Time-Inconsistent Preferences, Borrowing Costs, and Social Security*

T. Scott Findley†
Utah State University
tscott.findley@usu.edu

Frank N. Caliendo
Utah State University
frank.caliendo@usu.edu

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Abstract

Can time-inconsistent preferences justify a mandatory saving program like social security? Previous studies have considered this research question specifically for the extreme assumptions of missing credit markets and perfect credit markets, but not for the intermediate case in which borrowing is costly. We consider a more general setting with a credit spread between the interest rates on borrowing and saving. Our setting nests the extremes of missing credit markets (infinite spread) and perfect credit markets (zero spread), and it also includes the full spectrum of credit spreads in between. We prove that a fully-funded social security arrangement is irrelevant **only** at the knife edge of perfect credit markets. In other words, if there are borrowing costs of any size then social security can improve the welfare of individuals with time-inconsistent preferences. We conclude that non-standard preferences provide a more compelling justification for the mandatory saving role of social security than previously supposed.

Keywords: *hyperbolic discounting, social security, time inconsistency, credit spread.*

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†Corresponding author.

1 Introduction

Intuition suggests that individuals who have trouble following through with previous saving goals, as in the case of hyperbolic discounting, will benefit from participating in a mandatory saving (social security) program.[1] A number of studies have attempted to verify this intuition in theoretical models (İmrohoroğlu, İmrohoroğlu, and Joines (2003), Gul and Pesendorfer (2004), Caliendo (2011, 2013), and Guo and Caliendo (2014)). These studies assume that credit markets are either totally missing so that borrowing is impossible, or they assume that credit markets are "perfect" or "complete" in the sense that households can borrow and save at the same interest rate. They find that participation in a fully-funded social security program can improve welfare if credit markets are totally missing, but social security is irrelevant and does not act as a commitment device when credit markets are complete.[2,3] Clearly, the assumptions about credit markets are of first-order importance in determining whether or not social security succeeds at helping hyperbolic discounters.

We investigate the role of mandatory saving in a more general setting with a credit spread between the interest rates on borrowing and saving. Our setting nests the extremes of missing credit markets (infinite spread) and perfect credit markets (zero spread), and it also includes the full spectrum of credit spreads in between. Our motivation for doing this is because the missing and perfect credit market assumptions are difficult to defend empirically (Davis, Kubler, and Willen (2006)), and because it is unclear *a priori* what the welfare effects of social security will be when we leave these extremes.[4] Therefore, in our view, there remains an open, fundamental question in public economics: Can time-inconsistent preferences justify a social security program under realistic assumptions about credit markets? Intuitively, it is very important to properly capture key features of credit markets because the success

[1]For example, Akerlof (1998, p.187) conjectures that "The hyperbolic model explains the uniform popularity of social security, which acts as a precommitment device to redistribute consumption from times when people would be tempted to overspend—during their working lives—to times when they would otherwise be spending too little—in retirement. Even with the distortions entailed in such taxation with hyperbolic discounting...such a transfer is most likely to improve welfare significantly." For similar statements, see Laibson (1998), Aaron (1999), Akerlof (2002), Fehr (2002), Diamond and Kőszegi (2003), McCaffery and Slemrod (2006), Pestieau and Possen (2008), Cremer and Pestieau (2011), Lazear (2011), and Boadway (2012).

[2]This is true whether the model is cast in three periods or continuous time and whether the individual is naive or sophisticated about his time-inconsistent preferences.

[3]Malin (2008) finds that a "savings floor" can improve the welfare of hyperbolic discounters. While this result is somewhat related to the social security literature, social security acts like a savings floor only in the special case where credit markets are completely missing (no borrowing).

[4]Although some households certainly face borrowing constraints, Davis, Kubler, and Willen (2006) show that typical households can and do borrow significant amounts and still have unused borrowing capacity. And at the other extreme, a single interest rate clearly lacks empirical support as well. The gap between the interest rates on prime bank loans and short-term treasuries during the Greenspan-Bernanke era in the US was 3 percentage points. And typical households face a credit spread on unsecured debt that is two or three times larger still.

of any mandatory saving program depends on how much individuals unwind forced saving through private dissaving. Neither the perfect credit market assumption nor the missing market assumption can be expected to properly capture a realistic amount of unwinding, because such assumptions imply all or nothing.

Our main finding is that, of the full spectrum of credit spreads ranging from zero (perfect markets) to infinity (missing markets), a fully-funded social security arrangement is irrelevant **only** at the knife edge of perfect credit markets. In other words, if there is a credit spread of *any* size then social security can improve welfare for individuals with time-inconsistent preferences. Once the analysis is expanded to include the broad spectrum of credit market assumptions between the extremes, non-standard preferences provide a more compelling justification for the mandatory saving role of social security than is implied by past studies.[5]

The intuition for these results is as follows. Unlike a setting with perfect credit markets in which social security contributions are just offset one-for-one with reductions in private asset holdings, the individual does not reduce asset holdings one-for-one in the presence of a credit spread because such a reduction can trigger high borrowing costs. And this means that social security causes consumption to fall during the early years to keep the instantaneous household budget constraint in balance and then rise later during the retirement years. Such a transfer can improve welfare under various definitions of welfare.[6]

For instance, if we assume that welfare is of the mean-variance variety (i.e., the policy-maker treats all the temporally sequenced selves of a single individual equally and dislikes inequality among the selves), then welfare unequivocally increases with the introduction of a fully-funded social security program at our baseline credit spread (3 percentage points). Social security causes mean consumption to increase slightly and it causes the variance of consumption to fall by almost 50%. If we instead assume the goal of the policymaker is to move the individual's actual consumption path as close to his first plan as possible (rather than optimizing a mean-variance welfare function), then social security is still welfare improving because it helps to close this gap. Social security can close as much as 25% of the gap between planned and actual consumption behavior in our baseline parameterization.

Finally, we adapt the computational algorithm developed in Caliendo and Guo (2014) to our setting with time-inconsistent preferences. The algorithm involves smoothing the discontinuity in the credit spread with a logistic approximation, which then allows us to solve a sequence of unconstrained Pontryagin problems. We analytically solve the model up to an unknown vector of initial values from the many planned consumption paths, and each

[5] Hurst and Willen (2007) also allow for a credit spread when calculating the welfare consequences of social security, but they do not consider individuals with time-inconsistent preferences.

[6] Of course, "welfare" is a nebulous concept when individuals have time-inconsistent preferences. We side-step this debate and instead use two welfare metrics that have received considerable attention.

element of this vector is the solution to a separate boundary value problem with a Volterra differential equation that depends on the complete history of past states.[7] We now turn to the details of our model.

2 Model

To investigate the mandatory saving role of social security as a potential remedy to time-inconsistent saving behavior, we abstract from income heterogeneity and from uncertainty about income and longevity. Although these other features are certainly important for studying asset accumulation in general, they are not essential to our question and abstracting from them avoids confounding the welfare effects that come from mandatory saving in isolation with the welfare effects of redistribution and risk-sharing through social security. We also focus on a fully-funded arrangement because the welfare gains and losses of unfunded social security in dynamically inefficient and efficient economies are already well understood, and we abstract from general equilibrium effects for the same reason. Therefore, we purposefully consider a deterministic, representative agent, fixed lifespan model with an efficiently-financed social security program and no general equilibrium effects so that any welfare gains from mandatory saving are due solely to the presence of time-inconsistent preferences.

For a stylized setting such as this, it is worth first reviewing what happens to welfare when individuals are forced to save in the standard setup with time-consistent preferences and a revealed preference welfare criterion. If credit markets are perfect, then mandatory saving has no effect on welfare. If credit markets are missing, then mandatory saving has a negative effect on welfare. This is because the individual was already maximizing his utility before being forced to save, and forced saving cannot generally be undone when borrowing is impossible. Alternatively, if borrowing is allowed but at a higher interest rate (as in our model), then social security again has a negative effect on welfare because forced saving cannot be undone one-for-one due to high borrowing costs, which leaves consumption and savings allocations distorted once again. But as soon as we leave the standard setup and instead assume that individuals have time-inconsistent preferences, we open the door for distortions caused by mandatory saving to improve welfare, because individual behavior is suboptimal by definition. Thus, whether a distortion improves welfare or not becomes a philosophical question that depends on how welfare is defined, and we show a number of

[7] An alternative approach is to break asset holdings into two separate state variables, one for debt and one for savings, each with its own state equation and inequality restriction (Davis, Kubler, and Willen (2006)). This method is computationally intensive even in standard models with time-consistent preferences because the complementary slackness conditions must be verified at each point in time. The complexity would sharply increase in a time-inconsistent problem that must be re-solved at each and every vantage point.

examples to this effect.

We assume that individuals are naive about their future time inconsistency. Naive individuals make financial plans for the future, but then they abandon those plans and end up with less assets than they had originally intended. We like the naiveté assumption because it allows us to study whether mandatory saving helps those who procrastinate saving for retirement, which is a well-documented phenomenon (e.g., O'Donoghue and Rabin (1999a)). This is an innocuous assumption in a qualitative sense because our theoretical results are also valid for the case of sophistication (see Remark 1 below), though we expect that the exact magnitude of the welfare effects in our numerical examples may differ.

2.1 Notation

Age is continuous and is indexed by t. The individual starts work at $t = 0$, retires at $t = T$, and dies at $t = \bar{T}$. The individual is endowed with wage income w during the working years. His consumption is $c(t)$ and his asset holdings are $k(t)$. He starts and stops the life cycle with no assets $k(0) = k(\bar{T}) = 0$.

The interest rate on assets $r(k(t))$ depends on the sign of assets

$$r(k(t)) = \begin{cases} r_B, & \text{if } k(t) < 0, \\ r_S, & \text{if } k(t) > 0, \end{cases} \tag{1}$$

where

$$r_B \geq r_S. \tag{2}$$

We follow Caliendo and Guo (2014) and approximate the discontinuity in $r(k(t))$ with a continuous function. This allows us to solve a sequence of unconstrained Pontryagin problems which require differentiability in the state equation,

$$r(k(t)) \approx r_S - \frac{r_S - r_B}{1 + \exp[\psi k(t)]}, \tag{3}$$

where ψ is a large, positive scalar. Note that

$$\lim_{\psi \longrightarrow \infty} \left[r_S - \frac{r_S - r_B}{1 + \exp[\psi k(t)]} \right] = \begin{cases} r_B, & \text{if } k(t) < 0, \\ r_S, & \text{if } k(t) > 0, \end{cases} \tag{4}$$

$$\lim_{k(t) \longrightarrow -\infty} \left[r_S - \frac{r_S - r_B}{1 + \exp[\psi k(t)]} \right] = r_B, \tag{5}$$

$$\lim_{k(t) \longrightarrow +\infty} \left[r_S - \frac{r_S - r_B}{1 + \exp[\psi k(t)]} \right] = r_S. \tag{6}$$

The individual pays taxes on wages at rate τ in exchange for a social security benefit flow b. We assume that social security is fully funded in the sense that it provides the same return r_S as private savings. Thus, benefits are

$$b = \frac{\int_0^T \tau w \exp[-r_S t] dt}{\int_T^{\bar{T}} \exp[-r_S t] dt}. \tag{7}$$

Finally, the individual's period utility function $u[c(t)]$ is strictly concave and the discount function for a delay of x is $F(x)$.

2.2 Time-Inconsistent Decision Making

Standing at age $t \in [0, \bar{T})$, the individual solves

$$\max_{\{c(v)\}} : \int_t^{\bar{T}} F(v-t) u[c(v)] dv, \tag{8}$$

subject to

$$\frac{dk(v)}{dv} = r(k(v))k(v) + y(v) - c(v), \tag{9}$$

$$y(v) = \begin{cases} (1-\tau)w, \text{ for } v \in [t, T], \\ b, \text{ for } v \in [T, \bar{T}], \end{cases} \tag{10}$$

$$r(k(v)) = r_S - \frac{r_S - r_B}{1 + \exp[\psi k(v)]}, \tag{11}$$

$$k(t) \text{ given}, \; k(\bar{T}) = 0. \tag{12}$$

The first-order necessary conditions include the Maximum Condition

$$F(v-t) u_c[c(v)] = \lambda(v), \tag{13}$$

and the multiplier equation

$$\frac{d\lambda(v)}{dv} = -\lambda(v)[r'(k(v))k(v) + r(k(v))], \tag{14}$$

which has the particular solution

$$\lambda(v) = \lambda(t) \exp\left[-\int_t^v \{r'(k(z))k(z) + r(k(z))\} dz\right]. \tag{15}$$

Combine with the Maximum Condition

$$\begin{aligned} c(v) &= u_c^{-1}\left[\frac{\lambda(t)}{F(v-t)}\exp\left[-\int_t^v \{r'(k(z))k(z)+r(k(z))\}dz\right]\right] \\ &= u_c^{-1}\left[\frac{u_c[c(t)]}{F(v-t)}\exp\left[-\int_t^v \{r'(k(z))k(z)+r(k(z))\}dz\right]\right], \text{ for } v\in[t,\bar{T}], \quad (16) \end{aligned}$$

where $c(v)_{v\in[t,\bar{T}]}$ is the planned consumption path from the perspective of age t and $c(t)$ is actual consumption at age t because the individual does in fact stick with his plan the moment it is made, but not afterwards. From this point forward, we use an asterisk ($*$) to distinguish actual quantities from planned quantities.

All that remains is to pin down the actual consumption profile, $c^*(t)$, which is the envelope of initial values from a continuum of planned consumption paths. This must be done numerically for all t, because actual consumption $c^*(t)$ is the solution to a continuum of boundary value problems where the (Volterra) differential equation from each problem depends on the complete history of past states,

$$\frac{dk(v)}{dv} = r(k(v))k(v)+y(v)-u_c^{-1}\left[\frac{u_c[c^*(t)]}{F(v-t)}\exp\left[-\int_t^v \{r'(k(z))k(z)+r(k(z))\}dz\right]\right], \quad (17)$$

$$k(t)=k^*(t) \text{ given}, \ k(\bar{T})=0. \quad (18)$$

We develop a computational procedure to efficiently solve this time-inconsistent problem.

2.3 Computational Procedure

Planned and actual consumption and saving behavior can be computed as follows:

Step 1. Guess the vector of actual consumption $c^*(t)_{t\in[0,\bar{T})}$.

Step 2. Using $c^*(t)$ from Step 1, compute the actual asset holdings $k^*(t)_{t\in(0,\bar{T}]}$ using $k^*(0)=0$ and the law of motion

$$\frac{dk^*(t)}{dt} = r(k^*(t))k^*(t)+y(t)-c^*(t), \quad (19)$$

$$y(t) = \begin{cases} (1-\tau)w, & \text{for } t\in[0,T], \\ b, & \text{for } t\in[T,\bar{T}], \end{cases} \quad (20)$$

$$r(k^*(t)) = r_S - \frac{r_S-r_B}{1+\exp[\psi k^*(t)]}. \quad (21)$$

Step 3. For all $t\in[0,\bar{T})$, use $\{c^*(t),k^*(t)\}$ from Steps 1 and 2 to simulate the *planned* asset holdings $k(v)_{v\in(t,\bar{T}]}$ from each vantage point t using the differential equation

$$\frac{dk(v)}{dv} = r(k(v))k(v) + y(v) - u_c^{-1}\left[\frac{u_c[c^*(t)]}{F(v-t)}\exp\left[-\int_t^v \{r'(k(z))k(z) + r(k(z))\}dz\right]\right], \quad (22)$$

and the initial value $k(t) = k^*(t)$.

Step 4. Store all of the terminal asset holdings from the many planned paths $k(\bar{T})$ in a vector $\mathcal{K}$.

Step 5. Compute the norm $\|\mathcal{K}\|$.

Step 6. Update the guess on the actual consumption vector $c^*(t)$ until $\|\mathcal{K}\| < \epsilon$.

3 Welfare

İmrohoroğlu, İmrohoroğlu, and Joines (2003), Gul and Pesendorfer (2004), Caliendo (2011, 2013), and Guo and Caliendo (2014) show that a fully-funded social security program is irrelevant to the welfare of individuals with time-inconsistent preferences when credit markets are perfect. Our first theorem is meant as a review of this concept, though here we generalize to any concave utility function and to any discount function. This sets the stage for our main result (Theorem 2) which breaks new ground.

Theorem 1. *A fully-funded social security arrangement is irrelevant for consumption allocations and welfare if credit markets are perfect* ($r_B = r_S$).

Proof. Set $r_B = r_S = r$. In this case, at any age $t \in [0, \bar{T})$, the planned consumption path $c(v)$ over the interval $[t, \bar{T}]$, as a function of actual consumption $c^*(t)$, is

$$c(v) = u_c^{-1}\left[\frac{u_c[c^*(t)]\exp[-r(v-t)]}{F(v-t)}\right], \text{ for } v \in [t, \bar{T}]. \quad (23)$$

From the budget constraints we know that planned consumption must obey

$$\int_t^{\bar{T}} \exp[-r(v-t)]c(v)dv = k^*(t) + \int_t^{\bar{T}} \exp[-r(v-t)]y(v)dv, \quad (24)$$

and hence actual consumption at age t, $c^*(t)$, is pinned down by the condition

$$\int_t^{\bar{T}} \exp[-r(v-t)]u_c^{-1}\left[\frac{u_c[c^*(t)]\exp[-r(v-t)]}{F(v-t)}\right]dv = k^*(t) + \int_t^{\bar{T}} \exp[-r(v-t)]y(v)dv, \quad (25)$$

for all $t \in [0, \bar{T})$. Note that actual savings at age t is

$$k^*(t) = \int_0^t \exp[r(t-v)]y(v)dv - \int_0^t \exp[r(t-v)]c^*(v)dv, \tag{26}$$

and hence

$$\begin{aligned} \int_t^{\bar{T}} \exp[-r(v-t)]u_c^{-1}\left[\frac{u_c[c^*(t)]\exp[-r(v-t)]}{F(v-t)}\right]dv &= \exp[rt]\int_0^{\bar{T}} \exp[-rv]y(v)dv \\ &\quad - \int_0^t \exp[r(t-v)]c^*(v)dv. \end{aligned} \tag{27}$$

Notice only the present value of the income flow $\int_0^{\bar{T}} \exp[-rv]y(v)dv$, and not the timing of that flow, matters in the determination of the actual consumption profile $c^*(t)$ under time-inconsistent preferences and perfect markets. This completes the proof that a fully-funded (zero net present value) program is irrelevant. ■

The above theorem helps to reinforce and generalize the known result that a fully-funded social security arrangement is welfare neutral when credit markets are complete. Notice that this result holds for any utility function, for any discount function (including but not limited to the typical exponential and hyperbolic functions), and for any consumption-based welfare function. This is because welfare cannot improve if consumption is unchanged, and consumption is unchanged because social security taxes crowd out private asset holdings one-for-one.

Next we go a step further by proving that a fully-funded social security arrangement is irrelevant only if credit markets are complete. This means that any credit spread (small or large) is sufficient to ensure that a fully-funded social security program redistributes consumption over the life cycle. The consumption path without social security is not feasible in a state of the world with social security; instead, social security taxes crowd out assets less than one-for-one and the individual must reduce consumption in response to taxation.

Theorem 2. *Of the full spectrum of credit spreads ranging from zero (perfect credit markets) to infinity (missing credit markets), a fully-funded social security arrangement is irrelevant* ***only*** *at the knife edge of perfect credit markets.*

Proof. We prove by contradiction. Suppose the theorem is false. If so, then fully-funded social security crowds out assets one-for-one when $r_B > r_S$. If so, then the individual holds consumption fixed at the no-social-security allocation. Pick any age $t \in [0, T)$ for which the individual borrows in a world without social security. A dollar of social security taxation at age t would accumulate into $\exp[r_S(T-t)]$ dollars of social security wealth by retirement and would provide a flow of retirement benefits $\exp[r_S(T-t)]/\int_T^{\bar{T}}\exp[-r_S(t-T)]dt$. And one-for-one crowding out implies taking on an extra dollar of debt that would grow into $\exp[r_B(T-t)]$ dollars by retirement, which implies a debt-service payment flow over the retirement period $\exp[r_B(T-t)]/\int_T^{\bar{T}}\exp[-r_B(t-T)]dt$. But

$$r_B > r_S \Longrightarrow \frac{\exp[r_B(T-t)]}{\int_T^{\bar{T}}\exp[-r_B(t-T)]dt} > \frac{\exp[r_S(T-t)]}{\int_T^{\bar{T}}\exp[-r_S(t-T)]dt}. \tag{28}$$

This means the individual's incremental debts are not fully retired at death, which is not allowed by definition. ■

Remark 1 (Naiveté and Sophistication). Theorems 1 and 2 hold whether the individual is naive or sophisticated. For example, see Gul and Pesendorfer (2004) or Caliendo (2013) for a proof of Theorem 1 for sophisticated individuals, and notice that Theorem 2 does not require any particular assumption about the self-awareness of the individual. Thus, whether the individual recognizes or fails to recognize his time-inconsistent preferences, mandatory saving is irrelevant only at the knife edge of perfect credit markets.

Now that we have established that a credit spread ($r_B > r_S$) is necessary to improve welfare (Theorem 1) and sufficient to alter the distribution of consumption over the life cycle (Theorem 2), we seek to *quantify* the welfare effects from social security. Of course, the direction and magnitude of the welfare effects depend on how welfare is defined. And there is no census yet on how to define welfare when individuals have time-inconsistent preferences. We side-step this philosophical debate and instead focus on quantifying the welfare gains using two metrics that have received considerable attention.

3.1 Welfare Metric 1: Mean-Variance Welfare

Let us assume that the policymaker treats all the temporally sequenced selves of a single individual equally. He wants all the selves to consume as much as possible, and he dislikes inequality among the selves (i.e., smooth consumption is desirable). Hence, the policymaker has mean-variance preferences over the individual's actual consumption allocations.

Let $c^*(t|\tau)$ be the actual consumption path conditional on τ, and let

$$C(\tau) \equiv \bar{T}^{-1}\int_0^{\bar{T}} c^*(t|\tau)dt, \tag{29}$$

$$VAR(\tau) \equiv \int_0^{\bar{T}} [c^*(t|\tau) - C(\tau)]^2 dt. \tag{30}$$

Welfare is

$$S(\tau) \equiv C(\tau) - \phi VAR(\tau), \tag{31}$$

where ϕ is the penalty for inequality (non-smoothness) among the selves.

Admittedly, this metric is not explicitly connected to time inconsistency per se. That is, there are other reasons, besides time inconsistency, for why individuals do not naturally maximize a mean-variance welfare function on their own. And yet, this metric provides useful information on how social security affects the moments of the consumption distribution over the life cycle. Next we discuss our preferred metric, which is directly tied to time inconsistency.

3.2 Welfare Metric 2: First-Plan is the Policy Target

Another popular welfare metric or policy target is to treat the first plan as the relevant goal (e.g., Laibson (1998), O'Donoghue and Rabin (1999b), Gruber and Kőszegi (2001), Rubinstein (2006), Cremer and Pestieau (2011)).[8] This is justifiable given that the first plan *strictly and robustly multiself Pareto dominates* the path that an individual with time-inconsistent preferences actually follows (Caliendo and Findley (2014)). Let $c^*(t|\tau)$ be the actual consumption path conditional on τ, and let $c_0(t)$ be the first plan for the specific case of $\tau = 0$. The Euclidean distance (gap) between the planned and actual consumption profiles is

$$g(\tau) \equiv \sqrt{\int_0^{\bar{T}} [c^*(t|\tau) - c_0(t)]^2 dt}. \tag{32}$$

Notice that the gap is always the distance from actual consumption $c^*(t|\tau)$ (for any τ) to the first plan without social security $c_0(t)$, because the latter is the policy target by definition. Because welfare is inversely related to the gap, the fraction of the gap that is closed by social security is the relevant welfare statistic

$$\Delta g \equiv \frac{g(0) - g(\tau)}{g(0)}. \tag{33}$$

4 Numerical Examples

We start with the following baseline parameter values, but we check the sensitivity of our results to these assumptions in the next section. We assume the individual starts work at

[8] See Gul and Pesendorfer (2004) for a critique of this approach.

age 25, retires at 65 and dies at 80. Hence, $T = 40$ and $\bar{T} = 55$. We assume the individual starts with no assets $k(0) = 0$ and faces a 3-point spread: $r_S = 1\%$ (which is approximately the return on short-term treasuries) and $r_B = 4\%$. A spread of 3 points matches the average spread in the US during the Greenspan-Bernanke era. We normalize $w = 1$ and social security is either non-existent or the tax matches the US program, $\tau = 0$ or $\tau = 10.6\%$, with $b = 0$ or $b = 0.3743$. Utility is of the isoelastic variety, $u[c(t)] = c(t)^{1-\sigma}/(1-\sigma)$ for $\sigma \neq 1$ and $u[c(t)] = \ln c(t)$ otherwise.[9] We set $\sigma = 1$ in our baseline calculations. We use a standard hyperbolic function $F(x) = [1 + \beta x]^{-1}$ with $\beta = 7\%$, and we set $\psi = 50$ from the logistic function. Finally, what actually goes into the computer is a discrete-time version of our model with a grid step size of one year.[10]

Figure 1 shows the goodness-of-fit of our approximated interest rate function. The larger the value of ψ the closer the fit, though we have found that our computational procedure can struggle if ψ is too large (presumably because the curvature becomes almost perfectly kinked and differential approximation then becomes problematic). A value of 50 is large enough to give a nice tight fit and yet is small enough that we don't encounter any computational difficulty.

Figure 2 shows the approximation error in the computational procedure that we outlined above. This figure plots the elements of the vector $\mathcal{K}$ from Step 4 (i.e., the terminal asset holdings from the many planned paths $k(\bar{T})$). The planned, terminal asset holdings should be very close to zero for all vantage points t, and indeed these values are small enough that we are confident our computational procedure is correctly identifying $c^*(t)$.

Figure 3 shows asset holdings over the life cycle without a social security program. One of the curves corresponds to the individual's initial plan and the other corresponds to the path that actually materializes. Notice that the individual borrows much more than initially planned and he falls dramatically short of his initial saving goals, just as we would expect of individuals with time-inconsistent preferences.

In Figure 4 we plot three consumption profiles over the life cycle. The initial consumption plan without social security and the actual consumption path without social security give us a feel for how far the individual's spending behavior deviates from his first plan in a world without any intervention. Then the third consumption path corresponds to an otherwise identical world with a fully-funded social security arrangement. Using the data from Figure

[9] For isoelastic utility, the law of motion for planned assets in Step 3 of the computational procedure is:

$$\frac{dk(v)}{dv} = r(k(v))k(v) + y(v) - c^*(t)F(v-t)^{1/\sigma} \exp\left[\frac{1}{\sigma}\int_t^v \{r'(k(z))k(z) + r(k(z))\}dz\right].$$

[10] We have verified that a finer grid (e.g., one-tenth of a year) does not change our quantitative results in a significant way, though it does dramatically slow down the speed of our computations.

4, we plot in Figure 5 the squared deviations of the actual consumption paths from the first plan.

From the perspective of Welfare Metric 1, social security unambiguously improves welfare. This is because a fully-funded social security program leaves the average value of the actual consumption path virtually the same as in a world without social security.[11] Yet, social security shrinks the variance of the actual consumption path by nearly 50%! Thus social security redistributes consumption across the life cycle in a mean-preserving (but variance-reducing) fashion, and this leaves welfare strictly higher. If we instead assume a single interest rate on borrowing and saving, then fully-funded social security leaves consumption and welfare completely unchanged for the hyperbolic discounter (see Theorem 1).

Welfare also goes up from the perspective of Welfare Metric 2. This is not as obvious to see in the graphs because there are some clear trade-offs. At some ages social security brings the actual consumption path closer to the first plan without social security, but at other ages the reverse is true. For instance, at $t = 0$ no government intervention is needed to keep the individual close to his first plan, because he has (trivially) had no time to deviate from it. And there are a few other instances where the individual's actual consumption spending inadvertently ends up close to his original plan. But overall, social security definitely helps bring consumption behavior into better alignment with the first plan. There are especially large gains late in life (see Figure 5). In total, social security closes slightly more than 25% of the gap between planned and actual consumption behavior at the baseline parameterization. And once again, fully-funded social security leaves consumption and welfare unchanged in the absence of a credit spread.

The intuition for these results is clear. Unlike a model with perfect credit markets (single interest rate on borrowing and saving) in which social security contributions are just offset one-for-one with reductions in private asset holdings, in the present model the individual does not reduce asset holdings one-for-one because such a reduction can trigger high borrowing costs. And this means that social security taxation causes consumption to fall during the early years to keep the instantaneous household budget constraint in balance and then rise

[11] In fact, fully-funded social security can actually increase average consumption. This is not a free lunch; it is the result of the credit spread and the fact that social security crowds out asset holdings less than one-for-one. When the government collects a dollar in social security taxation at age t, consumption and asset holdings both fall at that moment. The decline in consumption represents forced saving at the fully-funded rate $r_S > 0$. The decline in asset holdings represents dissaving at rate r_B. If the decline in consumption is large enough relative to the decline in asset holdings, then the extra social security benefits will be large enough to not only service the incremental debt but also large enough to more than offset the initial reduction in consumption, thereby pulling up average consumption. This can be proven rigorously by assuming the decline in asset holdings is infinitesimally small and the decline in consumption approaches unit (almost like a borrowing constraint): in this case, social security is sure to increase average consumption. Of course, it is a quantitative question whether average consumption goes up or not, but it is clearly possible.

later during the retirement years.

Notice that we are basically just carrying over the intuition that we already know holds true in a model with missing credit markets. If borrowing is not allowed, then social security causes less than one-for-one crowding out (or even zero crowding out) of private asset holdings and therefore social security has similar effects on the distribution of consumption allocations over the life cycle. But the key contribution of our paper is to show that we do not need to make the counterfactual assumption that credit markets are totally missing to get this result. We only need a small borrowing cost to generate welfare gains from social security participation.

5 Robustness

In this section we systematically check the robustness of our results to other reasonable assumptions about parameter values.

5.1 Unobservable Preference Parameters

The quantitative magnitude and sign of the welfare effect of social security participation depends on the unobservable preference parameters β and σ. Table 1 shows the effect of social security on the components of the mean-variance welfare function (Welfare Metric 1). Table 2 shows the effect of social security on the gap between planned and actual consumption (Welfare Metric 2).

Panel A of Table 1 reports the percentage change in mean consumption $[C(\tau)-C(0)]/C(0)$ and Panel B reports the percentage change in the variance of consumption $[VAR(\tau) - VAR(0)]/VAR(0)$. In the context of Welfare Metric 1, there is an unequivocal increase in welfare when the mean goes up and the variance goes down. This is often the case. But at other parameterizations social security can either lower the mean or increase the variance, implying a trade-off that would depend on the value of the penalty parameter ϕ.

Panel A of Table 2 reports the Euclidean gap between planned consumption without social security and actual consumption without social security, $g(0)$. The gap is largest when the discount parameter β is largest and when the utility function is closest to linear (small σ). Panel B shows the fraction of this gap (Δg) that is closed by social security. At some parameterizations social security actually reduces welfare a little. But at most parameterizations the effect of social security on welfare is not only positive but can be very large.

5.2 Alternative Social Security Arrangements

In Tables 3 and 4 we replicate Tables 1 and 2 under the alternative assumption that the social security system has a tax rate of 21%. This matches the average tax rate across OECD public pension systems. We continue to assume that benefits are fully funded. In terms of the broad lessons to learn, Tables 3 and 4 look about the same as Tables 1 and 2, though now social security reduces the variance of consumption at all parameterizations (and this reduction can be huge). Also, social security sometimes closes an even larger share of the gap between planned and actual consumption behavior.

While our computational method typically works well in the sense that $\|\mathcal{K}\|$ falls below an acceptable threshold, the method does not always work perfectly. We struggled to get our algorithm to converge at two particular points in the parameter space. These points are denoted with a triple asterisk in Tables 3 and 4 ($***$).

We have also re-calculated the welfare effects under the alternative assumption that social security is unfunded with an internal rate of return that is less than the return on private savings. As anticipated, it is more difficult though not impossible to find welfare gains. But this is due to the inefficiency in financing the arrangement rather than a fundamental problem with the intergenerational transfer itself. In contrast, an unfunded social security program always reduces welfare in a setting with a single interest rate on borrowing and saving.

5.3 Credit Spread

Tables 5 and 6 report similar calculations but for different credit spreads. In Table 5 we see that a fully-funded social security arrangement tends to reduce average consumption when the interest rate on saving r_S is low or when the spread is small, and social security reduces the variance of consumption at all parameterizations. Hence, social security is either unambiguously welfare improving (when the mean goes up and the variance goes down) or it has an ambiguous effect on welfare (when both the mean and variance go down). But there are no parameterizations in which social security unambiguously reduces welfare. Table 6 shows that social security typically closes a significant share of the gap between planned and actual consumption paths. As before, however, this result is not perfectly robust because we can find some parameterizations in which the reverse is true.

5.4 Initial Debt, $k(0) < 0$

Some households already have significant debt when they enter the workforce at $t = 0$. Student loans, car loans, home loans, and credit card debt are common examples. Our baseline welfare results do not change much when we build this feature into the model. For example, suppose we set $k(0) = -1$, which is a year's worth of wage income. In this case, social security causes the mean of consumption to increase and the variance of consumption to fall by about the same proportions as in the baseline parameterization. And social security closes slightly more of the gap between planned and actual consumption behavior. Therefore, adding initial debt to the model (by even a significant amount as we have done here) does not have an important effect on the direction of the results.

6 Concluding Discussion

Common intuition suggests that a mandatory saving program will help individuals who fall short of their retirement saving goals due to possessing time-inconsistent preferences. Yet a number of theoretical studies show that a fully-funded social security arrangement has no effect on welfare when credit markets are complete and improves welfare only if credit markets are totally missing. Because it is difficult to defend the assumption that credit markets are missing, hyperbolic discounting has not yet provided a particularly compelling justification for the mandatory saving function of social security. This apparent inability to explain the existence and persistence of such a fundamental economic institution has remained a challenge to hyperbolic discounting as a theory of intertemporal choice.

However, we show that the disconnect between intuition and theory vanishes under more realistic assumptions about credit markets: a mandatory saving program is able to improve the well-being of individuals with time-inconsistent preferences if the model includes an empirically reasonable credit spread between the interest rates on borrowing and saving. In fact, we analytically prove that a fully-funded social security program can improve welfare if there is a credit spread of *any* size. We also demonstrate with numerical simulations that the welfare gains can be large even when the credit spread is small. Hence, while it is commonly believed that innovation in credit markets "eliminates the commitment properties of illiquid assets [like social security]" (Laibson (1998, p.869)), the key contribution of our paper is to show that even in a setting where individuals can borrow as much as they want, social security still provides partial commitment as long as there is a credit spread.

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Table 1. Robustness to Unobservables: Welfare Metric 1 under US Tax

Panel A. Change in mean consumption, $[C(\tau) - C(0)]/C(0)$

	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 3.0$	$\sigma = 4.0$	$\sigma = 5.0$	$\sigma = 6.0$
$\beta = 1\%$	0.02%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%
$\beta = 4\%$	1.99%	1.15%	0.16%	0.01%	0.01%	0.01%	0.01%
$\beta = 7\%$	0.56%	0.66%	0.65%	0.27%	0.06%	0.02%	0.01%
$\beta = 10\%$	-2.24%	-1.17%	-0.08%	0.28%	0.20%	0.08%	0.02%

Panel B. Change in variance, $[VAR(\tau) - VAR(0)]/VAR(0)$

	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 3.0$	$\sigma = 4.0$	$\sigma = 5.0$	$\sigma = 6.0$
$\beta = 1\%$	0.01%	-0.06%	-0.16%	-0.22%	-0.26%	-0.28%	-0.30%
$\beta = 4\%$	-49.21%	-40.08%	-10.06%	0.43%	0.25%	0.17%	0.13%
$\beta = 7\%$	-50.17%	-45.98%	-36.59%	-17.52%	-3.51%	0.16%	0.20%
$\beta = 10\%$	-28.19%	-38.13%	-38.04%	-30.49%	-16.28%	-5.31%	-0.43%

Table 2. Robustness to Unobservables: Welfare Metric 2 under US Tax

Panel A. Euclidean gap between planned and actual consumption, $g(0)$

	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 3.0$	$\sigma = 4.0$	$\sigma = 5.0$	$\sigma = 6.0$
$\beta = 1\%$	0.23	0.11	0.06	0.04	0.03	0.02	0.02
$\beta = 4\%$	1.00	0.69	0.38	0.26	0.20	0.16	0.14
$\beta = 7\%$	1.14	0.97	0.68	0.48	0.37	0.30	0.26
$\beta = 10\%$	1.71	1.15	0.88	0.67	0.52	0.43	0.36

Panel B. Fraction of the gap closed by social security, Δg

	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 3.0$	$\sigma = 4.0$	$\sigma = 5.0$	$\sigma = 6.0$
$\beta = 1\%$	0.05%	0.10%	0.18%	0.26%	0.33%	0.40%	0.46%
$\beta = 4\%$	5.62%	2.18%	0.58%	-0.12%	-0.01%	0.02%	0.04%
$\beta = 7\%$	52.05%	26.54%	12.15%	3.45%	0.49%	-0.07%	-0.01%
$\beta = 10\%$	18.92%	35.82%	23.87%	13.44%	5.09%	1.17%	0.004%

Table 3. Robustness to Unobservables: Welfare Metric 1 under OECD Tax

Panel A. Change in mean consumption, $[C(\tau) - C(0)]/C(0)$

	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 3.0$	$\sigma = 4.0$	$\sigma = 5.0$	$\sigma = 6.0$
$\beta = 1\%$	0.03%	0.02%	0.02%	0.03%	0.03%	0.03%	0.03%
$\beta = 4\%$	***	3.72%	1.90%	1.16%	0.80%	0.58%	0.44%
$\beta = 7\%$	***	0.88%	1.27%	1.04%	0.83%	0.68%	0.56%
$\beta = 10\%$	-5.65%	-4.01%	-1.99%	-0.97%	-0.59%	-0.39%	-0.26%

Panel B. Change in variance, $[VAR(\tau) - VAR(0)]/VAR(0)$

	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 3.0$	$\sigma = 4.0$	$\sigma = 5.0$	$\sigma = 6.0$
$\beta = 1\%$	-1.38%	-5.62%	-10.93%	-14.14%	-16.24%	-17.68%	-18.72%
$\beta = 4\%$	***	-98.82%	-96.39%	-92.39%	-87.24%	-81.22%	-74.68%
$\beta = 7\%$	***	-89.98%	-93.53%	-93.12%	-92.04%	-90.38%	-87.98%
$\beta = 10\%$	-46.82%	-63.91%	-77.89%	-81.45%	-81.15%	-80.40%	-79.61%

Table 4. Robustness to Unobservables: Welfare Metric 2 under OECD Tax

Panel A. Euclidean gap between planned and actual consumption, $g(0)$

	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 3.0$	$\sigma = 4.0$	$\sigma = 5.0$	$\sigma = 6.0$
$\beta = 1\%$	0.23	0.11	0.06	0.04	0.03	0.02	0.02
$\beta = 4\%$	***	0.69	0.38	0.26	0.20	0.16	0.14
$\beta = 7\%$	***	0.97	0.68	0.48	0.37	0.30	0.26
$\beta = 10\%$	1.71	1.15	0.88	0.67	0.52	0.43	0.36

Panel B. Fraction of the gap closed by social security, Δg

	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 3.0$	$\sigma = 4.0$	$\sigma = 5.0$	$\sigma = 6.0$
$\beta = 1\%$	0.01%	-0.36%	-1.16%	-1.78%	-2.25%	-2.60%	-2.87%
$\beta = 4\%$	***	-23.48%	-10.64%	1.59%	8.52%	12.07%	13.23%
$\beta = 7\%$	***	28.12%	21.51%	24.48%	30.25%	34.26%	36.62%
$\beta = 10\%$	26.39%	41.56%	45.05%	43.43%	45.58%	48.19%	50.03%

Table 5. Robustness to Credit Spread: Welfare Metric 1 under US Tax

Panel A. Change in mean consumption, $[C(\tau) - C(0)]/C(0)$

	$r_B = 2.0\%$	$r_B = 3.0\%$	$r_B = 4.0\%$	$r_B = 5.0\%$	$r_B = 6.0\%$	$r_B = 7.0\%$
$r_S = 0.0\%$	-1.88%	-1.53%	-0.87%	-0.27%	-0.03%	-0.02%
$r_S = 0.5\%$	-1.31%	-0.90%	-0.13%	0.57%	0.86%	0.88%
$r_S = 1.0\%$	-0.74%	-0.27%	0.66%	1.47%	1.83%	1.84%
$r_S = 1.5\%$	-0.25%	0.32%	1.46%	2.41%	2.85%	2.87%

Panel B. Change in variance, $[VAR(\tau) - VAR(0)]/VAR(0)$

	$r_B = 2.0\%$	$r_B = 3.0\%$	$r_B = 4.0\%$	$r_B = 5.0\%$	$r_B = 6.0\%$	$r_B = 7.0\%$
$r_S = 0.0\%$	-23.02%	-34.27%	-40.56%	-41.67%	-41.19%	-41.17%
$r_S = 0.5\%$	-21.43%	-35.46%	-43.04%	-44.54%	-43.96%	-43.95%
$r_S = 1.0\%$	-18.59%	-36.19%	-45.98%	-47.99%	-47.30%	-47.29%
$r_S = 1.5\%$	-12.47%	-36.28%	-49.27%	-52.17%	-51.39%	-51.38%

Table 6. Robustness to Credit Spread: Welfare Metric 2 under US Tax

Panel A. Euclidean gap between planned and actual consumption, $g(0)$

	$r_B = 2.0\%$	$r_B = 3.0\%$	$r_B = 4.0\%$	$r_B = 5.0\%$	$r_B = 6.0\%$	$r_B = 7.0\%$
$r_S = 0.0\%$	0.80	0.74	0.75	0.81	0.84	0.85
$r_S = 0.5\%$	0.90	0.84	0.85	0.90	0.93	0.93
$r_S = 1.0\%$	1.04	0.97	0.97	1.01	1.03	1.04
$r_S = 1.5\%$	1.21	1.13	1.13	1.16	1.17	1.18

Panel B. Fraction of the gap closed by social security, Δg

	$r_B = 2.0\%$	$r_B = 3.0\%$	$r_B = 4.0\%$	$r_B = 5.0\%$	$r_B = 6.0\%$	$r_B = 7.0\%$
$r_S = 0.0\%$	29.04%	30.44%	13.55%	-0.84%	-6.11%	-6.29%
$r_S = 0.5\%$	24.52%	32.36%	20.74%	7.36%	1.70%	1.48%
$r_S = 1.0\%$	18.68%	31.53%	26.54%	15.41%	9.80%	9.56%
$r_S = 1.5\%$	10.40%	28.31%	30.06%	22.60%	17.63%	17.40%

Figure 1. Continuous Approximation (Caliendo and Guo (2014))

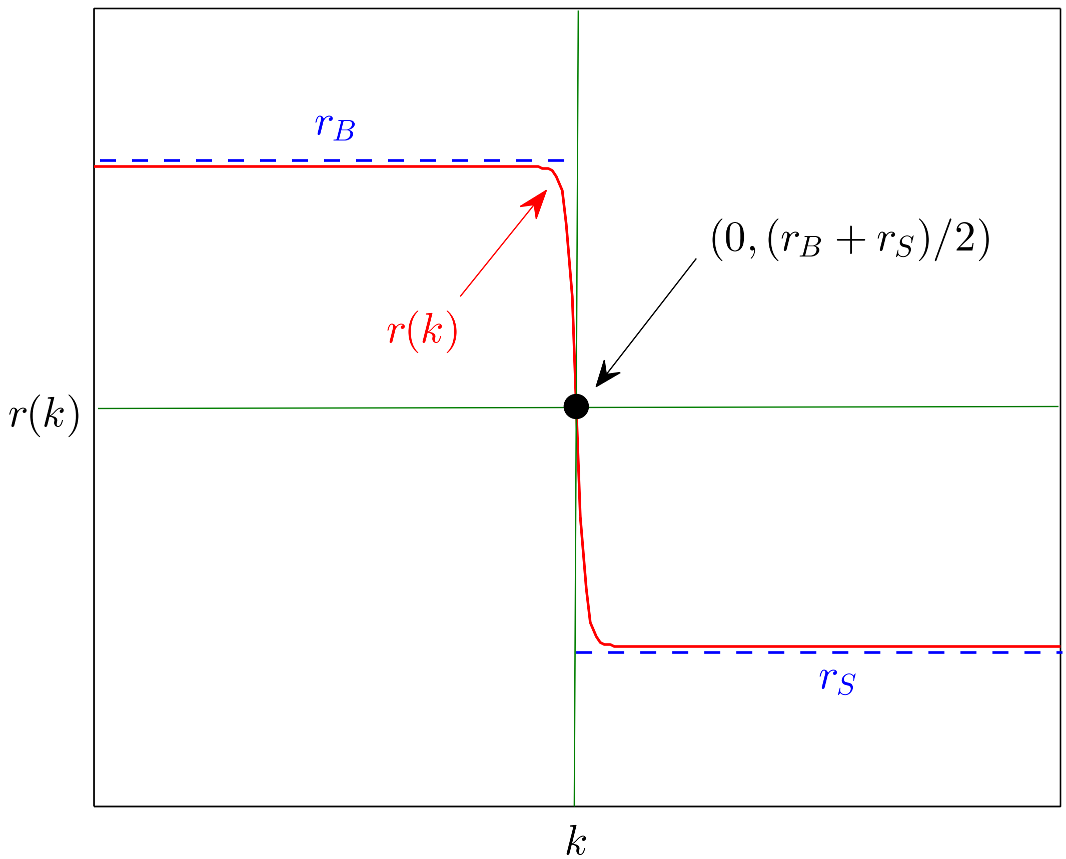
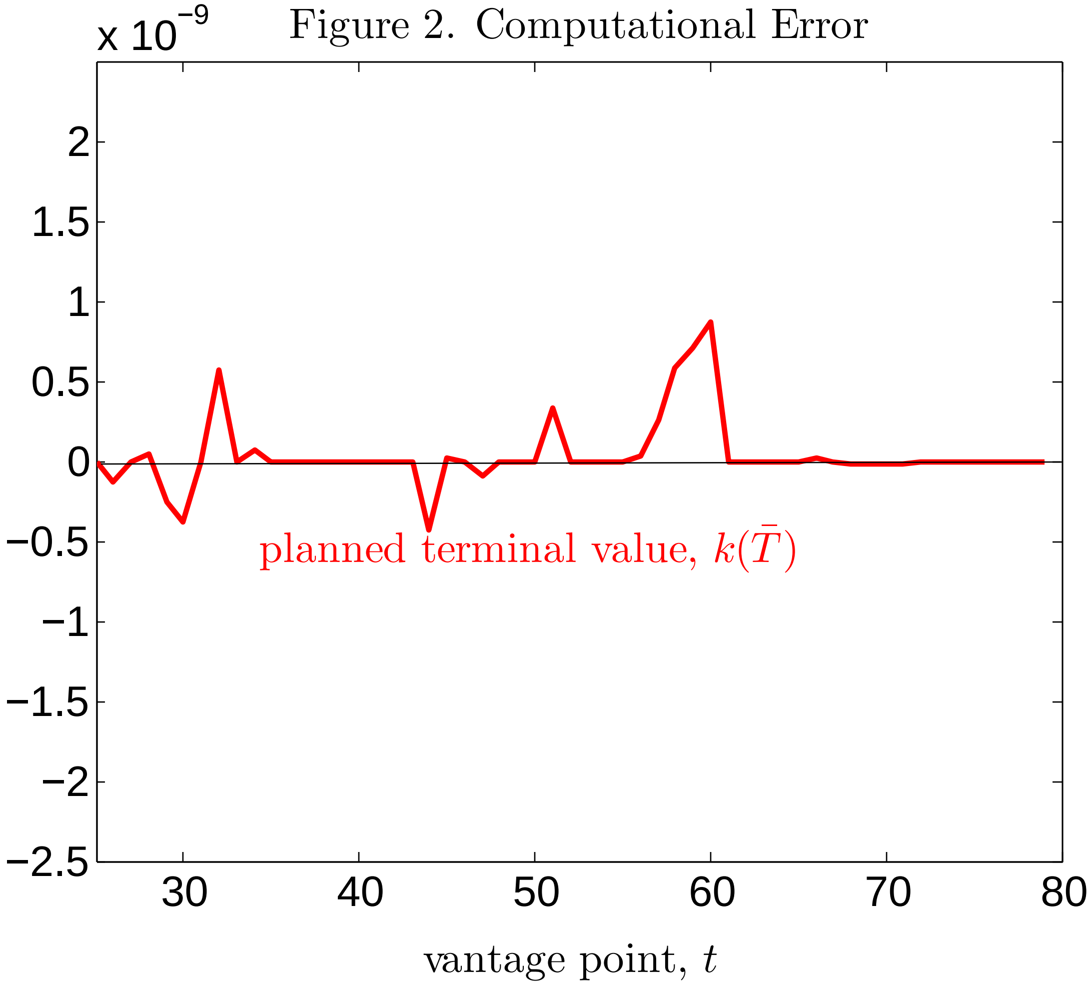
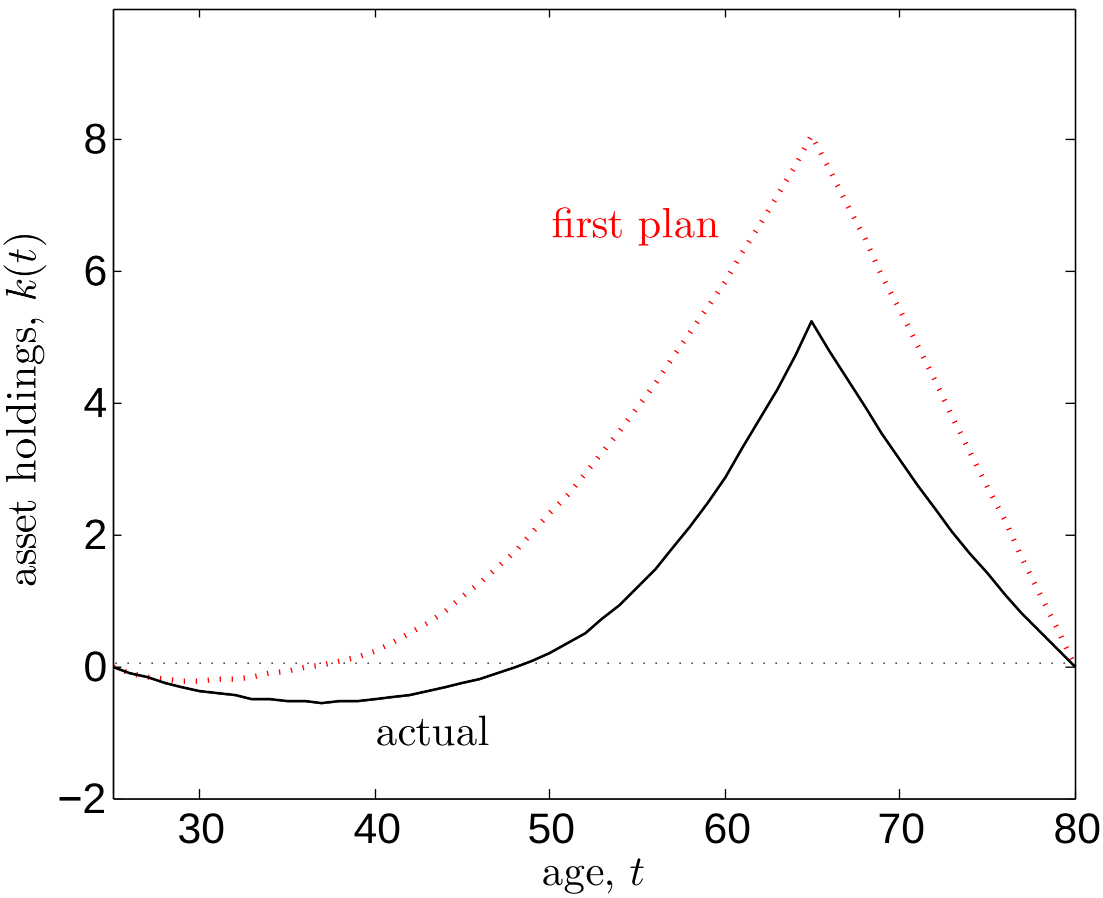


Figure 2. Computational Error



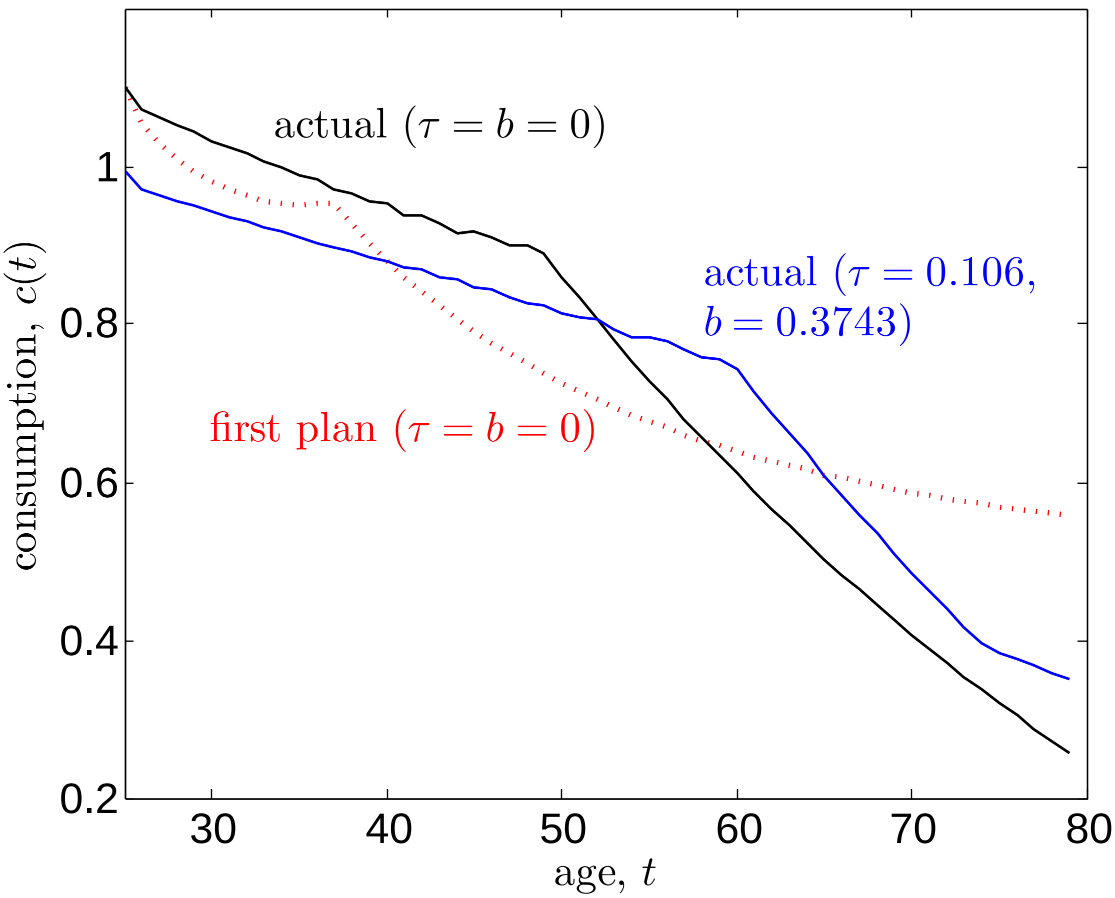
Parameters: $k(0) = 0$, $r_S = 0.01$, $r_B = 0.04$, $\tau = b = 0$, $w = 1$, $\psi = 50$, $T = 40$, $\bar{T} = 55$, $\beta = 0.07$, $\sigma = 1$.

Figure 3. Asset Holdings over the Life Cycle: No Social Security



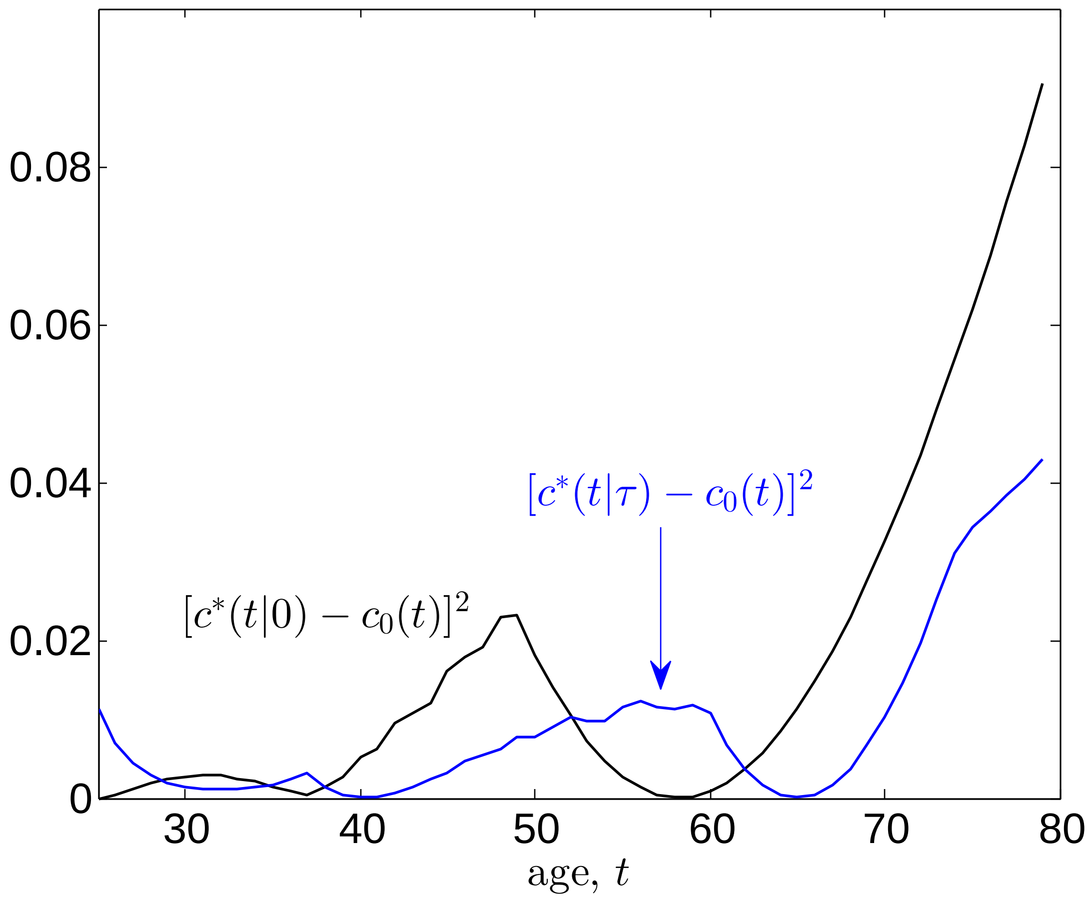
Parameters: $k(0) = 0$, $r_S = 0.01$, $r_B = 0.04$, $\tau = b = 0$, $w = 1$, $\psi = 50$, $T = 40$, $\bar{T} = 55$, $\beta = 0.07$, $\sigma = 1$.

Figure 4. Consumption over the Life Cycle



Parameters: $k(0) = 0$, $r_S = 0.01$, $r_B = 0.04$, $w = 1$, $\psi = 50$, $T = 40$, $\bar{T} = 55$, $\beta = 0.07$, $\sigma = 1$.

Figure 5. Squared Distance from First Plan



Parameters: $k(0) = 0$, $r_S = 0.01$, $r_B = 0.04$, $w = 1$, $\psi = 50$, $T = 40$, $\bar{T} = 55$, $\beta = 0.07$, $\sigma = 1$.