Discussion of paper: Chen-Beetsma-Broeders

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1 The processes

Let me first summarise the notation of the authors. We have three intervals for the participating cohorts s:

- I_t are the all cohorts $t - t_D \leq s \leq t$
- I_t^w are the working cohorts $t - t_R \leq s \leq t$
- I_t^w are the retired cohorts $t - t_D \leq s \leq t - t_R$

The investment portfolio evolves (under the measure $\mathbb{Q}$) according to a standard Black-Scholes model as

$$dP_t = rP_t\,dt + \sigma P_t\,dW_{P,t} \tag{1.1}$$

with a constant risk-free interest rate r.

The asset dynamics (under the measure $\mathbb{Q}$) are given by

$$dA_t = \frac{dP_t}{P_t}A_t + (C_t - B_t^{TOT})dt \tag{1.2}$$

which is portfolio return plus total contributions C_t minus total payments B_t^{TOT}.

Actuarial fair value of pension entitlements is

$$\Pi_t(B_{s,t}) = R_{t-s}B_{s,t}, \tag{1.3}$$

where R_{t-s} is the **annuity factor** for the pension benefits $B_{s,t}$. For a cohort of age $v = t - s$ we can compute this annuity factor as (given the constant interest rate r) as:

$$R_v = \int_{t_R \vee v}^{t_D} e^{-r(u-v)}\,du = \frac{e^{rv}}{r}\left(e^{-r(t_R \vee v)} - e^{-rt_D}\right) \tag{1.4}$$

for $v \leq t_D$.

The pension fund's actuarial value of the total liabilities is given by:

$$L_t = \int_{t-t_D}^{t} R_{t-s}B_{s,t}ds. \tag{1.5}$$

The pension entitlement for cohort s at time t evolves as:

$$dB_{s,t} = \left(\psi \mathbb{1}_{t-t_R \leq s \leq t} + (\gamma_t - 1)B_{s,t}\right)dt \tag{1.6}$$

where ψ denotes the increase in entitlements for every dt worked and γ_t is the indexation at time t. After retirement, the ψ-term disappears.

The total benefit payments at time t are

$$B_t^{TOT} = \int_{t-t_D}^{t-t_R} B_{s,t}ds. \tag{1.7}$$

I have removed γ_t in front of integral!

The aggregate contributions over all working cohorts are:

$$C_t = \int_{t-t_R}^{t} \psi R_{t-s} + \pi_{s,t}\,ds, \tag{1.8}$$

where $\pi_{s,t}$ denotes a possible recovery contribution on top of the basic contribution ψR_{t-s}. The integral $\int R ds$ can be expressed as

$$\bar{R} := \int_{t-t_R}^{t} R_{t-s} ds = \frac{(e^{-rt_R} - e^{-rt_D})(e^{rt_R} - 1)}{r^2}. \tag{1.9}$$

For ease of notation we also introduce the following quantity

$$\bar{\pi}_t := \int_{t-t_R}^{t} \pi_{s,t}\, ds. \tag{1.10}$$

Therefore, we can express C_t as $C_t = \psi \bar{R} + \bar{\pi}_t$.

2 Funding Ratio Process

In this section I give an alternative derivation of the dynamics of the funding ratio process. The funding ratio of the pension fund is defined as:

$$F_t := \frac{A_t}{L_t}. \tag{2.1}$$

The liability process L_t is a finite variation process (since $B_{s,t}$ is FV), and the asset process A_t is a stochastic process. Therefore we can express the dynamics of F_t as

$$dF_t = \left(\frac{1}{L_t}\right) dA_t - A_t \left(\frac{dL_t}{L_t^2}\right) = F_t \left(\frac{dA_t}{A_t} - \frac{dL_t}{L_t}\right). \tag{2.2}$$

This expression has a nice interpretation: the funding grows with the return on the asset portfolio, minus the growth in the liabilities.

The first term dA/A can be expanded as:

$$\frac{dA_t}{A_t} = \frac{dP_t}{P_t} + \frac{C_t - B_t^{TOT}}{A_t} dt = \left(r + \frac{\bar{\pi}_t + \psi \bar{R} - B_t^{TOT}}{A_t}\right) dt + \sigma dW_{P,t}. \tag{2.3}$$

To analyse the second term dL/L we first take a look at dL. This differential (of a FV process) can be expressed as

$$dL_t = (R_0 B_{t,t} - R_{t_D} B_{t-t_D,t}) dt + \int_{t-t_D}^{t} \left(B_{s,t} dR_{t-s} + R_{t-s} dB_{s,t}\right) ds. \tag{2.4}$$

The first two terms are both equal to zero (since $B_{t,t} = 0$ and $R_{t_D} = 0$), and we split the integral at the retirement age t_R to obtain:

$$dL_t = \int_{t-t_D}^{t-t_R} \left(B_{s,t} dR_{t-s} + R_{t-s} dB_{s,t}\right) ds + \int_{t-t_R}^{t} \left(B_{s,t} dR_{t-s} + R_{t-s} dB_{s,t}\right) ds. \tag{2.5}$$

For the derivative dR/dt we get

$$\frac{dR_{t-s}}{dt} = \begin{cases} rR_{t-s} & \text{for } t - s < t_R \\ rR_{t-s} - 1 & \text{for } t_R \le t - s \le t_D \end{cases} \tag{2.6}$$

Note, that the derivative is discontinuous at $t-s=t_R$. For $t-s<t_R$ the derivative is positive, as we have less discounting if we move forward in time. For $t_R \le t-s \le t_D$ the derivative is negative, as we then see the effect of each cohort dying at age t_D.

For the derivative dB/dt we get

$$\frac{dB_{s,t}}{dt} = \begin{cases} \psi + (\gamma_t - 1)B_{s,t} & \text{for } t-s<t_R \\ (\gamma_t - 1)B_{s,t} & \text{for } t_R \le t-s \le t_D \end{cases} \tag{2.7}$$

Using both these results, we obtain for the first integral in (2.5):

$$\left(\int_{t-t_D}^{t-t_R} \left(R_{t-s}(r+\gamma_t - 1) - 1 \right) B_{s,t} ds \right) dt \tag{2.8}$$

and for the second integral in (2.5)

$$\left(\psi \int_{t-t_R}^{t} R_{t-s} ds + \int_{t-t_R}^{t} R_{t-s}(r+\gamma_t-1)B_{s,t} ds \right) dt. \tag{2.9}$$

Combining both integrals and simplifying, we obtain for (2.5):

$$dL_t = \left(\psi \bar{R} - B_t^{TOT} + (r+\gamma_t - 1)L_t \right) dt \tag{2.10}$$

And we arrive at the following equation for the funding ratio process:

$$dF_t = \left((1-\gamma_t)F_t + \frac{\bar{\pi}_t - (B_t^{TOT} - \psi \bar{R})(1-F_t)}{L_t} \right) dt + \sigma F_t dW_{P,t}. \tag{2.11}$$

A necessary condition to arrive at an equilibrium (in $\mathbb{Q}$-expectation!), is that for $t \to \infty$ we must have that the drift term is zero in (2.11). If we set $\bar{\pi}_\infty = 0$ and , $\gamma_\infty = 1$ then the only possible solution is $F_\infty = 1$.

The equilibrium value for L_∞ is then given by the condition that the drift of dL must also be zero in (2.10) which implies $L_\infty = (B_\infty^{TOT} - \psi \bar{R})/r$. For $\gamma_t \equiv 1$ we calculate $B_\infty^{TOT} = \bar{B}^{TOT} = \psi t_R(t_D - t_R)$ and therefore we find the expression

$$L_\infty = \frac{\psi}{r} \left(t_R(t_D - t_R) - \bar{R} \right). \tag{2.12}$$

What happens if we choose values $\gamma \ne 1$ and $\pi \ne 0$. Can we then also find equilibria? Suppose we take $\gamma_t \ne 1$ as a constant. Then the pension benefits become a function B_v of $v = t-s$. We can solve B from the differential equation (2.7) and we get

$$\frac{dB_v}{dv} = \begin{cases} \psi + (\gamma - 1)B_v & \text{for } v<t_R \\ (\gamma - 1)B_v & \text{for } t_R \le v \le t_D \end{cases} \tag{2.13}$$

with the boundary condition $B_0 = 0$. The solution for B_v is

$$B_v = \begin{cases} \psi \frac{e^{(\gamma-1)v}-1}{\gamma-1} & \text{for } v<t_R \\ \psi \frac{e^{(\gamma-1)t_R}-1}{\gamma-1} e^{(\gamma-1)(v-t_R)} & \text{for } t_R \le v \le t_D \end{cases} \tag{2.14}$$

and we find for $\bar{B}^{TOT}$ the expression

$$\bar{B}^{TOT} = \int_{t_R}^{t_D} B_v dv = \psi \frac{(e^{(\gamma-1)t_R} - 1)(e^{(\gamma-1)(t_D - t_R)} - 1)}{(\gamma-1)^2}. \tag{2.15}$$

The equilibrium liabilities are given by

$$L_\infty = \frac{\bar{B}^{TOT} - \psi \bar{R}}{(r + \gamma - 1)}. \tag{2.16}$$

The zero-drift condition for the funding ratio now leads to:

$$(1-\gamma)F_\infty + \frac{\bar{\pi}}{L_\infty} - (r+\gamma-1)(1-F_\infty) = 0 \tag{2.17}$$

If we solve this linear equation we find

$$F_\infty = \frac{(r+\gamma-1)L_\infty - \bar{\pi}}{rL_\infty}. \tag{2.18}$$

If we take $\gamma = 1$ and $\pi = 0$ we recover our previous answer $F_\infty = 1$.

The question we have not answered, is whether these equilibria are stable. In the stochastic differential equation (2.11) we see that there is a stochastic term that will always push the funding ratio away from it's equilibrium level. We will therefore only have a stable equilibrium if the stochastic process is mean-reverting. For the constant coefficient case, the mean-reversion term is equal to rF_t. The funding ratio process is mean-reverting iff $r < 0$. For $r > 0$ this condition is not satisfied which leads us to the conclusion that any equilibrium for F_∞ is *unstable*.

Important conclusion: there is no set of constant values (γ, π) for which the funding ratio process is intrinsically stable. Therefore, we need to stabilise the funding ratio by actively steering the funding ratio towards the target level $F_\infty = \bar{F} = 1$.

3 Control Policies for the Funding Ratio

If we seek to control the funding ratio process F_t we have two control instruments: the indexation level γ_t and the recovery contribution $\bar{\pi}_t$. In the sub-sections below I will somewhat deviate from the exposition of the paper by Chen et al.

3.1 Control via indexation

Suppose we want to control the funding ratio via a "linear control policy" on the indexation γ_t. Let us consider the following decision rule:

$$\gamma_t = 1 + \left(1 - \frac{1}{F_t}\right) g \quad \text{with } g > 0. \tag{3.1}$$

With this decision rule we have indexation $\gamma_t < 1$ for $F_t < 1$ and indexation $\gamma_t > 1$ for $F_t > 1$. The parameter g determines the "steepness" of the indexation ladder.

We also assume that $\pi_{s,t} \equiv 0$ for simplicity. If we substitute the control policies into the funding ratio process (2.11) we find for the drift term:

$$\left(g - \frac{B_t^{TOT} - \psi \bar{R}}{L_t}\right)(1 - F_t). \tag{3.2}$$

This drift-term is zero for $F_t = 1$, which implies that the level $F_t = 1$ is indeed an equilibrium point. Furthermore, the condition to obtain a stable equilibrium at $F_t = 1$ is $g > (B_t^{TOT} - \psi\bar{R})/L_t$. If we substitute the equilibrium level for $\gamma_t \equiv 1$ (approximation only!!) we get $L_t = (\bar{B}^{TOT} - \psi\bar{R})/r$ and the stability condition simplifies to $g > r$, which seems economically reasonable.

3.2 Control via indexation and recovery contribution

We now want to use the recovery contribution $\bar{\pi}_t$ as an additional control instrument. Let us consider the following decision rule:

$$\bar{\pi}_t = p(L_t - A_t) = p(1 - F_t)L_t \quad \text{with } p > 0. \tag{3.3}$$

Note that $\bar{\pi}_t$ is an amount of money (and not a ratio as γ_t), therefore we express $\bar{\pi}_t$ as the difference between the value of the liabilities L_t and the assets A_t.

If we substitute the control policies (3.1) and (3.3) into the funding ratio process (2.11) we find for the drift term:

$$\left(g + p - \frac{B_t^{TOT} - \psi\bar{R}}{L_t}\right)(1 - F_t). \tag{3.4}$$

Again, we find that the level $F_t = 1$ is indeed an equilibrium point and that the condition to obtain a stable equilibrium at $F_t = 1$ is $g + p > (B_t^{TOT} - \psi\bar{R})/L_t \approx r$.

3.3 DB Pension Scheme

For the DB pension scheme we have $g = 0$ which implies $\gamma_t \equiv 1$ and we have $L_t = (\bar{B}^{TOT} - \psi\bar{R})/r$. If we use the control policy (3.3) we obtain the funding ratio process

$$dF_t = (p - r)(1 - F_t)dt + \sigma F_t dW_{P,t}. \tag{3.5}$$

This is a "geometric Ornstein-Uhlenbeck" model (a.k.a. the Dixit & Pindyck model) with mean-reversion parameter $(p - r)$. The characteristic function of this process can be obtained analytically, and we can price European options in closed form. Also note that our mean-reversion parameter $(p - r)$ is similar as the paper's parameter $-\ln\alpha$.

4 Conclusion

Very nice paper, with nice results. We can drastically simplify the derivations in the appendices (currently almost 40 pages...) which will greatly improve the readability of the results in the appendix.