Sets of Indistinguishable Models for Robust Optimisation

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1 October, 2015

Literature Uncertainty

- Robust Optimisation
- Set of models needed
- Research on quantification of uncertainty
- Literature overview
  - Hansen and Sargent Robust Optimal Control
  - ϕ-Divergence
  - Model Confidence Set
  - Confidence Interval for Parameters

Problem

- Distance measures
- How far is too far?
- By only α and β we get explicit size of set

Model

- SDEs of form
$$dX = \mu(t,\omega)\,dt + \sigma(t,\omega)\,dW(t)$$
- Possible alternative models $dW_t + \lambda(t,\omega)\,dt$
- Ex ante $(t = 0)$
- Those $\lambda(t,\omega)$ indistinguishable from $\lambda = 0$
- $dW_t + \lambda(t,\omega)dt$ not only adjusting the mean of the probability distribution
- Example cosh

Stochastic Example

- Let $\lambda(t, \omega) = a \tanh(aW(t))$
- Implies

$$R(T) = e^{-\frac{1}{2}a^2 T} \cosh(aW(T)) = \frac{1}{2}\left(e^{-\frac{1}{2}a^2 T + aW(T)} + e^{-\frac{1}{2}a^2 T - aW(T)}\right)$$

- Under $\mathbb{Q}$ a mixture of $N(aT, T)$ and $N(-aT, T)$, together not normal, mean 0 and variance $T + (aT)^2$
- Under $\mathbb{P}$ always $W(T) \sim N(0, T)$

Model (2)

- Test $H_0 : \mathbb{P}$ versus $H_A : \mathbb{Q}$
- Hence $R(T) = 1/\text{likelihood}$
- Radon-Nikodym

$$R(T) = \exp\left\{ -\frac{1}{2} \int_0^T \lambda(t,\omega)^2 dt + \int_0^T \lambda(t,\omega) dW^{\mathbb{P}}(t,\omega) \right\}$$

- Value $R(T,\omega)$ determined by realisation ω
- Test if model $\mathbb{P}$ should be rejected in favour of model $\mathbb{Q}$
- Two simple hypotheses, Neyman-Pearson Lemma most powerful test is likelihood ratio test

Model (3)

- Type I error: incorrectly rejecting model $\mathbb{P}$

$$\mathbb{P}[R(T) \geq \gamma] = \alpha$$

- Type II error: incorrectly rejecting model $\mathbb{Q}$

$$\mathbb{Q}[R(T) < \gamma] = \beta$$

- Power: probability of accepting model $\mathbb{Q}$ when model $\mathbb{Q}$ is the true model

$$\begin{aligned} \mathbb{Q}[R(T) \geq \gamma] &= 1 - \beta \\ &= \mathbb{E}^{\mathbb{P}}\left[R(T)\mathbb{1}(R(T) \geq \gamma)\right] \end{aligned}$$

Deterministic

- Radon-Nikodym, without ω, log normal
- E.g. $\alpha = 0.05$, then $\Phi^{-1}(\alpha) = -1.64$
- If we take $\beta = 0.20$ then power is 0.80 and we have $\Phi^{-1}(0.80) = 0.84$
- Hence, the class of all indistinguishable models is then given by all models that satisfy

$$\left(\int_0^T \lambda(t)^2 \, dt\right)^{\frac{1}{2}} \leq 0.84 - (-1.64) = 2.48$$

Time-consistency

- For time-consistent coherent risk measures
  - $|\lambda(t,\omega)| \leq k$
- Optimisation problem

$$\max_{\gamma, |\lambda(t,\omega)| \leq k} \mathbb{E}\left[R(T)\mathbb{1}(R(T) \geq \gamma)\right]$$
$$\text{s.t. } \mathbb{E}\left[\mathbb{1}(R(T) \geq \gamma)\right] = \alpha$$
$$dR = \lambda(t,\omega) R dW$$

- Maximum for $|\lambda(t,\omega)| \equiv k \Rightarrow$ log Normal

Stochastic

- Given that the optimal $R^*(T)$ is a lognormal martingale with volatility k
- The optimised power at time $t = 0$ and $R(0) = 1$ is therefore equal to

$$\mathbb{E}\left[R^*(T)\mathbb{1}\left(R^*(T) > \gamma^*\right)\right] = \mathbb{Q}\left[R^*(T) > \gamma^*\right] = \Phi\left(\Phi^{-1}(\alpha) + k\sqrt{T}\right)$$

- Set of indistinguishable models $|\lambda(t,\omega)^*| \leq k = 2.48/\sqrt{T}$

Bounds on Divergences

Divergence	$\phi(t)$	for $\lvert\lambda(t,\omega)\rvert = k$	$k\sqrt{T} = 2.48$
Kullback-Leibler	$t \ln t - t + 1$	$\frac{1}{2}k^2 T$	3.08
Burg entropy	$-\ln t + t - 1$	$\frac{1}{2}k^2 T$	3.08
J-divergence	$(t-1)\ln t$	$k^2 T$	6.15
χ^2-divergence	$\frac{1}{t}(t-1)^2$	$e^{k^2 T} - 1$	467.90
Modified χ^2-divergence	$(t-1)^2$	$e^{k^2 T} - 1$	467.90
Hellinger distance	$(\sqrt{t}-1)^2$	$2 - 2e^{-\frac{1}{8}k^2 T}$	1.07
Variation distance	$\lvert t-1\rvert$	$4N(\frac{1}{2}k\sqrt{T}) - 2$	1.57
χ-divergence of order $\theta > 1$	$\lvert t-1\rvert^\theta$	–	–
Cressie-Read $\theta \neq 0, 1$	$\frac{1-\theta+\theta t-t^\theta}{\theta(1-\theta)}$	–	–

Conclusion

- Quantify uncertainty
- Most powerful test
- Ex ante
- For given size and power
- Stochastic deviation from the drift