Voluntary Participation in a Defined Benefit Pension Scheme: An Option Pricing Approach

Damiaan H.J. Chen*

University of Amsterdam, MN, Tinbergen Institute

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Abstract

This paper evaluates an American pension option, whereby participants have the option to convert their defined benefit (DB) pension entitlements of a collective scheme to an individual defined contribution (DC) plan, using contingent claim analysis. This way, we can evaluate the participation decisions under a voluntary collective pension scheme. We approximate the value of this option with risky investment returns by applying Least Squares Monte Carlo simulations as proposed by Longstaff and Schwartz (2001). When more decision dates are included, generations are more willing to participate in the collective pension scheme. If the funding rate falls below a critical value, some young generations will exercise the option. As a result, other generations might be willing to leave as well, which results in a collapse of the collective pension scheme. In the absence of mandatory participation, it is only a matter of time before such a break down occurs.

Keywords Pension funds, participation decisions, valuing American options, Least Squares Monte Carlo simulations, stability.
JEL Codes C61, G23, J32.

*MInt/Amsterdam School of Economics, University of Amsterdam, Valckenierstraat 65–67 1018 XE Amsterdam, The Netherlands; e-mail: D.H.J.Chen@UvA.nl. In particular I thank Roel Beetsma for his insights and helpful comments. This paper also benefited from discussions with Ward Romp and Sweder van Wijnbergen.

1 Introduction

Past decades many pension systems are being revised worldwide, particularly because of ageing problems, low interest rates and the financial crisis. Changes from defined benefit (DB) to defined contribution (DC) pension schemes are frequently observed for a number of reasons. First, the increasing workforce mobility makes a DC pension scheme more attractive, since DB pension schemes are typically back-loaded. Second, the worsened financial health of pension funds makes a DB pension scheme less attractive for young participants, as their expected benefits might be lower than under a DC arrangement in case they have to make up for the shortages in the DB pension scheme. Third, employers are typically responsible for the payout under DB arrangements and they are required to put these risks on their balance sheet under the revised accounting frameworks, making DC arrangements preferable from the employer's viewpoint (Broadbent, Palumbo, & Woodman, 2006). Despite this current tendency from DB to DC, there are still many funded DB pension plans worldwide. These are particularly the public sector pension plans, such as those in Australia, Canada, Germany, the Netherlands, Norway, Switzerland, the U.K. and the sub-national civil servants' plans in the U.S. (Ponds, Severinson, & Yermo, 2011). This paper explores the sustainability of DB pension schemes when participants have the option to convert their DB pension entitlements into individual DC capital.

There are several real life examples of transfers between DB and DC pension schemes. In the United Kingdom, DB to DC transfers are permitted for private sector pensions and funded public pensions (HM Treasury, 2014). The Italian National Pension Plan has offered an option to switch from a DB pension to DC for a limited class of workers (Bacinello, 2000). Similarly, some Australian pension plans offer a "greater of" benefit at retirement, that is the maximum of a DB and DC pension scheme (Sherris, 1993). Finally, the State of Florida provides a related example for the United States. In 2002, 600,000 public employees were given the choice of converting their collective DB plan to individual DC. For those who elected the DC plan, the additional option to switch back was granted by the State as well. A number of authors, such as Lachance, Mitchell, and Smetters (2003); Milevsky and Promislow (2004) investigate the value of this option. They show that this value can be substantial for rational participants by applying the optimal exercise decision.

In some countries, participation in funded pension schemes is mandatory, which can be motivated by a number of reasons, e.g. lower costs, irrational individual behaviour and risk-sharing (D. Chen & Beetsma, 2013). Such risk-sharing can be ex-ante beneficial for all participants (Ball & Mankiw, 2007; D. Chen, Beetsma, Ponds, & Romp, 2014; Cui, Jong, & Ponds, 2011; Gollier, 2008; Gordon & Varian, 1988; Shiller, 1999; Teulings & De Vries, 2006). However, when the funding rate is too low, the scope for risk-sharing is likely to be only one-way traffic, i.e. from the new participant to existing participants (R. Beetsma, Romp, & Vos, 2012; R. M. W. J. Beetsma & Romp, 2013; Siegmann, 2011). Hence, mandatory participation in collective pension schemes has gained less support over the last years, as risk-sharing pension schemes are to a lower extent ex-ante beneficial for all participants. Under the pension option considered in this paper, participants have a once-and-for-all option to leave the collective pension fund. Since DB pension schemes are typically collective arrangements and individual pension arrangements are typically of the DC type, this specific option can be seen as providing a compromise between mandatory and voluntary participation. For example, most pension schemes in the Netherlands are subject to mandatory participation and are typically collective funded DB pension schemes, while the voluntary arrangements are typically individual DC (D. Chen & Beetsma, 2013). The model can be used to answer whether a collective pension system is sustainable once mandatory participation is abolished or made less stringent.

We focus on pure DC and pure DB pension schemes, whereby each participant has the option to convert DB pension entitlements into DC pension assets. Under the DC pension scheme, a fixed rate of an individual's salary is contributed to the pension fund. This way, the individual builds up personal assets which accumulate according to the generated returns from the pension fund's investment portfolio. Once retired, the participant uses the accumulated assets as pension income or to buy an annuity. Under the DB pension scheme, pension entitlements are built up on the basis of a fixed accrual rate. Once retired, the individual receives a fixed benefit based on his accumulated pension entitlements until death. In this case, the contribution rate is determined by the financial health of the pension fund (the so-called "funding rate"). Under the DB-DC option, the individual starts under the DB pension scheme and has the opportunity to convert his accumulated pension rights to individual DC pension assets. The option is of the American type, hence it may be exercised at each date until retirement. Hence, this model differs from "greater of" benefit options, where at maturity the best of two outcomes can be chosen. Approximation methods need to be applied for this class of options. Hence, in order to approximate the option value we use the method proposed by Longstaff and Schwartz (2001). This method is known as the Least Squares Monte Carlo (LSMC) approach.

We obtain several interesting insights from our analysis. First, ceteris paribus young workers are more inclined to exercise the option than older workers, since the uniform contribution rate is relatively unattractive for young generations. Second, participants are more willing to enter the collective pension scheme when more flexibility is provided by means of more exercise dates of the option. Third, the entry funding rate threshold is lower when investment risk is larger and when the recovery window is longer. However, investment risk and smoothing of funding rate recovery also increase the volatility of the funding rate, which in turn increases the uncertainty about future participation. Fourth, if a relatively small negative shock occurs, some young generations will leave, while other generations will stay in the collective pension scheme. However, when the negative shock is large, it is initially optimal for a large group of young generations to exercise the option.

As a result, quitting becomes beneficial for the remaining cohorts as well, because their recovery contributions become too large. Finally, we find that it is only a matter of time before the collective scheme collapses. The system survives the coming 10 years with less than 40% probability, while all participants have exercised the option almost surely after 20 to 40 years, depending on the parameter settings.

Our results connect to different findings from the existing literature. For example, Siegmann (2011) analyses what is the threshold on the funding rate for which an individual would still voluntarily participate in a DB pension fund, based on expected utility. He finds that the threshold for a DB fund is between 87% and 120% in nominal terms, depending on participants' risk aversion and level of sophistication. Contrary to his analysis, we model the participation decision as an American option based on risk neutral valuation. Under our benchmark parameter setting, we find that entry generations have a funding rate threshold between 84.4% and 103.7% in real terms, depending on the flexibility to leave the collective DB pension scheme in the future. Furthermore, Molenaar, Peijnenburg, and Ponds (2011) explore the funding rates for which participants are willing to leave the pension plan. They find that the youngest and the oldest active members have the largest incentive to opt-out, since the uniform contribution is relatively unattractive for the young, while the old are mostly affected by indexation reductions, as they have the largest accumulated pension entitlements. The effect on the young generations is in line with our results, while the effect on the old generations is not confirmed by our findings, as we do not consider fluctuations in the indexation policy.

Other literature about the incentive to participate in a collective pension fund with intergenerational risk-sharing are the following. Van Hemert (2005) investigates intergenerational risk-sharing in a two-period overlapping generations model with four correlated risks on human and financial capital. He shows that when no risk free asset is present, it can be substituted by social security transfers, which enhances welfare. As a result, cooperation by young generations is only a minor issue. However, if the risk free asset is available, imposing an incentive constraint for the young leads to a collapse of the social security scheme. Van Bommel (2007) finds that both over- and underfunding may limit the scope of intergenerational risk-sharing, in the former case because the existing participants prefer to liquidate the pension fund and keep its assets for themselves. In our model, liquidation only occurs in case of underfunding, since we assume that participants never get more than the actuarial value of their pension rights when leaving the collective pension fund and, hence, the threat of liquidation in the case of overfunding is not an issue. R. Beetsma et al. (2012) and R. M. W. J. Beetsma and Romp (2013) analyse participation constraints in a two-period overlapping generations model with voluntary participation. Individuals only participate in the collective pension scheme with risk-sharing when it is more attractive than autarky, which holds for sufficiently strong financial market risk and sufficiently high risk aversion. Furthermore, the collective scheme can only continue to exist if the current young believe that the future young are prepared to participate, so that they guarantee the future pensions of the current young. They find that under many standard parameter settings mandatory participation is needed to sustain a funded pension pillar with intergenerational risk-sharing. In contrast to the above-mentioned papers which focus on two-period overlapping generations, we apply option pricing theory in a stochastic framework with a continuum of overlapping generations, whereby individuals have a continuum of participation decisions.

Several other studies have applied option pricing theory to model pension schemes. Among others, Blake (1998) and Timmermans, Schumacher, and Ponds (2011) treat different pension schemes as combinations of put and call options. Broeders, Chen, and Rijsbergen (2013) obtain a market-consistent valuation formula for liabilities of a hybrid pension plan. Friedman and Shen (2002) and Chevalier (2006) analyse the American option of early retirement, whereby they obtain existence, uniqueness and the properties of the optimal retirement. Fung and Chan (1995) model the termination decision of a pension fund as an American call option and offer an explanation why sponsors typically do not terminate overfunded pension plans. Kocken (2006) applies risk management and option pricing theory in order to analyse the risks for corporate pension funds. He identifies the different stake holders of the contract and their different risk preferences. Conflicts arise as certain risks cannot be fully hedged. In contrast to our model, he assumes a closed setting, i.e. a maturing pension fund without new inflow of participants. More related to the current paper are studies applying the LSMC approach to pensions and life insurance products. A number of examples are Bernard and Lemieux (2008); Boyer and Stentoft (2013); Cathcart and Morrison (2009); Pelsser, Cao, and Iseger (2007). A particular new feature of the current paper is that we derive an algorithm to determine the individual's optimal continuum of participation decisions, by modelling it as an American option. Our model reveals that this flexibility makes participants less inclined to leave the collective pension scheme.

This paper also connects to the literature on stability of pension plans. Dufresne (1989) analyses how the fluctuations in contributions, assets and liabilities can be lowered. D. H. J. Chen and Romp (2015) show how to model the policies of funded pension schemes by implementing regulation to the policy instruments. This way, global stability is assured, regardless the degree of risk-sharing and the type of pension plan (DB, DC or hybrid). This paper applies this method to prevent exploding simulations.

The remainder of this current paper is as follows. Section 2 lays out the model and Section 3 presents the benchmark parameter settings. In Section 4, in order to gain some basic insights, we first investigate the decision to participate in the pension scheme for a fixed number of exercise dates. Then, Section 5 presents the approximation algorithm for the American option based on Longstaff and Schwartz (2001), together with some preliminary results. Section 6 explores the interrelation of participation decisions, where we find a critical threshold for which the collective pension scheme completely breaks down in case the funding rate falls below this threshold. In Section 7, we investigate under what conditions cohorts newly entering the labour market are willing to participate in the DB funded pension scheme. Section 8 explores the distribution of the pension fund's life-length. Section 9 concludes the main text of this paper. Some technical details and additional figures are relegated to the Appendix.

2 The Model

In this section, we lay out the model. Section 2.1 describes the economy and its underlying processes. In Section 2.2, we discuss the valuation method. Section 2.3 explains the different pension plans.

2.1 Economy and Individual Lifetime

All processes in the model are specified under a risk neutral measure $\mathbb{Q}$ and we assume the only source of risk is the return on investments. The investment portfolio of the pension fund follows a geometric Brownian motion:

$$dP_t = rP_t dt + \sigma_P P_t dW_{P,t}.$$

Hence, the drift term is exactly the risk free rate r under the risk neutral measure, where P_t is the value of the investment portfolio.

One period corresponds to a year. An individual's working period is from age t_0 to t_R and the retirement period is from age t_R to t_D. We assume that his wage is constant and normalized to one per period:

$$w_s = \begin{cases} 1, & \text{for } s \in [t_0, t_R], \\ 0, & \text{otherwise.} \end{cases}$$

A fraction c_t of wage in period t is contributed to the pension fund.

2.2 Valuation Method

At time t, the price of a security with random pay off X_s at $s \geq t$ is given by $\Pi_t(X_s)$. According to the martingale representation theory, we can price securities with respect to the expectation under the risk neutral measure $\mathbb{Q}$. Hence, we obtain

$$\Pi_t(X_s) = \exp\{-r(s-t)\} E_t^{\mathbb{Q}}[X_s],$$

where $E_t^{\mathbb{Q}}$ is the expectation under the risk neutral measure $\mathbb{Q}$ conditional on information up to time t. We assume that the market is complete and, therefore, the nominal interest rate is the unique numéraire. We take this valuation method as the "actuarially fair pricing" method.

2.3 The Pension Schemes

There are three types of pension schemes: (i) the individual DC pension, (ii) the collective DB pension, and (iii) the DB-DC pension option.

2.3.1 The Individual Defined Contribution (DC) Pension

For a generation born at date t, the accumulated pension assets at age s are given by

$$A_{t,s}^{DC} = \int_0^s c^{DC} \frac{P_{t+s}}{P_{t+u}} du, \quad \text{for } s \in [t_0, t_R],$$

where c^{DC} is a fixed contribution rate. Retirement takes place at the age of t_R, where the amount of assets given by A_{t,t_R}^{DC} are used to buy an annuity according to the market price of annuities, such that a fixed retirement income is obtained until the age of death t_D. The constant annuity benefit is given by B^{DC}, which are obtained by

$$\begin{aligned} A_{t,t_R}^{DC} =& E_{t+t_R}^{\mathbb{Q}} \left\{ \int_{t_R}^{t_D} \exp\left[-r\left(s-t_R\right)\right] B^{DC} ds \right\} \\ \Longleftrightarrow \quad B^{DC} =& r A_{t,t_R}^{DC} / \left\{1 - \exp\left[-r\left(t_D - t_R\right)\right]\right\}. \end{aligned}$$

Furthermore, this pension scheme is actuarially fair by construction:

$$\Pi_{t+s}\left(A_{t,s}^{DC}\right) = A_{t,s}^{DC}.$$

2.3.2 Collective Defined Benefit (DB) Pension

Denote I_t as the set of all age groups participating in the collective pension scheme at time t. Under full participation $I_t = \{s : s \in [t_0, t_D]\}, \forall t$. The total mass of working agents in the collective pension scheme at time t is given by

$$m_t^w = \int_{\{s: s \in [t_0, t_R]\} \cap I_t} 1 ds$$

and the total mass of retirees at time t is

$$m_t^r = \int_{\{s: s \in [t_R, t_D]\} \cap I_t} 1 ds.$$

Workers accrue pension entitlements at a constant rate, until they retire. Hence, the pension entitlements of a participant of age s are:

$$
\begin{aligned}
B_s =& \begin{cases} \psi s, & \text{for } s \in [t_0, t_R], \\ B_{t_R}, & \text{for } s \in (t_R, t_D), \end{cases} \\
=& \min(s, t_R)\, \psi,
\end{aligned}
$$

where ψ is the accrual rate and B_{t_R} is the constant benefit that the participant receives during retirement. Note that accrued entitlements are not indexed, i.e. the indexation rate is zero. The total volume of pensions paid by the pension fund is:

$$B_t^{TOT} = m_t^r B_{t_R}.$$

The discount factor of DB pension entitlements at age s is given by

$$
\begin{aligned}
R_s =& \begin{cases} \exp\left[-r(t_R - s)\right] \int_{t_R}^{t_D} \exp\left[-r(u - t_R)\right] du, & \text{for } s \in [t_0, t_R], \\ \int_s^{t_D} \exp\left[-r(u - s)\right] du, & \text{for } s \in (t_R, t_D), \end{cases} \\
=& \frac{1}{r} \exp\left\{-r\left[t_R - \min(t_R, s)\right]\right\} \left(1 - \exp\left\{-r\left[t_D - \max(t_R, s)\right]\right\}\right).
\end{aligned}
$$

Hence, the actuarially fair price for DB pension entitlements at time t for a participant with age s is:

$$\Pi_t(B_s) = R_s B_s,$$

while the liabilities of the pension fund under the DB pension scheme are given by:

$$L_t = \int_{I_t} R_s B_s ds.$$

Then, the process of the liabilities is as follows:

$$dL_t = L_{t+dt} - L_t = \int_{I_{t+dt} \setminus I_t} R_s B_s ds - \int_{I_t \setminus I_{t+dt}} R_s B_s ds,$$

where $I_{t+dt} \setminus I_t$ is the inflow of age groups and $I_t \setminus I_{t+dt}$ is the outflow of age groups in period $t + dt$. Here, $I_{t+dt} \setminus I_t$ is the set of participants in $t + dt$ that were not yet participants in t and $I_t \setminus I_{t+dt}$ is the set of participants in t that are no longer participant in $t + dt$. Under full participation we have $I_{t+dt} \setminus I_t = \emptyset$ and $I_t \setminus I_{t+dt} = \emptyset$, so the liabilities are constant ($dL_t = 0$).

The process of the pension fund's assets is denoted by

$$dA_t = \frac{dP_t}{P_t} A_t + \left(C_t - B_t^{TOT}\right) dt.$$

Hence, the pension assets grow according to the portfolio returns $\left(\frac{dP_t}{P_t}\right)$ plus the total volume of contributions $(C_t = m_t^w c_t)$ minus the total volume of benefit payments.

The pension fund is subject to investment risk, which is partly absorbed by the funding rate and partly covered by the participants through contribution rate fluctuations. The funding rate is given by:

$$F_t = A_t / L_t.$$

The pension fund is required to fulfil regulation imposed by the regulator. The regulator requires the pension fund to reduce the gap with the target funding rate $\bar{F}$ by setting

$$E_t^{\mathbb{Q}} \left(F_{t+dt} - \bar{F}\right) = \alpha^{dt} \left(F_t - \bar{F}\right), \quad 0 < \alpha < 1, \tag{1}$$

where α denotes the smoothing parameter. This rule ensures that the funding rate gravitates towards $\bar{F}$. For α close to one, funding rate recovery is smoothed out over a longer horizon, and vice versa. Appendix A.1 shows that the regulation policy can be rewritten to:

$$E_t^{\mathbb{Q}} \left(F_{t+s} - \bar{F}\right) = \alpha^{s} \left(F_t - \bar{F}\right), \quad \forall s \geq 0.$$

Because the indexation rate and the accrual rate are constant, the only instrument through which the pension fund can respond to risks is the contribution rate, which is endogenously determined by Eq. (1). Appendix A.2 shows how Eq. (1) can be rewritten to the following contribution rate policy:

$$c_t = \frac{1}{m_t^w} \left[(\log \alpha) \left(A_t - \bar{F} L_t\right) + F_t E_t^{\mathbb{Q}} \left(\frac{dL_t}{dt}\right) - r A_t + m_t^r B_{t_R} \right].$$

2.3.3 DB-DC Option

The individual starts accumulating pension rights according to the DB pension scheme. This is the default situation. Participants are allowed to opt out from the moment that they start their working life until the retirement age t_R.[1] If at age t_R the opting-out possibility has not yet been exercised, then the participant receives an annual retirement benefit B_{t_R}

[1] A thorough analysis of default options for pension plans is provided by Madrian and Shea (2001), who find that only a few people decide to opt out.

Table 1: Choice of parameter values

Description	Symbol	Value
Entry age	t_0	0
Retirement age	t_R	40
Age of death	t_D	60
Target funding rate	$\bar{F}$	1
Funding rate smoothing	α	0.5
Interest rate (risk free)	r	0.02
Portfolio return volatility	σ_P	0.15
Wage	w	1
Accrual rate	ψ	$70\%/t_R$

until death. If the option is exercised by an agent with age $s < t_R$ and at time $t+s$, then the pension rights B_s are converted into personal assets $A_{t,s}^{DC}$. The amount of personal assets obtained for somebody with age s at time $t+s$ depends on the actuarial price of the accumulated pension entitlements and the funding rate:

$$A_{t,s}^{DC} = \min\left(1, F_{t+s}\right)\Pi_{t+s}\left(B_s\right).$$

Hence, in the case of underfunding the market value of the pension entitlements is multiplied by the funding rate. We assume that at most an amount $\Pi_{t+s}\left(B_s\right)$ can be taken out of the pension fund by exercising the option. Otherwise, if $F_{t+s} > 1$ at retirement date, the participant would always exercise his option and be able to buy a larger annuity than he would get as retired participant of the pension fund.

After exercising, the pension arrangement continues as an individual DC pension scheme from time $t+s$ onwards, whereby the personal assets further accumulate according to

$$A_{t,v}^{DC} = A_{t,s}^{DC}\frac{P_{t+v}}{P_{t+s}} + \int_s^v c^{DC}\frac{P_{t+v}}{P_{t+u}}du, \quad \text{for } v \in [s, t_R].$$

3 Parametrization

Table 1 reports the choice of the benchmark parameter values. As a robustness check, we will later also explore other parameter settings. We assume full participation, i.e. $I_t = \{s : s \in [t_0, t_D]\}, \forall t$. Our analysis is based on $Q = 10^4$ simulation runs. Time steps need to be small to approximate continuous time. We set the time steps at $\delta = 0.1$, which means that there are 10 exercise dates per annum. For convenience, the benchmark calculations are based on a generation who starts working at time $t_0 = 0$, such that time equals age.

4 European and Bermudan Option

Before analysing the American option and to obtain a better understanding of the results, we start by studying the European option, which can only be exercised at maturity date, and the Bermudan option, which can be exercised at a prespecified set of dates.

4.1 The European Option

If the only exercise date is at age $s = t_R$, then it is never beneficial to exercise the option, since:

$$\min\left(1, F_{t_R}\right)\Pi_{t_R}\left(B_{t_R}\right) \leq \Pi_{t_R}\left(B_{t_R}\right).$$

Clearly, for $F_{t_R} < 1$ it is not optimal to exercise, while for $F_{t_R} \geq 1$, the participant would be indifferent between exercising or not exercising. We adopt the convention that in case of indifference the option will not be exercised.

4.2 The Bermudan Option

Suppose there is a fixed set of exercise dates $t \in \{t_0, t_1, t_2, \ldots, t_R\}$. At each exercise date we explore what is the threshold level on the funding rate at which it becomes profitable to exercise the option. Because the DB pension entitlements become more valuable as one gets closer to retirement and the contribution rate is equal across all working generations, i.e. it is "uniform", one might expect the threshold on the funding rate at which an individual prefers to quit the pension fund to fall as one gets older.

Step 1: two exercise dates $(t \in \{t_0, t_R\})$ As shown above for the European option, the individual will not exercise the option at time t_R. At age $t_0 = 0$ the individual exercises the option to exit the collective DB pension scheme if the funding rate falls below some threshold, denoted by Γ_0, that we determine below.

The DB pension scheme is attractive when discounted benefits exceed expected discounted contributions:

$$\begin{aligned}
&\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right) \geq E_0^{\mathbb{Q}}\left[\int_0^{t_R} c_s \exp\left(-rs\right) ds\right] \\
\Longleftrightarrow\; &\left(F_0 - \bar{F}\right)\left(1 - \exp\left\{-t_R\left[r - (\log\alpha)\right]\right\}\right) \geq \\
&\left[\frac{1}{r\bar{L}}\left(t_D - t_R\right) B_{t_R} - \bar{F}\right]\left[1 - \exp\left(-rt_R\right)\right] - \frac{t_R}{\bar{L}}\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right). \qquad (2)
\end{aligned}$$

Appendix A.4 presents the derivation of the last inequality. The right-hand side (RHS) of the last line is constant, while the left-hand side (LHS) depends on the initial funding rate F_0. In Figure 1, the LHS minus the RHS is shown for different starting funding rates F_0. We

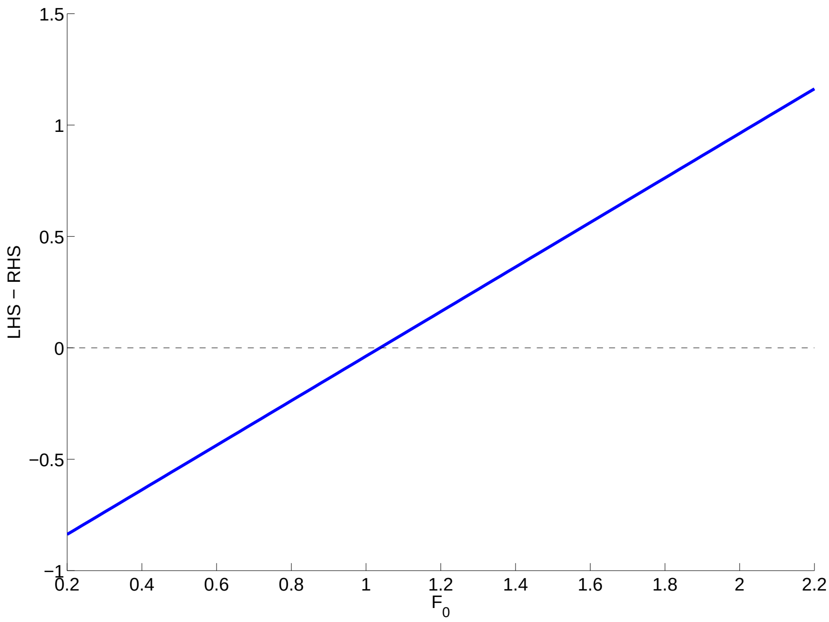


Figure 1: Participation is beneficial at entry ($t_0 = 0$) if $LHS - RHS > 0$ in Eq. (2).

observe that the DB pension scheme is more attractive when $F_0 \geq \Gamma_0 = 103.7\%$. Hence, if the funding rate falls below Γ_0, the expected future contribution payments exceed the value of the future benefits, which makes the individual unwilling to enter the collective DB plan.

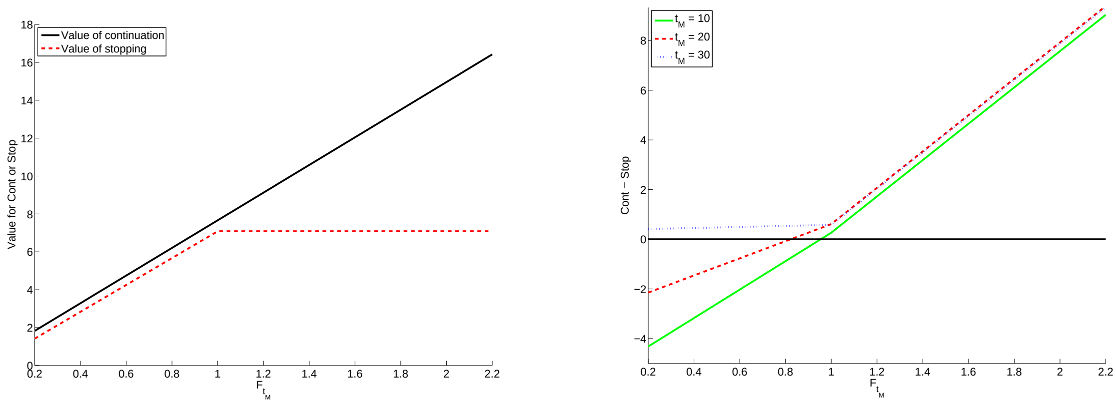
The actuarially-fair contribution rate is lower than the contribution rate under the collective pension scheme that is in line with the target funding rate $F_t = \bar{F} = 1$. Appendix A.5 provides the proof. Under the benchmark parametrization, these contribution rates are $c^{ac} = 18.83\%$ and $c^{eq} = 19.86\%$, respectively. Hence, when the funding rate is at 100%, an entry participant expects to pay larger contributions under the collective pension scheme than the market value of his future retirement benefits. This way, the entry threshold Γ_0 is larger than 100%.

Step 2: three exercise dates ($t \in \{t_0, t_M, t_R\}$) Again, the individual will not exercise the option at age $s = t_R$ as explained above. At the age of t_M the individual exercises the option when the value of stopping exceeds the value of continuation. The value of stopping at age t_M is:

$$Stop_{t_M} = \min\left(1, F_{t_M}\right) \Pi_{t_M}\left(B_{t_M}\right) \tag{3}$$

and the value of continuation at age t_M is:

$$Cont_{t_M} = \exp\left[-r\left(t_R - t_M\right)\right] \Pi_{t_R}\left(B_{t_R}\right) - E^{\mathbb{Q}}_{t_M}\left\{\int_{t_M}^{t_R} c_s \exp\left[-r\left(s - t_M\right)\right] ds\right\}. \tag{4}$$



(a) The value of continuation and the value of stopping at $t_M = \frac{3}{4}t_R$.

(b) The value of continuation minus the value of stopping at different t_M.

Figure 2: The value of continuation (Eq. (4)) versus the value of stopping (Eq. (3)) at age t_M.

For the age of $t_M = \frac{3}{4}t_R$, the values of stopping and continuation are plotted in Figure 2a. The value of stopping is constant at $\Pi_{t_M}(B_{t_M})$ for $F_{t_M} \geq 1$ and falls with a fall in the funding rate if $F_{t_M} < 1$. The value of continuation is linearly increasing in the funding rate, which is formally shown in Appendix A.6. In Step 1, we have that $Cont_0 = Stop_0$ for a funding rate $F_0 = 103.7\%$ at age 0, while in Figure 2 we observe that $Cont_{t_M} > Stop_{t_M}$ for $F_{t_M} = 103.7\%$ and $t_M > 0$, which is due to the uniform contribution rate. In other words, the value of continuation is strictly larger than the value of stopping for a funding rate of $F_{t_M} = 103.7\%$ at age $t_M > 0$, since the funding rate threshold is $\Gamma_0 = 103.7\%$ at entry and decreases with age.

In Figure 2b, we plot the differences between the value of continuation and the value of stopping for ages $t_M = \frac{1}{4}t_R$, $t_M = \frac{1}{2}t_R$ and $t_M = \frac{3}{4}t_R$. Low funding rates mean that, for given pension benefit, participants have to pay high future contributions if they do not leave the pension fund, while the opposite is the case for high funding rates. Due to the uniform contribution rate, it is strictly optimal to continue when the funding rate is exactly 103.7%, as explained above. We observe that it is never optimal to exercise the option at ages $s \geq \frac{3}{4}t_R$. However, for ages $s \leq \frac{1}{2}t_R$ the value of continuation can be lower than the value of stopping in case of low funding rates. Hence, the option of leaving the pension fund only adds some value when the participant is not too advanced in his career.

The exercise decision at an age $t_M > 0$ might lower the participation threshold at entry (Γ_0). Without the option to convert to DC at the age of t_M, the individual enters the collective DB pension scheme only when $F_0 \geq 103.7\%$. However, if we consider for example the option to convert after one year of entry $(t_M = 1)$, the individual already participates for funding rates $F_0 \geq 96.6\%$. The reason is that the funding rate might recover during the first year. If not, the participant still has the option to leave at age $t_M = 1$.

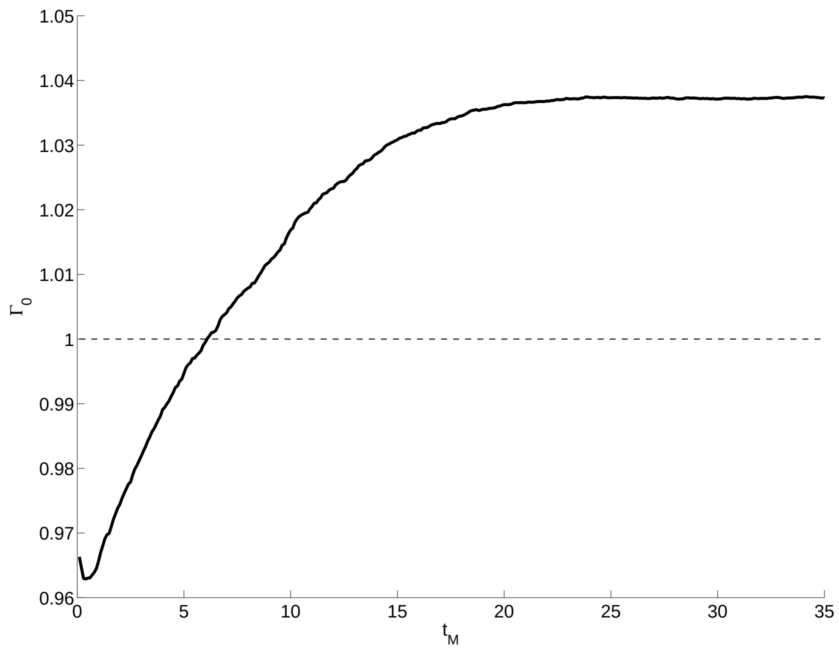


Figure 3: Funding rate participation threshold for an entry generation when one exercise date is included at age t_M.

Figure 3 shows the participation thresholds Γ_0 for an entry generation as a function of t_M, given that there is exactly one exercise date $t_M > 0$ of the option in addition to the entry and retirement date. The option does not add much value when $t_M \geq 15$, since the entry threshold is still above 103%. For extremely low t_M (say $t_M < 0.1$) the option is almost similar to the decision at t_0 and, therefore, it does not change the entry threshold much either. When the exercise decision is at age $t_M \in [0.1, 6]$, the entry threshold is less than 100% and, therefore, the option is most valuable in that interval. This is particularly due to the upside potential of the option: barely any pension entitlements have been accumulated, such that not much would be lost by leaving at $t_M \in [0.1, 6]$, but there is a possibility that the funding rate improves during $t \in [t_0, t_M]$, thereby benefiting the participant.

Step 3: multiple exercise dates $(t \in \{t_0, t_1, \ldots, t_R\})$ So far, we only considered one intermediate exercise date at age t_M. Including additional exercise dates makes the the option to leave more valuable. For example, with only one intermediate exercise date $t_M = 0.4$ we found that $\Gamma_0 = 96.3\%$. Adding additional exercise dates pushes the threshold Γ_0 below 96.3%. Below we will approximate the American option by adding more and more exercise dates.

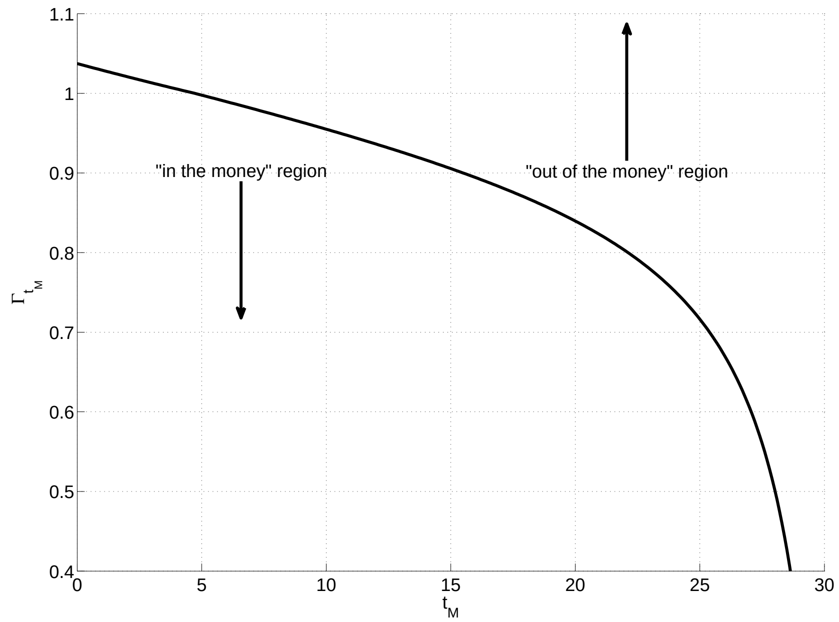


Figure 4: "In-the-money" funding rate thresholds (Γ_{t_M}) at different ages t_M.

5 Approximation Method American Pension Option

We use the Least Squares Monte Carlo (LSMC) approximation method proposed by Longstaff and Schwartz (2001) to determine the option value by solving the funding rate thresholds at each exercise date using backward recursion. The American Option is approximated by choosing a small partition $t \in \{t_0, t_1, t_2, \ldots, t_R\}$ for the exercise dates of the Bermudan Option. Here we describe the approximation method.

Step 1: define "in the money" thresholds We denote "in the money" when the funding rate is such that exercising would be optimal, in case there is no such option in the future. However, the actual exercise thresholds are lower than these "in the money" thresholds, because the participant can exercise the option in the future as well. In other words, when the funding rate is in the "out of the money" region, which holds for funding rates above the "in the money" thresholds Γ, then it is definitely optimal to not exercise the option. For funding rates "in the money" it *may* be optimal to exercise, but not necessarily due to the future possibilities to exercise.

These funding rate thresholds at different ages t_M, given that there is no option to exercise at age $s > t_M$, are shown in Figure 4. We observe that the thresholds decrease over time, as quitting becomes less beneficial when older. We ignore exercise ages $s > 28$, since it is extremely unlikely that the funding rate will fall below $\Gamma_{28} = 34.4\%$.

Step 2: define values at final decision date We need to determine the optimal exercise decisions using backward recursion. We start at maturity, which takes place at the retirement age t_R, and determine whether it is optimal to exercise and register the corresponding payout. However, as we have seen above, we can ignore option decisions at ages $s > 28$, since it is extremely unlikely that the value of quitting is larger than the value of continuation at these ages. In order to save computation time, we take $t_{end} = 28$ as the final decision date. Then, we run Q simulation paths and set up a $[Q \times 1]$ −vector $ExTime$, where each element equals t_{end}. Similarly, we define a $[Q \times 1]$ −vector $Payout$, which equals max $(Cont, Stop)$ at age t_{end}. We obtain $Payout = Cont_{t_{end}}$, since the "in the money" threshold at that age is 34.4% (see Figure 4) and we have no simulations below this threshold. We initialise by setting $t_{old} = t_{end}$

Step 3: move one step back in time with step size δ We now consider the simulation values at age $t = t_{old} - \delta$. In case the funding rate is "out of the money", which holds for funding rates higher than the "in the money" thresholds shown in Figure 4, then continuation is optimal. Otherwise, we apply the LSMC method. Denote Λ_t as the set of the simulation runs, for which the funding rate at time t is "in the money", i.e. $\Lambda_t = \{i \in \{1, 2, \ldots, Q\} : F_{t,i} \leq \Gamma_t\}$. The integer n_t is defined as the number of elements in Λ_t. The highest possible value for n_t is Q, which holds when the funding rates of all simulation runs are "in the money" at time t, while the lowest possible value for n_t is zero, which holds when none of the funding rate simulations at time t are "in the money". Then, we define a new vector $\hat{F}_t$ which consists of the period t funding rates of the "in the money" simulations only:

$$\hat{F}_t = \left(F_{t,\Lambda_{t,1}}, F_{t,\Lambda_{t,2}}, \ldots, F_{t,\Lambda_{t,n_t}}\right).$$

We apply the following regression model over the "in the money" paths:

$$\begin{bmatrix} Cont_{t,\Lambda_{t,1}} \\ Cont_{t,\Lambda_{t,2}} \\ \vdots \\ Cont_{t,\Lambda_{t,n_t}} \end{bmatrix} = \begin{bmatrix} 1_{n_t} & \hat{F}_t & \hat{F}_t^2 \end{bmatrix} \beta + \varepsilon,$$

where the left-hand side is the vector of continuation values at t of the "in the money" simulation paths. Hence, we try to fit the value of continuation with a quadratic function of the underlying asset (here: $\hat{F}_t$). The symbol 1_{n_t} denotes a vector of ones of length n_t.

The value of continuation at time t for simulation run i is given by:

$$Cont_{t,i} = Payout_i \exp\left[-r\left(ExTime_i - t\right)\right] - \int_t^{ExTime_i} c_{s,i} \exp\left[-r\left(s-t\right)\right] ds,$$

i.e., the discounted payout of this simulation run at the exercise date minus the discounted future contribution payments along this path.

Using the estimated regression coefficient vector $\hat{\beta}$, we calculate the fitted values of continuation as:

$$\hat{Cont}_t = \left[1_{n_t} \quad \hat{F}_t \quad \hat{F}_t^2\right] \hat{\beta}_t.$$

Step 4: update vectors when stopping is optimal The value of stopping at age $s > t$ over the "in the money" paths is obtained by:

$$Stop_t = \min\left(1, \hat{F}_t\right) \Pi_t\left(B_s\right).$$

Then, for each of the "in the money" paths, we can determine whether the value of stopping is larger than the approximated value of continuation, i.e. $Stop_{t,j} > \hat{Cont}_{t,j}$. For each simulation run where stopping is optimal, the corresponding elements in $ExTime$ are updated to t and the corresponding elements in $Payout$ are updated to $Stop_t$. For example, suppose simulation run i is "in the money" and element j of Λ_t corresponds to i, i.e. $\Lambda_{t,j} = i$. Then, if $Stop_{t,j} > \hat{Cont}_{t,j}$, we update the elements for simulation run i by setting $ExTime_i := t$ and $Payout_i := Stop_{t,j}$.

Step 5: backward recursion If $t > 0$, then set $t := t_{old}$ and go back to Step 3. If $t = 0$, then go to Step 6.

Step 6: determine the value of participation at entry Now we have the exercise times along each path and the corresponding payouts at these dates. Hence, we can calculate the value of continuation at the entry age t_0. The value of stopping is equal to zero, because no pension entitlements are accumulated yet. Then, the value of participation is the average of the values of continuation over all simulation runs:

$$Participation = \frac{1}{Q} \sum_{i=1}^{Q} Cont_{t_0,i}.$$

Step 7: determine the funding rate threshold for participation Figure 5a shows the values of participation at age $t_0 = 0$ under this American option at different starting funding rates F_0. The value of participation is positive for large funding rates. Due to the

American option, the funding rate threshold has become $\Gamma_0 = 84.41\%$, which is a substantially lower threshold compared to the case without an option to convert to DC (i.e. $84.41\% < 103.7\%$). The "American-type" of the option stimulates participation in the beginning when upside risk is relatively large. Since exercising at a later date is always a possibility, the participant has some kind of downward protection, while it has relatively large upside potential. At the beginning of the participant's career barely any pension entitlements are accumulated, while gains can be obtained after a large positive shock. Hence, the participant has an incentive to delay exercising the option and hope for large gains.

Step 8: funding rate thresholds over life Figure 5b shows the funding rate thresholds at other ages. Obviously, these thresholds are strictly below the "in the money" thresholds, because under the American option continuation is more valuable. The gap between these thresholds dies out, since the added value of having future exercise dates decreases over time and, hence, the thresholds converge.

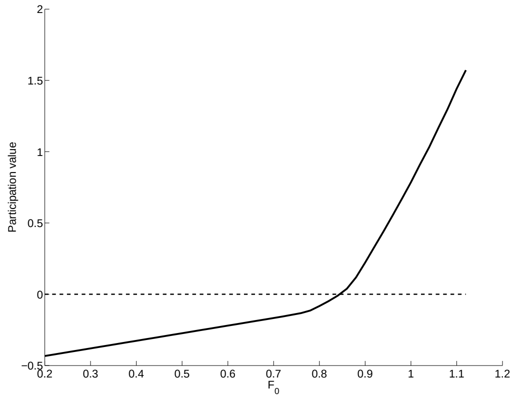
Sensitivity Analysis In order to get insights about the parameters, we explore the effects of changes in the parameter settings. First, if we decrease the standard deviation of the portfolio returns by taking $\sigma_F = 0.10$, the funding rates become less volatile. The threshold at entry increases to $\Gamma_0 = 90.2\%$, since the option becomes less valuable when risk is low – see Figure 5c. The thresholds are also higher at later ages – see Figure 5d. Figure 6a shows the participation thresholds at entry for a larger range of portfolio volatilities σ_P, whereby the negative relation is clearly observed. Hence, the participation thresholds are lower when the investment portfolio is more risky.

Second, if we decrease the extent of smoothing the funding rate recovery by taking $\alpha = 0.3$, then the participation threshold increases to $\Gamma_0 = 87.6\%$, as shown in Figure 5e. The thresholds increase at later ages as well – see Figure 5f. With less smoothing contribution rates are raised more in the case of underfunding and, therefore, we obtain higher participation thresholds when α is smaller. Figure 6b shows the participation thresholds for a larger range of smoothing parameters α, whereby the negative relation is clearly observed.

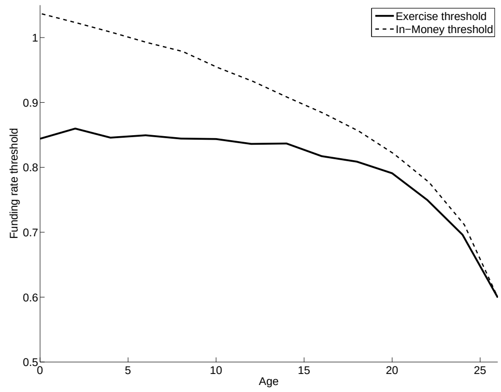
Finally, Figures 6c and 6d show how the funding rate thresholds vary for different target funding rates and accrual rates. As expected, the target funding rate positively affects the participation funding rate threshold, while the accrual rate is essentially uncorrelated with the participation threshold.

6 Interrelation of Participation Decisions

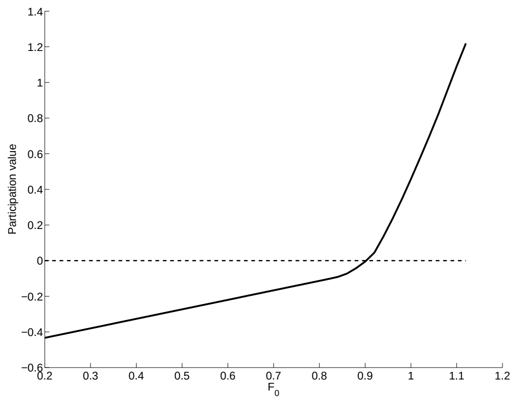
So far, we assumed full participation. For the American option to quit this holds when funding rates are above 85% over the last t_D years and, therefore, nobody had an incentive



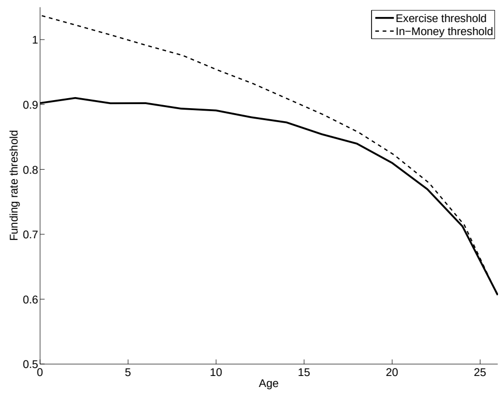
(a) benchmark: $\sigma_P = 0.15$, $\alpha = 0.5$



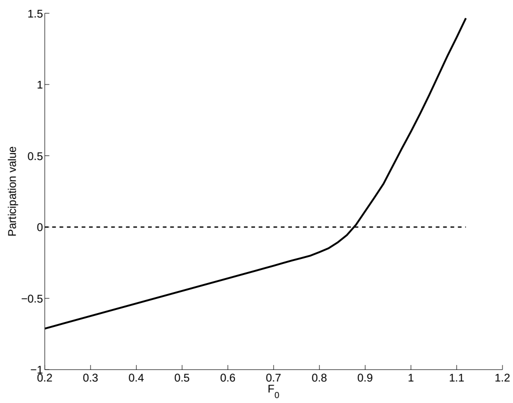
(b) benchmark: $\sigma_P = 0.15$, $\alpha = 0.5$



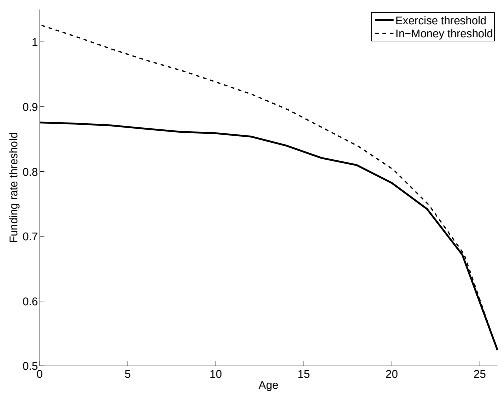
(c) $\sigma_P = 0.10$



(d) $\sigma_P = 0.10$

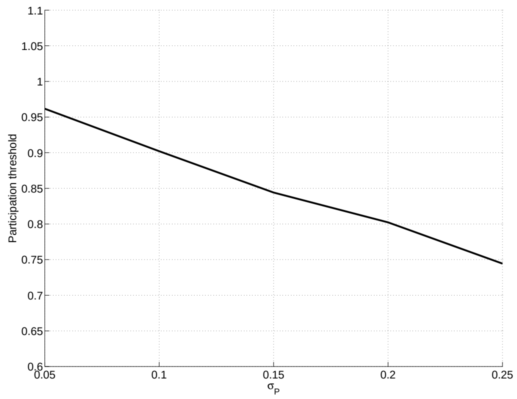


(e) $\alpha = 0.3$

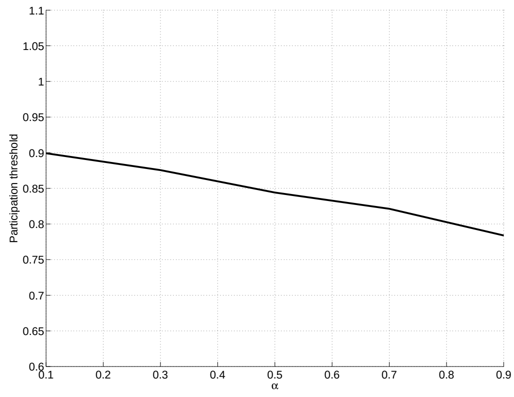


(f) $\alpha = 0.3$

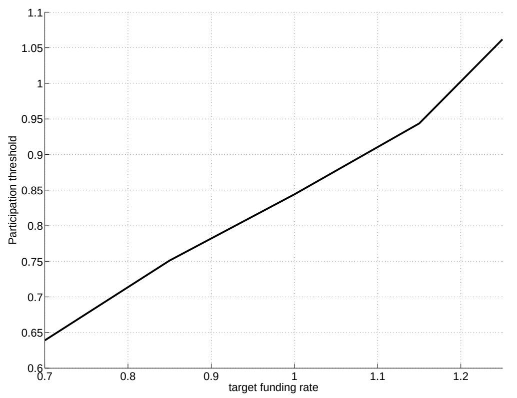
Figure 5: The value of participation for an entry generation (left panels) and the funding rate thresholds over lifetime (right panels) under the American DB-DC pension option.



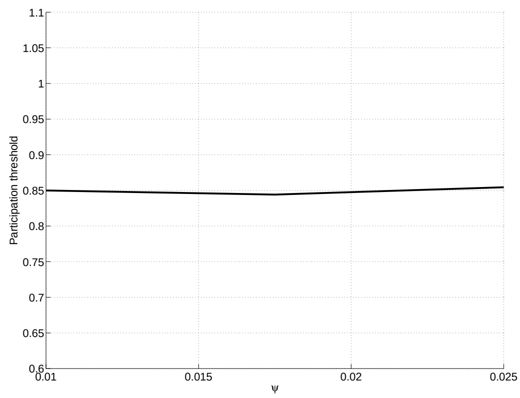
(a) varying portfolio risk σ_P



(b) varying smoothing parameter α



(c) varying target funding rate $\bar{F}$



(d) varying accrual rate ψ

Figure 6: Participation threshold of an entry generation under different parameter settings.

to exercise the option. However, if investment returns are particularly low, then the funding rate falls and generations up to a certain age will leave the collective pension scheme by exercising the option.

Suppose that at a certain time $t = \hat{t}$ the funding rate is $F_{\hat{t}} < \Gamma_0$, the threshold funding rate at entry, such that it is optimal for generations up to age q to exercise the option. Then, a participant of age $s < q$ obtains the following amount of personal assets from the pension fund when leaving:

$$A^{DC}_{\hat{t}-s,s} = F_{\hat{t}}\Pi_{\hat{t}}\left(B_s\right) = F_{\hat{t}}R_sB_s.$$

The departure of the generations up to age q affects the assets and liabilities of the pension fund. The funding rate at time $\hat{t}$ before the departure of generations, i.e. under full participation, is $F^{\text{old}}_{\hat{t}} = A^{\text{old}}_{\hat{t}}/L^{\text{old}}_{\hat{t}}$. The total amount of assets taken out of the pension fund by the departing generations is $F^{\text{old}}_{\hat{t}}\int_0^q R_sB_sds$. Hence, the assets become

$$\begin{aligned}
A^{\text{new}}_{\hat{t}} =& A^{\text{old}}_{\hat{t}} - F^{\text{old}}_{\hat{t}}\int_0^q R_sB_sds \\
=& F^{\text{old}}_{\hat{t}}\left(L^{\text{old}}_{\hat{t}} - \int_0^q R_sB_sds\right) \\
=& F^{\text{old}}_{\hat{t}}\left(\int_0^{t_D} R_sB_sds - \int_0^q R_sB_sds\right) \\
=& F^{\text{old}}_{\hat{t}}\int_q^{t_D} R_sB_sds
\end{aligned}$$

and the liabilities become:

$$\begin{aligned}
L^{\text{new}}_{\hat{t}} =& L^{\text{old}}_{\hat{t}} - \int_0^q R_sB_sds \\
=& \int_0^{t_D} R_sB_sds - \int_0^q R_sB_sds \\
=& \int_q^{t_D} R_sB_sds.
\end{aligned}$$

Then, the funding rate after the departure of generations is:

$$F^{\text{new}}_{\hat{t}} = A^{\text{new}}_{\hat{t}}/L^{\text{new}}_{\hat{t}} = F^{\text{old}}_{\hat{t}}.$$

Hence, the funding rate is unaffected, as it remains equal to $F_{\hat{t}}$. However, the future participation thresholds might be affected, since a smaller group of participants cover the burden of contribution adjustments.

Denote $\hat{I}_u\left(I_t\right)$ as the belief about the future participation setting at time $u > t$ when the current participation setting is I_t. Suppose each participant believes that nobody leaves the collective pension scheme in the future. This includes the belief that new generations enter at the start of their career, since they participate by default. So far, this assumption was implicitly already made, since we did not incorporate effects from other people exercising the option. Then, given current participation setting $I_{\hat{t}} = \{s : s \in [q, t_D]\}$, the belief about the participation setting at time $u > \hat{t}$ becomes:

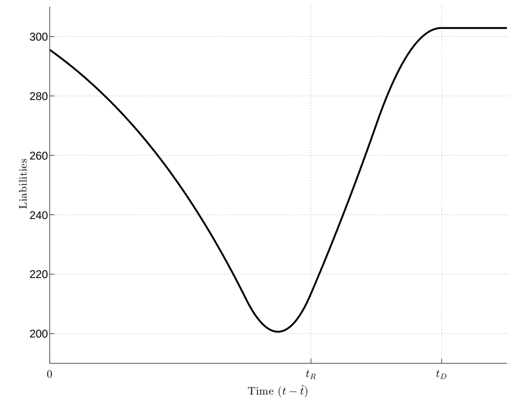
$$\hat{I}_u\left(I_{\hat{t}}\right) = \begin{cases} \left\{s : s \in \left[0, u-\hat{t}\right] \cup \left[q + u - \hat{t}, t_D\right]\right\}, & \text{for } u - \hat{t} \in (0, t_D - q), \\ \left\{s : s \in \left[0, u-\hat{t}\right]\right\}, & \text{for } u - \hat{t} \in [t_D - q, t_D], \\ \{s : s \in [0, t_D]\}, & \text{for } u - \hat{t} \in (t_D, \infty). \end{cases}$$

In the first case, the belief about the future participation setting consists of the union of two sets. The first set consists of new age groups up to age $u - \hat{t}$, while the second set consists of participants who have decided to not exercise the option at time $\hat{t}$ and are still alive at time u. In the second case, the belief about the participation setting at time u consists of new age groups up to age $u - \hat{t}$ only, since all generations who participated before the departure of generations at time $\hat{t}$ have passed away. In the last case, there is full participation again, which holds for time $\hat{t} + t_D$ onward.

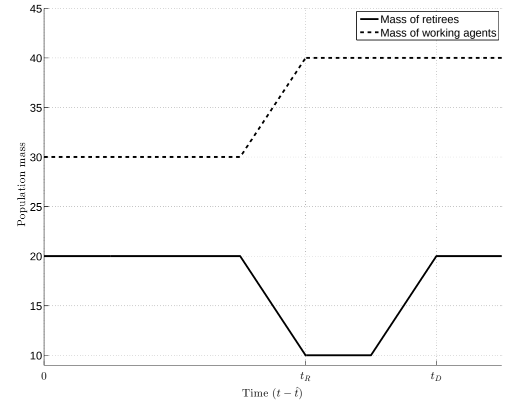
In order to see the effects on the participation thresholds of the remaining generations in the collective scheme, we consider a scenario whereby the funding rate falls to $F_{\hat{t}} = 84.36\%$. Therefore, everybody younger than $q = 10$ leaves the pension fund and the new participation setting becomes $I_{\hat{t}} = \{s : s \in [10, t_D]\}$. Then, the liabilities and population mass become non-constant for $t < t_D + \hat{t}$. In Figure 7a, we observe that the liabilities are decreasing first, because old people pass away, but at a certain point in time the liabilities start increasing, since participation increases again. After t_D years, the liabilities converge to a constant level, i.e. full participation. Figure 7b shows the mass of retirees and the mass of working agents over time, which are also converged after t_D years.

At time $t = \hat{t}$, the youngest participant has the age of q. In order to determine the participation threshold of this generation, we need to apply the LSMC method again as explained in Section 5. First, we derive the "in the money" funding rates, which are the funding rates where stopping is more beneficial than continuing in case this is the final decision date. These new "in the money" thresholds are shown in Figure 7d (dashed line). Then, by applying our approximation method, we obtain the value of participation for different funding rates, as shown in Figure 7c. The participation threshold for generation q at time $t = \hat{t}$ is obtained at 83.89%. Figure 7d provides the participation thresholds over lifetime of the generation with age q at time $t = \hat{t}$ (solid line).

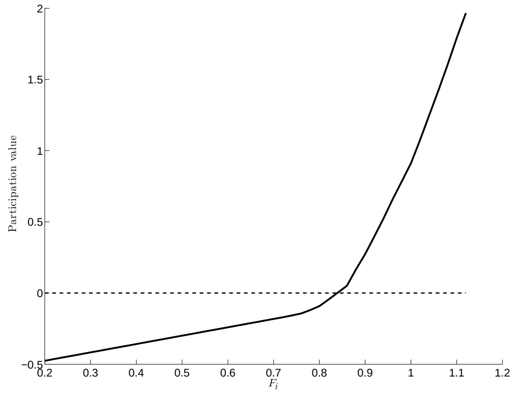
There are two forces affecting the contribution rate and participation thresholds. First, the contribution rate becomes larger when working cohorts leave the collective scheme and, hence, participation thresholds increase. Second, liabilities decrease for about $t_R - q$ periods, which results in lower contribution rates and participation thresholds. In Figure



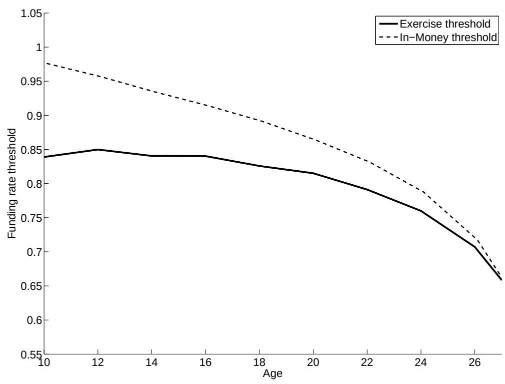
(a) The level of liabilities from time $t = \hat{t}$ onward.



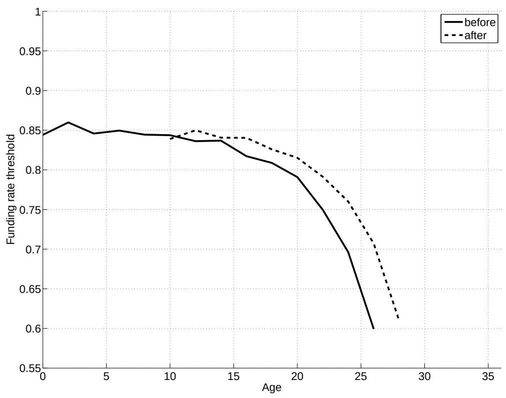
(b) The mass of working agents and mass of retirees from time $t = \hat{t}$ onward.



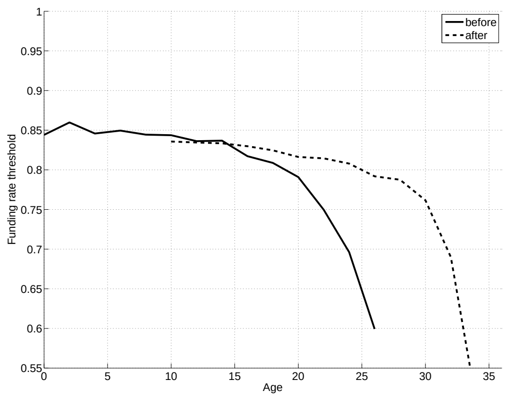
(c) Values of participation for generation with age q.



(d) Participation thresholds over remaining life for generation with age q.



(e) Participation thresholds over remaining life for generation with age q before and after outflow of participants.



(f) Participation thresholds for generations older than q before and after outflow of participants.

Figure 7: Future liabilities, population mass and participation thresholds when agents younger than $q = 10$ do not participate at time $t = \hat{t}$.

7e and 7f, we compare the new thresholds, after the departure of generations up to age q, to the old situation with full participation. Figure 7e shows the thresholds of a participant with age q over his remaining lifetime, while Figure 7f shows the thresholds for different generations at time $t = \hat{t}$. From Figure 7e we can conclude that the participation thresholds for generation with age $q = 10$ have not changed much for the next 5 years. Similarly, for slightly older generations, the thresholds have not changed much either, as shown in Figure 7f. Only for ages above 15, we observe an increase in the participation thresholds, which basically means that their value of participation has decreased as a result of generations younger than q having left the system causing an increase in the uniform contribution rate. Furthermore, we can say that for ages between $q = 10$ and 15, the immediate increase in contributions and the forthcoming decrease in liabilities roughly cancel out each other.

For the results above we considered $q = 10$, while the dashed line in Figure 8 shows the participation thresholds of the youngest generation when cohorts up to age q leave the pension fund. Hence, the youngest generation still participating is the generation with age q. The threshold of the youngest generation is typically around 84% when cohorts up to age $q = 30$ have left the pension fund. However, when older generations exercise the option as well, the contribution burden on the remaining participants becomes so large that their participation threshold substantially increases.

The solid line in Figure 8 are the thresholds under the full participation setting as obtained earlier. The lines start at the same point at age $q = 0$, since this point represents the entry threshold under full participation in both cases. Starting from this point, the solid line slightly increases. The reason is that the agent has zero pension entitlements at at age $q = 0$ and, hence, he would not lose much by exercising the option shortly thereafter. This is not the case at later ages, since the positive amount of accumulated pension entitlements is multiplied by the funding rate when exercising the option. Hence, participants are more willing to give up their exiting option at entry, resulting in a lower participation threshold than at slightly older ages. After a few years, the effect that the uniform contribution rate is more beneficial for older participants starts to dominate. Hence, the participation threshold is decreasing in age over the remaining life, and the solid line in Figure 8 crosses the dashed line exactly once at a threshold of 83.79% and age 11.5.

We can consider two scenarios. First, when the funding rate falls between 83.79% and $\Gamma_0 = 84.41$%, which is the funding rate threshold of an entry generation as we have obtained in Step 7 of Section 5, then some young generations will leave, but the collective pension scheme can continue thereafter with the remaining participants. Second, when the funding rate falls below 83.79%, young people exercise the option up to a certain age $q > 11.5$. As a result, the thresholds of older generations increase and, therefore, they prefer to leave the collective scheme as well. In order to better understand the latter feature, consider a fall in the funding rate to 79.09%. Then, it is optimal for all participants up

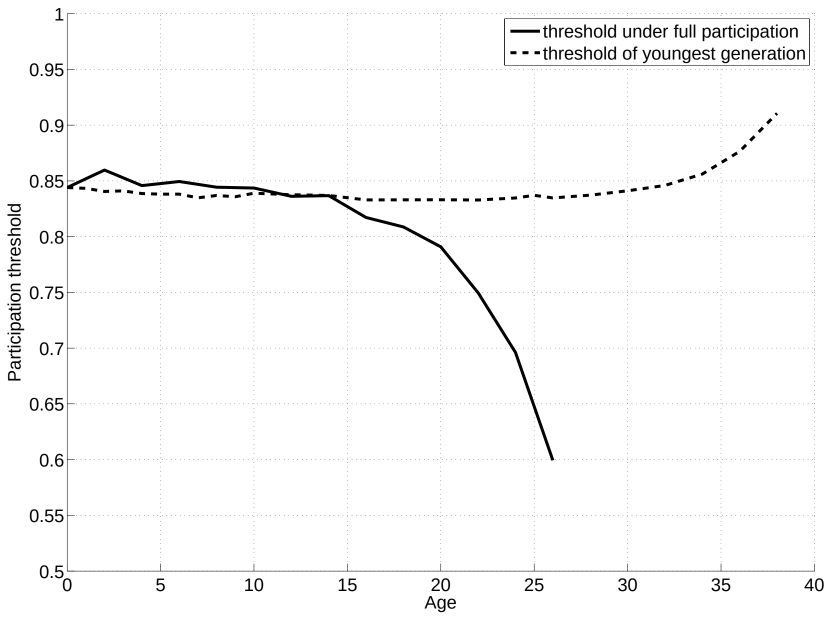
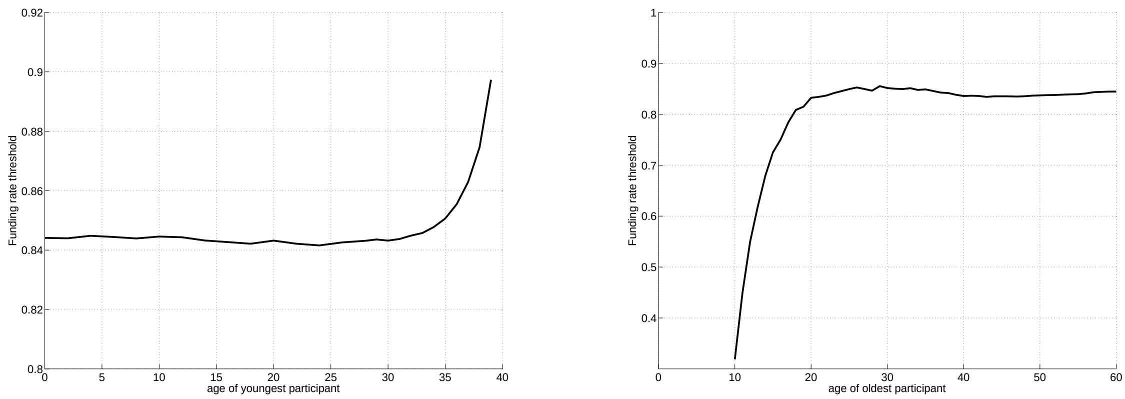


Figure 8: Participation thresholds under full participation and after outflow of young generations. The solid line represents the participation thresholds under full participation as a function of age. The dashed line represents the thresholds of the youngest participating generation as a function of age. Under the latter, the participation setting is $I = \{s : s \in [q, t_D]\}$, where q is the age of youngest participant. For funding rates for which the solid line lies below the dashed line, which is the case for thresholds below 83.79%, the initial outflow of young cohorts causes an outflow of all other working cohorts.

to age 20 to exercise the option – see the solid line in Figure 8. As a result, the thresholds of the remaining participants change. In particular, the new threshold of the youngest generation (with age 20) increases from 79.09% (the point on the solid line in Figure 8 at age 20) to 83.31% (the point on the dashed line in Figure 8 at age 20). This way, it has become optimal for this specific cohort to leave as well. Then, a slightly older generation becomes the youngest generation for which the participation threshold is even higher, since the dashed line in Figure 8 is increasing. In other words, a chain reaction of cohorts exercising the option occurs and the collective system collapses.

To conclude, the contribution rate goes up as a result of generations leaving the collective scheme, but starts decreasing thereafter because the liabilities decrease as well. When the negative shock is small causing only a small group to leave, the pension system can continue with the remaining participants. In case the negative shock is large causing a large group to leave, then this results in a chain reaction of other cohorts leaving as well. This way, the system breaks down and everybody gets the present value of their pension rights multiplied by the funding rate.



(a) Age of youngest generation is q: participation setting $I = \{s : s \in [q, t_D]\}$.

(b) Age of oldest generation is q: participation setting $I = \{s : s \in [0, q]\}$.

Figure 9: Participation thresholds for entry generations when there is no full participation.

7 Entry Generations

This section investigates to what extent generations are willing to participate when existing participation is not full.

In case the current funding rate is between 83.8% and 84.4%, there are some generations leaving, but the system does not break down. However, we still need to test whether new participants would be willing to enter the pension scheme at the start of their career in case of no full participation.

Suppose the current participation setting is $I_0 = \{s : s \in [q, t_D]\}$, which means that the youngest participant is q years old and everybody older than q participates as well. We have obtained that for $q = 0$, the participation threshold of an entry generation is 84.4%. For larger values of q, the thresholds for entry generations are shown in Figure 9a, where we observe that the thresholds are fairly stable up to $q = 30$, but sharply increase when the current youngest participant of the pension fund is older than 30. As we have seen above, the system breaks down for $q > 11.5$, so the interesting part is $q < 11.5$. For this region, the collective pension fund will attract young cohorts again, when the funding rate recovers to 84.4% or higher.

After a breakdown of the collective system, cohorts can start a new collective pension scheme again, when they enter the labour force. Suppose the current participation setting is $I_0 = \{s : s \in [0, q]\}$, which means that the oldest participating cohort is q years old. Figure 9b shows these participation thresholds for entry generations. Full participation holds for $q = t_D$ with corresponding threshold 84.4%. When the oldest generation is younger, the collective pension scheme becomes more attractive for entry generations, which is translated into lower participation thresholds. This is shown in Figure 9b, where the thresholds are particularly lower when the oldest generation is younger than 20 years.

From these results we can conclude that entrance is more attractive when the existing participants are relatively young on average and vice versa when they are relatively old. Old generations have large liabilities associated with them, while they are expected to make little or no contributions any more. Hence, when the existing participants are relatively young on average, the build up of a new collective scheme after a break down is easier than under a full participation setting.

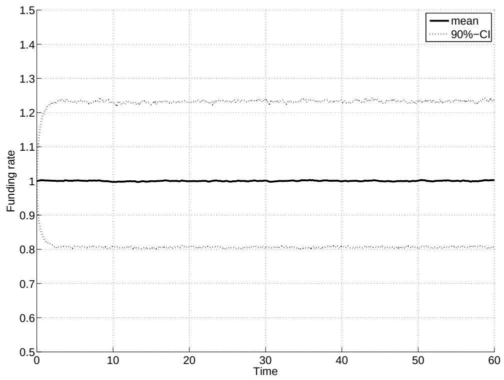
8 Break Down Distributions

Figure 10 shows 90% confidence intervals for the funding rate and the contribution rate given that the system converges towards a full participation setting. The system starts with a funding rate of $\bar{F} = 100\%$. We consider three different participation settings I_0 at time $t = 0$. The participation setting does not affect the funding rate distributions as shown in the left panels of the figure. This is clearly not the case for the contribution rates. Regulation requires the funding rate to converge in expectation to its target level by adjusting contributions. The spread in the contribution rate is less when pension fund participants are relatively young. Under a relatively old participation setting, however, liabilities are large which requires a similar amount of assets to fulfil the funding rate requirements. Larger fluctuations in contributions are needed to close a given gap with the target funding rate.

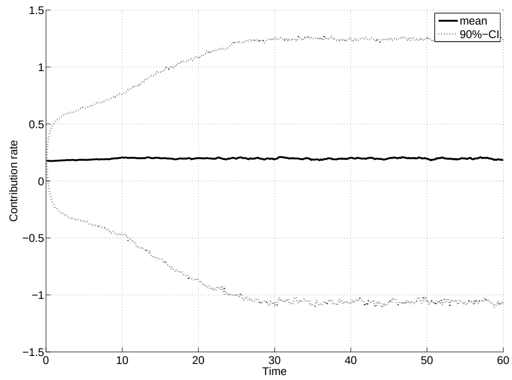
From the three panels with the funding rate distributions we observe that 5% of the funding rate simulations are below 81%, which is lower than the break down threshold of 83.8% as we have found in Section 6. Figure 11 shows the frequency distribution of the number of years until breakdown for initial funding rates of $F_0 = 100\%$ (left panels) and $F_0 = 150\%$ (right panels). From the cumulative distributions (bottom panels), we observe that the probability of the pension scheme surviving at least 10 years is only 21.5% and 32.5% for a starting funding rate of $F_0 = 100\%$ and $F_0 = 150\%$, respectively. Furthermore, a 90% survival certainty holds only for 0.9 and 2.9 years for the two starting funding rates, respectively. Hence, a collapse of the system would be frequently observed under the benchmark parameter settings.

Threshold breakdown rates for a variety of values of the smoothing parameter α and the parameter for the portfolio risk σ_P are shown in Figure 12.[2] The breakdown threshold decreases for a longer recovery window and for higher investment risk. However, these parameter changes also raise the funding rate volatility. From the statistics shown in the middle and bottom panels of Figure 12, we observe that a break down occurs less frequently when the recovery window increases. Increasing investment risk does not have a large impact on the stability of the collective pension scheme. It is only detrimental when

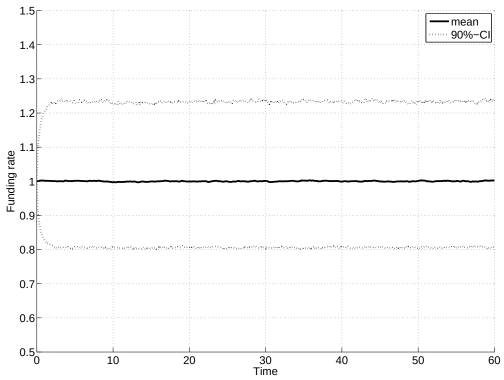
[2] More general results are shown in Appendix B, where we present the figures of funding rate equilibria and break down distributions for the different parameter choices α and σ_P.



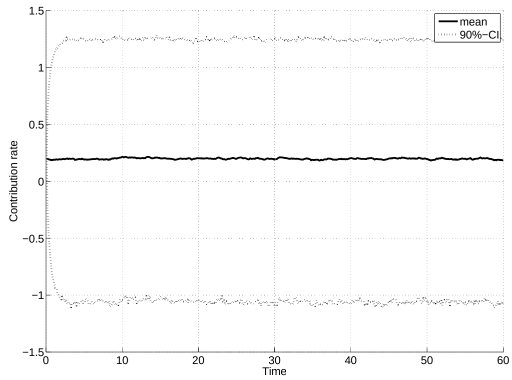
(a) Funding rate distribution with participation setting $I_0 = \left\{s : s \in \left[0, \frac{1}{2}t_D\right]\right\}$.



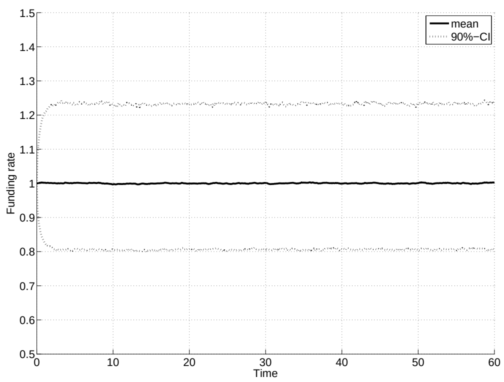
(b) Contribution rate distribution with participation setting $I_0 = \left\{s : s \in \left[0, \frac{1}{2}t_D\right]\right\}$.



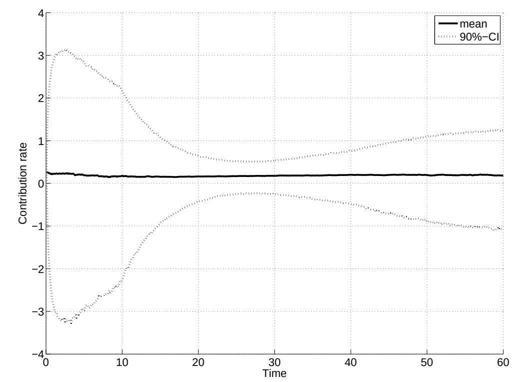
(c) Funding rate distribution with full participation $I_0 = \{s : s \in [0, t_D]\}$.



(d) Contribution rate distribution with full participation $I_0 = \{s : s \in [0, t_D]\}$.

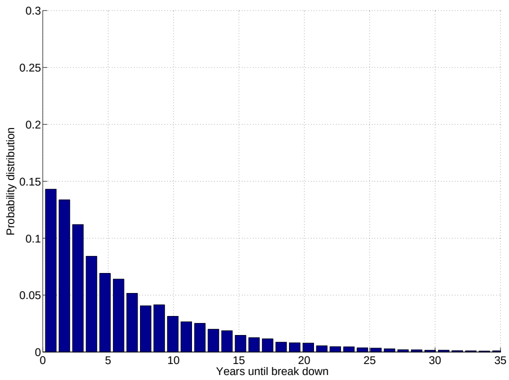


(e) Funding rate distribution with participation setting $I_0 = \left\{s : s \in \left[\frac{1}{2}t_D, t_D\right]\right\}$.

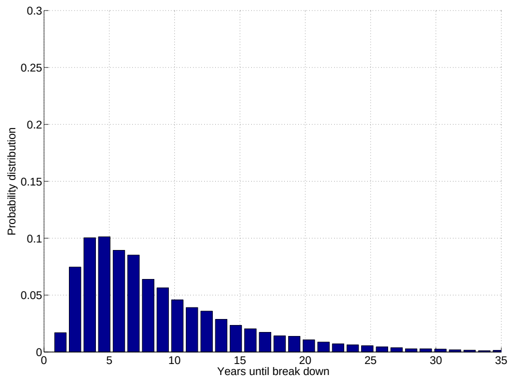


(f) Contribution rate distribution with participation setting $I_0 = \left\{s : s \in \left[\frac{1}{2}t_D, t_D\right]\right\}$.

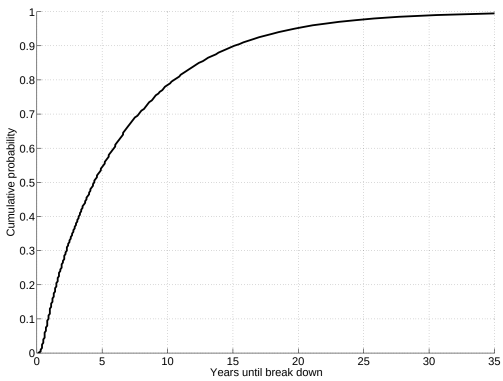
Figure 10: Distributions of the funding rate (left panels) and contribution rate (right panels) under different participation settings I_0 at time $t = 0$.



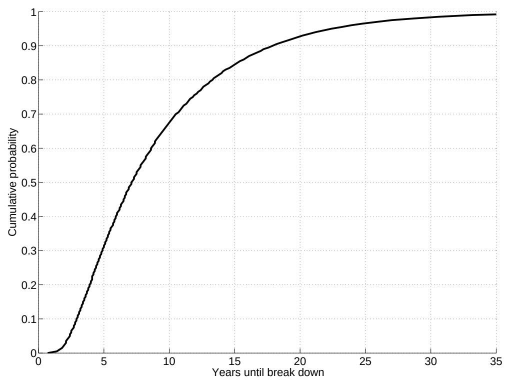
(a) Frequency distribution with starting funding rate $F_0 = 100\%$.



(b) Frequency distribution with starting funding rate $F_0 = 150\%$.



(c) Cumulative distribution with starting funding rate $F_0 = 100\%$.



(d) Cumulative distribution with starting funding rate $F_0 = 150\%$.

Figure 11: Distributions of a break down.

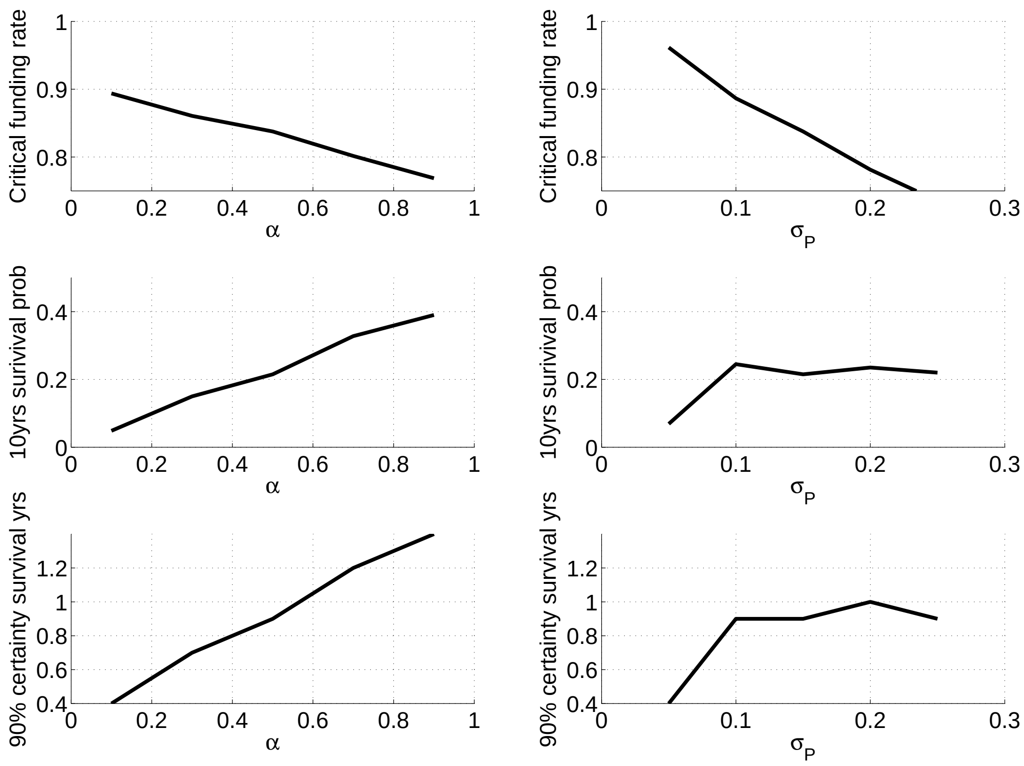


Figure 12: Break down level and probabilities for different levels of smoothing α (left panels) and different levels of investment risk σ_P (right panels).

investment risk is particularly low, say $\sigma_P < 10\%$.

9 Conclusion

We have explored the sustainability of a DB pension fund when its participants possess the option to leave the pension fund in favour of participation in an individual DC funded pension scheme. We paid particular attention to the case in which this option was of the American type. Using relevant option pricing valuation techniques, we obtained a number of interesting insights. First, young generations have a relatively strong incentive to exercise their option to quit the DB fund. Second, entrance funding rate thresholds become lower when the number of exercise dates of the Bermudan option increases. Third, when a small negative shock hits the funding rate, some young cohorts exercise their option and the collective pension scheme continues with the remaining generations. However, when the negative shock is large, the number of young generations leaving the pension fund is so large that also the other generations prefer to quit the pension fund. Hence, the pension fund unravels with all remaining generations exercising their option. Finally, we have found that the participation thresholds of new cohorts of workers become lower when investment risks are larger and the recovery window is longer. However, the changes in the underlying parameters also raise the funding rate volatility. The system survives the coming 10 years with less than 40% certainty, while all participants will have exercised the option almost surely after 20 to 40 years, depending on the parameter settings. Hence, in the absence of mandatory participation, it is only a matter of time before the collective pension scheme collapses.

The latter result raises several questions about the structure of a collective pension scheme without mandatory participation. Since young generations are typically more inclined to leave, this is an argument for limiting their contribution to stabilising the funding rate. This yields some interesting implications about intergenerational risk-sharing and pension system design, which we leave for future research.

The model can also be extended into a variety of directions, for example by modelling the collective pension scheme as a hybrid plan with both DB and DC elements. Furthermore, the LSMC approach is particularly powerful for valuing options that depend on multiple factors. Hence, an interesting enrichment of the setting would be to allow for additional sources of risk, such as demographic, interest rate and wage risks. However, not all types of risks are hedgeable, which contradicts the assumption of market completeness and, as a result, makes the risk-neutral pricing approach inapplicable. Hence, an interesting extension would be to apply utility indifference pricing, whereby a reasonable set of option prices can be obtained, according to a variety of risk preferences.

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A Derivations

A.1 Regulation Policy

Suppose $n \in \mathbb{N}$. Then, we can use Eq. (1) and the law of iterated expectations to derive:

$$\begin{aligned}
E_t^{\mathbb{Q}}\left(F_{t+ndt}-\bar{F}\right) =& E_t^{\mathbb{Q}}\left[E_{t+(n-1)dt}^{\mathbb{Q}}\left(F_{t+ndt}-\bar{F}\right)\right] \\
=& \alpha^{dt} E_t^{\mathbb{Q}}\left(F_{t+(n-1)dt}-\bar{F}\right) \\
=& \alpha^{dt} E_t^{\mathbb{Q}}\left[E_{t+(n-2)dt}^{\mathbb{Q}}\left(F_{t+(n-1)dt}-\bar{F}\right)\right] \\
=& \alpha^{2dt} E_t^{\mathbb{Q}}\left(F_{t+(n-2)dt}-\bar{F}\right) \\
=& \ldots \\
=& \alpha^{ndt} E_t^{\mathbb{Q}}\left(F_t-\bar{F}\right) \\
=& \alpha^{ndt}\left(F_t-\bar{F}\right) \\
\Rightarrow E_t^{\mathbb{Q}}\left(F_{t+s}-\bar{F}\right) =& \alpha^{s}\left(F_t-\bar{F}\right).
\end{aligned}$$

A.2 Contribution Rate

Rewriting Eq. (1) yields:

$$\begin{aligned}
E_t^{\mathbb{Q}}\left(\frac{A_t+dA_t}{L_t+dL_t}\right)-\bar{F} =& \alpha^{dt}\left(\frac{A_t}{L_t}-\bar{F}\right) \\
\Longleftrightarrow \frac{E_t^{\mathbb{Q}}\left(A_t+dA_t\right)}{E_t^{\mathbb{Q}}\left(L_t+dL_t\right)} =& \alpha^{dt}\left(\frac{A_t}{L_t}-\bar{F}\right)+\bar{F} \\
\Longleftrightarrow A_t+E_t^{\mathbb{Q}}\left(dA_t\right) =& \left[\alpha^{dt}\left(\frac{A_t}{L_t}-\bar{F}\right)+\bar{F}\right]\left[L_t+E_t^{\mathbb{Q}}\left(dL_t\right)\right] \\
\Longleftrightarrow \left(rA_t+C_t-B_t^{TOT}\right)dt =& \left(\alpha^{dt}-1\right)\left(A_t-\bar{F}L_t\right)+\left[\alpha^{dt}\left(F_t-\bar{F}\right)+\bar{F}\right]E_t^{\mathbb{Q}}\left(dL_t\right) \\
\Longleftrightarrow rA_t+C_t-B_t^{TOT} =& \left(\frac{\alpha^{dt}-1}{dt}\right)\left(A_t-\bar{F}L_t\right)+\left[\alpha^{dt}\left(F_t-\bar{F}\right)+\bar{F}\right]E_t^{\mathbb{Q}}\left(\frac{dL_t}{dt}\right) \\
\Longleftrightarrow rA_t+C_t-B_t^{TOT} =& \left(\log\alpha\right)\left(A_t-\bar{F}L_t\right)+F_tE_t^{\mathbb{Q}}\left(\frac{dL_t}{dt}\right) \\
\Longleftrightarrow c_t =& \frac{1}{m_t^w}\left[\left(\log\alpha\right)\left(A_t-\bar{F}L_t\right)+F_tE_t^{\mathbb{Q}}\left(\frac{dL_t}{dt}\right)-rA_t+m_t^rB_{t_R}\right],
\end{aligned}$$

where, going from the first to the second line, we have used the independence of the change in assets from the change in liabilities conditional on period t information, and, going from the next-next-to-last to the next-to-last line, we have used that $dt \downarrow 0$ as well as l'Hôpital's rule to get $\frac{\alpha^{dt}-1}{dt} = \lim_{x \downarrow 0} \frac{\alpha^{x}-1}{x} = \log \alpha$.

A.3 Discounted Funding Rates

We can use the result from Appendix A.1 to derive the following expression:

$$
\begin{aligned}
& E_u^{\mathbb{Q}} \left[\int_u^{t_R} F_s \exp(-rs)\, ds \right] \\
=& E_u^{\mathbb{Q}} \left[\int_u^{t_R} \left(F_s - \bar{F} + \bar{F}\right) \exp(-rs)\, ds \right] \\
=& \int_u^{t_R} E_u^{\mathbb{Q}} \left(F_s - \bar{F}\right) \exp(-rs) + \bar{F} \exp(-rs)\, ds \\
=& \int_u^{t_R} \alpha^{s-u} \left(F_u - \bar{F}\right) \exp(-rs) + \bar{F} \exp(-rs)\, ds \\
=& \exp(-ru) \int_0^{t_R - u} \left[\alpha^s \left(F_u - \bar{F}\right) + \bar{F}\right] \exp(-rs)\, ds \\
=& \exp(-ru) \left\{ \left(F_u - \bar{F}\right) \int_0^{t_R-u} \alpha^s \exp(-rs)\, ds + \bar{F} \int_0^{t_R-u} \exp(-rs)\, ds \right\} \\
=& \exp(-ru) \left[\left(F_u - \bar{F}\right) \int_0^{t_R-u} \exp\left\{-s\left[r - (\log\alpha)\right]\right\} ds + \bar{F} \int_0^{t_R-u} \exp(-rs)\, ds \right] \\
=& \left(F_u - \bar{F}\right) \frac{\exp(-ru)}{r - (\log\alpha)} \left(1 - \exp\left\{(u - t_R)\left[r - (\log\alpha)\right]\right\}\right) + \frac{\bar{F}}{r} \left\{\exp(-ru) - \exp\left[-r t_R\right]\right\} \\
=& F_u \frac{\exp(-ru)}{r - (\log\alpha)} \left(1 - \exp\left\{(u - t_R)\left[r - (\log\alpha)\right]\right\}\right) + \ldots \\
& \ldots \bar{F} \left(\frac{\exp(-ru) - \exp(-r t_R)}{r} - \frac{\exp(-ru)}{r - (\log\alpha)} \left(1 - \exp\left\{(u - t_R)\left[r - (\log\alpha)\right]\right\}\right) \right).
\end{aligned}
$$

Hence, $E_u^{\mathbb{Q}} \left[\int_u^{t_R} F_s \exp(-rs)\, ds \right]$ satisfies the Markov property, as its value is only dependent on time u's funding rate (F_u).

A.4 Eq. (2)

Under full participation we have that $m_t^w = t_R$, $m_t^r = (t_D - t_R)$, $dL_t = 0$ and $L_t = \bar{L}$, so we can write:

$$
\begin{aligned}
&E_0^{\mathbb{Q}}\left[\int_0^{t_R} c_s \exp\left(-rs\right) ds\right] \\
=&E_0^{\mathbb{Q}}\left\{\int_0^{t_R} \frac{1}{m_t^w}\left[\left(\log\alpha\right)\left(A_s - \bar{F}L_s\right) + F_s E_s^{\mathbb{Q}}\left(\frac{dL_s}{dt}\right) - rA_s + m_s^r B_{t_R}\right]\exp\left(-rs\right) ds\right\} \\
=&E_0^{\mathbb{Q}}\left\{\int_0^{t_R} \frac{1}{t_R}\left[\left(\log\alpha\right)\left(A_s - \bar{F}\bar{L}\right) - rA_s + \left(t_D - t_R\right) B_{t_R}\right]\exp\left(-rs\right) ds\right\} \\
=&\frac{1}{t_R}\left\{-\left[r - \left(\log\alpha\right)\right] E_0^{\mathbb{Q}}\left[\int_0^{t_R} A_s \exp\left(-rs\right) ds\right] + \left[\left(t_D - t_R\right) B_{t_R} - \left(\log\alpha\right)\bar{F}\bar{L}\right]\int_0^{t_R}\exp\left(-rs\right) ds\right\} \\
=&-\frac{r - \left(\log\alpha\right)}{t_R} E_0^{\mathbb{Q}}\left[\int_0^{t_R} A_s \exp\left(-rs\right) ds\right] + \frac{\left(t_D - t_R\right) B_{t_R} - \left(\log\alpha\right)\bar{F}\bar{L}}{r t_R}\left[1 - \exp\left(-r t_R\right)\right].
\end{aligned}
$$

From Appendix A.3 we obtain

$$
E_0^{\mathbb{Q}}\left[\int_0^{t_R} F_s \exp\left(-rs\right) ds\right] = \left(F_0 - \bar{F}\right)\frac{1 - \exp\left\{-t_R\left[r - \left(\log\alpha\right)\right]\right\}}{r - \left(\log\alpha\right)} + \bar{F}\frac{1 - \exp\left(-r t_R\right)}{r}.
$$

For an entry generation the DB pension scheme is attractive when discounted benefits exceed expected discounted contributions, which we can rewrite using our derivations above as follows:

$$
\begin{aligned}
&\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right) \geq E_0^{\mathbb{Q}}\left[\int_0^{t_R} c_s \exp\left(-rs\right) ds\right] \\
\iff &\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right) \geq \\
&-\frac{r-(\log\alpha)}{t_R}E_0^{\mathbb{Q}}\left[\int_0^{t_R} A_s \exp\left(-rs\right) ds\right] + \frac{\left(t_D-t_R\right)B_{t_R}-(\log\alpha)\bar{F}\bar{L}}{rt_R}\left[1-\exp\left(-rt_R\right)\right] \\
\iff &\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right) \geq \\
&-\frac{r-(\log\alpha)}{t_R}E_0^{\mathbb{Q}}\left[\int_0^{t_R} A_s \exp\left(-rs\right) ds\right] + \frac{\left(t_D-t_R\right)B_{t_R}-(\log\alpha)\bar{F}\bar{L}}{rt_R}\left[1-\exp\left(-rt_R\right)\right] \\
\iff &\left[r-(\log\alpha)\right]E_0^{\mathbb{Q}}\left[\int_0^{t_R} A_s \exp\left(-rs\right) ds\right] \geq \\
&\frac{1}{r}\left[\left(t_D-t_R\right)B_{t_R}-(\log\alpha)\bar{F}\bar{L}\right]\left[1-\exp\left(-rt_R\right)\right] - t_R\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right) \\
\iff &E_0^{\mathbb{Q}}\left[\int_0^{t_R} A_s \exp\left(-rs\right) ds\right] \geq \\
&-\frac{1}{r-(\log\alpha)}\left\{t_R\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right)+\frac{1}{r}\left[1-\exp\left(-rt_R\right)\right]\left[(\log\alpha)\bar{F}\bar{L}-\left(t_D-t_R\right)B_{t_R}\right]\right\} \\
\iff &E_0^{\mathbb{Q}}\left[\int_0^{t_R} \frac{A_s}{\bar{L}} \exp\left(-rs\right) ds\right] \geq \\
&-\frac{1}{\left[r-(\log\alpha)\right]\bar{L}}\left\{t_R\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right)+\frac{1}{r}\left[1-\exp\left(-rt_R\right)\right]\left[(\log\alpha)\bar{F}\bar{L}-\left(t_D-t_R\right)B_{t_R}\right]\right\} \\
\iff &E_0^{\mathbb{Q}}\left[\int_0^{t_R} F_s \exp\left(-rs\right) ds\right] \geq \\
&-\frac{1}{\left[r-(\log\alpha)\right]\bar{L}}\left\{t_R\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right)+\frac{1}{r}\left[1-\exp\left(-rt_R\right)\right]\left[(\log\alpha)\bar{F}\bar{L}-\left(t_D-t_R\right)B_{t_R}\right]\right\} \\
\iff &\left(F_0-\bar{F}\right)\frac{1-\exp\left\{-t_R\left[r-(\log\alpha)\right]\right\}}{r-(\log\alpha)}+\bar{F}\frac{1-\exp\left(-rt_R\right)}{r} \geq \\
&-\frac{1}{\left[r-(\log\alpha)\right]\bar{L}}\left\{t_R\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right)+\frac{1}{r}\left[1-\exp\left(-rt_R\right)\right]\left[(\log\alpha)\bar{F}\bar{L}-\left(t_D-t_R\right)B_{t_R}\right]\right\} \\
\iff &\left(F_0-\bar{F}\right)\left(1-\exp\left\{-t_R\left[r-(\log\alpha)\right]\right\}\right) \geq \\
&\left[1-\exp\left(-rt_R\right)\right]\left[\frac{1}{r\bar{L}}\left(t_D-t_R\right)B_{t_R}-\bar{F}\right]-\frac{t_R}{\bar{L}}\exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right).
\end{aligned}
$$

Note that $(\log\alpha)<0$, since $\alpha\in(0,1)$ and, therefore, $r-(\log\alpha)>0$.

A.5 Proof for $c^{eq} \geq c^{ac}$

Proof. Suppose $\bar{F} = 1$, $t_R > 0$ and $r > 0$. Then, the actuarially fair contribution rate (c^{ac}) is obtained by:

$$\begin{aligned}
E_0^{\mathbb{Q}}\left[\int_0^{t_R} c^{ac}\exp\left(-rs\right)ds\right] &= \exp\left(-rt_R\right)\Pi_{t_R}\left(B_{t_R}\right)\\
\iff c^{ac}\int_0^{t_R}\exp\left(-rs\right)ds &= \exp\left(-rt_R\right)R_{t_R}B_{t_R}\\
\iff c^{ac}\frac{1}{r}\left[1-\exp\left(-rt_R\right)\right] &= \exp\left(-rt_R\right)R_{t_R}t_R\psi\\
\iff c^{ac} &= rt_R\psi R_{t_R}\frac{\exp\left(-rt_R\right)}{1-\exp\left(-rt_R\right)}\\
\iff c^{ac} &= t_R\psi\frac{\exp\left(-rt_R\right)-\exp\left(-rt_D\right)}{1-\exp\left(-rt_R\right)}.
\end{aligned}$$

The contribution rate of the collective pension scheme is given by

$$c_t = \frac{1}{m_t^w}\left[\left(\log\alpha\right)\left(A_t-\bar{F}L_t\right)+F_tE_t^{\mathbb{Q}}\left(\frac{dL_t}{dt}\right)-rA_t+m_t^rB_{t_R}\right].$$

In equilibrium we have that $m_t^w = t_R$, $m_t^r = t_D - t_R$, $F_t = \bar{F} = 1$, $dL_t = 0$ and $L_t = \bar{L}$. Hence, the equilibrium contribution rate (c^{eq}) is obtained by:

$$\begin{aligned}
c^{eq} &= \frac{1}{t_R}\left[\left(\log\alpha\right)\left(\bar{F}\bar{L}-\bar{F}\bar{L}\right)+\bar{F}E_t^{\mathbb{Q}}\left(\frac{0}{dt}\right)-r\bar{F}\bar{L}+\left(t_D-t_R\right)B_{t_R}\right]\\
&= \frac{\left(t_D-t_R\right)B_{t_R}-r\bar{L}}{t_R}\\
&= \left(t_D-t_R\right)\psi-\frac{r\bar{L}}{t_R}.
\end{aligned}$$

Furthermore, we can write the liabilities as:

$$
\begin{aligned}
\bar{L} =& \int_0^{t_D} R_s B_s ds \\
=& \int_0^{t_D} \frac{1}{r} \exp\left\{-r\left[t_R - \min\left(t_R, s\right)\right]\right\} \left(1 - \exp\left\{-r\left[t_D - \max\left(t_R, s\right)\right]\right\}\right) \psi \min\left(s, t_R\right) ds \\
=& \frac{\psi}{r}\left(\left\{1 - \exp\left[r\left(t_R - t_D\right)\right]\right\} \int_0^{t_R} s \exp\left[r\left(s - t_R\right)\right] ds + t_R \int_{t_R}^{t_D} \left\{1 - \exp\left[r\left(s - t_D\right)\right]\right\} ds\right) \\
=& \frac{\psi}{r}\left\{1 - \exp\left[r\left(t_R - t_D\right)\right]\right\} \exp\left(-r t_R\right) \int_0^{t_R} s \exp\left(rs\right) ds + \ldots \\
\ldots& \frac{\psi}{r} t_R \left\{t_D - t_R - \exp\left(-r t_D\right) \int_{t_R}^{t_D} \exp\left(rs\right) ds\right\} \\
=& \frac{\psi}{r}\left\{1 - \exp\left[r\left(t_R - t_D\right)\right]\right\} \exp\left(-r t_R\right) \frac{1}{r^2}\left[\left(r t_R - 1\right) \exp\left(r t_R\right) + 1\right] + \ldots \\
\ldots& \frac{\psi}{r} t_R \left(t_D - t_R - \frac{1}{r}\left\{1 - \exp\left[-r\left(t_D - t_R\right)\right]\right\} ds\right) \\
=& \frac{\psi}{r}\left\{1 - \exp\left[r\left(t_R - t_D\right)\right]\right\} \frac{1}{r^2}\left[r t_R - 1 + \exp\left(-r t_R\right)\right] + \ldots \\
\ldots& \frac{\psi}{r} t_R \left(t_D - t_R + \frac{1}{r}\left\{\exp\left[-r\left(t_D - t_R\right)\right] - 1\right\} ds\right),
\end{aligned}
$$

whereby we used:

$$
\int_0^{t_R} s \exp\left(rs\right) ds = \frac{1}{r^2}\left[\left(r t_R - 1\right) \exp\left(r t_R\right) + 1\right].
$$

Then, we can rewrite the equilibrium contribution rate as follows:

$$
\begin{aligned}
c^{eq} =& \left(t_D - t_R\right) \psi - \psi \left\{1 - \exp\left[r\left(t_R - t_D\right)\right]\right\} \frac{1}{r^2}\left\{r + \frac{1}{t_R}\left[\exp\left(-r t_R\right) - 1\right]\right\} - \ldots \\
\ldots& \psi\left(t_D - t_R + \frac{1}{r}\left\{\exp\left[r\left(t_R - t_D\right)\right] - 1\right\}\right) \\
=& \frac{\psi}{r}\left(\left\{1 - \exp\left[r\left(t_R - t_D\right)\right]\right\}\left\{\frac{1}{r t_R}\left[1 - \exp\left(-r t_R\right)\right] - 1\right\} + \left\{1 - \exp\left[r\left(t_R - t_D\right)\right]\right\}\right) \\
=& \frac{\psi}{r^2 t_R}\left\{1 - \exp\left[r\left(t_R - t_D\right)\right]\right\}\left[1 - \exp\left(-r t_R\right)\right] \\
=& \frac{\psi}{r^2 t_R}\left\{1 - \exp\left[-r\left(t_D - t_R\right)\right] - \exp\left(-r t_R\right) + \exp\left(-r t_D\right)\right\}.
\end{aligned}
$$

This way, we get:

$$
\begin{aligned}
& 0 < c^{eq} - c^{ac} \\
\iff & 0 < \frac{\psi}{r^2 t_R} \left\{1 - \exp\left[-r\left(t_D - t_R\right)\right] - \exp\left(-rt_R\right) + \exp\left(-rt_D\right)\right\} - t_R \psi \frac{\exp\left(-rt_R\right) - \exp\left(-rt_D\right)}{1 - \exp\left(-rt_R\right)} \\
\iff & 0 < 1 - \exp\left[-r\left(t_D - t_R\right)\right] - \exp\left(-rt_R\right) + \exp\left(-rt_D\right) - \left(rt_R\right)^2 \frac{\exp\left(-rt_R\right) - \exp\left(-rt_D\right)}{1 - \exp\left(-rt_R\right)} \\
\iff & 0 < \left[1 - \exp\left(-rt_R\right)\right]\left[1 - \exp\left[-r\left(t_D - t_R\right)\right] - \exp\left(-rt_R\right) + \exp\left(-rt_D\right)\right] - \left(rt_R\right)^2 \left[\exp\left(-rt_R\right) - \exp\left(-rt_D\right. \\
\iff & 0 < 1 - \exp\left[-r\left(t_D - t_R\right)\right] + \exp\left(-2rt_R\right) - \exp\left[-r\left(t_D + t_R\right)\right] + \left[2 + \left(rt_R\right)^2\right]\left[\exp\left(-rt_D\right) - \exp\left(-rt_R\right)\right] \\
\iff & 0 < \left[\exp\left(-rt_R\right) - \exp\left(-rt_D\right)\right]\left[\exp\left(-rt_R\right) + \exp\left(rt_R\right)\right] + \left[2 + \left(rt_R\right)^2\right]\left[\exp\left(-rt_D\right) - \exp\left(-rt_R\right)\right] \\
\iff & 0 < \exp\left(-rt_R\right) + \exp\left(rt_R\right) - 2 - \left(rt_R\right)^2 .
\end{aligned}
$$

This holds for all $r > 0$ and $t_R > 0$, which is shown below.

If we take $r = 0$, then $\exp\left(-rt_R\right) + \exp\left(rt_R\right) - 2 - \left(rt_R\right)^2 = 0$. Furthermore:

$$
\begin{aligned}
\frac{\partial\left(\exp\left(-rt_R\right) + \exp\left(rt_R\right) - 2 - \left(rt_R\right)^2\right)}{\partial r} &= t_R\left[\exp\left(rt_R\right) - \exp\left(-rt_R\right) - 2rt_R\right] \\
\text{for } r = 0: \ & t_R\left[\exp\left(rt_R\right) - \exp\left(-rt_R\right) - 2rt_R\right] = 0 \\
\frac{\partial^2\left(\exp\left(-rt_R\right) + \exp\left(rt_R\right) - 2 - \left(rt_R\right)^2\right)}{\partial r^2} &= t_R^2\left[\exp\left(rt_R\right) + \exp\left(-rt_R\right) - 2\right] \\
\text{for } r = 0: \ & t_R^2\left[\exp\left(rt_R\right) + \exp\left(-rt_R\right) - 2\right] = 0 \\
\frac{\partial^3\left(\exp\left(-rt_R\right) + \exp\left(rt_R\right) - 2 - \left(rt_R\right)^2\right)}{\partial r^3} &= t_R^3\left[\exp\left(rt_R\right) - \exp\left(-rt_R\right)\right] \\
\text{for } r = 0: \ & t_R^3\left[\exp\left(rt_R\right) - \exp\left(-rt_R\right)\right] = 0 \\
\frac{\partial^4\left(\exp\left(-rt_R\right) + \exp\left(rt_R\right) - 2 - \left(rt_R\right)^2\right)}{\partial r^4} &= t_R^4\left[\exp\left(rt_R\right) + \exp\left(-rt_R\right)\right] > 0 \text{ for } r, t_R > 0.
\end{aligned}
$$

Hence, $\exp\left(-rt_R\right) + \exp\left(rt_R\right) - 2 - \left(rt_R\right)^2$ is zero for $r = 0$ and increasing in r for $r, t_R > 0$. Similarly, we can show that $\exp\left(-rt_R\right) + \exp\left(rt_R\right) - 2 - \left(rt_R\right)^2$ is zero for $t_R = 0$ and increasing in t_R for $r, t_R > 0$. This proves that:

$$
\begin{aligned}
& 0 < r, t_R \\
\implies & 0 < \exp\left(-rt_R\right) + \exp\left(rt_R\right) - 2 - \left(rt_R\right)^2 \\
\iff & c^{ac} < c^{eq}.
\end{aligned}
$$

□

A.6 Value of Continuation Linearly Increasing in F_t

Under full participation we have that $m_t^w = t_R$, $m_t^r = (t_D - t_R)$, $dL_t = 0$ and $L_t = \bar{L}$, so we can write the contribution rate as:

$$c_t = \frac{1}{t_R}\left[(\log\alpha)\left(A_t - \bar{F}\bar{L}\right) - rA_t + (t_D - t_R)\,B_{t_R}\right].$$

Then, we can write the value of continuation as:

$$\begin{aligned}
Cont_{t_M} =& \exp\left[-r\left(t_R - t_M\right)\right]\Pi_{t_R}\left(B_{t_R}\right) - E^{\mathbb{Q}}_{t_M}\left\{\int_{t_M}^{t_R} c_s \exp\left[-r\left(s - t_M\right)\right]ds\right\}\\
=& \exp\left[-r\left(t_R - t_M\right)\right]\Pi_{t_R}\left(B_{t_R}\right) - \dots\\
&\dots E^{\mathbb{Q}}_{t_M}\left\{\int_{t_M}^{t_R}\frac{1}{t_R}\left[(\log\alpha)\left(A_s - \bar{F}\bar{L}\right) - rA_s + (t_D - t_R)\,B_{t_R}\right]\exp\left[-r\left(s - t_M\right)\right]ds\right\}\\
=& \exp\left[-r\left(t_R - t_M\right)\right]\Pi_{t_R}\left(B_{t_R}\right) + \dots\\
&\dots\frac{\exp\left(rt_M\right)}{t_R}E^{\mathbb{Q}}_{t_M}\left[\int_{t_M}^{t_R}\left\{\left[r - (\log\alpha)\right]A_s + (\log\alpha)\,\bar{F}\bar{L} - (t_D - t_R)\,B_{t_R}\right\}\exp\left(-rs\right)ds\right]\\
=& \exp\left[-r\left(t_R - t_M\right)\right]\Pi_{t_R}\left(B_{t_R}\right) + \frac{\exp\left(rt_M\right)\left[(\log\alpha)\,\bar{F}\bar{L} - (t_D - t_R)\,B_{t_R}\right]}{t_R}\int_{t_M}^{t_R}\exp\left(-rs\right)ds + \dots\\
&\dots\frac{\exp\left(rt_M\right)\left[r - (\log\alpha)\right]}{t_R}E^{\mathbb{Q}}_{t_M}\left[\int_{t_M}^{t_R}A_s\exp\left(-rs\right)ds\right]\\
=& \exp\left[-r\left(t_R - t_M\right)\right]\Pi_{t_R}\left(B_{t_R}\right) + \frac{\left[(\log\alpha)\,\bar{F}\bar{L} - (t_D - t_R)\,B_{t_R}\right]\left\{1 - \exp\left(\left[-r\left(t_R - t_M\right)\right]\right)\right\}}{rt_R} + \dots\\
&\dots\frac{\exp\left(rt_M\right)\left[r - (\log\alpha)\right]\bar{L}}{t_R}E^{\mathbb{Q}}_{t_M}\left[\int_{t_M}^{t_R}F_s\exp\left(-rs\right)ds\right]
\end{aligned}$$

From Appendix A.3 we obtain

$$\begin{aligned}
&E^{\mathbb{Q}}_{t_M}\left[\int_{t_M}^{t_R}F_s\exp\left(-rs\right)ds\right]\\
=&F_{t_M}\frac{\exp\left(-rt_M\right)}{r - (\log\alpha)}\left(1 - \exp\left\{\left(t_R - t_M\right)\left[(\log\alpha) - r\right]\right\}\right) + \dots\\
\dots&\bar{F}\left[\frac{\exp\left(-rt_M\right) - \exp\left(-rt_R\right)}{r} - \frac{\exp\left(-rt_M\right)}{r - (\log\alpha)}\left(1 - \exp\left\{\left(t_R - t_M\right)\left[(\log\alpha) - r\right]\right\}\right)\right],
\end{aligned}$$

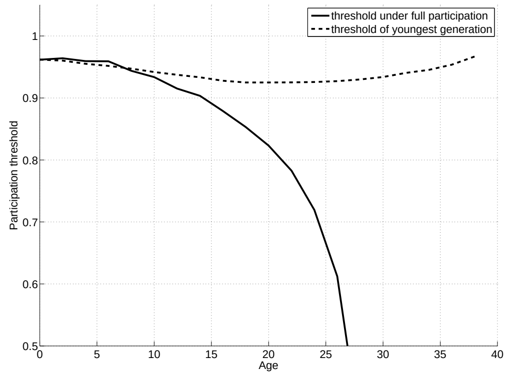
which we can use to further rewrite the value of continuation as follows:

$$
\begin{aligned}
Cont_{t_M} =& \exp\left[-r\left(t_R - t_M\right)\right]\Pi_{t_R}\left(B_{t_R}\right) + \frac{\left[\left(\log\alpha\right)\bar{F}\bar{L} - \left(t_D - t_R\right)B_{t_R}\right]\left\{1 - \exp\left[-r\left(t_R - t_M\right)\right]\right\}}{rt_R} + \ldots \\
&\ldots\frac{\bar{L}\bar{F}}{t_R}\left\{\left[1 - \frac{\left(\log\alpha\right)}{r}\right]\left\{1 - \exp\left[-r\left(t_R - t_M\right)\right]\right\} + \exp\left\{-\left(t_R - t_M\right)\left[r - \left(\log\alpha\right)\right]\right\} - 1\right\} + \ldots \\
&\ldots F_{t_M}\frac{\bar{L}}{t_R}\left(1 - \exp\left\{-\left(t_R - t_M\right)\left[r - \left(\log\alpha\right)\right]\right\}\right) \\
Cont_{t_M} =& \exp\left[-r\left(t_R - t_M\right)\right]\Pi_{t_R}\left(B_{t_R}\right) - \frac{\left(t_D - t_R\right)B_{t_R}\left\{1 - \exp\left[-r\left(t_R - t_M\right)\right]\right\}}{rt_R} + \ldots \\
&\ldots\frac{\bar{L}\bar{F}}{t_R}\left(\exp\left\{-\left(t_R - t_M\right)\left[r - \left(\log\alpha\right)\right]\right\} - \exp\left[-r\left(t_R - t_M\right)\right]\right) + \ldots \\
&\ldots F_{t_M}\frac{\bar{L}}{t_R}\left(1 - \exp\left\{-\left(t_R - t_M\right)\left[r - \left(\log\alpha\right)\right]\right\}\right) \\
\frac{\partial Cont_{t_M}}{\partial F_{t_M}} =& \frac{\bar{L}}{t_R}\left(1 - \exp\left\{-\left(t_R - t_M\right)\left[r - \left(\log\alpha\right)\right]\right\}\right),
\end{aligned}
$$

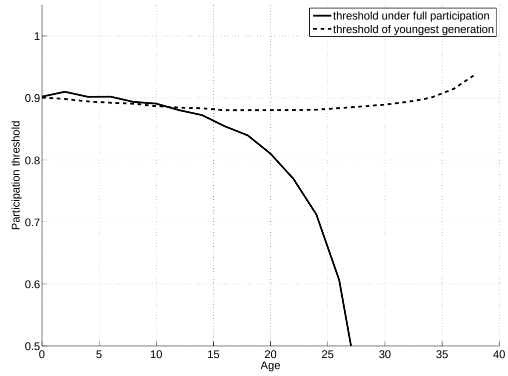
which is constant and strictly positive. Hence, the value of continuation at age t_M is linearly increasing in the funding rate F_{t_M}.

B Figures of Funding Rate Equilibria and Distributions

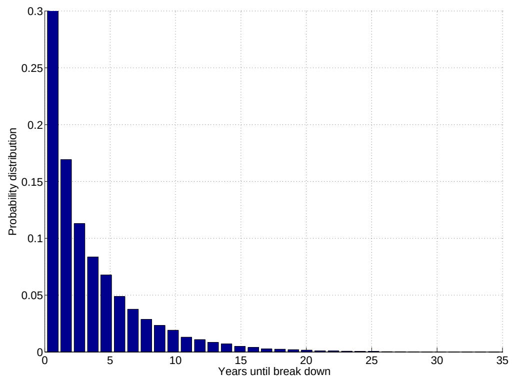
This Appendix presents the figures of funding rate equilibria and break down distributions for a variety of smoothing levels α and investment risk levels σ_P.



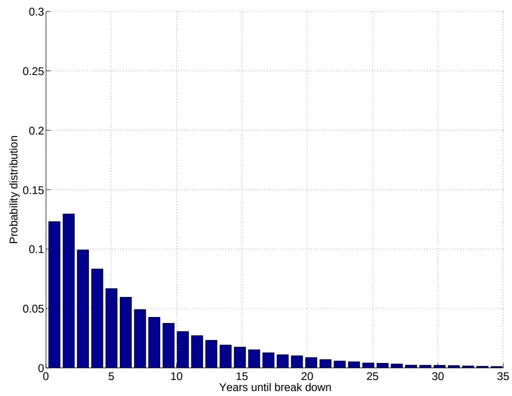
(a) Equilibrium threshold for $\sigma_P = 5\%$.



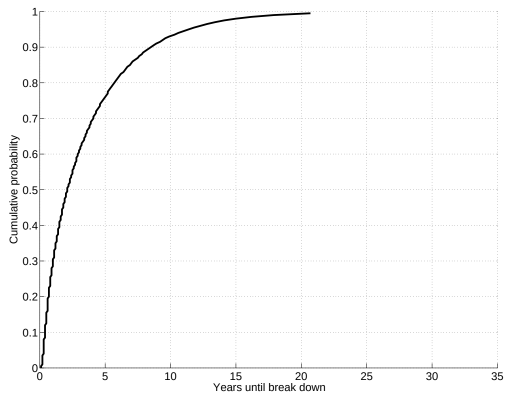
(b) Equilibrium threshold for $\sigma_P = 10\%$.



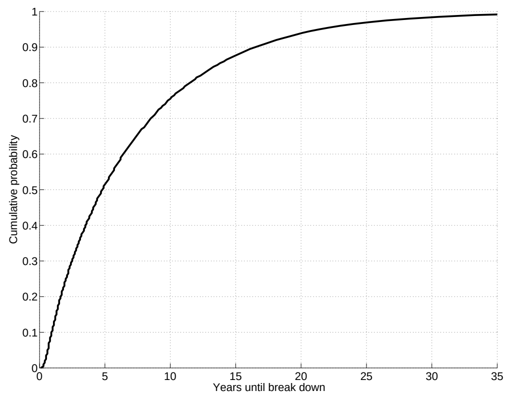
(c) Probability distribution for $\sigma_P = 5\%$.



(d) Probability distribution for $\sigma_P = 10\%$.

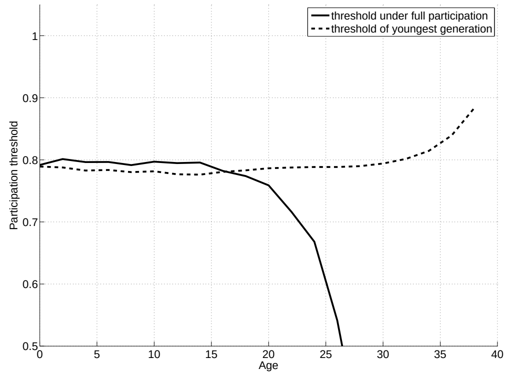


(e) Cumulative distribution for $\sigma_P = 5\%$.

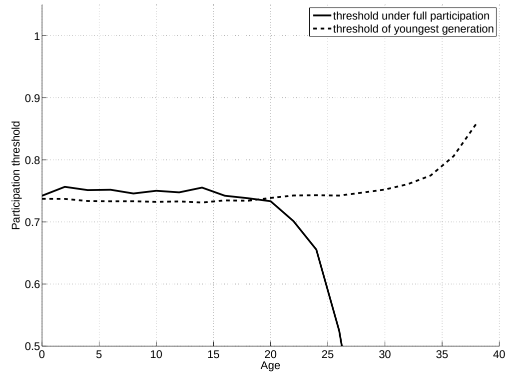


(f) Cumulative distribution for $\sigma_P = 10\%$.

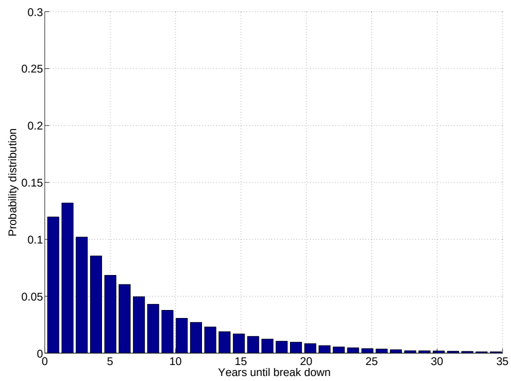
Figure 13: Equilibrium and distributions of a break down for different investment risk levels σ_P.



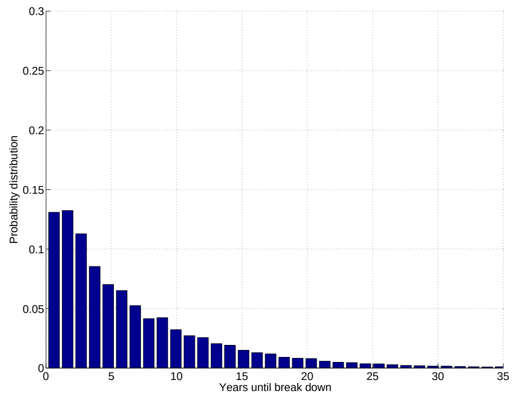
(a) Equilibrium threshold for $\sigma_P = 20\%$.



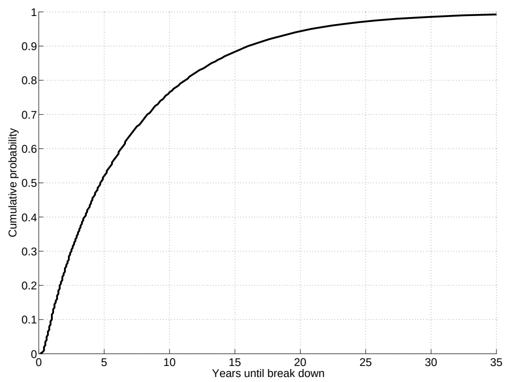
(b) Equilibrium threshold for $\sigma_P = 25\%$.



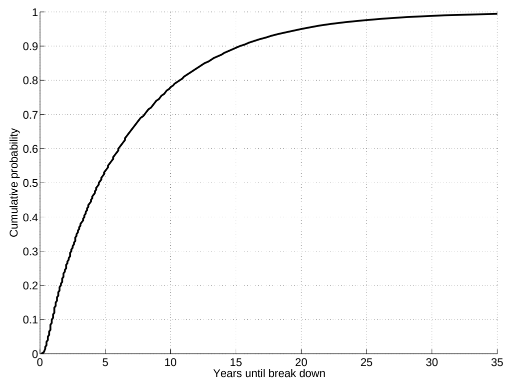
(c) Probability distribution for $\sigma_P = 20\%$.



(d) Probability distribution for $\sigma_P = 25\%$.

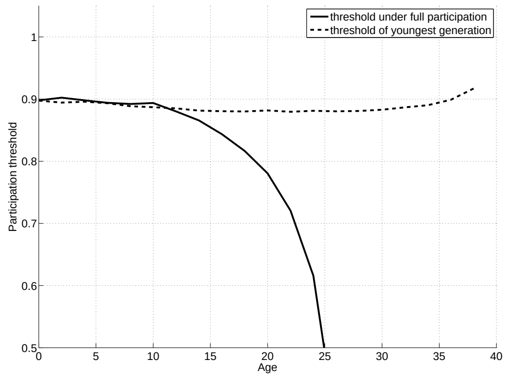


(e) Cumulative distribution for $\sigma_P = 20\%$.

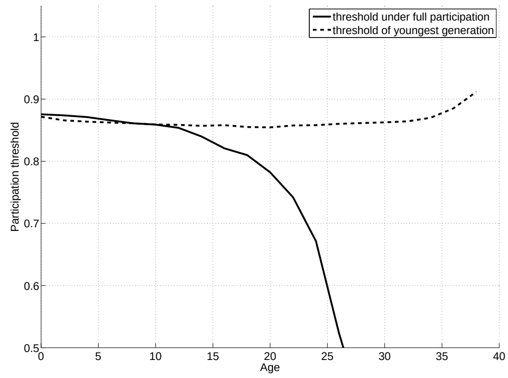


(f) Cumulative distribution for $\sigma_P = 25\%$.

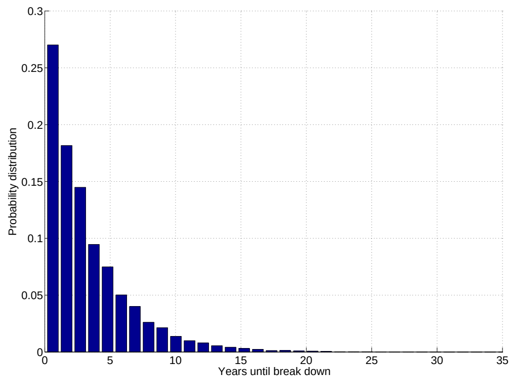
Figure 14: Equilibrium and distributions of a break down for different investment risk levels σ_P.



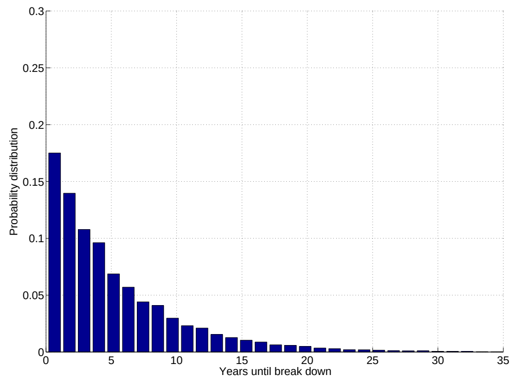
(a) Equilibrium threshold for $\alpha = 0.1$.



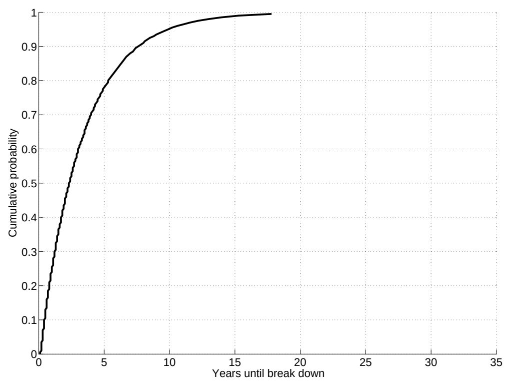
(b) Equilibrium threshold for $\alpha = 0.3$.



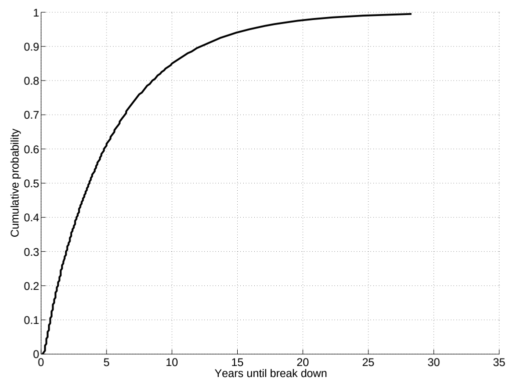
(c) Probability distribution for $\alpha = 0.1$.



(d) Probability distribution for $\alpha = 0.3$.

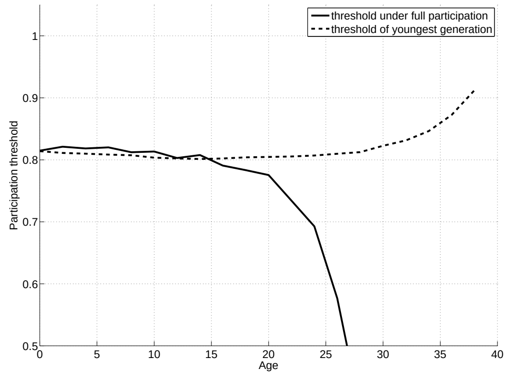


(e) Cumulative distribution for $\alpha = 0.1$.

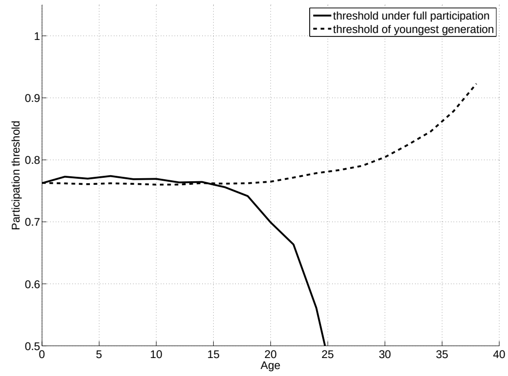


(f) Cumulative distribution for $\alpha = 0.3$.

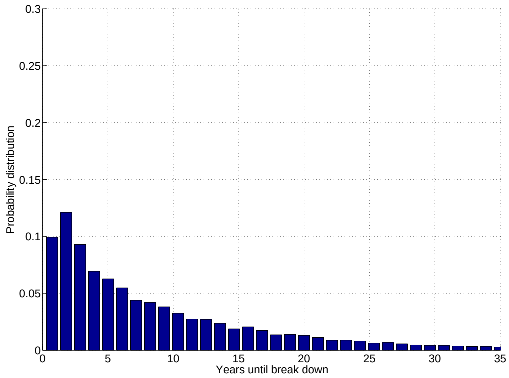
Figure 15: Equilibrium and distributions of a break down for different smoothing levels α.



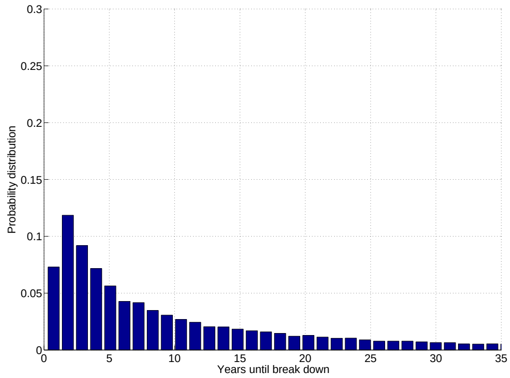
(a) Equilibrium threshold for $\alpha = 0.7$.



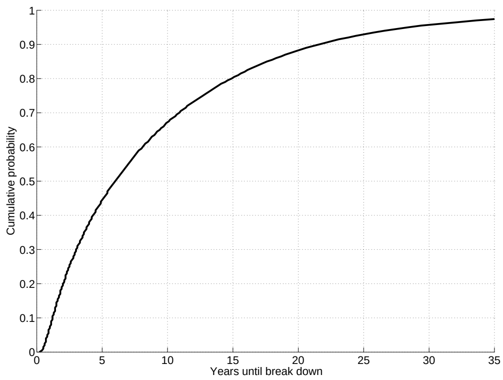
(b) Equilibrium threshold for $\alpha = 0.9$.



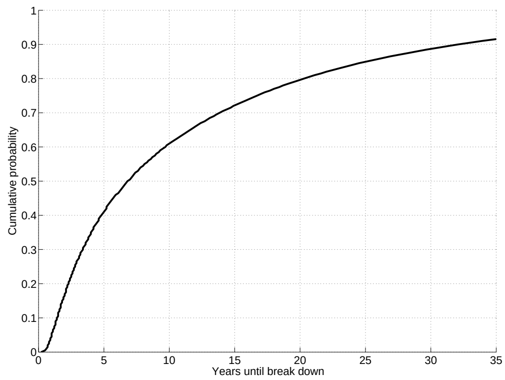
(c) Probability distribution for $\alpha = 0.7$.



(d) Probability distribution for $\alpha = 0.9$.



(e) Cumulative distribution for $\alpha = 0.7$.



(f) Cumulative distribution for $\alpha = 0.9$.

Figure 16: Equilibrium and distributions of a break down for different smoothing levels α.