a standard Brownian motion independent of W_S and W_H. Unless $\hat{\rho}_Y = 0$, the income risk is unspanned by traded assets so the individual faces an incomplete market. The income-house and income-stock correlations ρ_{HY} and ρ_{YS} are constant, and

$$\hat{\rho}_{HY} \equiv \frac{\rho_{HY} - \rho_{YS}\rho_{HS}}{\sqrt{1-\rho_{HS}^2}}, \qquad \hat{\rho}_Y \equiv \sqrt{1 - \rho_{YS}^2 - \hat{\rho}_{HY}^2}.$$

Since income contains unspanned risk and is not bounded from below by a positive level, non-negative terminal wealth can only be insured by keeping tangible wealth non-negative throughout, i.e., $X_t \geq 0$ for all $t \in [0, T]$. Furthermore, we impose the constraints

$$\Pi_S \geq 0, \quad \Pi_H \geq 0, \quad \Pi_S + \kappa \Pi_H \leq 1, \tag{23}$$

which rule out short-selling and limits borrowing to a fraction $(1-\kappa)$ of the current value of the housing investment, where $\kappa \in [0, 1]$.

An admissible strategy $a = (c, q, \Pi_S, \Pi_H)$ satisfies standard integrability conditions and the above constraints, and it generates the expected utility

$$J(t, x, y, h, \bar{q}; a) = E_t \left[\int_t^T e^{-\delta(s-t)} U(c_s, q_s, \bar{q}_s)\, ds \right], \tag{24}$$

where the expectation is conditional on $X_t = x$, $Y_t = y$, $H_t = h$, and $\bar{q}_t = \bar{q}$. If $\mathcal{A}$ denotes the set of all admissible strategies, the value function is defined as

$$J(t,x,y,h,\bar{q}) = \sup_{a\in\mathcal{A}} J(t,x,y,h,\bar{q};a).$$

Because of incomplete markets and portfolio constraints, we cannot solve the problem in closed form. Due to the high number of state variables, grid-based methods are cumbersome to implement and lead to high computation times when used with the grid sizes required for high precision. Below we outline the numerical solution method we apply.

4.2 Outline of the solution approach

We apply the SAMS (Simulation of Artificial Markets Strategies) approach introduced by Bick, Kraft, and Munk (2013), henceforth BKM, and illustrated in Figure 6. The method exploits that in each of various *artificial* markets a closed-form solution to the utility maximization problem exists; the solution is similar to those presented in Theorems 1 and 2.[16] In any of the artificial markets the agent is unconstrained, has access to the same assets (with identical or higher returns) as in the true market, plus an additional asset completing the market. Hence, the agent can achieve at least as high an expected utility as in the true market. In Figure 6, the points marked to the right on the axis indicate the maximal utility in different artificial markets denoted by θ_1, θ_2, etc. The lowest expected utility among these artificial markets—indicated by $\bar{\theta}$ on the axis—is still at least as large as the unknown maximum in the true market.

The explicit, optimal strategy in an artificial market is infeasible in the true market, but can be *feasibilized*—transformed into a feasible strategy—and its expected utility in the true market can be evaluated by Monte Carlo simulation. In this way we can generate the points on the left part of the axis in Figure 6. Maximizing over these feasibilized strategies, we obtain the expected utility indicated by θ^* in the figure. The corresponding *near-optimal* strategy is the strategy suggested by the SAMS approach.[17]

[16] Artificial markets were introduced by Karatzas, Lehoczky, Shreve, and Xu (1991) and Cvitanić and Karatzas (1992). Already Haugh, Kogan, and Wang (2006) observed that artificial markets produce a utility loss bound on any given feasible strategy. BKM search for the best possible strategy in a parameterized family of strategies derived from artificial markets and search over a parameterized family of upper utility bounds to find the tightest possible bound.

[17] The most time-consuming part of the method is the maximization over the feasibilized strategies. In fact, the feasibilized strategy corresponding to the least-favorable artificial market (indicated by $\bar{\theta}$) performs almost as well as the near-optimal strategy and can be computed without the maximization and thus much faster. BKM refer to this as the parsimonious SAMS method.

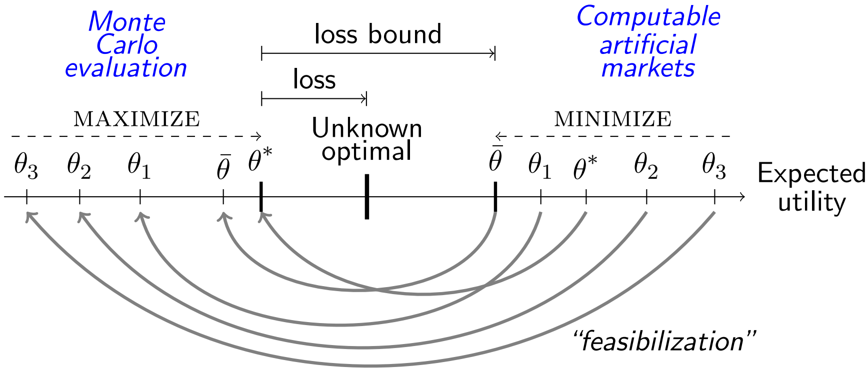

Figure 6: Our solution technique. The axis shows the agent's expected utility. "Unknown optimal" represents the indirect utility in the true market, i.e., the expected utility generated by the unknown optimal consumption-investment strategy. Each point to the right corresponds to the indirect utility in an artificial market with deterministic modifiers characterized by some parameter set θ. The corresponding strategy is transformed into a feasible strategy in the true market which generates an expected utility on the left part of the axis. The best of these strategies is derived from the optimal strategy in an artificial market for some θ^*. The arrows above the axis indicate the unknown utility loss and a computable upper bound on the loss the agent suffers by following the best of the considered feasible strategies instead of the unknown optimal strategy.

As with other numerical methods, the suggested strategy is unlikely to be identical to the truly optimal strategy so the agent suffers a welfare loss by applying the suggested strategy. We derive an upper bound on the welfare loss by comparing the expected utility generated by the near-optimal strategy to the expected utility in the worst of the artificial markets considered. This upper bound represents a measure of precision of the approach. In wealth-equivalent terms, the loss bound in our baseline case below is only 1.1% of the agent's wealth. In the examples of BKM, the true loss is significantly smaller than the loss bound, and we expect the same in our model.

Relative to alternative numerical methods, this approach distinguishes itself by being relatively easy to implement, being based on closed-form consumption and investment strategies, and providing a measure of its accuracy. The following subsections provide details on the SAMS method applied to our problem.

4.3 A family of artificial markets

Intuitively we search for artificial markets in which (i) we can solve the utility maximization problem and (ii) the optimal consumption-investment strategy is close to what we expect to be a good, feasible strategy in the true market. Because the portfolio weights in an artificial market are unconstrained, whereas they must satisfy (23) in the true market, we adjust the risk-free rate or the expected excess returns on stocks and houses. Following

Cvitanić and Karatzas (1992) and BKM (Sec. 8), and using $\nu^- = \max(-\nu, 0)$, we define

$$\tilde{\mu}_S(t) = \mu_S + \nu_S(t), \quad \tilde{\mu}_H(t) = \mu_H + \nu_H(t), \quad \tilde{r}(t) = r + \max\left(\nu_S(t)^-, \tfrac{1}{\kappa}\,\nu_H(t)^-\right).$$

The dynamics of stock and house prices in the artificial market are thus

$$d\tilde{S}_t = \tilde{S}_t \left[(\tilde{r}(t) + \tilde{\mu}_S(t))\, dt + \sigma_S \, dW_{St} \right], \ \tilde{S}_0 = S_0, \tag{25}$$

$$d\tilde{H}_t = \tilde{H}_t \left[(\tilde{r}(t) + \tilde{\mu}_H(t))\, dt + \sigma_H \left(\rho_{HS} \, dW_{St} + \sqrt{1 - \rho_{HS}^2}\, dW_{Ht} \right) \right], \ \tilde{H}_0 = H_0. \tag{26}$$

Irrespective of the signs and magnitudes of ν_S and ν_H, the risk-free rate and the expected returns of stocks and houses are at least as big in the artificial as in the true market.

To complete the market we introduce an income derivative, i.e., an asset sensitive to the income-specific shock W_Y. The price dynamics of this asset are of the form

$$dI_t = I_t \left[(\tilde{r}(t) + \nu_I(t))\, dt + \rho_{IS}\, dW_{St} + \hat{\rho}_{IH}\, dW_{Ht} + \hat{\rho}_I \, dW_{Yt} \right], \tag{27}$$

where

$$\hat{\rho}_{IH} \equiv \frac{\rho_{IH} - \rho_{IS}\rho_{HS}}{\sqrt{1 - \rho_{HS}^2}}, \qquad \hat{\rho}_I \equiv \sqrt{1 - \rho_{IS}^2 - \hat{\rho}_{IH}^2},$$

and we require $\hat{\rho}_I > 0$. The assumption of a unit volatility is without loss of generality. The asset is characterized by the excess expected return $\nu_I(t)$ and the correlations ρ_{IS}, ρ_{IH}. Let Π_{It} be the fraction of tangible wealth invested in this asset.

The housing consumption constitutes a challenge not covered by the single-good case in which BKM introduced the SAMS approach. The house prices in the artificial market and the true market are related through

$$\tilde{H}_t = \omega(t) H_t, \quad \omega(t) = e^{\int_0^t r_\omega(u)\, du}, \quad r_\omega(u) = \nu_H(u) + \max\left(\nu_S(u)^-, \tfrac{1}{\kappa}\nu_H(u)^-\right) \geq 0.$$

Hence, if we let the rental rate be the same as in the true market, the consumption of a given number of housing units would be more expensive (and thus maybe not feasible) in the artificial market. Therefore we set the rental rate and the maintenance rate in $\mathcal{M}_\theta$ to

$$\tilde{\chi}(t) = \frac{\chi}{\omega(t)}, \qquad \tilde{m}(t) = m + \tilde{\chi}(t) - \chi$$

which sustains the unit rental price $\tilde{\chi}(t)\tilde{H}_t = \chi H_t$, as well as the net rental rate after maintenance costs, $\tilde{\chi}(t) - \tilde{m}(t) = \chi - m$. The Sharpe ratios of the stock and the house investment (including the net rental rate) are

$$\tilde{\lambda}_S(t) = \frac{\tilde{\mu}_S(t)}{\sigma_S}, \qquad \tilde{\lambda}_H(t) = \frac{\tilde{\mu}_H(t) + \chi - m}{\sigma_H},$$

while $\nu_I(t)$ is the Sharpe ratio of the income derivative.

An artificial market $\mathcal{M}_\theta$ corresponds to a given choice of

$$\theta = (\nu_S(t), \nu_H(t), \nu_I(t), \rho_{IS}, \rho_{IH})\,. \tag{28}$$

Let $J_\theta(t, x, y, h, \bar{q})$ denote the value function in the artificial market $\mathcal{M}_\theta$, that is the expected utility of the remaining life maximized over all strategies $(c, q, \Pi_S, \Pi_H, \Pi_I)$. In this market the expected utility of any strategy is still determined by (24), but the relevant dynamics are now given by (2), (22), (25), (26), and (27). We verify in Appendix A.3 that any strategy admissible in the true market leads to at least the same utility in each artificial market. Since the set of admissible strategies in the artificial complete market is greater than in the incomplete market, the next lemma follows.

Lemma 1 *For any artificial market $\mathcal{M}_\theta$, the following inequality holds:*

$$J_\theta(t, x, y, h, \bar{q}) \geq J(t, x, y, h, \bar{q}).$$

Let Θ represent the set of θ's with deterministic modifiers $\nu_S(t), \nu_H(t), \nu_I(t)$. For $\theta \in \Theta$ we derive below closed-form expressions for J_θ and the corresponding optimal strategy. Minimizing over $\theta \in \Theta$, we get an upper bound on the true-market value function:

$$\bar{J}(t, x, y, h, \bar{q}) = \min_{\theta \in \Theta} J_\theta(t, x, y, h, \bar{q}) \tag{29}$$

4.4 Optimal decisions in the artificial markets

Any artificial market $\mathcal{M}_\theta$, $\theta \in \Theta$, is very similar to that of Section 3.2, so the value function and optimal strategies have the same form as found in that model. The only differences are that there are now three risky assets instead of two, and that various quantities are now time dependent.

Theorem 3 *For $\theta \in \Theta$, the value function in the artificial market $\mathcal{M}_\theta$ is*

$$J_\theta(t,x,y,h,\bar{q}) = \frac{1}{1-\gamma}(\chi h)^{k\gamma} G_\theta(t)^\gamma (x + yF_\theta(t) - \bar{q}\chi h B_\theta(t))^{1-\gamma}, \tag{30}$$

where

$$\begin{aligned}
G_\theta(t) &= \hat{b}\int_t^T e^{-\int_t^u r_G(s)\,ds}\,(1+\alpha B_\theta(u))^k\,du,\\
F_\theta(t) &= \begin{cases} \int_t^T e^{-\int_t^u r_F(s)\,ds}\,du & t \in (\tilde{T}, T],\\ \int_t^{\tilde{T}} e^{-\int_t^u r_F(s)\,ds}\,du + \Upsilon \int_{\tilde{T}}^T e^{-\int_t^u r_F(s)\,ds}\,du & t \in [0,\tilde{T}],\end{cases}\\
B_\theta(t) &= \int_t^T e^{-\int_t^u r_B(s)\,ds}\,du,\\
r_G(t) &= \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}\tilde{r}(t) + \frac{\gamma-1}{2\gamma^2}\tilde{\Lambda}(t)^2\\
&\quad - k\left[\tilde{r}(t) - r_\omega(t) + \frac{\tilde{\mu}_H(t)}{\gamma} - \frac{\gamma-1}{\gamma}(\chi - m) + \frac{k-1}{2}\sigma_H^2\right],\\
r_F(t) &= \tilde{r}(t) - \mu_Y(t) + \sigma_Y(t)\left(\psi_S\tilde{\lambda}_S(t) + \psi_H\tilde{\lambda}_H(t) + \psi_Y\nu_I(t)\right),\\
r_B(t) &= r_\omega(t) + \varepsilon - \alpha + \chi - m,
\end{aligned}$$

and ψ_S, ψ_H, ψ_Y are defined by the correlations, cf. (61) *in Appendix A.4, and $\tilde{\Lambda}(t)^2$ is defined by the Sharpe ratios and the correlations, cf.* (55)*. In terms of disposable wealth $\widehat{X}_t = X_t + Y_tF_\theta(t) - \bar{q}_t\chi H_t B_\theta(t)$, the optimal consumption and investment strategy is*

$$c_t = b\hat{b}\frac{(1+\alpha B_\theta(t))^k}{G_\theta(t)}\widehat{X}_t, \tag{31}$$

$$q_t = \bar{q}_t + (1-b)\hat{b}\,\frac{(1+\alpha B_\theta(t))^{k-1}}{\chi H_t G_\theta(t)}\widehat{X}_t, \tag{32}$$

$$\Pi_{St} = \frac{\xi_S(t)}{\gamma\sigma_S}\frac{\widehat{X}_t}{X_t} - \frac{\sigma_Y(t)\psi_S}{\sigma_S}\frac{Y_tF_\theta(t)}{X_t}, \tag{33}$$

$$\Pi_{Ht} = \frac{\xi_H(t)}{\gamma\sigma_H}\frac{\widehat{X}_t}{X_t} - \frac{\sigma_Y(t)\psi_H}{\sigma_H}\frac{Y_tF_\theta(t)}{X_t} + k\frac{\widehat{X}_t}{X_t} + \bar{q}_t\chi B_\theta(t)\frac{H_t}{X_t}, \tag{34}$$

$$\Pi_{It} = \frac{\xi_I(t)}{\gamma}\frac{\widehat{X}_t}{X_t} - \sigma_Y(t)\psi_Y\frac{Y_tF_\theta(t)}{X_t} \tag{35}$$

with $\xi_S(t), \xi_H(t), \xi_I(t)$ being defined in terms of the Sharpe ratios and the correlation structure, cf. (58)–(60) *in Appendix A.4.*

Despite the rich setting, the solution has the same structure as in the special cases in Section 3. Therefore, the optimal consumption strategy and the optimal stock and house investment strategy have the same interpretation as given after Theorems 1 and 2. The optimal investment in the income derivative consists of a speculative demand and an income-adjustment term. If the income derivative is assumed uncorrelated with both stock and house prices ($\rho_{IS} = \rho_{IH} = 0$ and thus $\hat{\rho}_{IH} = 0$ and $\hat{\rho}_I = 1$), then $\xi_I(t) = \nu_I(t)$ and $\psi_Y = \hat{\rho}_Y$ so that the speculative demand is determined by the Sharpe ratio $\nu_I(t)$ and the income-adjustment term by the unspanned income coefficient $\hat{\rho}_Y$.

Note that the elements of θ characterizing the artificial market enter most terms in the optimal consumption and investment strategy (31)-(35) in this market. Recall that the optimal strategy in any artificial market is infeasible in the true market. Intuitively, we are looking for artificial markets in which the optimal investment in the artificial income derivative is near zero and where the optimal strategies satisfy or are close to satisfying all constraints in the true market. The choice of the functional form of ν_S, ν_H, and ν_I influences the wealth loss. The affine specification

$$\nu_S(t) = \upsilon_0^S + \upsilon_1^S t, \quad \nu_H(t) = \upsilon_0^H + \upsilon_1^H t, \quad \nu_I(t) = \upsilon_0^I + \upsilon_1^I t$$

provides a good trade off between precision and computational complexity. The artificial markets are then characterized by the six constants in these functions plus ρ_{IS} and ρ_{IH}.

4.5 Consumption and investment strategies in the true market

The optimal strategy in any artificial market is infeasible in the true market, but we transform it into a feasible strategy. The time t true-market housing habit buffer $\bar{q}_t \chi H_t B(t)$ is at least as large as the artificial-market buffer $\bar{q}_t \chi H_t B_\theta(t)$ since $r_\omega(t) \geq 0$. The agent must ensure that tangible wealth exceeds the true-market buffer, i.e., that $X_t \geq \bar{q}_t H_t B(t)$. When wealth is close to the buffer, current consumption cannot be implicitly financed by future income, which is therefore less valuable. Following BKM, we replace $F_\theta(t)$ by

$$\widetilde{F}_\theta(t) = F_\theta(t) \left(1 - e^{-\eta[X_t - \bar{q}_t \chi H_t B(t)]}\right),$$

which parsimoniously captures this reduction in human capital. Here η is a positive constant to be determined experimentally. Hence, we adjust the disposable wealth to

$$\widetilde{X}_t^{(\theta)} = X_t + Y_t \widetilde{F}_\theta(t) - \bar{q}_t \chi H_t B(t),$$

and, following (21), we measure the relative risk aversion by

$$\mathrm{RRA} = \gamma \frac{X_t + Y_t \widetilde{F}_\theta(t)}{X_t + Y_t \widetilde{F}_\theta(t) - \bar{q}_t \chi H_t B(t)}. \tag{36}$$

The adjusted consumption strategy derived from the artificial market $\mathcal{M}_\theta$ is

$$c_t^{(\theta)} = b\hat{b} \frac{(1 + \alpha B_\theta(t))^k}{G_\theta(t)} \widetilde{X}_t^{(\theta)}, \qquad q_t^{(\theta)} = \bar{q}_t + (1-b)\hat{b}\, \frac{(1 + \alpha B_\theta(t))^{k-1}}{\chi H_t G_\theta(t)} \widetilde{X}_t^{(\theta)}.$$

The stock-house investment strategy must respect the constraints (23). Starting from (33)–(34), we adjust the housing habit buffer and the disposable wealth as explained above and take the positive part of the resulting expressions to get

$$\widetilde{\Pi}_{St}^{(\theta)} = \left(\frac{\xi_S(t)}{\gamma \sigma_S} \frac{\widetilde{X}_t^{(\theta)}}{X_t} - \frac{\sigma_Y(t) \psi_S}{\sigma_S} \frac{Y_t \widetilde{F}_\theta(t)}{X_t} \right)^+,$$

$$\widetilde{\Pi}_{Ht}^{(\theta)} = \left(\frac{\xi_H(t)}{\gamma \sigma_H} \frac{\widetilde{X}_t^{(\theta)}}{X_t} - \frac{\sigma_Y(t) \psi_H}{\sigma_H} \frac{Y_t \widetilde{F}_\theta(t)}{X_t} + k \frac{\widetilde{X}_t^{(\theta)}}{X_t} + \bar{q}_t \chi B(t) \frac{H_t}{X_t} \right)^+,$$

where $x^+ = \max(x, 0)$. If $\widetilde{\Pi}_{St}^{(\theta)} + \kappa \widetilde{\Pi}_{Ht}^{(\theta)} \leq 1$, we simply use the strategy

$$\Pi_{St}^{(\theta)} = \widetilde{\Pi}_{St}^{(\theta)}, \qquad \Pi_{Ht}^{(\theta)} = \widetilde{\Pi}_{Ht}^{(\theta)}.$$

In other cases we trim the portfolio weights as explained in Cvitanić and Karatzas (1992, Example 14.9) to

$$\left(\Pi_{St}^{(\theta)}, \Pi_{Ht}^{(\theta)}\right) = \begin{cases} (1, 0), & \text{if } \widetilde{\Pi}_{St}^{(\theta)} - \kappa \widetilde{\Pi}_{Ht}^{(\theta)} \geq 1, \\ \left(0, \frac{1}{\kappa}\right), & \text{if } \widetilde{\Pi}_{St}^{(\theta)} - \kappa \widetilde{\Pi}_{Ht}^{(\theta)} \leq -1, \\ \left(\frac{1 + \widetilde{\Pi}_{St}^{(\theta)} - \kappa \widetilde{\Pi}_{Ht}^{(\theta)}}{2}, \frac{1 - \widetilde{\Pi}_{St}^{(\theta)} + \kappa \widetilde{\Pi}_{Ht}^{(\theta)}}{2\kappa} \right), & \text{if } \left|\widetilde{\Pi}_{St}^{(\theta)} - \kappa \widetilde{\Pi}_{Ht}^{(\theta)}\right| < 1. \end{cases}$$

The transformation of the portfolio weights is illustrated in Figure 7.

For each $\theta \in \Theta$, the transformed strategy $a_\theta = \left(c^{(\theta)}, q^{(\theta)}, \Pi_S^{(\theta)}, \Pi_H^{(\theta)}\right)$ is a feasible

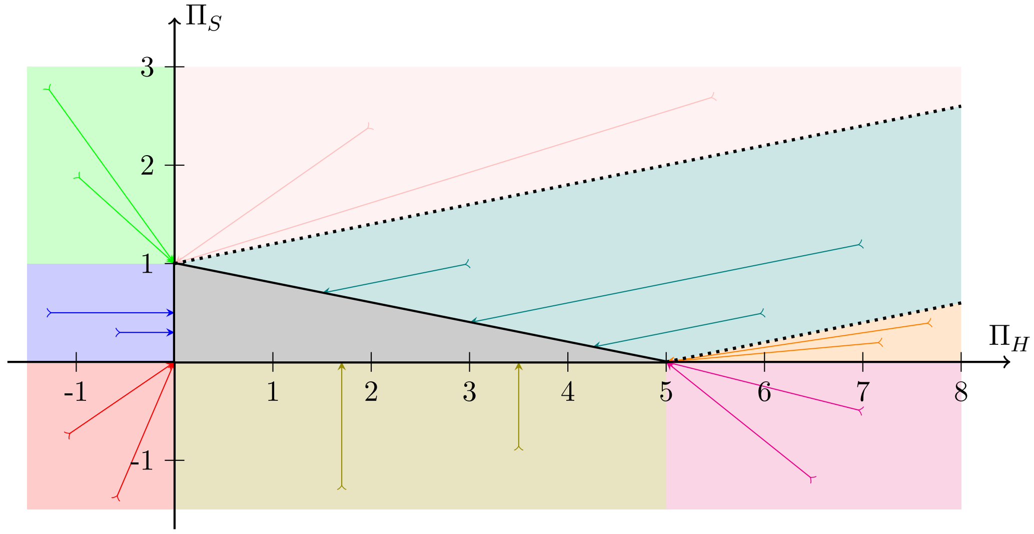

Figure 7: The transformation of portfolio weights. The gray triangle is the feasible region. The graph is drawn for $\kappa = 0.2$. The transformation depends on which colored area the artificial market portfolio is located in. The arrows show examples of the transformation.

strategy in the true market and generates an expected utility $J(t, x, y, h, \bar{q}; a_\theta)$ that we can estimate via Monte Carlo simulation. The performance of the strategy a_θ can be quantified by the wealth equivalent loss L_θ relative to the upper bound on the expected utility, $\bar{J}$, defined in (29). The loss is implicitly defined by

$$J\left(t, x, y, h, \bar{q}; a_\theta\right) = \bar{J}(t, x[1-L], y[1-L], h, \bar{q}), \tag{37}$$

so L is an upper bound on the relative reduction in initial wealth x and income y the agent would accept to get access to the truly optimal (but unknown) strategy. Note that L has to be determined by a numerical search routine. The upper utility bound $\bar{J}$ is the value function in some artificial market $\bar{\theta}$ that we find by a numerical minimization of (30). The left-hand side in (37) is determined by Monte Carlo simulation so to avoid a simulation bias we also compute the right-hand side by simulation of the optimal strategy in the market corresponding to the previously determined $\bar{\theta}$.

5 Results with baseline parameter values

5.1 Parameter values

Table 1 summarizes our baseline parameter values. We first estimate the parameter values for the stock, house, and income dynamics using quarterly U.S. data from 1953q1 to 2010q4 on the stock market index, the national house price index, and aggregate labor income. Subsequently some values are adjusted to be more representative of individual house prices and labor income, and other values are slightly rounded. For stocks we use the returns on the CRSP value-weighted market portfolio inclusive of the NYSE, AMEX, and NASDAQ markets. The risk-free rate is the three month Treasury bill yield from the Risk Free File on CRSP Bond tape. The house price is represented by the national Case-Shiller home price index with data taken from Robert Shiller's homepage.[18] We obtain quarterly U.S. data for aggregate disposable personal income from the NIPA tables and divide by the population size to compute the disposable labor income per capita. All time series are deflated using the CPI taken from CRSP.

The stock price volatility is estimated at 17%, a standard value. We reduce the estimated equity premium from 5.3% to 4% to account for the survivorship bias (Brown, Goetzmann, and Ross 1995) and the decline in discount rates and the implied unexpected capital gains over the sample period (Fama and French 2002). Furthermore, the closely related papers of Cocco, Gomes, and Maenhout (2005) and Yao and Zhang (2005) also assume a 4% equity premium.

For the house price, the estimated expected real growth rate is $r + \mu_H = 0$ as there has been virtually no growth in real house prices over the full sample period, and the same value was used, e.g., by Yao and Zhang (2005). We increase the volatility of the house series from 6.1% to 12%, in line with Flavin and Yamashita (2002) and Yao and Zhang (2005). The initial house price $H_0 = 0.25$ corresponds to \$250 per square foot, i.e., \$250,000 for a home of 1000 square feet of average quality and location.[19] We use a rental rate of 6.7% as estimated by Fischer and Stamos (2013) and a maintenance rate of 3.5% which includes property taxes that constitute 1-2% in many U.S. states. We let $\kappa = 0.4$

[18] `http://www.econ.yale.edu/~shiller/data.htm`

[19] Home prices vary a lot across states and regions. According to `www.zillow.com` the March 2015 median sale price per square foot was \$260 in California so a housing unit in our model is roughly corresponding to a square foot of housing of an average Californian location and quality. In Illinois, for example, the median sale price was \$122 per square foot so one housing unit is about two square foot of housing of an average Illinois location and quality.

Parameter	Description	Value
δ	time preference rate	0.05
γ	risk aversion coefficient	3
b	perishable/housing utility weight	0.69
α	habit scaling parameter	0.8
ε	habit persistence parameter	0.9
$\bar{q}_0$	initial housing habit level	200
T	remaining life time	50
$\tilde{T}$	time until retirement	35
X_0	initial financial wealth	20
r	risk-free rate	0.01
μ_S	excess expected stock return	0.04
σ_S	stock volatility	0.17
h_0	initial unit house price	0.25
μ_H	excess expected house price growth	-0.01
σ_H	house volatility	0.12
χ	rental rate	0.067
m	maintenance cost	0.035
κ	collateral parameter	0.4
y_0	initial income per year	20
μ_Y	expected income growth	0.01
σ_Y	income volatility	0.1
Υ	replacement ratio	0.6
ρ_{HS}	house-stock correlation	0.25
ρ_{YS}	income-stock correlation	0.22
ρ_{HY}	income-house correlation	0.16

Table 1: Baseline parameter values. Monetary quantities such as initial wealth and income are measured in thousands of USD. The text explains how the parameter values are determined.

so that homeowners can borrow up to 60% of the home value (at the risk-free rate).

The average growth rate of the aggregate income series is 1.7% per year, but this does not reflect the income growth that an individual can expect. As our benchmark we assume an expected income growth rate of 1% throughout the working life. Over the 35-year working period the income is then expected to grow by a factor $\exp(0.01 \times 35) \approx 1.42$, which seems reasonable and is close to the 38% reported as the median individual's income growth by Guvenen, Karahan, Ozkan, and Song (2015). We consider age- and education-dependent income growth rates in Section 6.5. Furthermore, we adjust the income volatility from 2.1% to 10% in line with Cocco, Gomes, and Maenhout (2005) and others. As in Yao and Zhang (2005) and Kraft and Munk (2011), the replacement ratio is

0.6, implying a 40% income drop at retirement. We allow income volatility in retirement as motivated earlier, and assume the same volatility as before retirement.

The pairwise correlations between stock prices, house prices, and labor income are all slightly positive and close to values used, e.g., by Cocco (2005), Yao and Zhang (2005), and Fischer and Stamos (2013). Other studies find income-stock correlations closer to zero, but obviously with some variation across individuals (e.g., Cocco, Gomes, and Maenhout 2005, Heaton and Lucas 2000). Benzoni, Collin-Dufresne, and Goldstein (2007) argue that the instantaneous correlation underestimates the longer-run stock-income correlation which is more important for long-term household decisions. In Section 6.6 we consider the effect of setting each of these correlation parameters to zero.

For concreteness, we study the decisions of a 30-year old who retires at 65 and dies at 80. We assume $\gamma = 3$, which implies an initial relative risk aversion of 3.28 as computed by (36). This is lower than values often used in this literature, but more in line with empirical estimates (Meyer and Meyer 2005). We assume an initial wealth and an initial annual income of \$20,000 as motivated in Section 3.1. The time preference rate of 5% is a standard choice in the literature. The initial housing habit is set at 200 housing units, which might represent a reasonable minimum home for a 30-year old individual. The habit parameters are $\alpha = 0.8$ and $\varepsilon = 0.9$ as in the simpler models of Section 3. The utility weight parameter b is set to 0.69 (0.65) with (without) habit formation in order to obtain the same average housing expenditure share in the two cases.

The results presented below are averages over 10,000 simulated paths with 50 time steps per year. With our baseline parameters, the loss bound is 1.1%, so by applying the consumption-investment strategy suggested by our method instead of the unknown optimal strategy, the agent's utility loss corresponds to at most 1.1% of her wealth.[20]

5.2 Results

Figure 8 illustrates various aspects of the optimal investments over the life cycle. First, we focus on the case with habit formation as illustrated by the blue curves. Initially the agent borrows 30 and invests that amount plus her initial wealth of 20 in housing units, and nothing in stocks. This corresponds to a portfolio weight of 2.5 for housing, 0 for stocks, and -1.5 for the risk-free asset. She invests nothing or very little in stocks early in life,

[20]The computer run time is approximately 90 minutes. With the parsimonious SAMS approach (cf. footnote 17), the run time is just a couple of minutes and the loss bound is 1.5%. Recall that the results of BKM indicate that the actual loss is considerably smaller than the loss bound.

but then gradually increases her stock investments (both amount and portfolio weight) until retirement, and then the amount invested in stocks decrease towards zero through retirement but at a roughly constant share.[21] Note that this is obtained for a low relative risk aversion. This pattern is in stark contrast to life-cycle models without housing and housing habits. For example, Cocco, Gomes, and Maenhout (2005) find that even with a risk aversion of 10, the agent typically starts out investing 100% of wealth in the stock market and after some years, the stock weight is gradually reduced to around 50% around retirement. Our model thus provides an explanation for the observed non-participation in stock markets of many young individuals without referring to stock market entry costs.

The portfolio is dominated by housing investments, partly funded by collateralized borrowing, throughout life. In the unconstrained case of Section 3.2, the speculative demands for stocks and housing were of the same order. The house price hedge and habit buffer components add to the importance of housing investments. Furthermore, housing investments can be levered through collateralized borrowing, whereas stock investments cannot. The housing investment peaks at around \$200,000 at retirement. As shown in the lower-right panel, the individual borrows throughout life, but only in the early years at the maximum amount possible (i.e., 60% of the housing investment).

By comparing to the case without habit formation (red curves in the figure), we see that the housing habits induce the individual to save more primarily through increased housing investments, partly financed by borrowing. The habit-forming agent enters the stock market at a later age and holds a somewhat lower fraction of wealth in stocks both early and late in life.

Figure 9 depicts the optimal consumption over the life cycle. The graphs are similar to those presented for the deterministic model in Section 3.1. The left panel shows that perishable consumption in the no-habit model is increasing through life, although relatively flat in retirement. With the housing habit, perishable consumption is hump shaped and peaks around retirement. Cocco, Gomes, and Maenhout (2005), among others, also report a consumption hump around retirement and explain it by a mix of mortality risk, borrowing constraints, and a hump-shaped income profile. We obtain the consumption hump without mortality risk or a hump-shaped income profile, and with much less tight borrowing constraints due to the access to borrowing collateralized by home ownership;

[21] We interpret the difference between the maximum and actual loan size as an amount invested in the risk-free asset. The stock weight in the financial portfolio is thus the amount in stocks divided by the sum of the amounts invested in stocks and the risk-free asset.

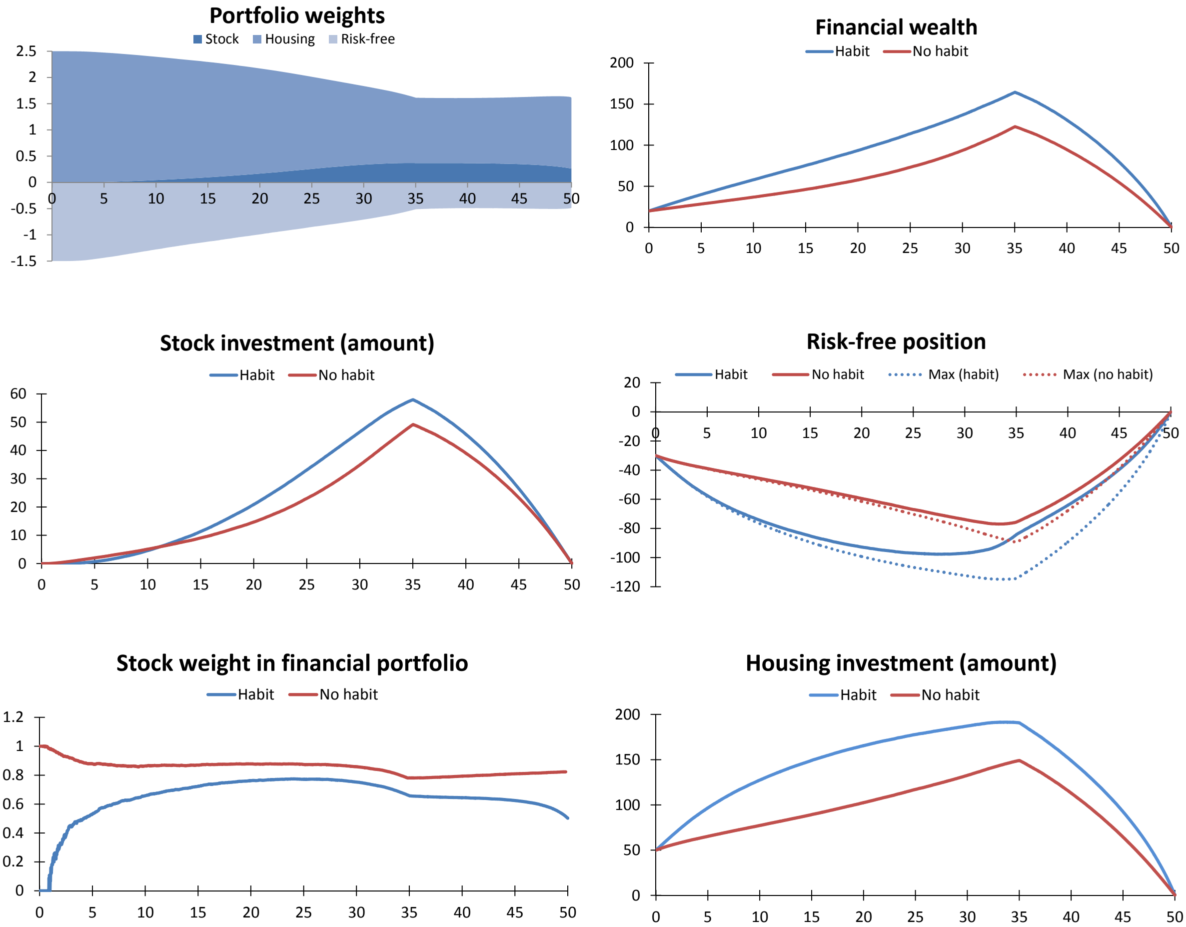

Figure 8: Investments over the life cycle in the full model. The graphs show averages across 10,000 simulations in which the consumption and investment strategy determined by our numerical method is used. The blue curves are for the case with a housing habit, the red curves are for the case without. The baseline parameters listed in Table 1 are used.

the determinants of the consumption hump is discussed further in Section 6. The right panel confirms that the housing habits produce a realistic life-cycle pattern in the housing expenditure share, unlike the constant share implied by the no-habit model.

Figure 10 compares optimal housing consumption and optimal housing investment over the life cycle. Without housing habits, the optimal housing consumption exceeds the optimal investment position at all ages. With the habit, housing consumption is first above the investment position, then below, and then above again in retirement. This can be interpreted as the agent preferring to rent her home early in life, then shifting to home ownership, and later shifting back to renting. Such a pattern is often observed in real life.

As in the deterministic model of Section 3.1 we consider the relative sensitivity of

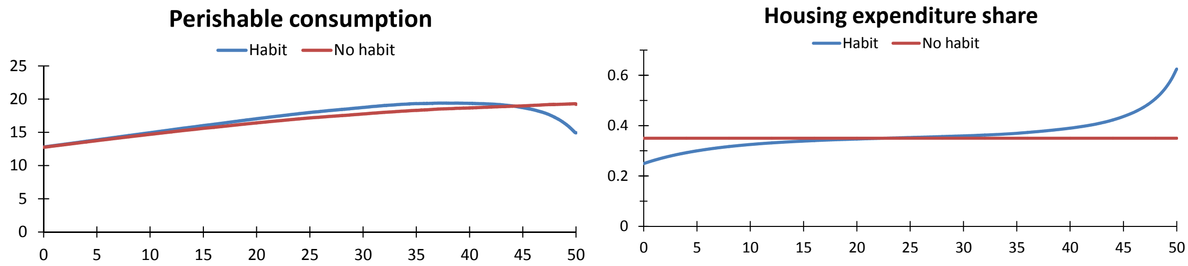

Figure 9: Consumption over the life cycle in the full model. The graphs show averages across 10,000 simulations in which the consumption and investment strategy determined by our numerical method is used. The blue curves are for the case with a housing habit, the red curves are for the case without. The baseline parameters listed in Table 1 are used.

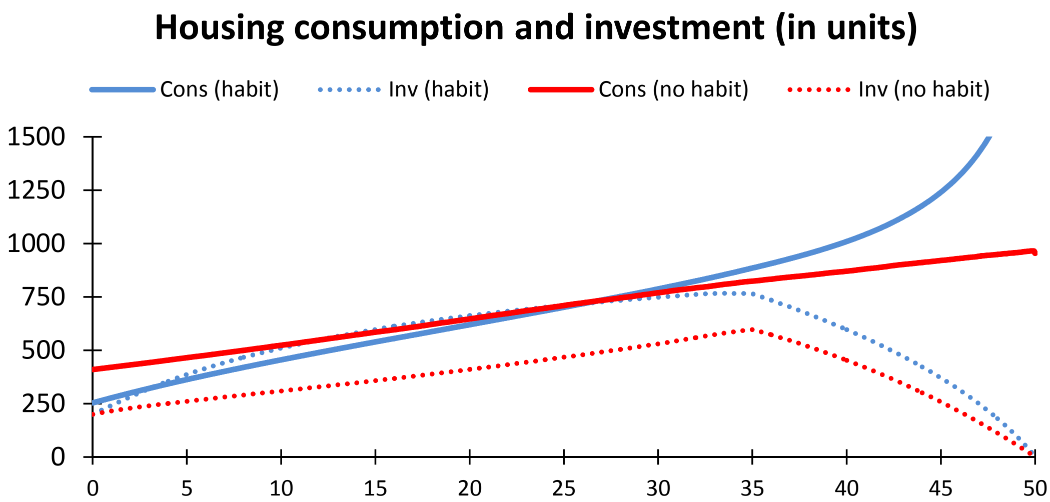

Figure 10: Housing consumption and investment over the life cycle in the full model. The graphs show averages across 10,000 simulations in which the consumption and investment strategy determined by our numerical method is used. The blue curves are for the case with a housing habit, the red curves are for the case without. The baseline parameters listed in Table 1 are used.

perishable and housing consumption to changes in wealth. For this purpose we have computed the marginal propensities to consume (MPC) out of wealth from the closed-form solution in the artificial market corresponding to our best feasible strategy. We find that the MPC of perishable consumption is 14.2 times bigger than the MPC of housing consumption early in life with this factor decreasing to 2.2 towards the end of life (our numerical solution produces identical results). These numbers are similar to the deterministic case and confirm that our model explains why households adjust perishable consumption much more than housing consumption in response to wealth shocks.

6 Robustness of results

Sections 6.1–6.6 illustrate the sensitivity of our results to the selected key parameters. Section 6.7 summarizes the results of an extension to mortality risk and bequest.

6.1 The habit strength

Recall that the habit dynamics involve the two parameters ε and α with $\varepsilon - \alpha$ representing the strength of the habit: if $\varepsilon - \alpha$ is small, the habit only declines very slowly even with minimum housing consumption and therefore restricts the agent more. Figure 11 illustrates the importance of the habit strength by fixing $\varepsilon = 0.9$ and considering $\alpha = 0.7$ (weak habit) and $\alpha = 0.88$ (strong habit) in addition to the baseline $\alpha = 0.8$.

The upper-left graph confirms that stronger habits imply more savings, and the upper-right graph shows that the extra saving comes mainly by larger housing investments. The amount invested in the stock is also increased, especially in the years leading up to retirement, as shown in the panels in the middle. A stronger habit leads to the portfolio share of the stock being higher before retirement and lower after retirement. The lower-left panel reveals that the life-cycle pattern in perishable consumption is similar for the different habit strengths, but with the strong habit the pattern is more curved and the peak appears earlier in life, as also noticed in the deterministic model of Section 3.1. The housing expenditure share in the lower-right panel is increasing through life, but with a weak or modest habit the share is almost constant for a long mid-life period.

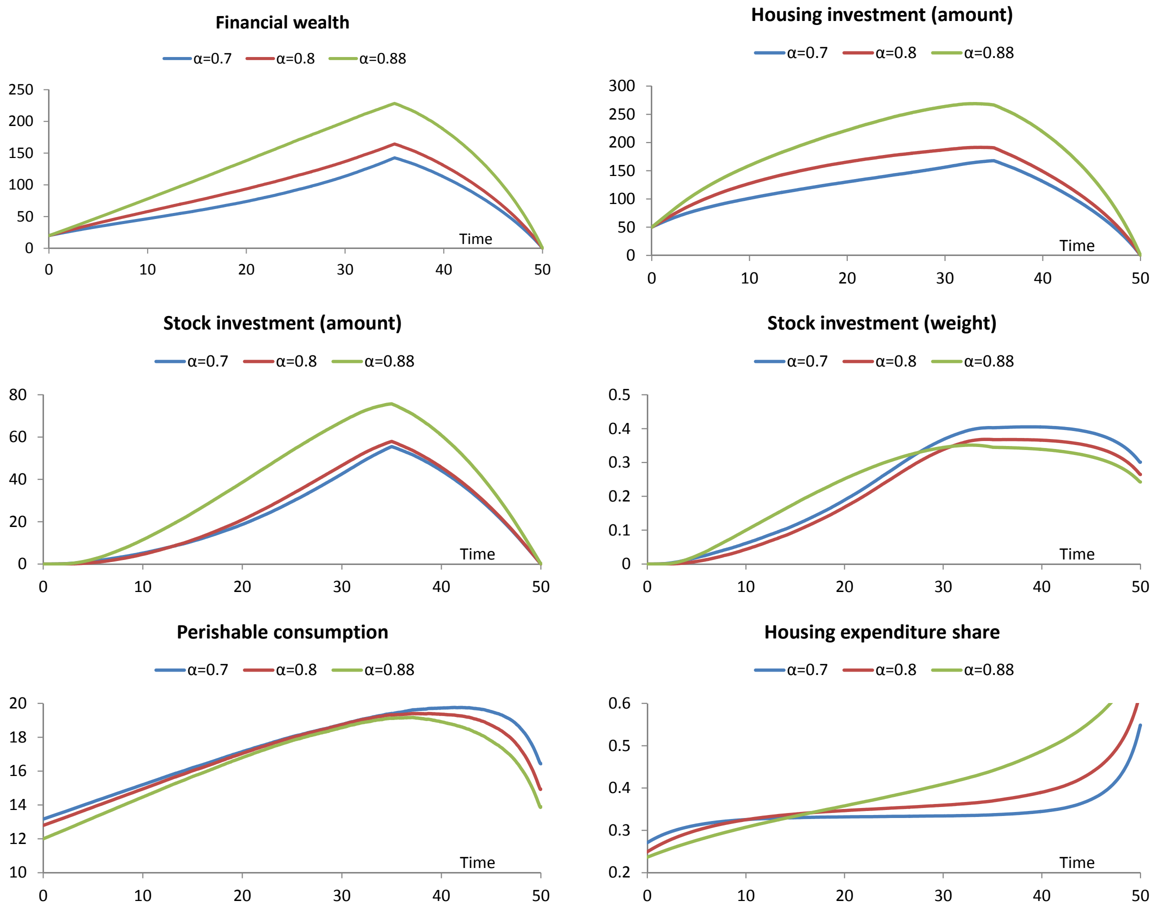

Figure 11: Consumption and investments for different habit strengths. The case of $\alpha = 0.8$ is our baseline case, whereas $\alpha = 0.7$ represents a weaker habit and $\alpha = 0.88$ a stronger habit. For all other parameters the baseline values in Table 1 are used.

6.2 The access to collateralized borrowing

In the baseline case the agent could borrow up to 60% of the value of her investment position in the housing market, corresponding to $\kappa = 0.4$. In Figure 12 we compare this with $\kappa = 0.2$ (borrowing up to 80% of house value) and $\kappa = 1$ (no borrowing at all).

In most situations, the agent optimally exploits the borrowing capacity early in life so that access to more borrowing also generates more borrowing in combination with a higher house investment. The upper-right panel confirms that the attractiveness of housing as an investment asset is to a large extent due to the associated access to credit. If the agent cannot borrow at all, she optimally invests a significant share of wealth in the stock market already in the early years as illustrated in the lower-left panel.

Perishable consumption is significantly affected by the access to collateralized borrowing. First, the level of consumption throughout life increases when the borrowing

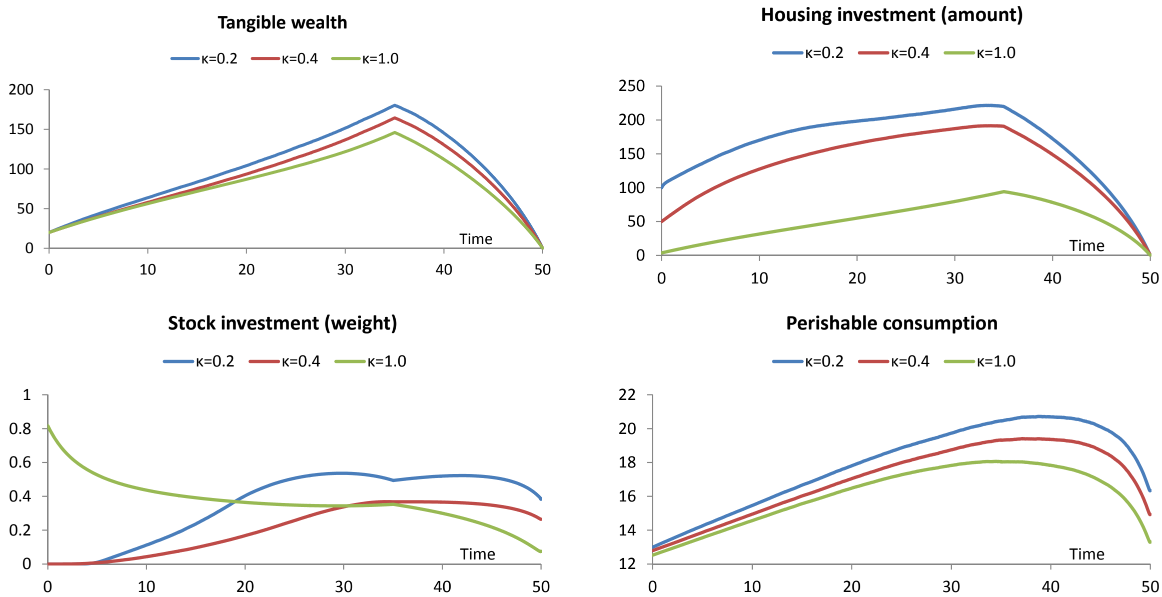

Figure 12: Consumption and investments for different borrowing limits. The case of $\kappa = 0.4$ is our baseline case, whereas $\kappa = 0.2$ allows for more borrowing and $\kappa = 1$ prohibits any borrowing. For all other parameters the baseline values in Table 1 are used.

constraint is relaxed. Secondly, perishable consumption peaks later when borrowing is allowed. Various papers explain the consumption hump by borrowing constraints and have formulated and calibrated models in which optimal consumption is indeed hump shaped typically with a peak around retirement, cf., e.g., Gourinchas and Parker (2002) and Cocco, Gomes, and Maenhout (2005). However, these models restrict the agent from any borrowing, whereas real-life homeowners have access to some collateralized borrowing. The lower-right panel shows that the access to borrowing postpones the peak age of consumption significantly. The housing habit in our model generates the hump-shaped perishable consumption with or without borrowing. As discussed earlier, a stronger habit leads to an earlier peak in consumption. Mortality risk can also contribute to a relatively early peak as was illustrated in the simple model in Section 3.1, and we confirm this for our full model in Section 6.7.

6.3 Risk aversion

Next we investigate how the risk aversion coefficient γ affects the results. As this depends on the access to borrowing, Figure 13 considers γ-values of 3 (baseline value) and 5 together with $\kappa = 0.4$ (baseline value, 60% borrowing) and $\kappa = 1$ (no borrowing). As expected a higher risk aversion coefficient induces the agent to save more and consume less early in

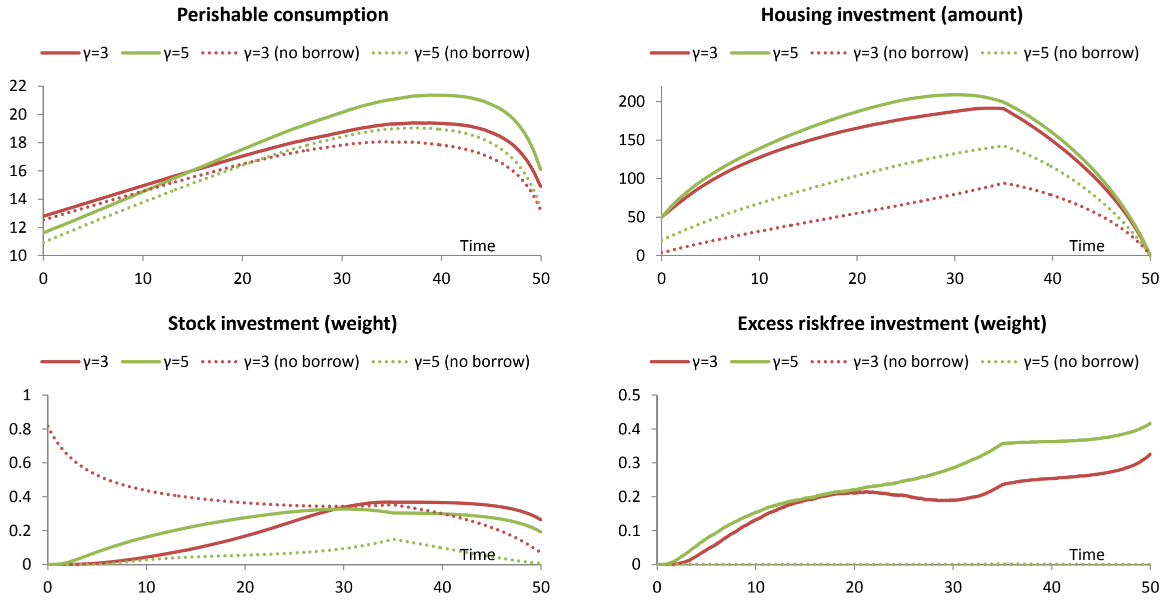

Figure 13: Consumption and investments for different degrees of risk aversion. Each panel considers the four combinations arising from (i) $\gamma = 3$ or $\gamma = 5$ and (ii) $\kappa = 0.4$ or $\kappa = 1$ (no borrowing). For all other parameters the baseline values in Table 1 are used.

life, whereas later in life the higher savings finance a larger consumption. Therefore, the perishable consumption profile is steeper and peaks later when the risk aversion coefficient is increased from 3 to 5. These observations hold with or without borrowing.

The impact on optimal investments of an increase in risk aversion is more intriguing. The amounts invested in housing and stocks increase at least before retirement and borrowing also increases in conjunction with the larger house investment. In terms of fractions of wealth invested, the weight of the stock increases, while the weight of the housing investment decreases. As shown by the lower-right panel, the difference between the portfolio weight of the riskfree asset and its minimum value (defined by $-(1-\kappa)$ times the weight of the housing investment) increases with the risk aversion, so in that sense borrowing is reduced in relative terms. The increase in risk aversion thus leads the agent to partially replace the leveraged housing investment by arguably less risky stock investments. Without borrowing, the same increase in risk aversion leads to a significantly higher housing investment and a significantly lower stock investment, both in absolute and relative terms. Here stock investments are replaced by the less risky non-leveraged housing investments. Again, it is striking to see how sensitive the optimal stock weight is to both the access to borrowing and the risk aversion coefficient.

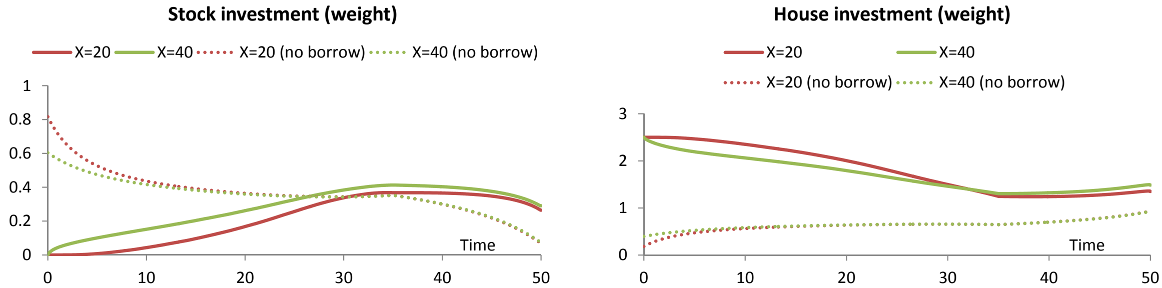

Figure 14: Portfolio weights and initial wealth. The red curves assume an initial financial wealth of $X_0 = 20$ as in the baseline case, whereas the green curves assume an initial wealth of $X_0 = 40$. The solid curves are for the baseline case of $\kappa = 0.4$ (60% borrowing of house value), whereas the dotted curves are for the case $\kappa = 1$ (no borrowing). For all other parameters the baseline values in Table 1 are used.

6.4 Initial wealth

How sensitive is the optimal portfolio to the initial financial wealth X_0? The left panel of Figure 14 shows that when collateralized borrowing is allowed, an increase in the initial wealth leads to an increase in the optimal stock weight. This relation is seen in the data (Wachter and Yogo 2010), but is inconsistent with the basic Samuelson-Merton model and various extensions thereof. The optimal house weight is decreasing in initial wealth. These effects can be explained by the fact that, when wealth is increased, the habit-induced wealth buffer seizes a smaller share of the sum of financial and human wealth. Notably, initial wealth has the opposite effect on the portfolio if collateralized borrowing is impossible as shown by the dotted curves in the figure. Higher wealth raises optimal future housing consumption, and hedging considerations may rationalize a higher housing investment.

6.5 Labor income depending on age and education

Above we assumed a constant expected income growth rate in the working phase, but income is known to grow faster for young adults than for more mature adults. We follow Cocco, Gomes, and Maenhout (2005) and use their estimated income profiles for three different educational levels labeled ‘no high school’, ‘high school’, and ‘college.’ Their estimated profiles determine $\mu_Y(t)$ in our model, cf. Munk and Sørensen (2010). In addition, the initial income is set to 20 for the agent with no high school, 25 with high school, and 29 with a college degree. These values roughly correspond to the average income of a 30-year old in the different educational groups according to Cocco, Gomes, and Maenhout

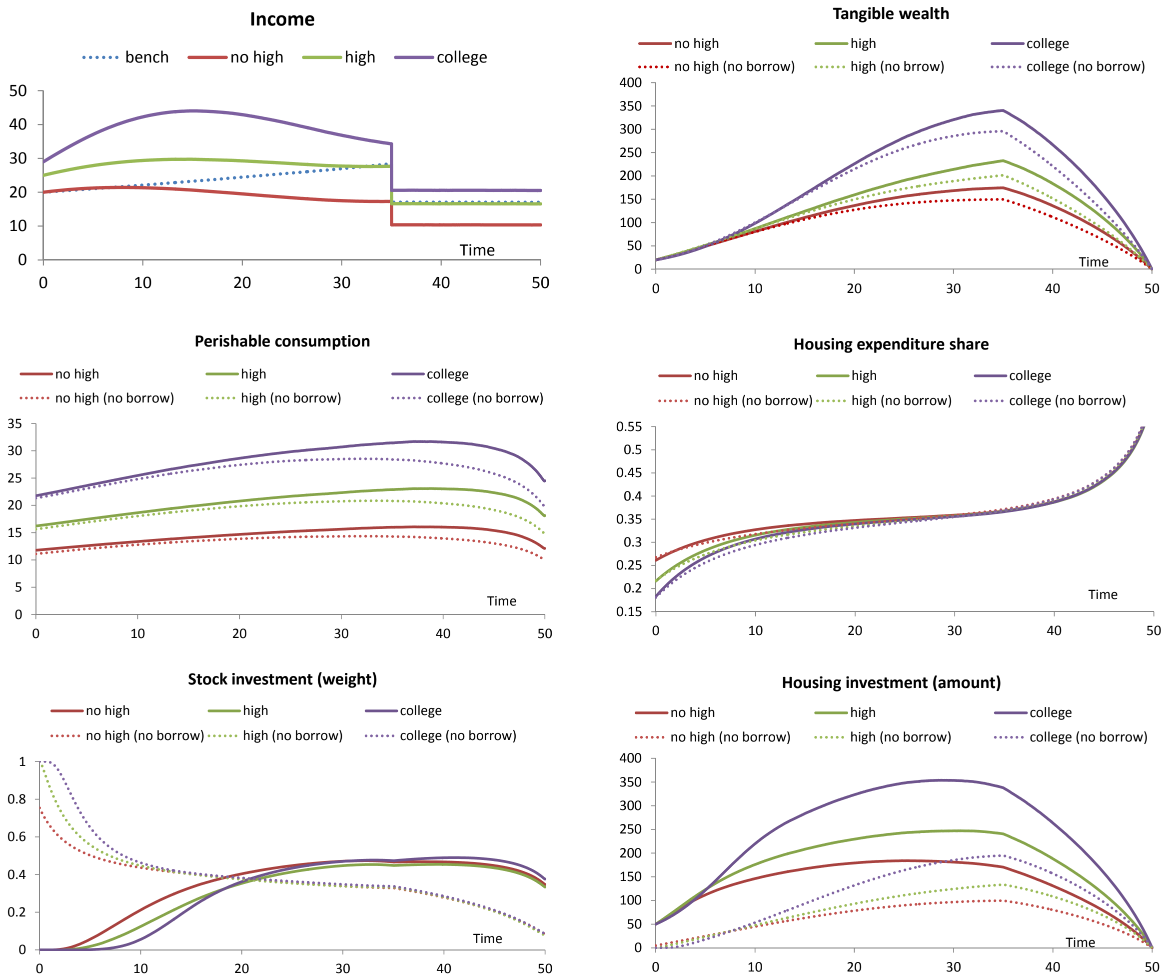

Figure 15: Consumption and investments for different educational levels. The average income profiles estimated by Cocco, Gomes, and Maenhout (2005) for three different educational groups are used. The solid curves are for the baseline case of $\kappa = 0.4$ (60% borrowing of house value), whereas the dotted curves are for the case $\kappa = 1$ (no borrowing). For all other parameters the baseline values in Table 1 are used.

(2005). The other parameters are unchanged compared to the baseline case.

Figure 15 depicts the average income profiles in the upper-left panel. The other panels show how wealth, consumption, and investments vary across the three profiles, both for the case where the agent can borrow up to 60% of house value (corresponding to $\kappa = 0.4$) and the case where the agent cannot borrow at all ($\kappa = 1$). The overall shape of the perishable consumption curve is the same for all three educational levels, but of course the level of consumption increases with income and thus education. The level of housing consumption also increases with education, but the expenditure share of housing is initially lowest for college graduates and highest for individuals without high school education, which is consistent with the shares across income quintiles reported in the Introduction. This can be understood from Eq. (11), which shows that the housing expenditure share in the deterministic version of the model is increasing in the ratio $\bar{q}_t/q_t$ of the habit level to current consumption. We assume the same initial habit level for all three educational groups, and since optimal housing consumption increases with education, this ratio and hence the housing expenditure share decrease with education. Eventually, the housing expenditure shares for the three groups converge. Of course, results may be different if the initial habit level or the habit strength depend on the education level.

In the early years, the college graduate saves very little and therefore invests less than the other individuals. After a few years, the college graduate starts saving more than the others, at first primarily through housing investments. In fact, the college graduate enters the stock market later than the other individuals, and while the college graduate eventually has the largest amount invested in stocks, the share of wealth invested in the stock is very similar (first slightly lower, then slightly higher) to the other individuals. Note again that the investment pattern is very different if borrowing is prohibited. In this case, all three individuals participate in the stock market from the beginning with the college graduate holding the largest share of wealth in stocks. Overall, with or without access to borrowing, the optimal portfolio weights vary only modestly with education level.

6.6 Correlations

In the baseline case we used the pairwise stock-house-income correlations estimated from the aggregate data in the solution of the individual's problem. Now we consider the effect of setting each of these correlations to zero. In Figure 16 the purple curves show the results of our model for $\rho_{YS} = 0$, a value often used in the literature, instead of the baseline value of 0.22. Since income is now less stock-like, the portfolio is more tilted towards stocks and away from real estate, although the individual still stays out of the stock market in the early years. Perishable consumption is somewhat lower and peaks a little earlier than in the benchmark case. If the baseline income-house correlation ρ_{HY} of 0.16 is replaced by zero, the portfolio is naturally tilted somewhat to houses and thus away from stocks. Finally, replacing the baseline house-stock correlation ρ_{HS} of 0.25 by zero improves diversification so the individual generally invests slightly more in both stocks and houses, and consumption is marginally increased.

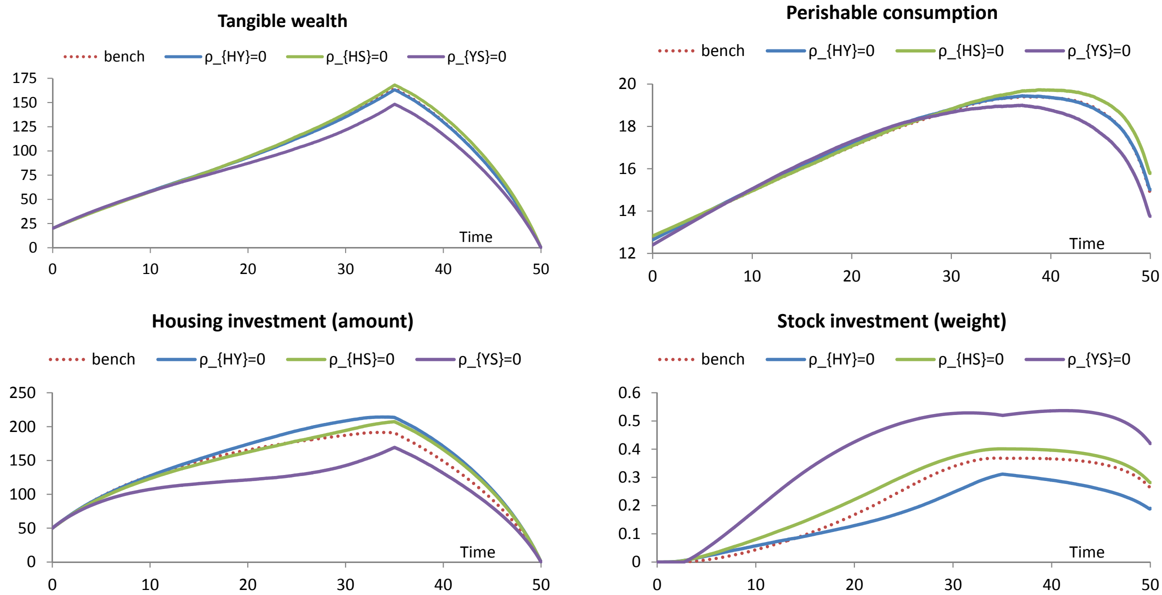

Figure 16: Consumption and investments for different correlations. Each panel compares the baseline case with the three cases where each of the correlation parameters ρ_{HY}, ρ_{HS}, and ρ_{YS} have been set to zero, but all other parameters assume the baseline values listed in Table 1. The dotted curve for the baseline case is hardly visible in the upper panels as it is almost indistinguishable from the blue curve.

6.7 An extension to mortality risk and bequests

We extend the main model to mortality risk and bequests just as we did for the simple model in Section 3.1. Appendix B presents a closed-form solution to the extended problem in any of the artificial markets with deterministic modifiers. In conjunction with the SAMS approach, this leads to the near-optimal strategies for the extended problem in the true market.

The extension of the model has a limited impact on the optimal behavior. With a preference for bequest, more wealth needs to be build up and maintained even late in retirement, which leads to larger amounts invested in both stocks and housing units, but the portfolio weights are very similar to the baseline case in Figure 8. Mortality risk and bequest preferences have more interesting effects on life-cycle consumption. Figure 17 shows six curves of perishable consumption over life. The three solid curves are for the case with collateralized borrowing ($\kappa = 0.4$) and the corresponding dotted curves for the no-borrowing case ($\kappa = 1$). The red curves are for no habit, the blue curves for the baseline habit strength ($\alpha = 0.8$), and the green curves for a stronger habit ($\alpha = 0.88$). We assume a bequest preference weight of $w = 10$, which leads to a sizeable bequest; for example, in

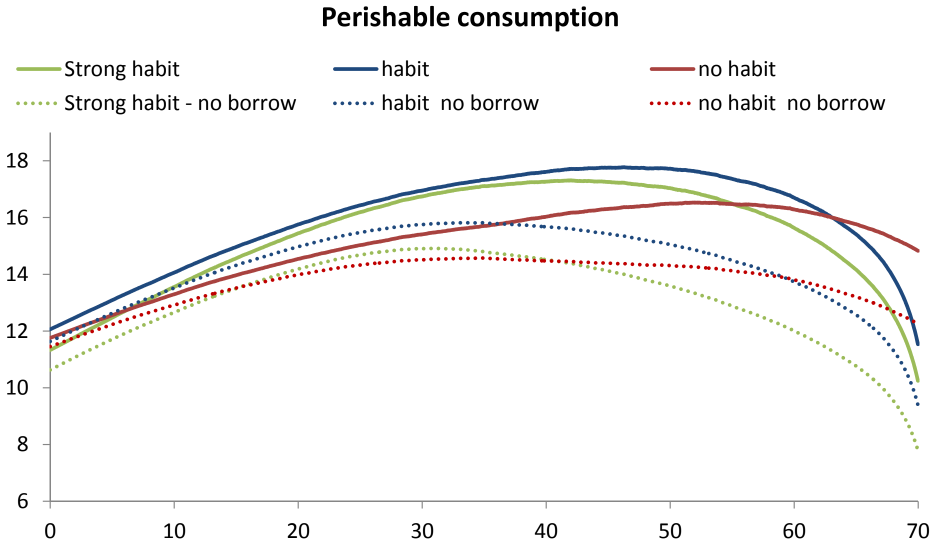

Figure 17: Perishable consumption over the life cycle with mortality risk and bequest. The graphs show averages across 10,000 simulations in which the consumption and investment strategy determined by our numerical method is used. The green curves are for a strong habit ($\alpha = 0.88$), the blue curves for the baseline habit strength ($\alpha = 0.8$), and the red curves for the case without habit formation. For each color the solid curve is for the case with collateralized borrowing ($\kappa = 0.4$) and the dotted curve for the case without borrowing ($\kappa = 1$). The mortality rates are derived from the 2009 life table of the total U.S. population. We assume an insurance price parameter of $\Gamma = 0.8$ and a preference weight on bequest of $w = 10$. For other parameters the baseline values listed in Table 1 are used.

the standard case with $\alpha = 0.8$ and $\kappa = 0.4$, the average bequest is 260.5 (thousand USD) should the agent survive until age 100. Comparing the three solid curves, we see again how a stronger housing habit leads to an earlier peak in consumption. This is also true if borrowing is disallowed as revealed by a comparison of the dotted curves. The figure also confirms that a relatively early hump can be obtained without habits if borrowing is impossible which, however, is a questionable premise. With access to borrowing, the housing habit can restore a hump around the retirement age.

7 Conclusion

We have solved for and investigated the optimal life-cycle consumption and investment decisions in a rich model with many realistic features. In particular, we include preferences for both perishable consumption and consumption of housing services with habit formation for housing. We have provided closed-form solutions for simplified settings to build intuition and used an innovative numerical method to solve the full model with portfolio constraints and unspanned income risk.

Our results show that the model can generate various stylized facts that seem puzzling through the lens of standard life-cycle models. First, stock investments are low or zero for many young agents and then gradually increasing over life. Housing investments crowd out stock investments, in particular for young agents, because housing investments (i) provide access to borrowing, (ii) hedge against increases in the price of housing consumption, (iii) ensure that the agent can obtain the future minimum housing consumption generated by her habits, and (iv) are not a bad investment from a purely speculative view. Secondly, our model generates an age- and wealth-dependent housing expenditure share instead of the counter-factually constant share found in standard models. Thirdly, in our model perishable consumption is much more sensitive to wealth and income shocks than housing consumption, in particular for young agents. Finally, the housing habit helps in explaining the hump-shaped life-cycle pattern in non-housing consumption.

A Proofs

A.1 Proof of Theorem 1

The Bellman equation is

$$0 = \mathcal{L}_1 J + \mathcal{L}_2 J,$$

where

$$\mathcal{L}_1 J = \sup_{c,q} \left\{ -cJ_x - (q-\bar{q})h\chi J_x + \frac{1}{1-\gamma} c^{b(1-\gamma)} (q-\bar{q})^{(1-b)(1-\gamma)} + \alpha(q-\bar{q})J_{\bar{q}} \right\}, \tag{38}$$

$$\mathcal{L}_2 J = J_t + rxJ_x + yJ_x + (r+\mu_H)hJ_h + \mu_Y yJ_y - \bar{q}h\chi J_x + (\alpha-\varepsilon)\bar{q}J_{\bar{q}} - \delta J. \tag{39}$$

The first-order conditions imply

$$\begin{aligned} bc^{b(1-\gamma)-1}(q-\bar{q})^{(1-b)(1-\gamma)} &= J_x, \\ (1-b)c^{b(1-\gamma)}(q-\bar{q})^{(1-b)(1-\gamma)-1} &= \chi h J_x - \alpha J_{\bar{q}}, \end{aligned}$$

and by dividing one by the other we see that

$$c = \frac{b}{1-b}(q-\bar{q})\left(\chi h - \alpha \frac{J_{\bar{q}}}{J_x}\right).$$

Substituting this relation back into one of the first-order conditions we obtain

$$c = b^{\frac{1}{\gamma}} \left(\frac{b}{1-b}\right)^{\frac{(1-b)(\gamma-1)}{\gamma}} \left(\chi h - \alpha \frac{J_{\bar{q}}}{J_x}\right)^{\frac{(1-b)(\gamma-1)}{\gamma}} J_x^{-\frac{1}{\gamma}} = b\,\hat{b}\left(\chi h - \alpha \frac{J_{\bar{q}}}{J_x}\right)^k J_x^{-\frac{1}{\gamma}},$$

and subsequently we get

$$q = \bar{q} + (1-b)\hat{b}\left(\chi h - \alpha \frac{J_{\bar{q}}}{J_x}\right)^{k-1} J_x^{-\frac{1}{\gamma}}.$$

After substitution of the optimal controls it follows that

$$\mathcal{L}_1 J = \frac{\gamma}{1-\gamma}\hat{b}\left(\chi h - \alpha \frac{J_{\bar{q}}}{J_x}\right)^k J_x^{\frac{\gamma-1}{\gamma}}. \tag{40}$$

Conjecture a value function of the form (4). With $\hat{x} = x + yF(t) - \bar{q}\chi hB(t)$, the relevant derivatives are

$$J_t = \hat{x}^{-\gamma}(\chi h)^{\gamma k} G^{\gamma-1}\left[\frac{\gamma}{1-\gamma}\hat{x}G' + GyF' - \bar{q}\chi hB'G\right], \quad J_{\bar{q}} = -(\chi h)^{1+\gamma k} B\hat{x}^{-\gamma}G^{\gamma},$$
$$J_x = \hat{x}^{-\gamma}G^{\gamma}(\chi h)^{\gamma k}, \quad J_y = \hat{x}^{-\gamma}(\chi h)^{\gamma k}G^{\gamma}F, \quad J_h = \hat{x}^{-\gamma}(\chi h)^{\gamma k}h^{-1}G^{\gamma}\left[\frac{k\gamma}{1-\gamma}\hat{x} - \bar{q}\chi hB\right],$$

from which (9) follows. Note that $J_{\bar{q}}/J_x = -h\chi B(t)$. Now substitute the derivatives into (39) and (40), replace rx by $r\hat{x} - ryF + r\bar{q}\chi hB$, and reduce to get

$$\mathcal{L}_1 J = \hat{x}^{1-\gamma}(\chi h)^{\gamma k}G^{\gamma-1}\frac{\gamma}{1-\gamma}\hat{b}(1+\alpha B(t))^k, \tag{41}$$

$$\begin{aligned}\mathcal{L}_2 J = \hat{x}^{-\gamma}(\chi h)^{\gamma k}G^{\gamma-1}\Bigg\{&\frac{\gamma}{1-\gamma}\hat{x}\left[G'(t) - \left(\frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}r - (r+\mu_H)k\right)G(t)\right] \\ &+ G(t)y\left[F'(t) + (\mu_Y(t) - r)F(t) + 1\right] \\ &- \bar{q}hG(t)\left[B'(t) - (\varepsilon - \alpha - \mu_H)B(t) + 1\right]\Bigg\}.\end{aligned} \tag{42}$$

After adding up and dividing by $\hat{x}^{-\gamma}(\chi h)^{\gamma k}G^{\gamma-1}$, each remaining term is either a multiple of $\hat{x}$ or does not involve $\hat{x}$. Collecting the terms involving $\hat{x}$, we conclude that G must satisfy

$$0 = G'(t) - r_G G(t) + \hat{b}(1+\alpha B(t))^k, \tag{43}$$

where r_G is given by (8). With $G(T) = 0$ (due to no bequests), (43) has the solution (6). The terms not involving $\hat{x}$ add up to zero if

$$0 = F'(t) - (r - \mu_Y(t))F(t) + 1, \qquad 0 = B'(t) - r_B B(t) + 1 \tag{44}$$

where r_B is given by (8). With $F(T) = B(T) = 0$, the solutions are as stated in (7) and (5).

Next we derive the dynamics of perishable consumption. By applying (44) as well as the optimal controls (9), we obtain

$$\begin{aligned} d\widehat{X}_t &= dX_t + F(t)\,dY_t + Y_tF'(t)\,dt - \chi H_tB(t)\,d\bar{q}_t - \bar{q}_t\chi B(t)\,dH_t - \bar{q}_t\chi H_tB'(t)\,dt \\ &= r\widehat{X}_t\,dt - c_t\,dt - (q_t - \bar{q}_t)\chi H_t(1+\alpha B(t))\,dt \\ &= \widehat{X}_t\left(r - \frac{\hat{b}(1+\alpha B(t))^k}{G(t)}\right)dt. \end{aligned}$$

From (9), it follows that

$$
\begin{aligned}
dc_t &= c_t \frac{d\widehat{X}_t}{\widehat{X}_t} + c_t \left(\frac{k\alpha B'(t)}{1+\alpha B(t)} - \frac{G'(t)}{G(t)} \right) dt \\
&= c_t \left(r - r_G + \frac{k\alpha B'(t)}{1+\alpha B(t)} \right) dt \\
&= c_t \left(\frac{r-\delta}{\gamma} + k \left[r + \mu_H + \frac{\alpha B'(t)}{1+\alpha B(t)} \right] \right) dt,
\end{aligned}
$$

where we have used (43). Note that $B'(t) = -e^{-r_B(T-t)} < 0$. The growth rate $\mu_c(t)$ is initially positive provided

$$
\frac{r-\delta}{\gamma} \geq k \left[-\frac{\alpha B'(0)}{1+\alpha B(0)} - (r+\mu_H) \right] = k \left[\frac{\alpha r_B}{(\alpha + r_B)e^{r_B T} - \alpha} - (r+\mu_H) \right].
$$

Since $B''(t) = r_B B'(t)$, we get

$$
\mu_c'(t) = k\alpha \frac{B''(t)[1+\alpha B(t)] - \alpha [B'(t)]^2}{(1+\alpha B(t))^2} = k\alpha \frac{(r_B + \alpha)B'(t)}{(1+\alpha B(t))^2} = k\alpha \frac{(\varepsilon - \mu_H)B'(t)}{(1+\alpha B(t))^2},
$$

which is negative if $\mu_H < \varepsilon$. Hence the consumption growth rate can eventually turn negative and will do so provided

$$
\frac{r-\delta}{\gamma} \leq -k \left(r + \mu_H + \frac{\alpha B'(T)}{1+\alpha B(T)} \right) = k(\alpha - [r+\mu_H]).
$$

This completes the proof.

A.2 Proof of Theorem 2

It is useful to introduce some vector-matrix notation. Let

$$
\Pi = \begin{pmatrix} \Pi_S \\ \Pi_H \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_S \\ \mu_H + \chi - m \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_S & 0 \\ \sigma_H \rho_{HS} & \sigma_H \sqrt{1-\rho_{HS}^2} \end{pmatrix},
$$

$$
\vec{\rho}_H = \begin{pmatrix} \rho_{HS} \\ \sqrt{1-\rho_{HS}^2} \end{pmatrix}, \quad \vec{\rho}_Y = \begin{pmatrix} \rho_{YS} \\ \sqrt{1-\rho_{YS}^2}. \end{pmatrix}
$$

For later use, note that

$$(\Sigma\Sigma^\top)^{-1}\mu = \frac{1}{1-\rho_{HS}^2}\begin{pmatrix} \frac{\lambda_S-\rho_{HS}\lambda_H}{\sigma_S} \\ \frac{\lambda_H-\rho_{HS}\lambda_S}{\sigma_H} \end{pmatrix}, \tag{45}$$

$$(\Sigma^\top)^{-1}\vec{\rho}_H = \begin{pmatrix} 0 \\ \frac{1}{\sigma_H} \end{pmatrix},$$

$$(\Sigma^\top)^{-1}\vec{\rho}_Y = \begin{pmatrix} \frac{\rho_{YS}-\rho_{HS}\tilde{\rho}_{HY}}{\sigma_S} \\ \frac{\tilde{\rho}_{HY}}{\sigma_H} \end{pmatrix}, \tag{46}$$

$$\Lambda^2 \equiv \mu^\top(\Sigma\Sigma^\top)^{-1}\mu = \lambda_S^2 + \frac{1}{1-\rho_{HS}^2}\left(\lambda_H - \rho_{HS}\lambda_S\right)^2, \tag{47}$$

$$\mu^\top(\Sigma^\top)^{-1}\vec{\rho}_H = \lambda_H,$$

$$\mu^\top(\Sigma^\top)^{-1}\vec{\rho}_Y = \lambda_S\rho_{YS} + \tilde{\rho}_{HY}\left(\lambda_H - \rho_{HS}\lambda_S\right). \tag{48}$$

The Hamilton-Jacobi-Bellman (HJB) equation in this case can then be written as

$$0 = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J + \mathcal{L}_4 J, \tag{49}$$

where $\mathcal{L}_1 J$ and $\mathcal{L}_2 J$ are given by (38) and (39), and

$$\mathcal{L}_3 J = \frac{1}{2}\sigma_H^2 h^2 J_{hh} + \frac{1}{2}\sigma_Y^2 y^2 J_{yy} + \rho_{HY}\sigma_H\sigma_Y hy J_{hy}, \tag{50}$$

$$\mathcal{L}_4 J = \sup_\Pi \left\{ J_x x \Pi^\top \mu + \frac{1}{2} J_{xx} x^2 \Pi^\top \Sigma\Sigma^\top \Pi + J_{xh} xh\sigma_H \Pi^\top \Sigma\vec{\rho}_H + J_{xy} xy\sigma_Y \Pi^\top \Sigma\vec{\rho}_Y \right\}. \tag{51}$$

The first-order condition for Π implies

$$\Pi = -\frac{J_x}{xJ_{xx}}(\Sigma\Sigma^\top)^{-1}\mu - \frac{yJ_{xy}}{xJ_{xx}}\sigma_Y(\Sigma^\top)^{-1}\vec{\rho}_Y - \frac{hJ_{xh}}{xJ_{xx}}\sigma_H(\Sigma^\top)^{-1}\vec{\rho}_H.$$

Substituting this back into $\mathcal{L}_4 J$, we find after long and tedious computations that

$$\begin{aligned}\mathcal{L}_4 J = &-\frac{1}{2}\frac{J_x^2}{J_{xx}}\mu^\top(\Sigma\Sigma^\top)^{-1}\mu - \frac{1}{2}\frac{y^2J_{xy}^2}{J_{xx}}\sigma_Y^2 - \frac{yJ_xJ_{xy}}{J_{xx}}\sigma_Y\mu^\top(\Sigma^\top)^{-1}\vec{\rho}_Y \\ &-\frac{1}{2}\frac{h^2J_{xh}^2}{J_{xx}}\sigma_H^2 - \frac{hJ_xJ_{xh}}{J_{xx}}\sigma_H\mu^\top(\Sigma^\top)^{-1}\vec{\rho}_H - \frac{yhJ_{xy}J_{xh}}{J_{xx}}\sigma_Y\sigma_H\rho_{HY}.\end{aligned}$$

As in the deterministic model we conjecture that J is of the form (4) for some functions G, F, B. The derivatives J_t, J_x, $J_{\bar{q}}$, J_y, and J_h are as stated in Appendix A.1. In addition, we now need the following derivatives:

$$
\begin{aligned}
J_{xx} &= -\gamma(\chi h)^{k\gamma} G^{\gamma} \hat{x}^{-\gamma-1}, \\
J_{yy} &= -\gamma(\chi h)^{k\gamma} G^{\gamma} \hat{x}^{-\gamma-1} F^2, \\
J_{hh} &= k\gamma(\chi h)^{k\gamma} G^{\gamma} \hat{x}^{-\gamma} h^{-2} \left(\frac{k\gamma - 1}{1-\gamma} \hat{x} - 2\bar{q}\chi h B + (\chi h \bar{q} B)^2 k^{-1} \hat{x}^{-1} \right), \\
J_{xy} &= -\gamma(\chi h)^{k\gamma} G^{\gamma} \hat{x}^{-\gamma-1} F, \\
J_{xh} &= \gamma(\chi h)^{k\gamma} G^{\gamma} \hat{x}^{-\gamma-1} h^{-1} \left(k\hat{x} + \bar{q}\chi h B \right), \\
J_{yh} &= \gamma(\chi h)^{k\gamma} G^{\gamma} \hat{x}^{-\gamma-1} h^{-1} F \left(k\hat{x} + \bar{q}\chi h B \right).
\end{aligned}
$$

Then

$$
\Pi = \frac{1}{\gamma}\frac{\hat{x}}{x}(\Sigma\Sigma^{\top})^{-1}\mu - \frac{yF}{x}\sigma_Y(\Sigma^{\top})^{-1}\vec{\rho}_Y + \left(k\frac{\hat{x}}{x} + \frac{\bar{q}\chi h B}{x} \right) \sigma_H (\Sigma^{\top})^{-1}\vec{\rho}_H,
$$

which leads to (18)–(19) by using (45)–(46). Furthermore,

$$
\begin{aligned}
\mathcal{L}_4 J = (\chi h)^{k\gamma} G^{\gamma} \hat{x}^{-\gamma-1} \Bigg\{ & \hat{x}^2 \Big[\frac{1}{2\gamma}\mu^{\top}(\Sigma\Sigma^{\top})^{-1}\mu + \frac{k^2\gamma}{2}\sigma_H^2 + k\sigma_H \mu^{\top}(\Sigma^{\top})^{-1}\vec{\rho}_H \Big] \\
& + \hat{x} \Big[\chi h \bar{q} B \sigma_H \left(k\gamma\sigma_H + \mu^{\top}(\Sigma^{\top})^{-1}\vec{\rho}_H \right) \\
& \qquad - yF\sigma_Y \left(k\gamma\sigma_H\rho_{HY} + \mu^{\top}(\Sigma^{\top})^{-1}\vec{\rho}_Y \right) \Big] \\
& + \frac{\gamma}{2}(yF)^2\sigma_Y^2 + \frac{\gamma}{2}\bar{q}\chi h B \sigma_H \left(\bar{q}\chi h B \sigma_H - 2yF\sigma_Y\rho_{HY} \right) \Bigg\}, \\
\mathcal{L}_3 J = (\chi h)^{k\gamma} G^{\gamma} \hat{x}^{-\gamma-1} \Bigg\{ & \hat{x}^2 \frac{k\gamma(k\gamma-1)}{2(1-\gamma)}\sigma_H^2 + \hat{x} k\gamma\sigma_H \Big[yF\rho_{HY}\sigma_Y - \bar{q}\chi h B \sigma_H \Big] \\
& - \frac{\gamma}{2}\sigma_H^2 \chi^2 h^2 \bar{q}^2 B^2 - \frac{\gamma}{2}\sigma_Y^2 y^2 F^2 + \gamma y F \chi h \bar{q} B \rho_{HY}\sigma_H\sigma_Y \Bigg\}.
\end{aligned}
$$

By adding up $\mathcal{L}_3 J$ and $\mathcal{L}_4 J$, various terms cancel, and we are left with

$$
\begin{aligned}
\mathcal{L}_3 J + \mathcal{L}_4 J = (\chi h)^{k\gamma} G^{\gamma} \hat{x}^{-\gamma} \Bigg\{ & \hat{x} \Big[\frac{1}{2\gamma}\mu^{\top}(\Sigma\Sigma^{\top})^{-1}\mu + \frac{k(k-1)\gamma}{2(1-\gamma)}\sigma_H^2 + k\sigma_H\mu^{\top}(\Sigma^{\top})^{-1}\vec{\rho}_H \Big] \\
& + \chi h \bar{q} B \sigma_H \mu^{\top}(\Sigma^{\top})^{-1}\vec{\rho}_H - yF\sigma_Y\mu^{\top}(\Sigma^{\top})^{-1}\vec{\rho}_Y \Bigg\}.
\end{aligned}
$$

If we substitute this as well as (41) and (42) into the HJB equation (49) and then divide by $(\chi h)^{k\gamma} G^{\gamma-1} \hat{x}^{-\gamma}$, each remaining term is either a multiple of $\hat{x}$ or does not involve $\hat{x}$ at all. Collecting the terms involving $\hat{x}$ we conclude that G must satisfy

$$0 = G'(t) - r_G G(t) + \hat{b}(1 + \alpha B(t))^k,$$

where r_G is now given by (15) if we use Eqs. (47)–(48). With $G(T) = 0$ (due to no bequests), the solution is (6). The terms not involving $\hat{x}$ add up to zero if

$$0 = F'(t) - r_F(t)F(t) + 1, \qquad 0 = B'(t) - r_B B(t) + 1, \tag{52}$$

where $r_F(t)$ and r_B are given by (16) and (17). With $F(T) = B(T) = 0$, the solutions are as stated in the theorem.

Before deriving the consumption dynamics, we consider the dynamics of wealth. By substitution of the optimal portfolio into the dynamics of tangible wealth, we find

$$\begin{aligned} dX_t &= \left\{X_t\left[r + \Pi_t^\top \mu\right] + Y_t - c_t - q_t \chi H_t\right\} dt + X_t \Pi_t^\top \Sigma\, dW_t \\ &= \Big\{rX_t + \frac{1}{\gamma}\widehat{X}_t \Lambda^2 - Y_t F(t)\sigma_Y \mu^\top (\Sigma^\top)^{-1}\vec{\rho}_Y \\ &\qquad + \left(k + \bar{q}_t \chi H_t B(t)\right)\sigma_H \lambda_H + Y_t - c_t - q_t \chi H_t\Big\}\, dt \\ &\qquad + \left\{\frac{1}{\gamma}\widehat{X}_t \mu^\top \Sigma^{-1} - Y_t F(t)\sigma_Y \vec{\rho}_Y^\top + \left(k\widehat{X}_t + \bar{q}_t \chi H_t B(t)\right)\sigma_H \vec{\rho}_H\right\} dW_t. \end{aligned}$$

By applying (52) as well as the optimal consumption decisions (9), we find that the dynamics of disposable wealth are

$$\begin{aligned} d\widehat{X}_t &= dX_t + F(t)\, dY_t + Y_t F'(t)\, dt - \chi H_t B(t)\, d\bar{q}_t - \bar{q}_t \chi B(t)\, dH_t - \bar{q}_t \chi H_t B'(t)\, dt \\ &= \widehat{X}_t\left(r + \frac{1}{\gamma}\Lambda^2 + k\sigma_H \lambda_H\right) dt + \widehat{X}_t\left(\frac{1}{\gamma}\mu^\top \Sigma^{-1} + k\sigma_H \vec{\rho}_H^\top\right) dW_t \\ &\quad - c_t\, dt - (q_t - \bar{q}_t)\chi H_t (1 + \alpha B(t))\, dt \\ &= \widehat{X}_t\left(r + \frac{1}{\gamma}\Lambda^2 + k\sigma_H \lambda_H - \hat{b}\frac{(1 + \alpha B(t))^k}{G(t)}\right) dt + \widehat{X}_t\left(\frac{1}{\gamma}\mu^\top \Sigma^{-1} + k\sigma_H \vec{\rho}_H^\top\right) dW_t. \end{aligned}$$

From (9) it follows that

$$\begin{aligned} dc_t &= c_t \frac{d\widehat{X}_t}{\widehat{X}_t} + c_t\left(\frac{k\alpha B'(t)}{1 + \alpha B(t)} - \frac{G'(t)}{G(t)}\right) dt \\ &= c_t \frac{d\widehat{X}_t}{\widehat{X}_t} + c_t\left(\frac{k\alpha B'(t)}{1 + \alpha B(t)} - r_G + \hat{b}\frac{(1 + \alpha B(t))^k}{G(t)}\right) dt \end{aligned}$$

$$= c_t \left(K_c + \frac{k\alpha B'(t)}{1+\alpha B(t)} \right) dt + \left(\frac{1}{\gamma} \mu^\top \Sigma^{-1} + k\sigma_H \vec{\rho}_H^\top \right) dW_t,$$

where we have used (43) and introduced

$$K_c = \frac{r-\delta}{\gamma} + \frac{\gamma+1}{2\gamma^2}\Lambda^2 + k\left(r + \mu_H + \frac{\mu_H + \chi - m}{\gamma} + \frac{k-1}{2}\sigma_H^2 \right).$$

The expected consumption is

$$\mathrm{E}[c_t] = c_0 e^{\int_0^t \mu_c(s)\,ds}, \quad \mu_c(t) = K_c + \frac{k\alpha B'(t)}{1+\alpha B(t)} = K_c - \frac{k\alpha r_B}{(r_B+\alpha)e^{r_B(T-t)} - \alpha}.$$

Note that

$$\frac{d\,\mathrm{E}[c_t]}{dt} = \mu_c(t)\,\mathrm{E}[c_t]\,dt$$

with the sign determined exclusively by $\mu_c(t)$. Since $B'(t) = -e^{-r_B(T-t)} < 0$ and $B''(t) = r_B B'(t) = -r_B e^{-r_B(T-t)}$, we get

$$\mu_c'(t) = k\alpha \frac{B''(t)[1+\alpha B(t)] - \alpha[B'(t)]^2}{(1+\alpha B(t))^2} = k\alpha \frac{(r_B+\alpha)B'(t)}{(1+\alpha B(t))^2} = k\alpha \frac{(\varepsilon+\chi-m)B'(t)}{(1+\alpha B(t))^2},$$

which is negative if $\varepsilon + \chi - m > 0$. Then we have a hump in expected consumption provided $\mu_c(0) > 0$ and $\mu_c(T) < 0$, which is the case exactly when (20) holds, and the hump occurs when $\mu_c(t) = 0$ which implies a hump at

$$t_{\text{hump}} = T - \frac{1}{r_B}\ln\left(\frac{\alpha(K_c + kr_B)}{K_c(\alpha + r_B)} \right).$$

A.3 Proof of Lemma 1

Let $\tilde{X}_t$ denote the tangible wealth in the artificial market. In analogy with (14), the wealth dynamics generated by a strategy $(c, q, \Pi_S, \Pi_H, \Pi_I)$ are

$$\begin{aligned} d\tilde{X}_t = &\Big\{ \tilde{X}_t \left[\tilde{r}(t) + \Pi_{St}\tilde{\mu}_S(t) + \Pi_{Ht}\left(\tilde{\mu}_H(t) + \tilde{\chi}(t) - \tilde{m}(t) \right) + \Pi_{It}\nu_I(t) \right] \\ &\quad + Y_t - c_t - q_t\tilde{\chi}(t)\tilde{H}_t \Big\}\, dt + \tilde{X}_t \left(\Pi_{St}\sigma_S + \Pi_{Ht}\sigma_H\rho_{HS} + \Pi_{It}\rho_{IS} \right) dW_{St} \\ &+ \tilde{X}_t \left(\Pi_{Ht}\sigma_H\sqrt{1-\rho_{HS}^2} + \Pi_{It}\hat{\rho}_{IH} \right) dW_{Ht} + \tilde{X}_t\Pi_{It}\hat{\rho}_I\, dW_{Yt} \\ = &\Big\{ \tilde{X}_t \left[\tilde{r}(t) + \Pi_{St}\tilde{\mu}_S(t) + \Pi_{Ht}\left(\tilde{\mu}_H(t) + \chi - m \right) + \Pi_{It}\nu_I(t) \right] \\ &\quad + Y_t - c_t - q_t\chi H_t \Big\}\, dt + \tilde{X}_t \left(\Pi_{St}\sigma_S + \Pi_{Ht}\sigma_H\rho_{HS} + \Pi_{It}\rho_{IS} \right) dW_{St} \end{aligned}$$

$$+ \tilde{X}_t \left(\Pi_{Ht}\sigma_H \sqrt{1-\rho_{HS}^2} + \Pi_{It}\hat{\rho}_{IH} \right) dW_{Ht} + \tilde{X}_t \Pi_{It}\hat{\rho}_I \, dW_{Yt},$$

where the equality follows from $\tilde{\chi}(t)\tilde{H}_t = \chi H_t$ and $\tilde{\chi}(t) - \tilde{m}(t) = \chi - m$, which are true by construction. In particular, if we take a strategy (c, q, Π_S, Π_H) which is feasible in the true market and let $\Pi_{It} \equiv 0$, we have

$$d\tilde{X}_t = \left\{ \tilde{X}_t \left[\tilde{r}(t) + \Pi_{St}\tilde{\mu}_S(t) + \Pi_{Ht} \left(\tilde{\mu}_H(t) + \chi - m \right) \right] + Y_t - c_t - q_t \chi H_t \right\} dt$$
$$+ \tilde{X}_t \left(\Pi_{St}\sigma_S + \Pi_{Ht}\sigma_H \rho_{HS} \right) dW_{St} + \tilde{X}_t \Pi_{Ht}\sigma_H \sqrt{1-\rho_{HS}^2} \, dW_{Ht}.$$

A comparison with (14) reveals that the strategy leads to at least the same wealth in the artificial market as in the true market provided that

$$\tilde{r}(t) + \Pi_{St}\tilde{\mu}_S(t) + \Pi_{Ht}\tilde{\mu}_H(t) \geq r + \Pi_{St}\mu_S + \Pi_{Ht}\mu_H$$

or, equivalently,

$$\Pi_{St}\nu_S(t) + \Pi_{Ht}\nu_H(t) + \max\left(\nu_S(t)^-, \tfrac{1}{\kappa}\nu_H(t)^- \right) \geq 0 \tag{53}$$

for any $(\nu_S(t), \nu_H(t))$. We verify (53) in the following cases that cover all situations:

(i) $\nu_S(t), \nu_H(t) \geq 0$: obvious as all terms on left-hand side are non-negative

(ii) $\nu_H(t) < 0 \leq \nu_S(t)$: the left-hand side is then

$$\Pi_{St}\nu_S(t) + \Pi_{Ht}\nu_H(t) - \tfrac{1}{\kappa}\nu_H(t) = \Pi_{St}\nu_S(t) - \tfrac{1}{\kappa}\nu_H(t) \left(1 - \kappa\Pi_{Ht} \right) \geq 0$$

(iii) $\nu_S(t) < 0 \leq \nu_H(t)$: the left-hand side is then

$$\Pi_{St}\nu_S(t) + \Pi_{Ht}\nu_H(t) - \nu_S(t) = -\nu_S(t) \left(1 - \Pi_{St} \right) + \Pi_{Ht}\nu_H(t) \geq 0$$

(iv) $\nu_S(t) \leq \frac{1}{\kappa}\nu_H(t) \leq 0$: the left-hand side is then

$$\Pi_{St}\nu_S(t) + \Pi_{Ht}\nu_H(t) - \nu_S(t)$$
$$= -\nu_S(t) \left(1 - \Pi_{St} - \kappa\Pi_{Ht} \right) + \kappa\Pi_{Ht} \left(\tfrac{1}{\kappa}\nu_H(t) - \nu_S(t) \right) \geq 0$$

(v) $\frac{1}{\kappa}\nu_H(t) \le \nu_S(t) \le 0$: the left-hand side is then

$$\begin{aligned}
&\Pi_{St}\nu_S(t) + \Pi_{Ht}\nu_H(t) - \tfrac{1}{\kappa}\nu_H(t) \\
&\qquad = \Pi_{St}\left(\nu_S(t) - \tfrac{1}{\kappa}\nu_H(t)\right) - \tfrac{1}{\kappa}\nu_H(t)\left(1 - \Pi_{St} - \kappa\Pi_{Ht}\right) \ge 0.
\end{aligned}$$

A.4 Proof of Theorem 3

In this appendix we consider a given artificial market as represented by a fixed θ, cf. (28). For simplicity we notationally suppress θ. The proof is a relatively straightforward extension of the proof of Theorem 2 presented in Appendix A.2. Compared to that case, we now have an additional risky asset, namely the income derivative, and some quantities are time dependent instead of constant.

A.4.1 Setting and notation

The investor has access to a risk-free asset with a instantaneous rate of return $\tilde{r}(t)$ and three risky assets with price dynamics

$$dP_t = \mathrm{diag}(P_t)\left[(\tilde{r}(t)\mathbf{1} + \tilde{\mu}(t)) + \Sigma\, dW_t\right],$$

where $W = (W_S, W_H, W_Y)^\top$ is a three-dimensional standard Brownian motion (with independent components), and

$$\tilde{\mu}(t) = \begin{pmatrix} \tilde{\mu}_S(t) \\ \tilde{\mu}_H(t) + \chi - m \\ \nu_I(t) \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_S & 0 & 0 \\ \sigma_H\rho_{HS} & \sigma_H\sqrt{1-\rho_{HS}^2} & 0 \\ \rho_{IS} & \hat{\rho}_{IH} & \hat{\rho}_I \end{pmatrix}.$$

The net rental rate $\tilde{\chi}(t) - \tilde{m}(t) = \chi - m$ is compounded into the house price dynamics to reflect the total return on a housing investment. We introduce the correlation vectors

$$\vec{\rho}_Y = \begin{pmatrix} \rho_{YS} & \hat{\rho}_{HY} & \hat{\rho}_Y \end{pmatrix}^\top, \quad \vec{\rho}_H = \begin{pmatrix} \rho_{HS} & \sqrt{1-\rho_{HS}^2} & 0 \end{pmatrix}^\top.$$

For future use, we note that

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_S} & 0 & 0 \\ -\frac{\rho_{HS}}{\sigma_S\sqrt{1-\rho_{HS}^2}} & \frac{1}{\sigma_H\sqrt{1-\rho_{HS}^2}} & 0 \\ \frac{\rho_{HS}\hat{\rho}_{HI}-\rho_{IS}\sqrt{1-\rho_{HS}^2}}{\sigma_S\sqrt{1-\rho_{HS}^2}\hat{\rho}_I} & -\frac{\hat{\rho}_{IH}}{\sigma_H\sqrt{1-\rho_{HS}^2}\hat{\rho}_I} & \frac{1}{\hat{\rho}_I} \end{pmatrix},$$

$$\tilde{\mu}(t)^\top(\Sigma\Sigma^\top)^{-1} = \begin{pmatrix} \frac{\xi_S(t)}{\sigma_S} & \frac{\xi_H(t)}{\sigma_H} & \xi_I(t) \end{pmatrix}, \tag{54}$$

$$\tilde{\Lambda}(t)^2 \equiv \tilde{\mu}(t)^\top(\Sigma\Sigma^\top)^{-1}\tilde{\mu}(t) = \xi_S(t)\tilde{\lambda}_S(t) + \xi_H(t)\tilde{\lambda}_H(t) + \xi_I(t)\nu_I(t), \tag{55}$$

$$\vec{\rho}_Y^\top\Sigma^{-1} = \begin{pmatrix} \frac{\psi_S}{\sigma_S} & \frac{\psi_H}{\sigma_H} & \psi_Y \end{pmatrix}, \tag{56}$$

$$\vec{\rho}_H^\top\Sigma^{-1} = \begin{pmatrix} 0 & \frac{1}{\sigma_H} & 0 \end{pmatrix}, \tag{57}$$

where

$$\xi_S(t) = \tilde{\lambda}_S(t)\frac{1-\rho_{IH}^2}{(1-\rho_{HS}^2)\hat{\rho}_I^2} - \tilde{\lambda}_H(t)\frac{\rho_{HS}-\rho_{IS}\rho_{IH}}{(1-\rho_{HS}^2)\hat{\rho}_I^2} - \nu_I(t)\frac{\rho_{IS}-\rho_{HS}\rho_{IH}}{(1-\rho_{HS}^2)\hat{\rho}_I^2}, \tag{58}$$

$$\xi_H(t) = -\tilde{\lambda}_S(t)\frac{\rho_{HS}-\rho_{IS}\rho_{IH}}{(1-\rho_{HS}^2)\hat{\rho}_I^2} + \tilde{\lambda}_H(t)\frac{1-\rho_{IS}^2}{(1-\rho_{HS}^2)\hat{\rho}_I^2} - \nu_I(t)\frac{\rho_{IH}-\rho_{HS}\rho_{IS}}{(1-\rho_{HS}^2)\hat{\rho}_I^2}, \tag{59}$$

$$\xi_I(t) = -\tilde{\lambda}_S(t)\frac{\rho_{IS}-\rho_{HS}\rho_{IH}}{(1-\rho_{HS}^2)\hat{\rho}_I^2} - \tilde{\lambda}_H(t)\frac{\rho_{IH}-\rho_{HS}\rho_{IS}}{(1-\rho_{HS}^2)\hat{\rho}_I^2} + \nu_I(t)\frac{1}{\hat{\rho}_I^2}, \tag{60}$$

$$\psi_Y = \frac{\hat{\rho}_Y}{\hat{\rho}_I}, \quad \psi_H = \frac{\hat{\rho}_{HY}-\hat{\rho}_{IH}\psi_Y}{\sqrt{1-\rho_{HS}^2}}, \quad \psi_S = \rho_{YS} - \rho_{HS}\psi_H - \rho_{IS}\psi_Y. \tag{61}$$

A.4.2 Human capital

The artificial market is complete so a unique risk-neutral measure $\mathbb{Q}$ exists, and the human capital can be valued as if the income was a dividend stream:

$$\mathcal{H}(t,y) = \mathrm{E}^{\mathbb{Q}}\left[\int_t^T Y_s e^{-\int_t^s \tilde{r}(u)\,du}\,ds \mid Y_t = y\right].$$

We need the risk-neutral dynamics of Y_t. The market price of risk is $\Sigma^{-1}\tilde{\mu}(t)$ so

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \Sigma^{-1}\tilde{\mu}(s)\,ds$$

defines a standard Brownian motion under $\mathbb{Q}$. Therefore

$$dY_t = Y_t\left[\left(\mu_Y(t) - \sigma_Y(t)\vec{\rho}^\top\Sigma^{-1}\tilde{\mu}(t)\right)\,dt + \sigma_Y(t)\vec{\rho}^\top\,dW_t^{\mathbb{Q}}\right].$$

It follows from standard asset pricing theory that $\mathcal{H}(t,y)$ satisfies the PDE

$$\tilde{r}(t)\mathcal{H} = \mathcal{H}_t + \mathcal{H}_y y \left(\mu_Y(t) - \sigma_Y(t)\vec{\rho}^\top \Sigma^{-1}\tilde{\mu}(t)\right) + \frac{1}{2}\mathcal{H}_{yy} y^2 \sigma_Y(t)^2 + y.$$

First consider the case after retirement, $t > \tilde{T}$, where the terminal value is $\mathcal{H}(T,y) = 0$. The solution is of the form $\mathcal{H}(t,y) = yF(t)$ if F satisfies the ODE

$$F'(t) - r_F(t)F(t) + 1 = 0 \tag{62}$$

and $F(T) = 0$, where

$$\begin{aligned} r_F(t) &= \tilde{r}(t) - \mu_Y(t) + \sigma_Y(t)\vec{\rho}^\top \Sigma^{-1}\tilde{\mu}(t) \\ &= \tilde{r}(t) - \mu_Y(t) + \sigma_Y(t)\left(\psi_S \tilde{\lambda}_S(t) + \psi_H \tilde{\lambda}_H(t) + \psi_Y \nu_I(t)\right), \end{aligned}$$

using (56). The solution is $F(t) = \int_t^T \exp\left\{-\int_t^u r_F(s)\,ds\right\}\,du$.

Before retirement, $t < \tilde{T}$, we need to take the income drop at retirement into account. The human capital still solves the above PDE but now with terminal condition $\mathcal{H}(\tilde{T},y) = \Upsilon Y_{\tilde{T}-} F(\tilde{T})$. The solution is then $\mathcal{H}(t,y) = yF(t)$ with

$$F(t) = \int_t^{\tilde{T}} \exp\left\{-\int_t^u r_F(s)\,ds\right\}\,du + \Upsilon \int_{\tilde{T}}^T \exp\left\{-\int_t^u r_F(s)\,ds\right\}\,du.$$

A.4.3 The HJB equation and its solution

We write the portfolio strategy in the artificial market compactly as $\Pi = (\Pi_S, \Pi_H, \Pi_I)^\top$. The wealth dynamics generated by a strategy $(c, q, \Pi_S, \Pi_H, \Pi_I)$ are

$$\begin{aligned} dX_t &= \left\{X_t\left[\tilde{r}(t) + \Pi_t^\top \tilde{\mu}(t)\right] + Y_t - c_t - q_t\tilde{\chi}(t)\tilde{H}_t\right\}\,dt + X_t\Pi_t^\top \Sigma\,dW_t \\ &= \left\{X_t\left[\tilde{r}(t) + \Pi_t^\top \tilde{\mu}(t)\right] + Y_t - c_t - q_t\chi H_t\right\}\,dt + X_t\Pi_t^\top \Sigma\,dW_t, \end{aligned}$$

where the equality is due to $\tilde{\chi}(t)\tilde{H}_t = \chi H_t$. We see that (t, X_t, Y_t, H_t) constitutes a controlled Markov diffusion so that these quantities are the relevant state variables for the agent with the addition of the housing habit $\bar{q}_t$ that directly affects the agent's utility. The dynamics of the state variables Y_t, H_t, and $\bar{q}_t$ are given by (22), (12), and (1), respectively.

The HJB equation for the value function $J(t,x,y,h,\bar{q})$ in the artificial market can be written as

$$0 = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J + \mathcal{L}_4 J,$$

where

$$\begin{aligned}
\mathcal{L}_1 J &= \sup_{c,q} \left\{ -cJ_x - (q-\bar{q})\chi h J_x + \frac{1}{1-\gamma} c^{b(1-\gamma)} (q-\bar{q})^{(1-b)(1-\gamma)} + \alpha(q-\bar{q}) J_{\bar{q}} \right\}, \\
\mathcal{L}_2 J &= J_t + \tilde{r} x J_x + y J_x + (r+\mu_H) h J_h + \mu_Y y J_y - \bar{q}\chi h J_x + (\alpha - \varepsilon)\bar{q} J_{\bar{q}} - \delta J, \\
\mathcal{L}_3 J &= \frac{1}{2}\sigma_H^2 h^2 J_{hh} + \frac{1}{2}\sigma_Y^2 y^2 J_{yy} + \rho_{HY}\sigma_H \sigma_Y h y J_{hy}, \\
\mathcal{L}_4 J &= \sup_{\Pi} \left\{ J_x x \Pi^\top \tilde{\mu} + \frac{1}{2} J_{xx} x^2 \Pi^\top \Sigma\Sigma^\top \Pi + J_{xh} x h \sigma_H \Pi^\top \Sigma \vec{\rho}_H + J_{xy} x y \sigma_Y \Pi^\top \Sigma \vec{\rho}_Y \right\}.
\end{aligned}$$

Note that $\mathcal{L}_1 J$ and $\mathcal{L}_2 J$ are exactly as in the simpler models, cf. (38)–(39), except that $\tilde{r}$ replaces r in front of xJ_x. Furthermore, $\mathcal{L}_3 J$ is exactly as in the spanned risk model of Section 3.2, cf. (50). In addition, $\mathcal{L}_4 J$ has the same structure as (51) in the spanned risk model, but μ is replaced by $\tilde{\mu}$, Π is now three-dimensional because of the additional risky asset, and $\vec{\rho}_H$ and $\vec{\rho}_Y$ are also three-dimensional.

Just as in the simpler models we conjecture a solution of the form

$$J(t,x,y,h,\bar{q}) = \frac{1}{1-\gamma}(\chi h)^{k\gamma} G(t)^\gamma \hat{x}^{1-\gamma}, \quad \hat{x} = x + yF(t) - \bar{q}\chi h B(t).$$

As in the proof of Theorem 1 (see Appendix A.1), this implies the optimal consumption policy (31)–(32) and, suppressing t-dependence for clarity,

$$\begin{aligned}
\mathcal{L}_1 J &= \hat{x}^{1-\gamma}(\chi h)^{\gamma k} G^{\gamma - 1} \frac{\gamma}{1-\gamma} \hat{b}(1+\alpha B)^k, \\
\mathcal{L}_2 J &= \hat{x}^{-\gamma}(\chi h)^{\gamma k} G^{\gamma-1} \Bigg\{ \frac{\gamma}{1-\gamma}\hat{x} \left[G' - \left(\frac{\delta}{\gamma} + \frac{\gamma - 1}{\gamma}\tilde{r} - (r+\mu_H)\, k \right) G \right] \\
&\qquad + Gy\left[F' + (\mu_Y - \tilde{r})F + 1\right] - \bar{q}\chi h G \left[B' - (\tilde{r} - r - \mu_H + \varepsilon - \alpha)\, B + 1 \right] \Bigg\}.
\end{aligned}$$

As in the proof of Theorem 2 (see Appendix A.2), we find

$$\begin{aligned}
\mathcal{L}_3 J = \hat{x}^{-\gamma-1}(\chi h)^{k\gamma} G^\gamma \Bigg\{ & \hat{x}^2 \frac{k\gamma(k\gamma - 1)}{2(1-\gamma)}\sigma_H^2 + \hat{x} k \gamma \sigma_H \left[yF\rho_{HY}\sigma_Y - \bar{q}\chi h B \sigma_H \right] \\
& - \frac{\gamma}{2}\sigma_H^2 h^2 \chi^2 \bar{q}^2 B^2 - \frac{\gamma}{2}\sigma_Y^2 y^2 F^2 + \gamma y F h \chi \bar{q} B \rho_{HY}\sigma_H \sigma_Y \Bigg\}.
\end{aligned}$$

Regarding $\mathcal{L}_4 J$, the first-order condition implies for Π implies

$$\Pi = -\frac{J_x}{xJ_{xx}}(\Sigma\Sigma^\top)^{-1}\tilde{\mu} - \frac{yJ_{xy}}{xJ_{xx}}\sigma_Y(\Sigma^\top)^{-1}\vec{\rho}_Y - \frac{\omega h J_{xh}}{xJ_{xx}}\sigma_H(\Sigma^\top)^{-1}\vec{\rho}_H.$$

By our conjecture for J, we get

$$-\frac{J_x}{xJ_{xx}} = \frac{\hat{x}}{\gamma x}, \qquad -\frac{yJ_{xy}}{xJ_{xx}} = -\frac{yF}{x}, \qquad -\frac{hJ_{xh}}{xJ_{xx}} = \frac{k\hat{x} + \bar{q}\chi h B}{x},$$

which, together with (54)–(57), implies that the optimal investment strategy can be written as (33)–(35). Substituting the above expression for Π back into $\mathcal{L}_4 J$, we obtain

$$\begin{aligned}\mathcal{L}_4 J = &-\frac{1}{2}\frac{J_x^2}{J_{xx}}\tilde{\mu}(t)^\top(\Sigma\Sigma^\top)^{-1}\tilde{\mu} - \frac{1}{2}\frac{y^2 J_{xy}^2}{J_{xx}}\sigma_Y^2 - \frac{yJ_xJ_{xy}}{J_{xx}}\sigma_Y\tilde{\mu}^\top(\Sigma^\top)^{-1}\vec{\rho}_Y \\ &-\frac{1}{2}\frac{h^2J_{xh}^2}{J_{xx}}\sigma_H^2 - \frac{hJ_xJ_{xh}}{J_{xx}}\sigma_H\tilde{\mu}^\top(\Sigma^\top)^{-1}\vec{\rho}_H - \frac{yhJ_{xy}J_{xh}}{J_{xx}}\sigma_Y\sigma_H\rho_{HY},\end{aligned}$$

and by substitution of our conjecture for J this leads to

$$\begin{aligned}\mathcal{L}_4 J = (\chi h)^{k\gamma}G^\gamma\hat{x}^{-\gamma-1}\Bigg\{&\hat{x}^2\Big[\frac{1}{2\gamma}\tilde{\mu}^\top(\Sigma\Sigma^\top)^{-1}\tilde{\mu} + \frac{k^2\gamma}{2}\sigma_H^2 + k\sigma_H\tilde{\mu}^\top(\Sigma^\top)^{-1}\vec{\rho}_H\Big] \\ &+\hat{x}\Big[\bar{q}\chi h B\sigma_H\left(k\gamma\sigma_H + \tilde{\mu}^\top(\Sigma^\top)^{-1}\vec{\rho}_H\right) \\ &\qquad - yF\sigma_Y\left(k\gamma\sigma_H\rho_{HY} + \tilde{\mu}^\top(\Sigma^\top)^{-1}\vec{\rho}_Y\right)\Big] \\ &+\frac{\gamma}{2}(yF)^2\sigma_Y^2 + \frac{\gamma}{2}\bar{q}\chi h B\sigma_H\left(\bar{q}\chi h B\sigma_H - 2yF\sigma_Y\rho_{HY}\right)\Bigg\}.\end{aligned}$$

By adding up $\mathcal{L}_3 J$ and $\mathcal{L}_4 J$, various terms cancel, and we are left with

$$\begin{aligned}\mathcal{L}_3 J + \mathcal{L}_4 J = (\chi h)^{k\gamma}G^\gamma\hat{x}^{-\gamma}\Bigg\{&\hat{x}\Big[\frac{1}{2\gamma}\tilde{\mu}^\top(\Sigma\Sigma^\top)^{-1}\tilde{\mu} + \frac{k(k-1)\gamma}{2(1-\gamma)}\sigma_H^2 + k\sigma_H\tilde{\mu}^\top(\Sigma^\top)^{-1}\vec{\rho}_H\Big] \\ &+\bar{q}\chi h B\sigma_H\tilde{\mu}^\top(\Sigma^\top)^{-1}\vec{\rho}_H - yF\sigma_Y\tilde{\mu}^\top(\Sigma^\top)^{-1}\vec{\rho}_Y\Bigg\}.\end{aligned}$$

By adding $\mathcal{L}_1 J$ and $\mathcal{L}_2 J$ to the above and equating the sum to zero, we have the full HJB equation. After division by $(\chi h)^{k\gamma}G^\gamma\hat{x}^{-\gamma}$, each remaining term is either a multiple of $\hat{x}$ or does not involve $\hat{x}$ at all. Collecting the terms with $\hat{x}$, we see that G must satisfy

$$0 = G'(t) - r_G(t)G(t) + \hat{b}\left(1 + \alpha B(t)\right)^k,$$

where

$$
\begin{aligned}
r_G(t) &= \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}\tilde{r}(t) + \frac{\gamma-1}{2\gamma^2}\tilde{\Lambda}(t)^2 - k\left[r + \mu_H + \frac{k-1}{2}\sigma_H^2 - \frac{\gamma-1}{\gamma}\sigma_H\tilde{\mu}^\top(\Sigma^\top)^{-1}\vec{\rho}_H\right] \\
&= \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}\tilde{r}(t) + \frac{\gamma-1}{2\gamma^2}\tilde{\Lambda}(t)^2 \\
&\quad - k\left[\tilde{r}(t) - r_\omega(t) + \nu_H + \mu_H + \frac{k-1}{2}\sigma_H^2 - \frac{\gamma-1}{\gamma}\left(\tilde{\mu}_H(t) + \chi - m\right)\right] \\
&= \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}\tilde{r}(t) + \frac{\gamma-1}{2\gamma^2}\tilde{\Lambda}(t)^2 \\
&\quad - k\left[\tilde{r}(t) - r_\omega(t) + \frac{\tilde{\mu}_H(t)}{\gamma} - \frac{\gamma-1}{\gamma}(\chi - m) + \frac{k-1}{2}\sigma_H^2\right].
\end{aligned}
$$

The terms not involving $\hat{x}$ add up to zero provided

$$
0 = F'(t) - r_F(t)F(t) + 1, \qquad 0 = B'(t) - r_B(t)B(t) + 1,
$$

where

$$
\begin{aligned}
r_F(t) &= \tilde{r}(t) - \mu_Y(t) + \sigma_Y(t)\tilde{\mu}(t)^\top(\Sigma^\top)^{-1}\vec{\rho}_Y \\
&= \tilde{r}(t) - \mu_Y(t) + \sigma_Y(t)\left(\psi_S\tilde{\lambda}_S(t) + \psi_H\tilde{\lambda}_H(t) + \psi_Y\nu_I(t)\right)
\end{aligned}
$$

as in Appendix A.4.2, and

$$
\begin{aligned}
r_B(t) &= \tilde{r}(t) - r - \mu_H + \varepsilon - \alpha + \sigma_H\tilde{\mu}(t)^\top(\Sigma^\top)^{-1}\vec{\rho}_H \\
&= r_\omega(t) - \nu_H(t) - \mu_H + \varepsilon - \alpha + \tilde{\mu}_H(t) + \chi - m \\
&= r_\omega(t) + \varepsilon - \alpha + \chi - m.
\end{aligned}
$$

Note that the ODE for F is satisfied according to our computation of the human capital, cf. Eq. (62). With the relevant boundary conditions $G(T) = B(T) = 0$, the solutions to the ODEs for G and B are as stated in the theorem.

B The extension to mortality risk and bequest utility

Let $\zeta(t)$ denote the mortality intensity. We assume the agent can invest in an insurance contract that pays the agent a continuous stream at the rate $\Gamma\zeta(t)N$ as long as the agent survives but, in return, when the agent dies the company receives an amount of N out or the wealth of the agent. Here N is a choice variable of the agent, whereas Γ is an exogenously given constant with $\Gamma = 1$ corresponding to a fairly priced contract.

In all our cases this adds two terms to the HJB equation:

(i) The term $\Gamma\zeta(t)NJ_x$ due to the insurance flow payment $\Gamma\zeta(t)N$.

(ii) The term $\zeta(t)\left(w^\gamma \frac{1}{1-\gamma}(x-N)^{1-\gamma} - J\right)$ reflecting the jump in utility upon death multiplied by the death intensity.

Here J and x refer to values immediately before time t. The agent cannot live longer than T. The optimal insurance goes to zero as time approaches T.

B.1 The model of Section 3.1

For the simple model of Section 3.1, the preferences are extended to include a power utility of terminal tangible wealth, representing a bequest:

$$J(t,x,h,y,\bar{q}) = \sup_{c,q,N} \mathrm{E}_t \left[\int_t^\tau e^{-\delta(s-t)} U(c_s, q_s, \bar{q}_s)\, ds + w^\gamma e^{-\delta(\tau-t)} \frac{1}{1-\gamma}(X_{\tau-} - N)^{1-\gamma}\right],$$

where the expectation is introduced because of the mortality risk and $\tau \leq T$ denotes the random terminal date. The terminal condition at T is $J(T,x,h,y,\bar{q}) = w^\gamma \frac{1}{1-\gamma} x^{1-\gamma}$.

Theorem 4 *The value function is*

$$J(t,x,h,y,\bar{q}) = \frac{1}{1-\gamma}\left(wA(t) + (\chi h)^k G(t)\right)^\gamma (x + yF(t) - \bar{q}h\chi B(t))^{1-\gamma},$$

where

$$A(t) = e^{-\int_t^T r_A(s)\, ds} + \Gamma^{1-\frac{1}{\gamma}} \int_t^T e^{-\int_t^u r_A(s)\, ds} \zeta(u)\, du,$$

$$B(t) = \int_t^T e^{-\int_t^s r_B(u)\, du}\, ds,$$

$$G(t) = \hat{b} \int_t^T e^{-\int_t^s r_G(u)\, du} (1 + \alpha B(s))^k\, ds,$$

$$F(t) = \begin{cases} \int_t^T e^{-\int_t^u r_F(s)\, ds}\, du, & t \in [\tilde{T}, T], \\ \int_t^{\tilde{T}} e^{-\int_t^u r_F(s)\, ds}\, du + \Upsilon \int_{\tilde{T}}^T e^{-\int_t^u r_F(s)\, ds}\, du, & t < \tilde{T}, \end{cases}$$

$$r_A(t) = \frac{\delta + \zeta(t)}{\gamma} + \frac{\gamma - 1}{\gamma}(r + \Gamma\zeta(t)), \qquad r_G(t) = r_A(t) - (r + \mu_H)k,$$

$$r_B(t) = \Gamma\zeta(t) + \varepsilon - \alpha - \mu_H, \qquad r_F(t) = r + \Gamma\zeta(t) - \mu_Y(t).$$

The optimal perishable and housing consumption rates are

$$c_t = b\hat{b}\,\frac{(\chi H_t)^k\,(1+\alpha B(t))^k}{wA(t)+(\chi H_t)^k G(t)}\widehat{X}_t, \quad q_t = \bar{q}_t + (1-b)\hat{b}\,\frac{(\chi H_t)^{k-1}\,(1+\alpha B(t))^{k-1}}{wA(t)+(\chi H_t)^k G(t)}\widehat{X}_t,$$

where disposable wealth $\widehat{X}_t$ *is defined as*

$$\widehat{X}_t = X_t + Y_t F(t) - \bar{q}_t \chi H_t B(t).$$

The proof can be found in the Internet Appendix at `http://bit.ly/1GNEwwJ`.

B.2 The full-fledged model of Section 4

For the full-fledged model of Section 4, we generalize preferences to bequest and mortality risk as follows:

$$J(t,x,h,y,\bar{q}) = \sup_{c,q,N,\Pi_S,\Pi_H} \mathrm{E}_t\left[\int_t^\tau e^{-\delta(s-t)}U(c_s,q_s,\bar{q}_s)\,ds + e^{-\delta(\tau-t)}\frac{w^\gamma}{1-\gamma}\,(X_{\tau-}-N)^{1-\gamma}\right],$$

where $\tau \le T$ is the random time of death with associated mortality intensity $\zeta(t)$, and N is the amount of insurance. The terminal condition is $J(T,x,h,y,\bar{q}) = w^\gamma \frac{1}{1-\gamma}x^{1-\gamma}$. We extend Theorem 3 to bequest and mortality risk:

Theorem 5 *For $\theta \in \Theta$, the value function in the artificial market $\mathcal{M}_\theta$ is*

$$J_\theta(t,x,h,y,\bar{q}) = \frac{1}{1-\gamma}\left(wA_\theta(t) + (\chi h)^k G_\theta(t)\right)^\gamma (x + yF_\theta(t) - \bar{q}\chi h B_\theta(t))^{1-\gamma},$$

where

$$\begin{aligned}
A_\theta(t) &= e^{-\int_t^T r_A(s)\,ds} + \Gamma^{1-\frac{1}{\gamma}}\int_t^T e^{-\int_t^u r_A(s)\,ds}\zeta(u)\,du,\\
B_\theta(t) &= \int_t^T e^{-\int_t^s r_B(u)\,du}\,ds,\\
G_\theta(t) &= \hat{b}\int_t^T e^{-\int_t^s r_G(u)\,du}(1+\alpha B_\theta(s))^k\,ds,\\
F_\theta(t) &= \begin{cases} \int_t^T e^{-\int_t^u r_F(s)\,ds}\,du, & t \in [\tilde{T},T],\\ \int_t^{\tilde{T}} e^{-\int_t^u r_F(s)\,ds}\,du + \Upsilon\int_{\tilde{T}}^T e^{-\int_t^u r_F(s)\,ds}\,du, & t < \tilde{T}, \end{cases}\\
r_A(t) &= \frac{\delta+\zeta(t)}{\gamma} + \frac{\gamma-1}{\gamma}(\tilde{r}(t)+\Gamma\zeta(t)) + \frac{\gamma-1}{2\gamma^2}\tilde{\Lambda}(t)^2,
\end{aligned}$$

$$
\begin{aligned}
r_B(t) &= r_\omega(t) + \Gamma\zeta(t) + \varepsilon - \alpha + \chi - m, \\
r_G(t) &= r_A(t) - k\left(\tilde r(t) - r_\omega(t) + \frac{\tilde\mu_H(t)}{\gamma} - \frac{\gamma-1}{\gamma}(\chi - m) + \frac{k-1}{2}\sigma_H^2\right), \\
r_F(t) &= \tilde r(t) + \Gamma\zeta(t) - \mu_Y(t) + \sigma_Y(t)\left(\psi_S\tilde\lambda_S(t) + \psi_H\tilde\lambda_H(t) + \psi_Y\nu_I(t)\right).
\end{aligned}
$$

Defining disposable wealth as $\widehat{X}_t = X_t + Y_tF_\theta(t) - \bar q_t\chi H_tB_\theta(t)$, *the optimal consumption and investment strategy is*

$$
\begin{aligned}
c_t &= b\hat b\,\frac{(\chi H_t)^k\,(1+\alpha B_\theta(t))^k}{wA_\theta(t) + (\chi H_t)^k G_\theta(t)}\widehat{X}_t, \\
q_t &= \bar q_t + (1-b)\hat b\,\frac{(\chi H_t)^{k-1}\,(1+\alpha B_\theta(t))^{k-1}}{wA_\theta(t) + (\chi H_t)^k G_\theta(t)}\widehat{X}_t, \\
\Pi_{St} &= \frac{\xi_S(t)}{\gamma\sigma_S}\frac{\widehat{X}_t}{X_t} - \frac{\sigma_Y(t)\psi_S}{\sigma_S}\frac{Y_tF_\theta(t)}{X_t}, \\
\Pi_{Ht} &= \frac{\xi_H(t)}{\gamma\sigma_H}\frac{\widehat{X}_t}{X_t} - \frac{\sigma_Y(t)\psi_H}{\sigma_H}\frac{Y_tF_\theta(t)}{X_t} + k\,\frac{(\chi H_t)^k G_\theta(t)}{wA(t) + (\chi H_t)^k G_\theta(t)}\,\frac{\widehat{X}_t}{X_t} + \bar q_t\chi B_\theta(t)\frac{H_t}{X_t}, \\
\Pi_{It} &= \frac{\xi_I(t)}{\gamma}\frac{\widehat{X}_t}{X_t} - \sigma_Y(t)\psi_Y\frac{Y_tF_\theta(t)}{X_t}
\end{aligned}
$$

with $\xi_S(t), \xi_H(t), \xi_I(t)$ *are defined in terms of the Sharpe ratios and the correlation structure, cf.* (58)–(60).

References

Andrews, D. and A. C. Sánchez (2011). The Evolution of Homeownership Rates in Selected OECD Countries: Demographic and Public Policy Influences. *OECD Journal: Economic Studies* 2011(1), 207–243.

Ang, A. (2014). *Asset Management*. Oxford University Press.

Aydilek, A. (2013). Habit Formation and Housing over the Life Cycle. *Economic Modelling* 33, 858–866.

Benzoni, L., P. Collin-Dufresne, and R. S. Goldstein (2007). Portfolio Choice over the Life-Cycle when the Stock and Labor Markets are Cointegrated. *Journal of Finance* 62(5), 2123–2167.

Bick, B., H. Kraft, and C. Munk (2013). Solving Constrained Consumption-Investment Problems by Simulation of Artificial Market Strategies. *Management Science* 59(2), 485–503.

Blanchard, O. J. (1985). Debt, Deficits, and Finite Horizons. *Journal of Political Economy* 93(2), 223–247.

Bricker, J., A. B. Kennickell, K. B. Moore, and J. Sabelhaus (2012). Changes in U.S. Family Finances from 2007 to 2010: Evidence from the Survey of Consumer Finances. *Federal Reserve Bulletin* 98(2), 1–80.

Brown, S., W. Goetzmann, and S. A. Ross (1995). Survival. *Journal of Finance* 50(3), 853–873.

Browning, M. and M. D. Collado (2007). Habits and Heterogeneity in Demands: A Panel Data Analysis. *Journal of Applied Econometrics* 22(3), 625–640.

Campbell, J. Y. (2006). Household Finance. *Journal of Finance* 61(4), 1553–1604.

Campbell, J. Y. and J. F. Cocco (2003). Household Risk Management and Optimal Mortgage Choice. *Quarterly Journal of Economics* 118(4), 1449–1494.

Campbell, J. Y. and J. H. Cochrane (1999). By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior. *Journal of Political Economy* 107(2), 205–251.

Carroll, C. D. (1997). Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis. *Quarterly Journal of Economics* 112(1), 1–55.

Chetty, R. and A. Szeidl (2007). Consumption Commitments and Risk Preferences. *Quarterly Journal of Economics* 122(2), 831–877.

Chetty, R. and A. Szeidl (2014, October). The Effect of Housing on Portfolio Choice. NBER Working paper 15998, available at http://www.rajchetty.com/.

Choi, H.-S., H. G. Hong, J. D. Kubik, and J. P. Thompson (2013, December). When Real Estate is the Only Game in Town. Available at SSRN: http://ssrn.com/abstract=2373179.

Cocco, J. F. (2005). Portfolio Choice in the Presence of Housing. *Review of Financial Studies* 18(2), 535–567.

Cocco, J. F., F. J. Gomes, and P. J. Maenhout (2005). Consumption and Portfolio Choice over the Life Cycle. *Review of Financial Studies* 18(2), 491–533.

Corradin, S., J. L. Fillat, and C. Vergara-Alert (2014). Optimal Portfolio Choice with Predictability in House Prices and Transaction Costs. *Review of Financial Studies* 27(4), 823–880.

Cotter, J. and R. Roll (2015). A Comparative Anatomy of Residential REITs and Private Real Estate Markets: Returns, Risks and Distributional Characteristics. *Real Estate Economics* 43(1), 209–240.

Cvitanić, J. and I. Karatzas (1992). Convex Duality in Constrained Portfolio Optimization. *Annals of Applied Probability* 2(4), 767–818.

Davis, S. J., F. Kubler, and P. Willen (2006). Borrowing Costs and the Demand for Equity over the Life Cycle. *Review of Economics and Statistics* 88(2), 348–362.

De Nardi, M., E. French, and J. B. Jones (2010). Why do the Elderly Save? The Role of Medical Expenses. *Journal of Political Economy* 118(1), 39–75.

Fama, E. F. and K. R. French (2002). The Equity Premium. *Journal of Finance* 57(2), 637–659.

Fischer, M. and M. Stamos (2013). Optimal Life Cycle Portfolio Choice with Housing Market Cycles. *Review of Financial Studies* 26(9), 2311–2352.

Flavin, M. and T. Yamashita (2002). Owner-Occupied Housing and the Composition of the Household Portfolio. *American Economic Review* 91(1), 345–362.

Gomes, F. and A. Michaelides (2003). Portfolio Choice with Internal Habit Formation:

A Life-Cycle Model with Uninsurable Labor Income Risk. *Review of Economic Dynamics* 6(4), 729–766.

Gourinchas, P.-O. and J. A. Parker (2002). Consumption Over the Life Cycle. *Econometrica* 70(1), 47–89.

Guiso, L. and P. Sodini (2013). Household Finance: An Emerging Field. Volume 2, Part B of *Handbook of the Economics of Finance*, Chapter 21, pp. 1397–1532. Elsevier.

Guvenen, F., F. Karahan, S. Ozkan, and J. Song (2015, February). What Do Data on Millions of U.S. Workers Reveal About Life-Cycle Earnings Risk. Available at SSRN: http://ssrn.com/abstract=2563279.

Haugh, M. B., L. Kogan, and J. Wang (2006). Evaluating Portfolio Policies: A Duality Approach. *Operations Research* 54(3), 405–418.

Heaton, J. and D. Lucas (2000). Portfolio Choice and Asset Prices: The Importance of Entrepreneurial Risk. *Journal of Finance* 55(3), 1163–1198.

van Hemert, O. (2010). Household Interest Rate Risk Management. *Real Estate Economics* 38(3), 467–505.

Hoesli, M. and E. Oikarinen (2012). Are REITs Real Estate? Evidence from International Sector Level Data. *Journal of International Money and Finance* 31(7), 1823–1850.

Ingersoll, Jr., J. E. (1992). Optimal Consumption and Portfolio Rules with Intertemporally Dependent Utility of Consumption. *Journal of Economic Dynamics and Control* 16(3-4), 681–712.

Karatzas, I., J. Lehoczky, S. Shreve, and G. Xu (1991). Martingale and Duality Methods for Utility Maximization in an Incomplete Market. *SIAM Journal on Control and Optimization* 29(3), 702–730.

Koijen, R. S. J., T. E. Nijman, and B. J. M. Werker (2010). When Can Life-cycle Investors Benefit from Time-varying Bond Risk Premia? *Review of Financial Studies* 23(2), 741–780.

Kraft, H. and C. Munk (2011). Optimal Housing, Consumption, and Investment Decisions over the Life-Cycle. *Management Science* 57(6), 1025–1041.

Lee, M.-L., M.-T. Lee, and K. Chiang (2008). Real Estate Risk Exposure of Equity Real Estate Investment Trusts. *Journal of Real Estate Finance & Economics* 36(2), 165–181.

Lynch, A. W. and S. Tan (2011). Labor Income Dynamics at Business-cycle Frequencies: Implications for Portfolio Choice. *Journal of Financial Economics* 101(2), 333–359.

Meyer, D. J. and J. Meyer (2005). Relative Risk Aversion: What Do We Know? *Journal of Risk and Uncertainty* 31(3), 243–262.

Munk, C. (2008). Portfolio and Consumption Choice with Stochastic Investment Opportunities and Habit Formation in Preferences. *Journal of Economic Dynamics and Control* 32(11), 3560–3589.

Munk, C. and C. Sørensen (2010). Dynamic Asset Allocation with Stochastic Income and Interest Rates. *Journal of Financial Economics* 96(3), 433–462.

Nagatani, K. (1972). Life Cycle Saving: Theory and Fact. *American Economic Review* 62(3), 344–353.

Pagliari, J. L., K. A. Scherer, and R. T. Monopoli (2005). Public Versus Private Real Estate Equities: A More Refined, Long-Term Comparison. *Real Estate Economics* 33(1), 147–187.

Piazzesi, M., M. Schneider, and S. Tuzel (2007). Housing, Consumption, and Asset Pricing. *Journal of Financial Economics* 83(3), 531–569.

Polkovnichenko, V. (2007). Life Cycle Portfolio Choice with Additive Habit Formation Preferences and Uninsurable Labor Income Risk. *Review of Financial Studies* 20(1), 83–124.

Ravina, E. (2007, November). Habit Formation and Keeping Up with the Joneses: Evidence from Micro Data. Available at SSRN: http://ssrn.com/abstract=928248.

Richard, S. F. (1975). Optimal Consumption, Portfolio and Life Insurance Rules for an Uncertain Lived Individual in a Continuous Time Model. *Journal of Financial Economics* 2(2), 187–203.

Sinai, T. and N. S. Souleles (2005). Owner-Occupied Housing as a Hedge Aganist Rent Risk. *Quarterly Journal of Economics* 120(2), 763–789.

Solnick, S. J. and D. Hemenway (2005). Are Positional Concerns Stronger in Some Domains than in Others? *American Economic Review* 95(2), 147–151.

Thurow, L. (1969). The Optimum Lifetime Distribution of Consumption Expenditures. *American Economic Review* 59(3), 324–330.

Wachter, J. A. and M. Yogo (2010). Why Do Household Portfolio Shares Rise in Wealth? *Review of Financial Studies* 23(11), 3929–3965.

Yang, F. (2009). Consumption over the Life Cycle: How Different is Housing? *Review of Economic Dynamics* 12(3), 423–443.

Yao, R. and H. H. Zhang (2005). Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints. *Review of Financial Studies* 18(1), 197–239.