Risk measures with volatility risk

Alessandro Pollastri*

Peter C. Schotman†

Stefan Straetmans‡

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Abstract

We study the properties of the volatility innovations on VaR and ES. Using a dynamic model for realized volatility we estimate the density of future volatility. Mixing this density with the conditional density of returns given the volatility we derive the predictive density of returns, which we use to estimate the risk measures. We find that higher uncertainty in the next day volatility leads to a fatter tailed distribution. Using a similar approach for the dynamics of the covariance matrix of a bivariate process we derive a bivariate predictive density. The latter helps us to disentangle the effects of volatility risk and correlation risk on the dependence structure of pairs of assets. Our results show that different structures of the covariance matrix of the residuals for our bivariate dynamic model lead to diverse degrees of dependence.

[1]Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. Email: a.pollastri@maastrichtuniversity.nl

[2]Email: p.schotman@maastrichtuniversity.nl

[3]Email: s.straetmans@maastrichtuniversity.nl

1 Introduction

It is common practice in financial risk management to forecast risk measures for portfolios of assets using one of the following approaches: historical simulation, extreme value theory based, parametric and semiparametric models. In the parametric approach Value at Risk can be expressed as:

$$VaR^{\alpha}_{t+1} = \sigma_{t+1}(\theta)Q^{\alpha}(\theta) \quad (1)$$

which, assuming zero mean, decomposes the risk measure in volatility and a distribution quantile. In this setup, it is evident how a forecast of a risk measure depends both on the uncertainty implied in the forecast of the volatility and the choice of a model to obtain the quantile. This paper contributes to the literature on risk measures forecasting using mixture models suggested by (Andersen et al., 2003) analyzing the effect of innovations. We investigate this from two different angles: first we consider, given a dynamic model for volatility, the effect of different distributional assumption of the innovations on the risk measures; second, we focus on a specific distribution for the innovations and check how different degrees of volatility uncertainty, also known as Vol of Vol, affects risk measures.

In order to perform our study, we rely on the usage of realized volatility as measure of σ_{t+1} in equation (1). The introduction of this estimator for the integrated variance of stock returns has made feasible to obtain $Q(\alpha)$ based on the normalized returns r_{t+1}/σ_{t+1}. In our approach, we consider at first the HAR model introduced by Corsi (2009) to forecast realized volatilies in order to retrieve the volatility of the forecast errors and hence we adopt a stochastic volatility model to make inference on VaR and ES. We show that taking into account that the variance is unknown for the next period leads to a different distribution of returns compared to the case where it is considered without uncertainty. This distinction leads to the so called Normal-Mixtures models. We consider several mixtures resulting from different distributional assumptions on the residuals of the volatility model whereas (Andersen et al., 2003) uses a Normal-Lognormal mixture.

Multivariate models to predict volatilities and covariances are extensively studied in the literature: some examples are given by multivariate GARCH models. Among them it is possible to distinguish between those which are derived as a linear combination of univariate GARCH model and nonlinear combination. An example of the second category is given by the CCC and DCC models.

Multivariate extensions to stochastic volatility models are also considered in the literature. Previous works have assumed the covariance matrix to be Wishart distributed. Among these papers, the ones from (Gourieroux, 2006) and (Gourieroux et al., 2009) where WAR processes are introduced to model volatilities. In (Gourieroux et al., 2009), the authors introduce the WAR model and the estimation method is based on nonlinear least squares with realized volatility. (Gourieroux, 2006) extends this setting to the continuous time version and he shows that it is also a multivariate extension of the CIR model. (Philipov and Glickman, 2006a) introduce a hierarchical model for the conditional distribution of returns given a covariance matrix characterized by a time varying shape parameter. (Philipov and Glickman, 2006b) modify the previous work using a factor model where factors are driven by Wishart processes.

In a bivariate setting we specify a dynamic model that also takes into account covariances. A bivariate mixture model is used to investigate the effects of volatility and covariance uncertainty on tail conditional probabilities:

$$\tau = \lim_{\alpha \to 1} \tau(\alpha) \tag{2}$$

$$\tau(\alpha) = \mathbb{P}\Big(r^1 > VaR^1(\alpha) | r^2 > VaR^2(\alpha)\Big) \tag{3}$$

Effects of volatility forecast errors on the dependence structure have been studied much less. For this probability we consider 3 different specifications of the bivariate stochastic volatility model: 1) two independent stochastic volatility processes, 2) two dependent volatility processes without stochastic covariance and 3) stochastic volatilities and stochastic covariances. Unlike the previously mentioned works, we do not assume the covariance matrices to follow Wishart-type processes, in fact such an assumption leads to a multivariate Student-t distribution of returns and hence strong tail dependence. Studying the three different cases extends the case considered by Embrechts et al. (1999) in chapter 3 where they consider the covariance matrix to be a multivariate normal and multiplied by a random variable. This implies that the volatilities are affected in the same way by any realization of this shock. Another recent example to model tail-dependence is given by (White et al., 2015), who propose an extension of the QAR model to multiple variables hence allowing for dependence between quantiles. Our methodology is agnostic with respect to the dependence that returns have in the tails.

The remainder of the paper is organized as follows: Section 2 explains the methodology that we use in order to assess the effect of volatility forecast errors on the tail risk measures both for the univariate and it is extended in order to study the effects on the dependence structures of returns. Section 3 presents the results obtained using the methodology explained in section 2. Section 4 concludes the paper.

2 Methodology

2.1 The Model

Let $p(r_{t+1}|\sigma^2_{t+1})$ be the density of a financial return conditional on σ^2_{t+1}. It is known that the variance is not observable and hence needs to be estimated from a sample data Y. In this framework the estimation result is a density $p(\sigma^2_{t+1}|Y_t)$. From this we obtain the prediction density:

$$p(r_{t+1}|Y_t) = \int p(r_{t+1}|\sigma^2_{t+1})p(\sigma^2_{t+1}|Y_t)d\sigma^2_{t+1} \tag{4}$$

From here on, we omit the time subscript for ease of notation. Risk measures based on $p(r)$ will generally be different from those resulting from $p(r|\sigma^2)$: the latter are conditional on σ^2 and therefore ignore the uncertainty arising from the estimation of σ^2.

Our interest is in tail risk measures, and in particular how these are affected by uncertainty in volatility. In the multivariate case, we will distinguish between volatility risk affecting the marginal distribution of returns, and correlation risk affecting the risk of portfolio return.

Since volatility varies over time, the estimate of volatility in the predictive density is the prediction of σ^2_{t+1} given a time series model for the log of realized volatility. As our model for volatility we use the following specification:

$$\begin{aligned} \ln \sigma^2_{t+1} &= f(X_t, \beta) + \eta_{t+1} && (5) \\ &= \mu_t + \omega_t z_{t+1} && (6) \end{aligned}$$

where μ_t and ω_t are the conditional mean and volatility of $\ln \sigma^2_{t+1}$. At this stage we consider a HAR model for volatility to obtain μ_t and ω_t and we do not consider possible misspecification or estimation risk, for more details about this model we refer to (Corsi, 2009). The HAR model states that the realized volatility of the next day is a linear function of past realized volatility. In our setting $X_t = \begin{bmatrix} 1 & \ln \sigma_t^2 & \ln \sigma_t^{2,w} & \ln \sigma_t^{2,m} \end{bmatrix}$ where the terms $\ln \sigma_t^{2,w}$ and $\ln \sigma_t^{2,m}$ are defined as the average log-realized variance of the previous 4 trading and as the average of the 21 preceeding days respectively. Hence $\mu_t = X_t \beta$ and ω is constant.

We obtain the predictive distribution of returns r marginalizing with respect to the variance uncertainty:

$$p(r) = \int p(r|\sigma^2)\ p(\sigma^2)\ \ d\sigma^2 \tag{7}$$

where $p(\sigma^2)$ is the density of σ^2. Furthermore we assume that the distribution of returns conditional to the variance is Normal, i.e. $r_{t+1}|\sigma_{t+1}^2 \sim \mathcal{N}(0, \sigma_{t+1}^2)$. Our first experiment allows for different specification of the model in equation (6) changing the distributional assumption for z_{t+1}. Specifically we let z_{t+1} be a Normal, a Student-t, a Weibull and finally a Pareto distribution. The volatility uncertainty will affect the predictive density and will have a significant impact on the tails. The only case in which (7) has a closed form is when σ follows an inverted Gamma distribution. In all the other cases we use numerical integration.

To exploit the closed form for equation (7) we still rely on the assumption that $p(r|\sigma^2)$ is Normally-distributed but we change our assumption on $p(\sigma^2)$: we assume in fact that σ is inverse-Γ distributed characterized by the parameter vector $\theta = \{\nu, s^2\}$. It is known that under these assumptions the predictive density of returns in (7) is a t-distribution with degrees of freedom ν and scale parameter s^2. From the HAR model we have first and second moments of $\ln \sigma^2$, which we use in order to calibrate the parameters of the inverted gamma density such that the data imply exactly the same moments of $\ln \sigma^2$. In order to obtain the parameters, we find θ such that $\mu(\theta) = \int_y y \cdot f(\theta)$ and $\omega^2(\theta) = \int_y y^2 \cdot f(\theta) - \mu(\theta)^2$ equal the moments implied by the HAR model.

2.2 Dependence

In order to study the effect of volatility uncertainty on the probability given by equation (3), let $\mathbf{\Sigma}$ be the covariance matrix with elements σ_{ij}. In the bivariate case $\mathbf{\Sigma}$ has three distinct elements, σ_{11}, σ_{22} and $\sigma_{12} = \rho\sqrt{\sigma_{11}\sigma_{22}}$, where ρ is the correlation. For the dynamic model of $\mathbf{\Sigma}$ consider the vector:

$$\boldsymbol{\zeta_t} = \begin{pmatrix} \ln \sigma_{11,t} \\ \ln \sigma_{22,t} \\ \ln \frac{1+\rho_t}{1-\rho_t} \end{pmatrix}$$

and the first order VAR:

$$\boldsymbol{\zeta}_{t+1} = \boldsymbol{\mu} + \boldsymbol{A}\Big(\boldsymbol{\zeta}_t - \boldsymbol{\mu}\Big) + \boldsymbol{\omega}\boldsymbol{Z}_{t+1} \tag{8}$$

where $\boldsymbol{\omega}\boldsymbol{\omega}' = \mathbf{\Omega}$ is the covariance matrix of $\boldsymbol{Z}_{t+1}$. The last element in the vector $\boldsymbol{\zeta}$ is a logit transformation of the realized correlation between a pair of stocks defined by:

$$\rho_{ij,t} = \frac{\sum_{s=1}^{M} r_{s,t}^{i} \cdot r_{s,t}^{j}}{\sqrt{\sum_{s=1}^{M} r_{s,t}^{2,i} \cdot \sum_{s=1}^{M} r_{s,t}^{2,j}}}$$

Given the model defined by equation (8), and the inverse transformation we can obtain $\mathbf{\Sigma}$. From it we get a bivariate density using the same methodology as in the univariate case:

$$p(\boldsymbol{r}) = \int p(\boldsymbol{r}|\mathbf{\Sigma})p(\mathbf{\Sigma})d\mathbf{\Sigma} \tag{9}$$

We want to study the effect of volatility risk on the tail correlations distinguishing three different specifications of model given by (8). It is worthwhile having a look at

$\boldsymbol{\Omega}$, omitting the time subscript:

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}$$

Each element is the (co-)variance associated to the residuals in the VAR, hence the first two elements in the diagonal can be interpreted as the risk of the variances of two assets, more interesting is the last element on the diagonal, which represents the correlation risk.

The first case that we cosider takes $\boldsymbol{\Omega}$ full as defined above. The second case, instead, considers only the upper left bidimensional sub-matrix of $\boldsymbol{\Omega}$, which means that only variances' risk and its covariance is present, written in the above notation:

$$\boldsymbol{\Omega}^1 = \left(\begin{array}{cc|c} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

Finally, the last case takes into account only variances' risk, in this circumstance the matrix above defined has the following form:

$$\boldsymbol{\Omega}^2 = \left(\begin{array}{cc|c} \omega_{11} & 0 & 0 \\ 0 & \omega_{22} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

To compute the coefficient given by equation (3) we first need the quantiles of the marginal distributions, $VaR^i(\alpha), i = 1, 2$. We obtain these quantiles from the model outlined in equation 6. We will deal with the case where $\boldsymbol{Z_{t+1}}$ is normal.

3 Results

In this section we present the results obtained using the methodology introduced previously: the first subsection focuses on the univariate case whereas the second subsection considers the bivariate one.

3.1 Univariate Case

We start estimating the HAR model, which we have introduced before using OLS on 12 high-frequency time series of log-returns: estimation results are shown in table 1. From the estimation we have obtained our measure of uncertainty as previously specified: values of such parameter are shown in appendix in table 1. The results from the estimation of our parameter allow us to differentiate between three cases: low, medium and high uncertainty.

TABLE 1 HERE

The meaning of the value of such parameter can be summarized as follows: a higher (lower) value of our parameter implies that we are less (more) confident about next day volatility prediction. In order to highlight the impact of such parameter, we have stressed its values and set these equal to 0.3, 0.5 and 0.7. As a particular case we have also considered when we have no uncertainty about volatility, i.e. ω equals 0.

Before we proceed to show the effect of parameter uncertainty we want to provide justification for the choice of the distributions as $p(\sigma^2)$. A test on the normality of the residuals of the HAR model leads to vastly reject the null. Hence, we can conclude that residuals have a leptokurtik but unknown distribution which justify our choice of fatter tailed distribution for the volatility.

FIGURE 1 HERE

Table 2 shows the results for the tail risk measures to the distributions obtained using (7). More specifically this table compares the risk measures at two levels α obtained from the different mixtures to a Normal that has the same variance of the mixed distribution. Panel A shows that at $\alpha = 95\%$ the *VaR* for the normal density is always higher compared to the mixture densities. However, the Normal and the mixture densities cross between 95% and 99%, Panel B in fact shows that mixture models have lead to higher estimates of *VaR*. This means that, going deeper in the tail of the distribution, if we do not take into account volatility uncertainty we are underestimating the risk measure. Panels C and D confirm the results obtained for *VaR*.

TABLE 2 HERE

The effect of VoV on risk measures have not been extensively analized in the literature. First, we focus on the results from the calibration of the gamma density to the moments implied by the HAR model. Values for the degrees of freedom parameter, ν are shown in table 3: this strongly highlights the importance of ω. This parameter controls the fatness of the tail of the Student-t density. Higher values of it lead to a distribution which is hard to distinguish from a Normal one whereas lower values of ν imply a more fat-tailed distribution with the extreme case with $\nu = 1$ of the Cauchy distribution. The lower the VoV, the higher the degrees of freedom the t-distribution has: this means that the tails of the predictive density are thinner.

TABLE 3 HERE

The last part of this section focuses on the graphical analysis of the obtained predictive densities when considering different uncertainty degrees about the parameter estimate.

Figure 2 and 3 show the effect of volatility uncertanty on the distributions: in particular, we focus on the effect for risk measures. Specifically, in figure 2 we present the ratio between the *VaR* of the models obtained using 7, where we have uncertainty in the variance, $\omega \neq 0$, and the *VaR* of the distribution without variance uncertainty, $\omega = 0$, that we denote VaR^0. Panel (a) considers the case where VoV is high: the distribution that present fatter tail, hence higher VaR, is the Student-t distribution obtained in the second experiment that we have previously explained. This fact is also confirmed by panels (b) and (c). The main difference among these plots is highlighted by the behaviour of the mixed distribution: in subfigure (a) models that consider a fatter tailed distribution show a significant difference in VaR compared to the log-normal one. This fact appears to have less and less importance when we move from a situation of high uncertainty to one where we are more confident about volatility. Figure 3 applies the same ratio described above considering instead the expected shortfall. Moreover, the analysis on this measure confirm what we have highlighted for Value at risk.

FIGURE 2 HERE

FIGURE 3 HERE

Given a specific mixture distribution, we also show the effect of VoV on risk measures in figures 4 and 5. In the case where we have the lowest value of ω for the normal-lognormal, normal-t and normal-weibull, the ratio stays flat meaning that the impact is not very strong when we move to the end of the distribution. The case where this parameter plays a major role is given by the normal-pareto mixture. When ω is the highest for each model, VoV has an impact on risk measures.

3.2 Dependence Analysis

In order to study the effect of uncertainty in the variance for a bivariate process of returns, we have simulated a bivariate time series with 1 billion draws. The high amount of draws is justified by the accuracy that we want to obtain for the estimates of $\tau(\alpha)$. In figure 6 and 7, we show the impact of uncertainty on variance on the tail probability coefficient when $\alpha \to 1$. In the first figure, we show the impact on the tail probabilities considering different specifications for $\mathbf{\Omega}$ in a log-log scale, whereas in the second figure, we show the ratio between the probabilities obtained with our model and the same probabilities obtained with a normal distribution. Both figures show that the case where we have all the elements of $\mathbf{\Omega} \neq 0$ coincides with higher values of $\tau(\alpha)$. Moreover $\mathbf{\Omega^2}$ presents the lowest values for the tail probabilities compared to the case represented by $\mathbf{\Omega^1}$. Standard errors in table 4 are small enough to determine the tail conditional probabilities at five decimal digits with the lowest being $\alpha = 0.0001$.

4 Conclusions

In this work we have analyzed the effect of variance uncertainty on tail risk measures employed by risk managers. In the univariate case, we have shown that the effect of uncertainty is evident and non-negligible: VaR and ES result to be very different when computed for different degrees of uncertainty in the variance. We have shown that higher VoV leads to higher estimates of risk measures given a distributional assumption for the innovations. Moreover, we have pointed out how distributional assumptions on the innovations drastically impact the predictive density of returns.

In a bivariate setting, the dependence analysis highlights that different specifications for the covariance matrix of innovations of (8) lead to different degrees of dependence between two assets. Specifically, results point out that the strongest contribution to the tail probabilities stems from uncertainty in correlations.

In this paper we have considered for our analysis a univariate model for volatility and the uncertainty in variance on risk measures. However, most practical questions related to risk management deal with portfolios composed by number of assets greater than two. We leave further work along these lines of research for future research.

References

T.G. Andersen, T. Bollerslev, F.X. Diebold, and P. Labys. Modeling and forecasting realized volatility. *Econometrica*, 2003.

F. Corsi. A simple approximate long-memory model of realized volatility. *Journal of Financial Econometrics*, 2009.

P. Embrechts, S. Resnick, and G. Samorodnitsky. Extreme value theory as a risk management tool. *North American Actuarial Journal*, 1999.

C. Gourieroux. Continuous time wishart process for stochastic risk. *Econometric Reviews*, 2006.

C. Gourieroux, J. Jasiak, and R. Sufana. The wishart autoregressive process of multivariate stochastic volatility. *Journal of Econometrics*, 2009.

A Philipov and A.E. Glickman. Multivariate stochastic volatility via wishart processes. *Journal of Business & Economic Statistics*, 2006a.

A Philipov and A.E. Glickman. Factor multivariate stochastic volatility via wishart processes. *Econometric Reviews*, 2006b.

H. White, T. Kim, and S. Manganelli. Var for var: Measuring tail dependence using multivariate quantile regressions. *Journal of Econometrics*, 2015.

A Dependence Probability Computation

In order to obtain the probability given by (3) we use an approach based on a simple non-parametric estimator. By simple rules of probability, (3) can be rewritten as:

$$\tau(\alpha) = \frac{\mathbb{P}\Big(r^1 > Q^1(\alpha), r^2 > Q^2(\alpha)\Big)}{1-\alpha} \tag{10}$$

In order to get the numerator in (10), we count the number of observations that lie in the following area:

$$\mathcal{A} = \left\{ \Big(r_i^1, r_i^2\Big)_{i=1,\cdots,n} \middle| \quad \Big|r_i^1\Big| > VaR^1(\alpha) \;\; and \;\; \Big|r_i^2\Big| > VaR^2(\alpha), \quad r_i^1 \cdot r_i^2 > 0 \right\} \tag{11}$$

If r_i is Bernoulli distributed, h_i takes value *1* if (11) is fullfilled. This random variable counts returns that are in the left or right tail that have the same sign. The numerator in (10) is obtained as:

$$\hat{\mathbb{P}}(-) = \frac{1}{2n}\sum_i^n h_i \tag{12}$$

The term *2* in front of the sum comes from the fact that the estimator uses n observations to exploit the symmetry. Moreover, for a Bernoulli random variable the standard error, SE($\hat{p}$) is given by:

$$SE(\hat{\mathbb{P}}) = \frac{1}{2}\sqrt{\frac{2\hat{\mathbb{P}}(1-2\hat{\mathbb{P}})}{n}} \tag{13}$$

B Inverted Wishart and SMM

B.1 Estimation

As we have done for the univariate case, we consider $\boldsymbol{\Sigma}$ to be inverted-Wishart distributed with some degrees of freedom ν and a certain shape matrix $\mathbf{S}$: we are interested in the first parameter which we retrieve for several covariance matrices $\boldsymbol{\Omega}$. We adopt the Method of Simulated Moments in order to get the estimate of the parameter of interest. This method proceeds as follows: first we simulate 10000 matrices $\Xi \sim iW(\nu, \boldsymbol{I}_2)$, we also know that $\boldsymbol{S} = \boldsymbol{CC'}$ and hence $\boldsymbol{\Sigma} = \boldsymbol{C'}\ \Xi\ \boldsymbol{C} \sim iW(\nu, \boldsymbol{S})$. Then, through the transformations used before, we get $\boldsymbol{\zeta}$ and its estimated moments, $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Omega}}$. For each value of the degrees of freedom parameter that we simulate from we minimize the distance between the estimated moments and the empirical ones implied by the VAR(1) estimation, $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}$ optimizing with respectd the elements of $\boldsymbol{C}$. More specifically we solve the following minimization problem:

$$\underset{\nu, \boldsymbol{C}}{\operatorname{argmin}} \quad \Big(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}\Big)' \cdot \boldsymbol{W_1} \cdot \Big(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}\Big) + \Big(vech(\boldsymbol{\Omega}) - vech(\hat{\boldsymbol{\Omega}})\Big)' \cdot \boldsymbol{W_2} \cdot \Big(vech(\boldsymbol{\Omega}) - vech(\hat{\boldsymbol{\Omega}})\Big)$$

where $\boldsymbol{W_1} = \boldsymbol{(V_1)^{-1}}$, with $\boldsymbol{V_1} = \frac{1}{N}\sum_i(\boldsymbol{\zeta_i} - \bar{\boldsymbol{\zeta}})(\boldsymbol{\zeta_i} - \bar{\boldsymbol{\zeta}})'$ and where $\boldsymbol{W_2} = \boldsymbol{(V_2)^{-1}}$, with $\boldsymbol{V_2} = \frac{1}{N}\sum_i(\boldsymbol{\xi_i} - \bar{\boldsymbol{\xi}})(\boldsymbol{\xi_i} - \bar{\boldsymbol{\xi}})'$ and $\xi_i = vech\Big((\boldsymbol{\zeta_i} - \bar{\boldsymbol{\zeta}})(\boldsymbol{\zeta_i} - \bar{\boldsymbol{\zeta}})'\Big)$.

B.2 IWE Results

We now focus on the results obtained using the MSM explained in the methodology section. Estimation is performed on 66 pairs of assets. Table 5 contains the average value of the degrees of freedom parameters obtained as the solution of the optimization problem. Moreover, this table shows the average errors between the estimated moments and the ones implied in the VAR(1) model. If we focus on the average difference between the first moments, we notice that the model has a good fit, however this is not the case for the average errors of the second moments, in particular for the one concerning the correlation. Tables 6, 7 and 8 focus on three different pairs of assets and show the estimate of the degree of freedom parameter and the difference between estimated and empirical moments. Regarding the fat tails parameter we confirm the results from the univariate analysis: a higher uncertainty in the covariance matrix leads to fatter tails. It is worth noticing that for the moment related to the correlation the fit deteriorates when ν is lower.

C Tables and Figures

$\hat{\theta}$	GM	WFC	$INTC$
α_0	0.0419	0.0140	0.0428
α_1	0.4044	0.4308	0.4640
α_2	0.2892	0.3750	0.2851
α_3	0.2736	0.1704	0.2182
ω	0.5703	0.4752	0.4340

Table 1: Parameter Estimates of HAR model for three different series of log returns

	Log-$\mathcal{N}$	$\mathcal{N}$	Student-t	$\mathcal{N}$	Weibull	$\mathcal{N}$	Pareto	$\mathcal{N}$
PANEL A: VaR 95%								
ω_{high}	0.273	0.293	0.289	0.317	0.376	0.414	0.518	0.614
ω_{medium}	0.259	0.264	0.269	0.284	0.330	0.347	0.439	0.551
ω_{low}	0.251	0.253	0.254	0.258	0.291	0.295	0.373	0.479
ω_{null}				0.246				
PANEL B: VaR 99%								
ω_{high}	0.435	0.395	0.506	0.448	0.654	0.583	1.267	0.881
ω_{medium}	0.391	0.370	0.431	0.402	0.516	0.488	1.111	0.783
ω_{low}	0.363	0.356	0.377	0.366	0.423	0.419	0.880	0.672
ω_{null}				0.346				
PANEL C: ES 95%								
ω_{high}	0.375	0.381	0.436	0.469	0.566	0.615	0.936	1.053
ω_{medium}	0.344	0.347	0.376	0.390	0.453	0.470	0.837	0.979
ω_{low}	0.321	0.322	0.334	0.338	0.378	0.383	0.693	0.855
ω_{null}				0.308				
PANEL D: ES 99%								
ω_{high}	0.542	0.497	0.715	0.636	0.925	0.832	1.576	1.289
ω_{medium}	0.474	0.450	0.566	0.527	0.673	0.635	1.483	1.215
ω_{low}	0.425	0.417	0.459	0.445	0.517	0.507	1.306	1.098
ω_{null}				0.397				

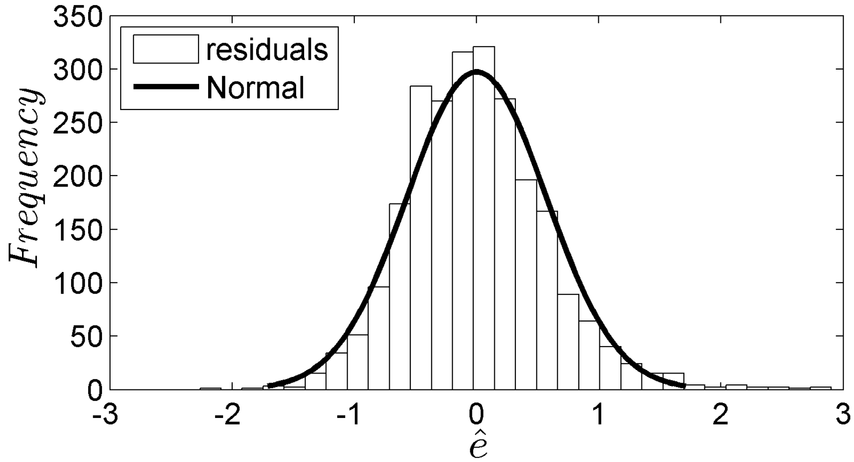
Table 2: Value at Risk at 95% and 99% for the predictive densities of returns obtained using equation 3

ω	ν
High	2.818
Medium	3.328
Low	4.314

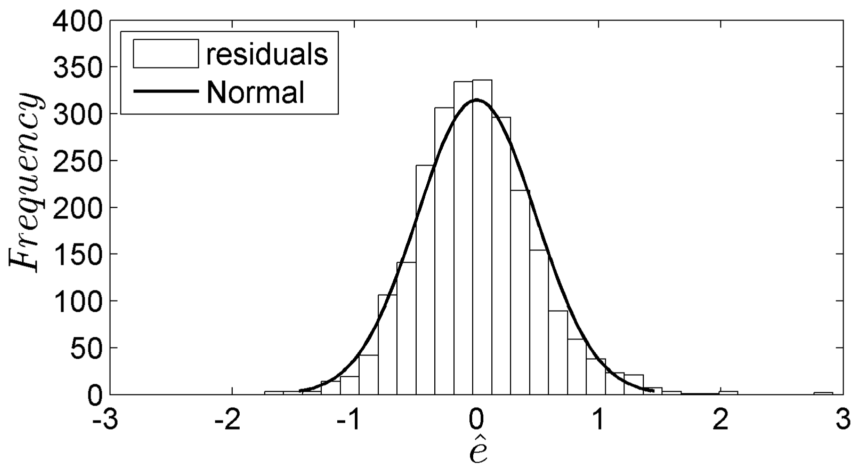
Table 3: Degrees of Freedom for Student-t predictive density obtained recovering parameters from of Γ-density.

α	0.05	0.03	0.02	0.01	0.005	0.001	0.0005	0.0001
$\mathbf{\Omega^0}$	4.7E-06	4E-06	3.5E-06	2.8E-06	2.2E-06	1.3E-06	1E-6	6E-07
$\mathbf{\Omega^1}$	4.3E-06	3.6E-06	3E-06	2.3E-06	1.7E-06	8E-07	6E-07	3E-07
$\mathbf{\Omega^2}$	4.5E-06	3.8E-06	3.3E-06	2.6E-06	2E-06	1.1E-06	9E-07	5E-07

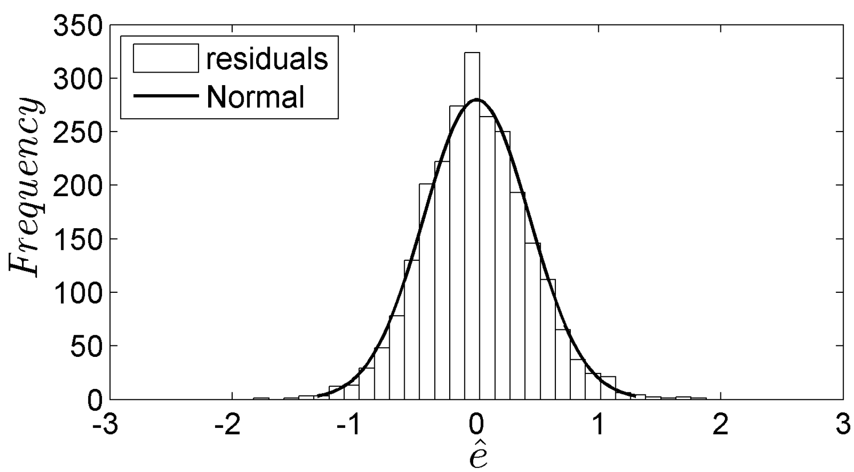
Table 4: Monte Carlo standard errors for $\tau(\alpha)$ for the different specifications of model 8



(a) High Uncertainty

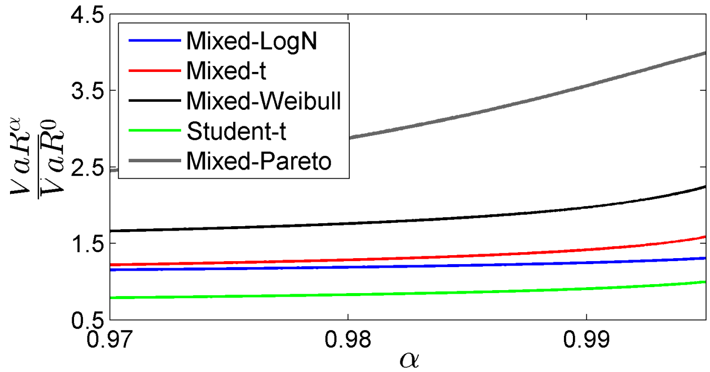


(b) Medium Uncertainty

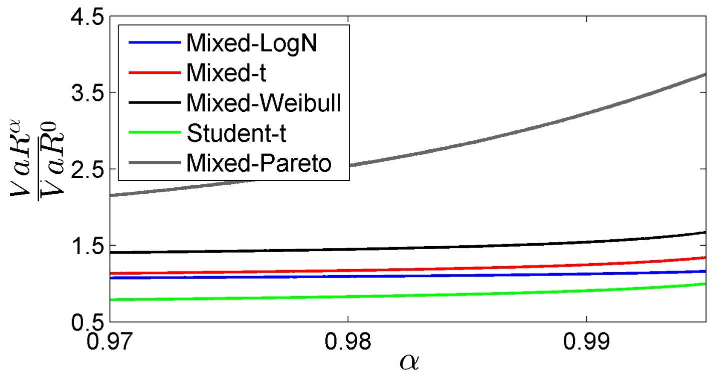


(c) Low Uncertainty

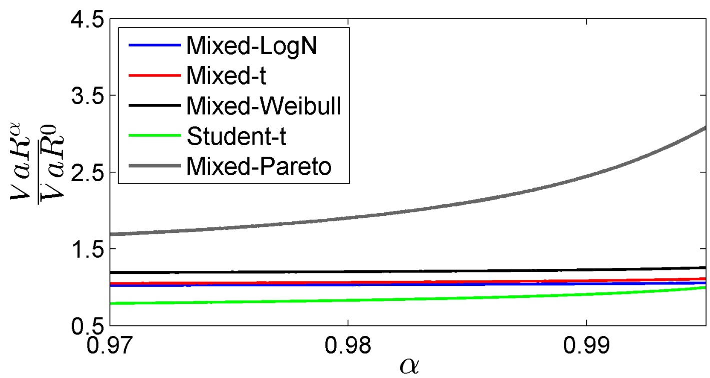
Figure 1: Distribution of the residuals of the estimated HAR model for different time series compared to a Normal distribution.



(a) High Uncertainty

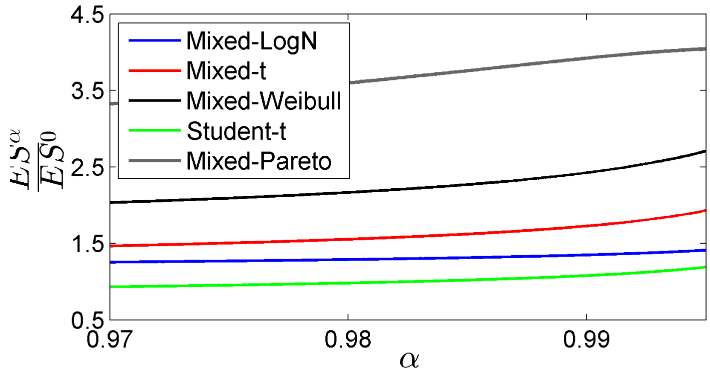


(b) Medium Uncertainty

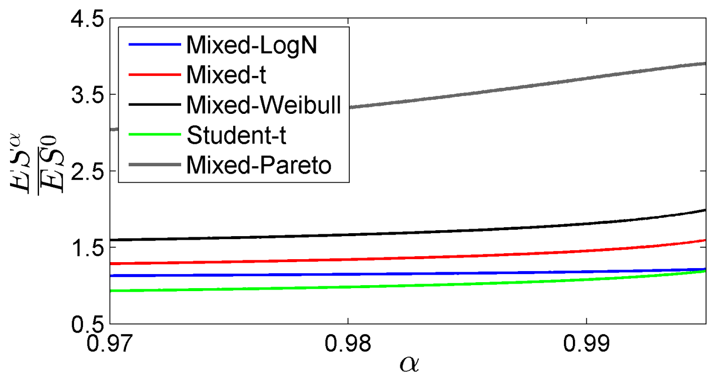


(c) Low Uncertainty

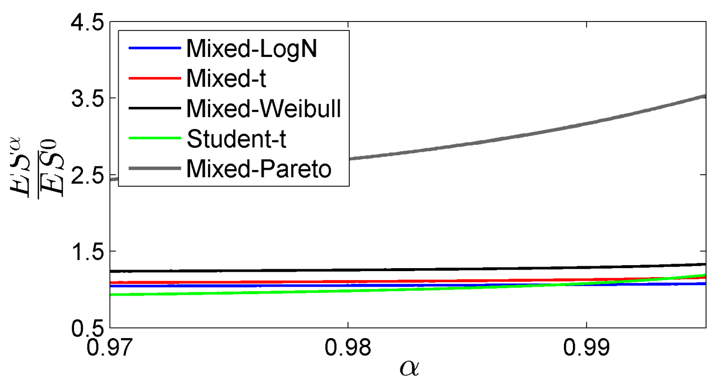
Figure 2: Ratio between the VaR at level α of the predictive distributions obtained using equation 3 and VaR of the distribution without uncertainty.



(a) High Uncertainty

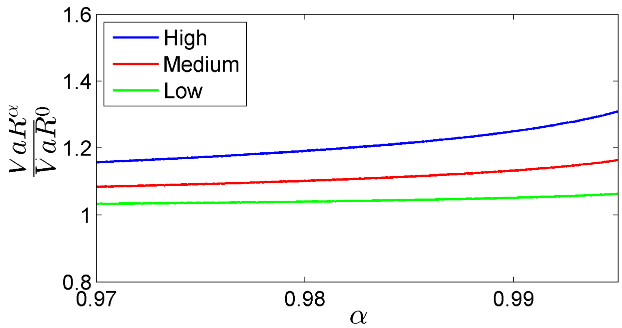


(b) Medium Uncertainty

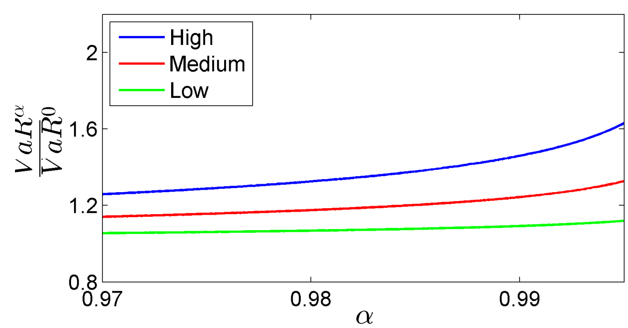


(c) Low Uncertainty

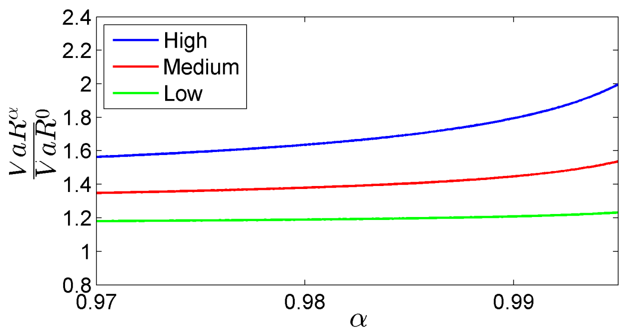
Figure 3: Ratio between the ES at level α of the predictive distributions obtained using equation 3 and ES of the distribution without uncertainty.



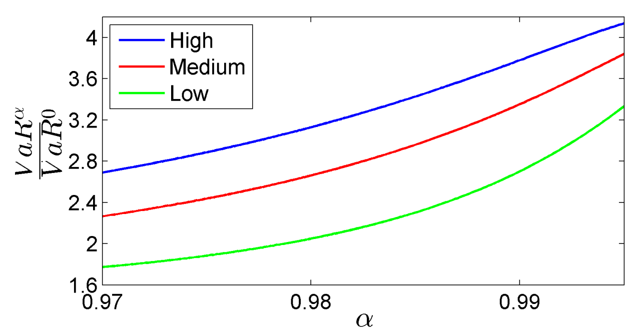
(a) Normal-Lognormal



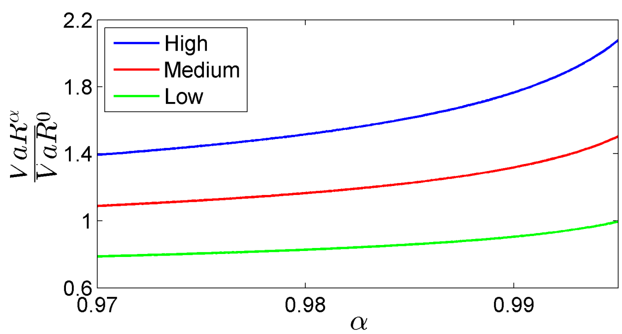
(b) Normal-t



(c) Normal-Weibull

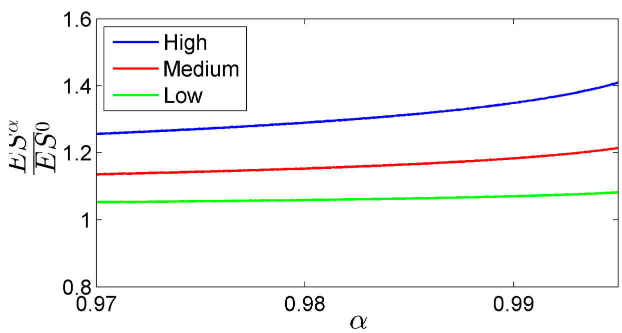


(d) Normal-Pareto

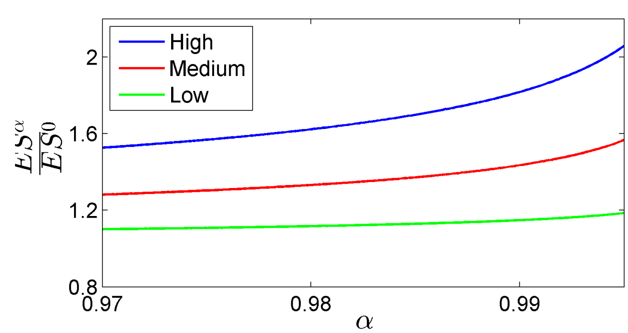


(e) Normal-iG

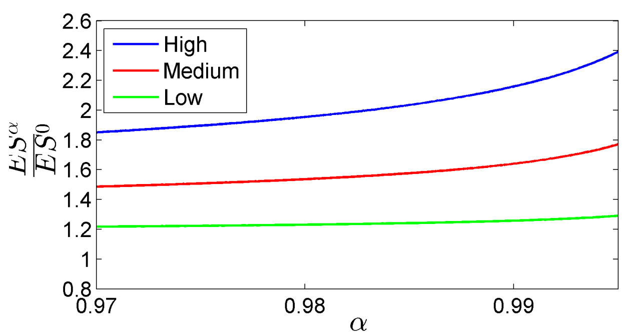
Figure 4: Ratio between the VaR at level α of the predictive distributions obtained using equation 3 and VaR of the distribution without uncertainty for each mixture and different ω.



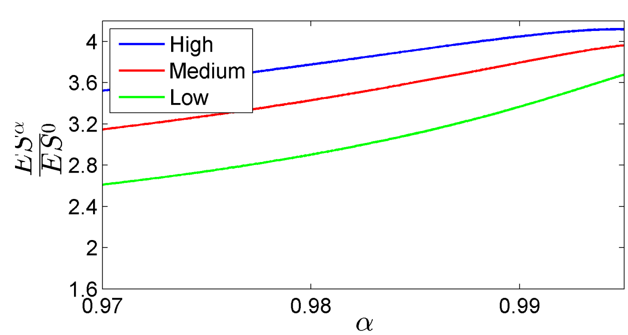
(a) Normal-Lognormal



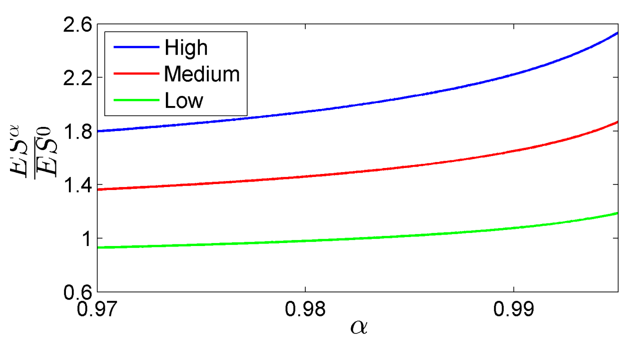
(b) Normal-t



(c) Normal-Weibull



(d) Normal-Pareto



(e) Normal-iG

Figure 5: Ratio between the ES at level α of the predictive distributions obtained using equation 3 and ES of the distribution without uncertainty for each mixture and different ω.

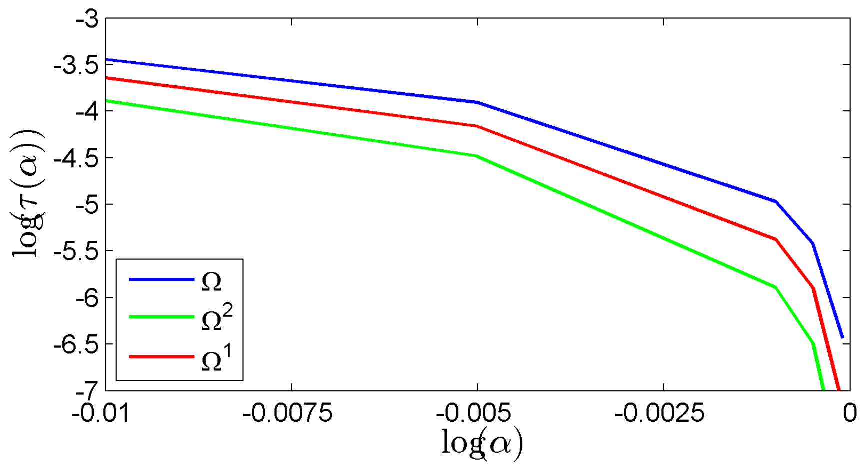


Figure 6: Tail conditional probability for the different specifications of model 8 in log-log scale.

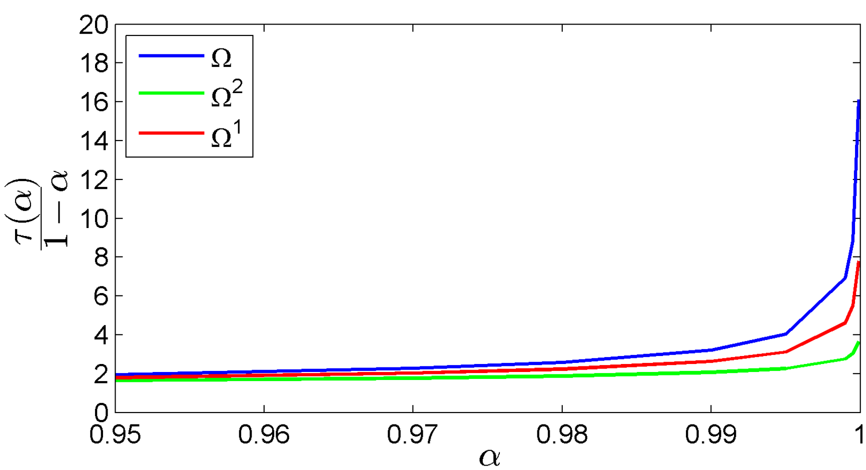


Figure 7: Tail conditional probability for the different specifications of model 8.

$\mathbb{E}[\hat{\nu}]$		4,590909	
	-0,01104	0,074558	-0,04654
$\mathbb{E}[\Omega - \hat{\Omega}]$	0,074558	-0,02115	-0,05108
	-0,04654	-0,05108	-0,58497
$\mathbb{E}[\mu - \hat{\mu}]$	-0,00231	-0,00232	-0,01958

Table 5: Average Moments Errors and degrees of Freedom

GM		HD	
ν		3	
	-0,00163	0,061995	-0,05221
$\Omega - \hat{\Omega}$	0,061995	-0,11634	-0,04419
	-0,05221	-0,04419	-0,85998
$\mu - \hat{\mu}$	-0,00117	-0,00116	-0,01438

Table 6: High Uncertainty: element by element difference between estimated moments and empirical ones

	HNZ		XOM
ν		5	
	0,042973	0,081983	-0,04023
$\Omega - \hat{\Omega}$	0,081983	-0,06253	-0,02301
	-0,04023	-0,02301	-0,50585
$\mu - \hat{\mu}$	-0,00117	-0,00112	-0,0136

Table 7: Medium Uncertainty: element by element difference between estimated moments and empirical ones

	MSFT		XOM
ν		8	
	0,051729	0,065144	-0,02315
$\Omega - \hat{\Omega}$	0,065144	0,023108	-0,02189
	-0,02315	-0,02189	-0,30147
$\mu - \hat{\mu}$	-0,00316	-0,00312	-0,02261

Table 8: Low Uncertainty: element by element difference between estimated moments and empirical ones