A financial scenario model with a zero lower bound

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Abstract

This document describes a model that generates financial scenarios. The term structure is modeled by imposing certain long-run restrictions on the zero lower bound model in Wu and Xia (2016). The inflation rate and the return on the MSCI World Index are modeled using an MA(1)-model and a GARCH-model, respectively.

1 Introduction

A typical procedure to assess the effects of a policy measure in the pension system is an evaluation of the effects in a certain simulation set. The simulation set is generated by some underlying model that aims to describe the dynamics of demographics, economic and financial variables. This document describes such a model for financial and economic scenarios. Following Koijen, Nijman, and Werker (2010), Draper (2014), and Muns (2015a), the central block of the model is a term structure model for the interest rates of different maturities.

The term structure model presented here adds to this literature in four ways. First, the model takes into account the zero (or slightly negative) lower bound (ZLB) on interest rates. Second, instead of imposing the Ultimate Forward Rate (UFR) of the Dutch Central Bank (DNB) as the asymptotic forward rate of a smooth curve, the UFR is implemented here in exactly the same way as DNB computes this rate.[1] Third, the effect of parameter uncertainty is taken into account. This is consistent with another block of scenario sets, the demographic scenarios.[2] Fourth, and related to the first point, the data sample ends in June 2016 with negative interest rates.

From the simulated set of term structures, two other essential macroeconomic variables for pensions are simulated, the inflation rate with an MA(1) model and the return on the MSCI World Index with a GARCH model.

Section 2 reviews the literature on term structure models. The ZLB term structure model is described in Section 3. Section 4 discusses the estimation methodology of the term structure model. The estimation results are presented in Section 5. Other macroeconomic variables are simulated in Section 6. The Appendix contains data sources and derivations.

2 Literature

A large strand of literature considers affine models for the U.S. term structure. This section summarizes this literature. Extensive overviews on term structure models are in Piazzesi (2009) and

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[1]For the methodology that determines the UFR, the interested reader is referred to `https://www.rijksoverheid.nl/binaries/rijksoverheid/documenten/rapporten/2013/10/11/advies-commissie-ufr/advies-commissie-ufr.pdf` (p.77, in Dutch). From July 2015 onwards, DNB sets $\alpha = 0$ at p.78.

[2]The model for the demographic scenarios is in Muns (2015b). The results in that paper indicate that at least the instantaneous dependence between the financial block and demographic block is low. Thus, the ZLB financial block outlined here can replace the term structure model in Muns (2015b).

Gürkaynak and Wright (2012). The first review considers term structure models from a finance perspective, the latter review focuses on the interaction between macroeconomics and monetary policy.

The seminal paper of Nelson and Siegel (1987) models the cross-sectional term structure with exponential functions.[3] Each time period has a level, slope, and curvature component. Svensson (1994) adds a second slope factor to the Nelson-Siegel model. This slope factor improves the modeling of Swedish forward interest rates. To enable forecasting procedures, Diebold and Li (2006) introduce an autoregressive setup in the Nelson and Siegel (1987) model to represent the time series dynamics of the latent level, slope and curvature. Diebold, Rudebusch, and Aruoba (2006) add macro-economic series to the explanatory variables in Diebold and Li (2006). Their forecasting exercise suggests strong evidence of macroeconomic effects on the future yield curve and somewhat weaker evidence of yield curve effects on future macroeconomic developments. The macro variables are capacity utilization, the Fed fund rate, and Consumer Price Inflation (CPI).

Though useful as a starting point, the standard Nelson-Siegel model has a conceptual shortcoming since it does not necessarily exclude arbitrage opportunities. Leaving aside a direct modelling of the level, slope and curvature factor, in a more general model Duffie and Kan (1996) impose no-arbitrage restrictions and specify continuous time relations for the latent factors. Compared to the Nelson-Siegel based models, the number of parameters is substantially higher for a given number of latent factors. Christensen, Diebold, and Rudebusch (2009, 2011) impose no-arbitrage restrictions on the Nelson-Siegel model. The restrictions show a good in-sample fit and can improve predictive performance. Krippner (2015a) shows that the Nelson-Siegel model arises from a low-order Taylor approximation of the generic Gaussian affine term structure model.

Building on the general model in Duffie and Kan (1996), de Jong (2000) finds that a three factor model provides the best fit of the cross-section and the time-series dynamics of the U.S. term structure between January 1970 and February 1991. Though not explicitly imposed, the three factors are very similar to a level, slope and curvature factor. Ang and Piazzesi (2003) estimate a similar VAR model for the joint dynamics of bond yields and macroeconomic variables. Macro factors explain up to 85% of the variation in bond yields, primarily movements in yields with short and medium term maturities. The macro variables are measures for inflation and real activity. Cochrane and Piazzesi (2005) consider predictability of the term structure. They find that a single factor, a single tent-shaped linear combination of forward rates, predicts excess returns on one- to five-year bonds with R^2 up to 44%. An important component of this single factor is unrelated to the level, slope, and curvature movements. The term structure model in Rudebusch and Wu (2008) suggests the empirical usefulness of CPI and capacity utilization. They end with a discussion on theoretical issues such as the necessity of the no-arbitrage condition in a combined macro-finance approach.

While macro factors are empirically uninformative for current bond returns, they might have predictive power for future excess bond returns. In that case, there exist unspanned macro risks. Wright (2011) investigates the term premium in an international panel dataset with Germany as the sole eurozone country. Macro-variables are unspanned factors which means that they are only used for forecasting the term structure. The decline in the term premium over the last 20 years is explained by the decline in inflation uncertainty as measured by the central bank's monetary policy. The quarterly dataset gives rise to an identification issue (Bauer, Rudebusch, and Wu (2014) and Wright (2014)). Joslin, Priebsch, and Singleton (2014) quantify for the U.S. the effect of economic activity and expected inflation from surveys on the market prices of level, slope, and curvature risk. Between 1985 and 2007, an economic activity index and expected inflation accounted for a large portion of the variation in forward term premiums.

Bauer and Rudebusch (2017) statistically reject the unspanned approach in Joslin, Priebsch, and Singleton (2014). Taking account for small measurement errors enables the inclusion of macro factors in the latent factors. The macro variables are (*i*) inflation series that determine the level of the yield curve, and (*ii*) activity measures that determine the slope of the yield curve. Bauer and

[3] An extensive review on the Nelson-Siegel model is in Diebold and Rudebusch (2013).

Hamilton (2018) reexamine evidence in previous literature that unspanned factors help in predicting the yield curve. By using bootstrap techniques that take the high persistence into account, they cannot find convincing evidence that macro factors predict excess bond returns.

At the time of writing, monetary policy rates are negative in the eurozone, Denmark, Sweden, Switzerland, and Japan. The yields on sovereign bonds and corporate bonds have also declined below zero for a growing number of assets. Costs for storage and transactions may explain why investors accept a negative yield on these assets. Nonetheless, there are some reasons to believe that interest rates are not unbounded in the negative territory. First, when yields on assets decrease to more negative levels, the zero nominal return on cash holdings becomes an increasingly attractive alternative investment. This principle holds also for investments in assets denominated in foreign currency. For instance, the expected return on an investment in Swiss ten-year bonds is currently substantially lower than holding Swiss franc in cash, though both investments may benefit from currency risk if the Swiss franc appreciates. Second, the implementation of asset purchase programmes of central banks have pushed interest rates in a downward direction. The decreasing trend is unlikely to continue in the far future since certain restrictions are imposed on the implemented asset purchase programmes. For instance, the European Central Bank is not permitted to buy any asset with a yield below minus 0.4%. In addition, the central bank can only buy an asset up to a certain fraction of the outstanding nominal value of this asset. In summary, the current financial market environment makes substantial negative interest rates highly unlikely. This suggests that current interest rates are stuck near the so-called zero lower bound (ZLB).

Since standard models do not assume a lower bound on interest rates, such models indicate that interest rates may further decline into deeper negative territories. As a consequence, the current ZLB episode is troublesome for affine models. Modelling the term structure with a binding ZLB on interest rates is therefore an active research area.[4] Key publications are Kim and Singleton (2012), Christensen and Rudebusch (2015, 2016), and Wu and Xia (2016). Both implement the ZLB by adding an option-based shadow rate to the standard affine term structure models. This literature is still in its infancy and it focuses on econometric modelling without a clear linkage to macro determinants. Bauer and Rudebusch (2016) document the benefits of including the macro factors inflation and unemployment gap as spanned factors for forecasting future Treasury rates.

Instead of studying the effect of macro factors on the ZLB term structure, Wu and Xia (2016) consider the reverse effect of the implied shadow rate from the ZLB term structure on the unemployment rate. They report that the Fed's actions which started in July 2009 decreased the unemployment rate in December 2013 with an additional 0.13%-point. However, Krippner (2017) demonstrates that the results are sensitive to the selected shadow rate model

3 Term structure model

The original idea of modeling a lower bound at zero on (nominal forward) interest rates as an option value is due to Black (1995). The more recent state-of-the-art term structure models in Krippner (2015b) and Wu and Xia (2016) have adopted this approach by implementing a lower bound on forward rates at a certain rate $\underline{r}$. As a consequence, observed interest rates are also bounded below by $\underline{r}$. The unobserved shadow rate can decrease below this lower bound. In such cases, the observed forward rate simply equals the lower bound:

$$f_{0,1,t}^{\mathrm{ZLB}} = \max\left(sr_t, 0\right) \tag{1}$$

The shadow rate at the end-of-month t is modeled by

$$sr_t = \delta_0 + X_{1,t} + X_{2,t}. \tag{2}$$

Since we impose additional restrictions, the model with two factors has too few parameters which results in unreported results into a linear term structure. This is in line with the findings in de Jong

[4]Krippner (2015b) contains an overview.

(2000) for the U.S. term structure. Therefore, we consider a model with three latent factors in X_t. Following the term structure literature, the third factor $X_{3,t}$ is the derivative of the second factor and does not enter (2) directly. The $\mathbb{P}$-dynamics are

$$X_t = \mu + \rho X_{t-1} + \Sigma e_t \qquad e_t \sim N(0, I) \tag{3}$$

where ρ is a 3 x 3 matrix and Σ a lower triangular Choleski 3 x 3 matrix. The unconditional mean of X_t is $\theta := E[X_t] = (I - \rho)^{-1}\mu$.

Following Wu and Xia (2016), the $\mathbb{Q}$-dynamics of X_t are determined by

$$X_t = \rho^{\mathbb{Q}} X_{t-1} + \Sigma e_t^{\mathbb{Q}} \qquad e_t^{\mathbb{Q}} \overset{\mathbb{Q}}{\sim} N(0, I) \tag{4}$$

and

$$\rho^{\mathbb{Q}} = \begin{pmatrix} \rho_1^{\mathbb{Q}} & & \\ & \rho_2^{\mathbb{Q}} & 1 \\ & & \rho_2^{\mathbb{Q}} \end{pmatrix}.$$

The absence of an intercept term implies $E^{\mathbb{Q}}[X_t] = [0\ 0]'$. The $\mathbb{Q}$-dynamics represent the risk-corrected beliefs of investors and thus determine the cross-sectional shadow term structure:

$$f_{n,n+1,t}^{sr} = a_n + b_n X_t \tag{5}$$

with

$$a_n = \delta_0 - \frac{\Delta}{2}\left(\sum_{j=0}^{n-1} b_j\right)\Sigma\Sigma^{'}\left(\sum_{j=0}^{n-1} b_j\right)^{'} \qquad b_n = \delta_1'\left(\rho^{\mathbb{Q}}\right)^n \tag{6}$$

$$\delta_1' = [1\ 1\ 0] \qquad \Delta = \frac{1}{12}.$$

The second term in a_n is the so-called volatility term in Diebold and Rudebusch (2013) and Krippner (2015b). The parameter Δ is the step size in years on a monthly grid. Note that $b_n X_t = \delta_1' E[X_{t+n}|X_t]$ such that

$$f_{n,n+1,t}^{sr} = E_t^{\mathbb{Q}}[sr_{t+n}]$$

and, in particular, $f_{0,1,t}^{sr} = a_0 + b_0 X_t = \delta_0 + X_{1,t} + X_{2,t} = sr_t$.

The shadow interest rate is the cumulative average of the shadow forward rates $f_{n,n+1,t}^{sr}$:

$$R_{n,t}^{sr} = \frac{1}{n}\sum_{i=0}^{n-1} f_{i,i+1,t}^{sr} = \bar{a}_n + \bar{b}_n X_t$$

where

$$\bar{a}_n = \frac{1}{n}\sum_{i=0}^{n-1} a_i \qquad \bar{b}_n = \frac{1}{n}\sum_{i=0}^{n-1} b_i. \tag{7}$$

Appendix A.1 derives explicit expressions for $\bar{a}_n$ and $\bar{b}_n$, provided $\delta_1' = [1\ 1\ 0]$. The forwards of the shadow interest rate determine the forwards of the ZLB term structure:

$$f_{n,n+1,t}^{\text{ZLB}} = \underline{r} + \sigma_n^{\mathbb{Q}}\ g\left(\frac{f_{n,n+1,t}^{sr} - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right)$$

with the lower bound $\underline{r}$, the function $g(z) := z\ \Phi(z) + f(z)$, $\Phi(z)$ the cdf of a Normal distribution, and

$$\left(\sigma_n^{\mathbb{Q}}\right)^2 := \text{Var}(sr_{t+n}) = \sum_{j=0}^{n-1} \delta_1'\left(\rho^{\mathbb{Q}}\right)^j \Sigma\Sigma'\left(\rho^{\mathbb{Q}'}\right)^j \delta_1 = \sum_{j=0}^{n-1} b_j \Sigma\Sigma' b_j'.$$

The nominal interest rate $R^{\text{ZLB}}_{n,t}$ is the cumulative average of the forward rates $f^{\text{ZLB}}_{0,1,t}, \ldots, f^{\text{ZLB}}_{n-1,n,t}$:

$$R^{\text{ZLB}}_{n,t} = \frac{1}{n}\sum_{i=0}^{n-1} f^{\text{ZLB}}_{i,i+1,t}$$

The appendix derives expressions for $f^{sr}_{n,t}$, $R^{sr}_{n,t}$, $f^{\text{ZLB}}_{n,t}$, and $R^{\text{ZLB}}_{n,t}$ in terms of the parameters for the case $\delta_1' = [1\ 1\ 0]$.

4 Methodology

This section describes the methodology for the estimation of the zero lower bound term structure model. Data sources are in Appendix B.

4.1 Dependent variable

Following Wu and Xia (2016) and Gürkaynak, Sack, and Wright (2007), the Svensson (1994) interpolation is used to estimate for each month t a parametric estimate of the forward curve of euroswap rates:

$$f_{n,n,t} = \beta_0 + \beta_1 \exp\left(-\frac{n}{t_1}\right) + \beta_2 \frac{n}{t_1}\exp\left(-\frac{n}{t_1}\right) + \beta_3 \frac{n}{t_2}\exp\left(-\frac{n}{t_2}\right)$$

This forward curve implies the euroswap curve

$$\begin{aligned} R^{\text{ZLB}}_{n,t} = \beta_0 + \beta_1 \frac{1-\exp\left(-n/t_1\right)}{n/t_1} + \beta_2\left[\frac{1-\exp\left(-n/t_1\right)}{n/t_1} - \exp\left(-\frac{n}{t_1}\right)\right] \\ + \beta_3\left[\frac{1-\exp\left(-n/t_2\right)}{n/t_2} - \exp\left(-\frac{n}{t_2}\right)\right]. \end{aligned}$$

We assume for the observed nominal interest rates

$$i_{n,t} = R^{\text{ZLB}}_{n,t} + \varepsilon_{n,t}$$

with measurement error variance $\text{Var}(\varepsilon_{n,t}) = \omega^2$ for each maturity n and each month t. The maximum likelihood maximizes the likelihood on the implied $\varepsilon_{n,t}$. Modeling forwards rates at short maturities is more important than those at longer maturities because incorrect short forward rates affect the whole term structure including long forward rates. Therefore, a proportional higher weight is assigned to the measurement error of the short forward rates. More specifically, the weight of the error that corresponds to the observed forward rate with maturity n_i equals the difference between the longest observed maturity and the average of the maturities n_{i-1} and n_i.

4.2 Restrictions

The lower bound of the forward rate is set at $\underline{r} = -0.25\%$. In our sample, the single observation that falls below this lower bound was the 1 month rate of -0.29% in January 2016. We explain this rate below $\underline{r}$ as a result of the measurement error ε. All other 5,405 observations exceed the lower bound $\underline{r} = -0.25\%$. Estimating the model without any restriction results in a steep slope, high interest rates $E\left[R^{\text{ZLB}}_{n,t}\right]$ for large maturities n, or even a diverging term structure.

A direct restriction on $E\left[R^{\text{ZLB}}_{n,t}\right]$ with n finite is computationally unattractive. Restrictions on $E\left[f^{\text{ZLB}}_{n,n+1,t}\right]$ and $E\left[R^{sr}_{n,t}\right]$ are also possible. However, an explicit expression is unavailable and the value of $E\left[R^{\text{ZLB}}_{n,t}\right]$ is still unclear from such restrictions. In contrast, the unconditional forward $E\left[f^{sr}_{n,n+1,t}\right]$ has an explicit expression in (5) since $E[X_t] = \theta$. This facilitates a relatively fast computation of the restrictions during the optimization procedure. Therefore, the restrictions are imposed on $E\left[f^{sr}_{n,n+1,t}\right]$. The restrictions are chosen in such a way that the unconditional term structure has a horizontal slope for large maturities with convergence towards a certain $E\left[R^{\text{ZLB}}_{\infty,t}\right]$.

The corresponding maximum likelihood estimate of the parameters leads to an overshooting of the shadow rate. The shadow rate (averaged over the simulations) starts at a negative rate that corresponds to the end-of-month rate of June 2016. During the first simulation years, this average shadow rate increases beyond the imposed long-run shadow rate $E[sr_t]$. After a number of years, the shadow rate decreases towards the imposed long-run value. This issue of overshooting of the expected shadow rate is resolved by setting $\rho_{31} = 0$ in (3). Then, the shadow rate converges gradually towards the long-run value.

The following six restrictions are imposed on the term structure:

$$\begin{array}{lll} E[sr_t] = E\left[f^{sr}_{0,1,t}\right] = 1.7\% & E\left[f^{sr}_{240,241,t}\right] = 2.5\% & E\left[f^{sr}_{720,721,t}\right] = 2.0\% \\ \sigma[sr_t] = 1.7\% & E\left[R^{\text{ZLB}}_{\infty,t}\right] = E\left[f^{\text{ZLB}}_{\infty,t}\right] = 3.2\% & \rho_{31} = 0 \end{array} \tag{8}$$

Appendix A derives how these restrictions can be written in explicit terms of the model parameters.

Four out of the six restrictions are on the level of the term structure. It is beyond the scope of this paper whether less restrictions, possibly imposed on other maturities, lead to a similar unconditional term structure as obtained in this paper.

Given δ_1 and $\rho^{\mathbb{Q}}$, the parameter δ_0 is determined by the restriction $E\left[R^{\text{ZLB}}_{\infty,t}\right] = 3.2\%$ (see (15) in Appendix A.3). The three parameters in μ fix the three restrictions on $E\left[f^{sr}_{i,t}\right]$. The Cholesky matrix Σ is scaled such that the restriction on $\sigma[sr_t]$ holds. The restriction $\rho_{31} = 0$ sets ρ_{31} directly. In total, the six restrictions determine the value of six parameters.

4.3 Parameters

The model contains the following 22 parameters:

1. 8 parameters for the 3×3 matrix ρ with $\rho_{31} = 0$
2. 3 parameters for the 3×1 vector θ
3. 2 parameters for the 3×3 matrix $\rho^{\mathbb{Q}}$
4. 6 parameters for the lower triangular 3×3 matrix Σ,
   1 scaling parameter and 5 normalized parameters
5. 1 parameter for the intercept δ_0
6. 1 parameter for the measurement error ω

Rather than estimating the intercept μ in (3), we estimate the unconditional mean of X, $\theta := (I - \rho)^{-1}\mu$. Taking into account the six restrictions in (8) leaves a total of $21 - 6 = 15$ free parameters in the model. The free parameters are simultaneously estimated.

5 Estimation results term structure model

5.1 Baseline results

The maximum likelihood estimates of the baseline model are in Table 1. The parameters ρ_{32} is insignificant. Setting this parameters to zero has a negligible impact on the simulation. We keep both parameters at the estimated nonzero values. Parameter $\Sigma_{11} = 0.3707$ is the scaling parameter of Σ. More formally, $\Sigma = 0.3707\Sigma_0$ where Σ_0 is a lower triangular matrix with a one at the left top entry and the other nonzero entries as in Table 1.

Table 1: **Parameter estimates.**
Standard deviations are determined with the outer product of gradients (OPG) estimator. The six parameters fixed by the six restrictions in (8) have no standard deviation.

		estimate	std.err	t
ρ	(1,1)	0.9972	$6.82\cdot10^{-3}$	146.3
	(2,1)	-0.02857	$6.48\cdot10^{-3}$	-4.410
	(3,1)	0		
	(1,2)	0.080843	$1.04\cdot10^{-2}$	7.786
	(2,2)	0.8877	$1.23\cdot10^{-2}$	71.92
	(3,2)	$-8.9024\cdot10^{-4}$	$1.96\cdot10^{-4}$	-4.543
	(1,3)	0.4940	0.415	1.191
	(2,3)	$1.14\cdot10^{-13}$	$3.08\cdot10^{-5}$	$3.71\cdot10^{-9}$
	(3,3)	0.9492	$1.38\cdot10^{-2}$	68.74
θ	(1)	-18.486		
	(2)	4.4428		
	(3)	0.03488		
$\log\left(1-\rho^{\mathbb{Q}}\right)$	(1)	-6.365	$4.24\cdot10^{-2}$	-150.1
	(2)	-4.697	0.011514	-408.0
Σ_{11}	(1,1)	0.3707		
Σ_0	(2,1)	-0.6079	0.081538	-7.456
	(3,1)	-0.0087	0.001122	-7.753
	(2,2)	1.1398	0.085746	13.29
	(3,2)	-0.01103	0.002083	-5.295
	(3,3)	-0.02497	0.001776	-14.057
δ_0		15.729		
ω		0.1842	0.000802	229.8

The matrices and vectors are

$$\rho = \begin{pmatrix} 0.9972 & 0.08084 & 0.4940 \\ -0.02857 & 0.8877 & 1.1422\cdot10^{-13} \\ 0 & -8.9024\cdot10^{-4} & 0.9492 \end{pmatrix} \qquad \theta = \begin{pmatrix} -18.49 \\ 4.443 \\ 0.03488 \end{pmatrix}$$

$$\rho^{\mathbb{Q}} = \begin{pmatrix} 0.99828 & & \\ & 0.99088 & 1 \\ & & 0.99088 \end{pmatrix} \qquad \delta_0 = 15.73$$

$$\Sigma = \begin{pmatrix} 0.3707 & & \\ -0.2254 & 0.4225 & \\ -3.2261\cdot10^{-3} & -4.089\cdot10^{-3} & 9.2570\cdot10^{-3} \end{pmatrix} \qquad \omega = 0.1842.$$

Figure 1 shows the results of 5,000 simulations at a simulation horizon of 150 years. A peak near the lower bound $\underline{r} = -0.25\%$ is desirable. However, the left plot in the second row in Figure 1 does not indicate a binding lower bound on the interest rate. To resolve this issue, the standard deviation of the shadow rate is scaled by a factor $c_\sigma^{\mathbb{Q}}$:

$$\left(\sigma_n^{\mathbb{Q}}\right)^2 := \mathrm{Var}(sr_{t+n}) = \left(c_\sigma^{\mathbb{Q}}\right)^2 \sum_{j=0}^{n-1} b_j \Sigma\Sigma' b_j'$$

We calibrate $c_\sigma^{\mathbb{Q}} = 0.7$ to generate a more persistent future shadow rate $E^{\mathbb{Q}}[sr_{t+n}]$ in the $\mathbb{Q}$-world. In the $\mathbb{P}$-world the persistence of sr_t is solely determined by Σ, not $c_\sigma^{\mathbb{Q}}$. As a consequence, the variance $\mathrm{Var}\left(R_{n,t}^{sr}\right)$ of the shadow term structure increases, which translates into an increase in the variance $\mathrm{Var}\left(R_{n,t}^{\mathrm{ZLB}}\right)$.

Figure 1: **5,000 simulations with parameters as given in Table 1.**
The plots are at 150 years of simulation, except for top left plot.

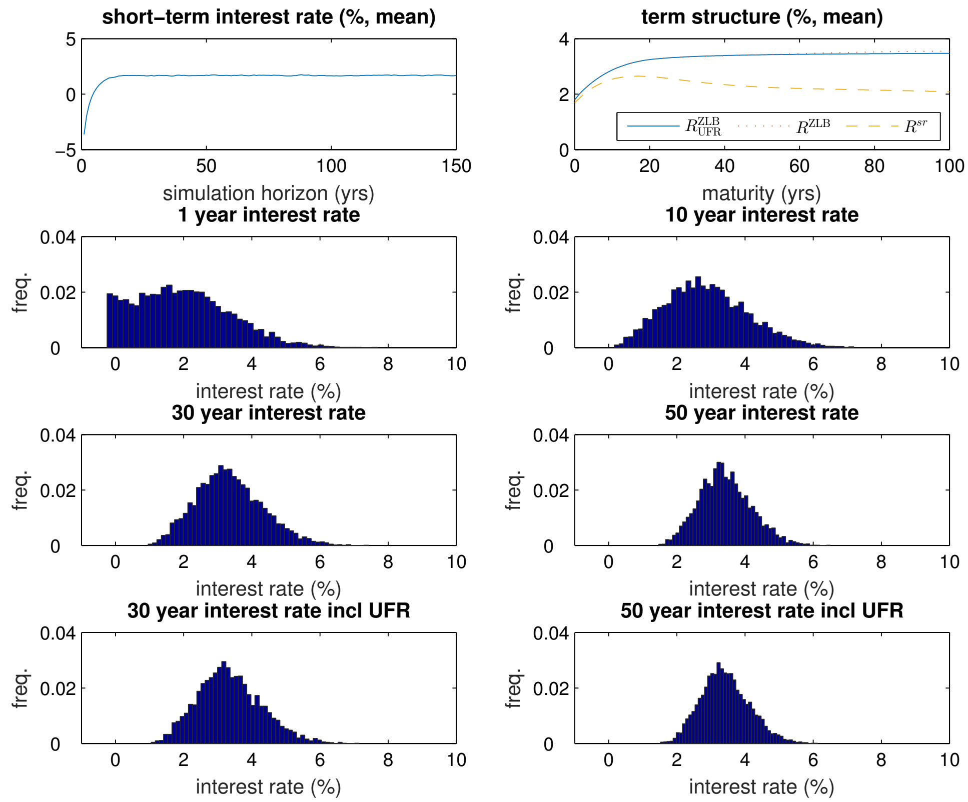


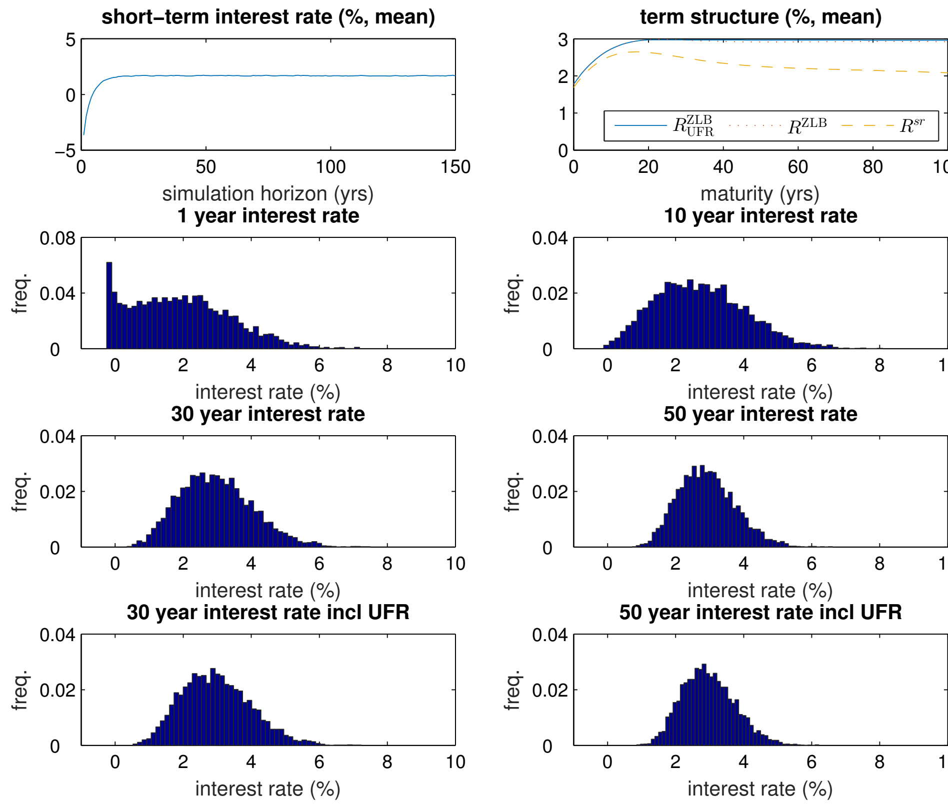
Figure 2 plots the results with $c_\sigma^{\mathbb{Q}} = 0.7$. The probability on a negative interest rate has increased from 5.3% to 7.7%. The factor $c_\sigma^{\mathbb{Q}} = 0.7$ also results in a downward shift of the ZLB term structure because the upside benefit of the ZLB decreases by the higher persistence of the shadow rate in the $\mathbb{Q}$-world. Table 2 reports the unconditional interest rates at several maturities.

The figures are not very different when including the effect of an Ultimate Forward Rate (UFR). First, the interest rates below 20 years are unaffected. Second, the mean term structure $R^{\mathrm{ZLB}}_{\mathrm{UFR}}$ and R^{ZLB} in the top right plot in Figure 2 are almost identical. Third, the two bottom rows of Figure 2 are not very different from each other because on average the UFR equals the prevailing 20 year forward rate. Thus, the UFR may sometimes lift longterm interest rates but in the long-run it is equally likely that the UFR brings longterm interest rates down.

5.2 Parameter uncertainty

From a Bayesian point of view, we can simulate the parameters from the corresponding asymptotic distribution. For each simulated set, the same six parameters are pinned down by the six restrictions in (8). Figure 3 indicates that the simulation results are qualitatively the same as in Figure 2.

Figure 2: **5,000 simulations with parameters as given in Table 1 and $c_\sigma = 0.7$.**
The plots are at 150 years of simulation, except for top left plot.



6 Other macroeconomic variables

The previous section considered the term structure. This section considers two other essential macro-economic variables for pension policy: the Dutch inflation rate and the MSCI World Index.

6.1 Inflation rate

In a purely empirical approach, a regression equation is estimated to model the association between the inflation rate π and the interest rate i. The following equation turns out to capture the dependence between the inflation rate and the interest rate (standard errors in parentheses[5]):

$$\widetilde{\pi}_t = \underset{(1.31\cdot 10^{-2})}{-1.28\cdot 10^{-2}}\,\tilde{i}_t^2 + \underset{(0.086)}{0.303}\,\tilde{i}_t + \xi_t \tag{9}$$

$$\xi_t = \underset{(0.0180)}{0.931}\,\xi_{t-1} + e_t \qquad e_t \sim \left(0, 0.230^2\right) \tag{10}$$

where $\widetilde{\pi}_t$ and $\tilde{i}_t$ is the deviation from the long-run mean $\bar{\pi}$= 1.7% and $\bar{i}$ = 1.8%, respectively. In unreported results, other explanatory variables are insignificant while other model setups did not improve upon the simple setup in (10).

[5]Equation (9) contains OPG standard errors, equation (10) contains HAC standard errors for the maximum likelihood parameters from (9).

Table 2: **Mean interest rate of several maturities.**
The superscript sr indicates the shadow rate, the superscript ZLB indicates the observed interest rate including the zero lower bound effect.

mat. (yrs)	n (mths)	$E\left[f^{sr}_{n,n+1,t}\right]$ (%)	$E\left[R^{sr}_{n,t}\right]$ (%)	$E\left[f^{\text{ZLB}}_{n,n+1,t}\right]$ (%)	$E\left[R^{\text{ZLB}}_{n,t}\right]$ (%)
0	0	1.69	1.69	1.69	1.69
1	12	1.97	1.83	1.98	1.83
2	24	2.20	1.96	2.22	1.97
3	36	2.39	2.07	2.42	2.08
4	48	2.55	2.17	2.59	2.19
5	60	2.68	2.26	2.73	2.28
10	120	2.95	2.56	3.12	2.62
20	240	2.47	2.66	3.02	2.87
30	360	1.99	2.50	2.80	2.88
40	480	1.87	2.36	2.80	2.86
50	600	1.93	2.26	2.88	2.85
60	720	1.98	2.21	2.94	2.86
∞	∞	0.93	0.93	2.34	2.34

Simulations show that the probability on a negative inflation rate is only 0.2%. To increase the probability on a negative interest rate we do two things:

1. the two parameters in (9) are scaled by $c_\pi = 1.2$.
2. the standard deviation of e_t is scaled by 0.7.

This gives the following specification

$$\widetilde{\pi}_t = -\underset{(1.58\cdot 10^{-2})}{1.53 \cdot 10^{-2}}\, \tilde{i}_t^2 + \underset{(0.103)}{0.363}\, \tilde{i}_t + \xi_t \tag{11}$$

$$\xi_t = 0.931\, \xi_{t-1} + e_t \qquad e_t \sim \left(0, 0.161^2\right) \tag{12}$$

Figure 4 shows the observed and estimated relation between i and π, with both choices for c_π. We do not model Dutch contractual wages separately because this series is very similar to the cumulated Dutch inflation rate (Figure 5).

6.2 MSCI World Index

The GARCH(1,1) model is the workhorse model in the literature for the time series of equity returns (Bollerslev (1986); Bollerslev et al. (1992)). Again, we impose a restriction to ensure that previously high equity returns do not result in a relatively high unconditional mean equity return. Therefore, an annual equity premium of 4% above the average shadow rate of 1.7% is imposed on the monthly (log) return on the MSCI World Index y_t:

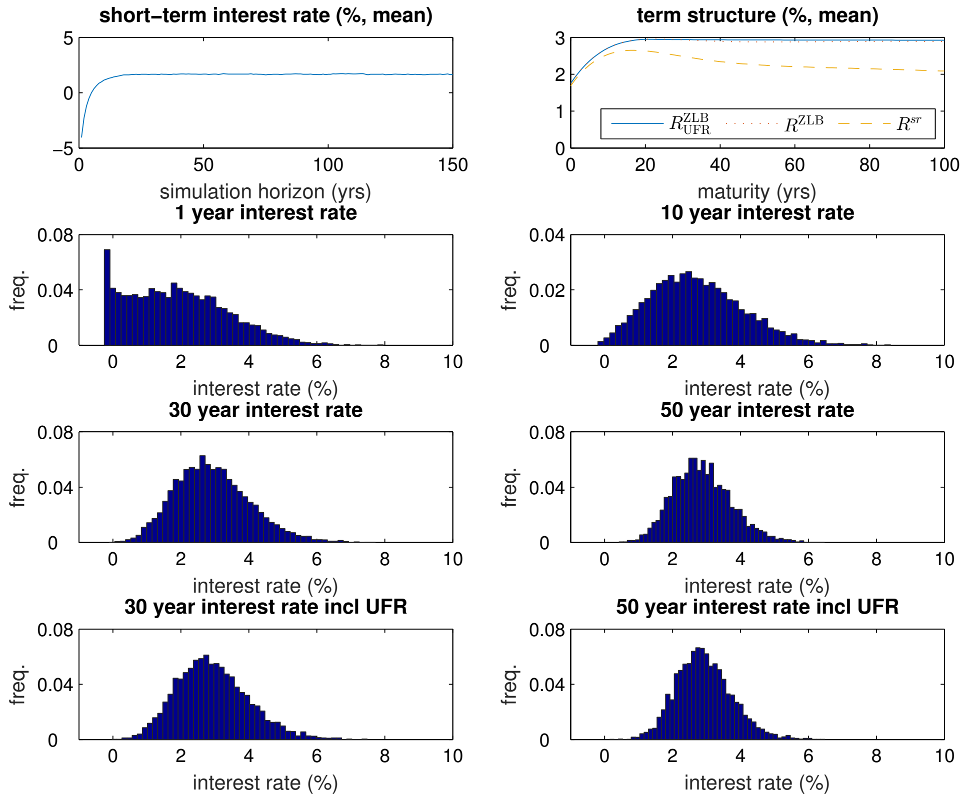
$$\widetilde{y}_t = y_t - \frac{1}{12}\,(1.7\% + 4\%)$$

The following GARCH(1,1) model is estimated for $\widetilde{y}_t$:

$$\widetilde{y}_t = 0.154\, \widetilde{y}_{t-1} + \text{res}_t \qquad \text{res}_t \sim (0, \sigma_t^2) \tag{13}$$

$$\sigma_t^2 = 0.815 + 0.812\, \sigma_{t-1}^2 + 0.146\, \text{res}_{t-1}^2 \tag{14}$$

Figure 3: **5,000 simulations with fixed parameters from Table 1 and $c_\sigma^{\mathbb{Q}} = 0.7$.**
The plots are at 150 years of simulation, except for top left plot.



In unreported results, alternative explanatory variables in the observation equation (13) as well as the volatility equation (14) are either insignificant, or have an unintuitive sign.[6]

6.3 Simulation results

Table 3 reports some observed data characteristics with fixed parameters, while Table 4 reports the same characteristics for simulated data with fixed parameters. Taking account of the imposed restrictions, the difference between Table 3 and Table 4 is relatively small. The correlation of the MSCI return is nonzero due to a finite sample effect. The correlation of multiple years of simulations is close to zero, as can be expected from (13) and (14).

From a Bayesian point of view, parameter uncertainty may produce additional uncertainty in the dynamics. To evaluate this source of uncertainty, we simulate the parameters in (11), (13), and (14) from the covariance matrix of the parameters. We obtain for each draw of (11) the maximum likelihood parameters in (12). Table 5 shows that the data characteristics hardly change when parameter uncertainty is included.

[6]The inflation rate had a negative impact on stock returns. We believe that this negative correlation is not representative for the future, because it is counterintuitive if dividend yields are positively correlated with the inflation rate.

Figure 4: **One year interest rate i and inflation rate π.**
The dotted line $c_\pi = 1$ is the least squares quadratic relation between i and π. The solid line $c_\pi = 1.2$ is the rotated relation that increases the incidence of deflation in the simulation. The sample period is from January 1990 to June 2016.

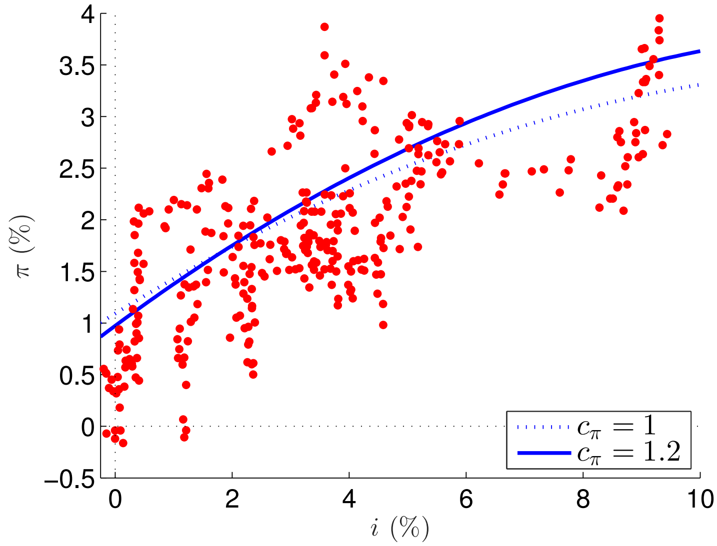


Figure 5: **Price index (Π_{price}, solid) and wage index (Π_{wage}, dotted).**

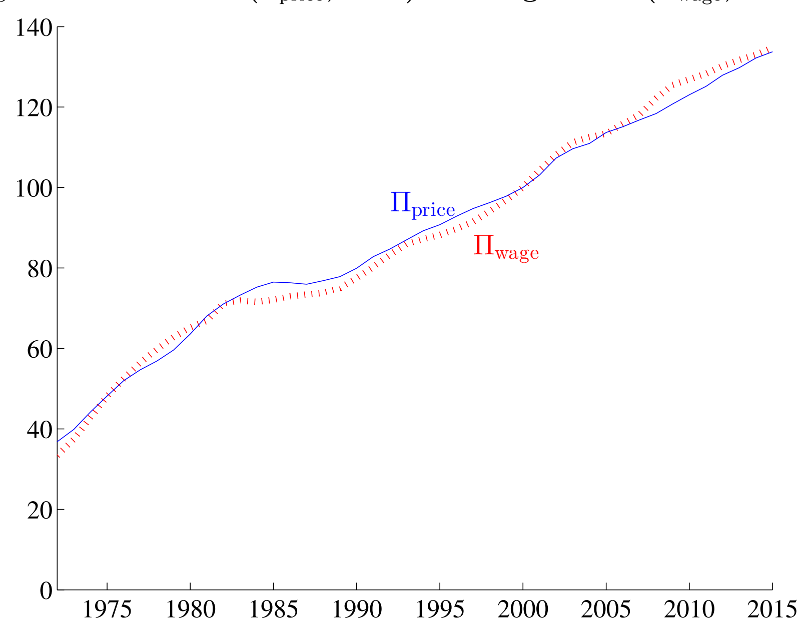


Table 3: **Observed annual correlations, means, and standard deviations**

		real sr	nom. sr	real 1 yr	nom. 1 yr	CPI afg	MSCI return
real sr	$R_0^{\mathrm{ZLB}}-\pi$	1	0.98	0.91	0.92	0.55	-0.15
nominal sr	R_0^{ZLB}	0.98	1	0.89	0.94	0.71	-0.19
real 1 yr	$R_1^{\mathrm{ZLB}}-\pi$	0.91	0.89	1	0.96	0.48	-0.13
nom.1 yr	R_1^{ZLB}	0.92	0.94	0.96	1	0.70	-0.18
CPI afg	π	0.55	0.71	0.48	0.70	1	-0.24
MSCI return	y	-0.15	-0.19	-0.13	-0.18	-0.24	1
mean (%)		0.71	2.55	1.56	3.40	1.84	6.53
st.dev. (%)		2.75	3.24	2.01	2.48	0.78	20.70

Table 4: **Simulation results, fixed parameters**

Correlations are at 150 years of 2,000 simulations.

		real sr	nom. sr	real 1 yr	nom. 1 yr	CPI afg	MSCI return
real sr	$R_0^{\mathrm{ZLB}}-\pi$	1	0.90	0.98	0.89	0.37	0.01
nominal sr	R_0^{ZLB}	0.90	1	0.88	0.99	0.73	0.03
real 1 yr	$R_1^{\mathrm{ZLB}}-\pi$	0.98	0.88	1	0.89	0.35	0.02
nom.1 yr	R_1^{ZLB}	0.89	0.99	0.89	1	0.74	0.03
CPI afg	π	0.37	0.73	0.35	0.74	1	0.05
MSCI return	y	0.01	0.03	0.02	0.03	0.05	1
mean (%)		0.16	1.87	0.30	2.01	1.71	7.62
st.dev. (%)		1.15	1.57	1.06	1.48	0.72	20.57

Appendix

A Derivations

Explicit expressions are derived for the case $\delta_1' = [1\ 1\ 0]$. First, some basic results are derived for geometric series. Use

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{i=0}^{n} i r^{i-1} = \frac{nr^{n+1}-(n+1)\,r^n+1}{(1-r)^2} = \frac{1-r^n}{(1-r)^2} - \frac{nr^n}{1-r}$$

$$\sum_{i=0}^{n} i\,(i-1)\,r^{i-2} = \frac{2\left(1-r^{n+1}\right)}{(1-r)^3} - \frac{2\,(n+1)\,r^n}{(1-r)^2} - \frac{(1+n)\,nr^{n-1}}{1-r}$$

$$\begin{aligned}\sum_{i=0}^{n} i^2 r^{i-1} &= r\sum_{i=0}^{n} i^2 r^{i-2} \\ &= r\sum_{i=0}^{n} i\,(i-1)\,r^{i-2} + \sum_{i=0}^{n} t r^{i-1} \\ &= \frac{2\left(r-r^{n+2}\right)}{(1-r)^3} - \frac{2\,(n+1)\,r^{n+1}-1+r^n}{(1-r)^2} - \frac{(2+n)\,nr^n}{1-r}\end{aligned}$$

Table 5: **Simulation results, parameter uncertainty**

Correlations are at 150 years of 2,000 simulations.

		real sr	nom. sr	real 1 yr	nom. 1 yr	CPI afg	MSCI return
real sr	$R_0^{\text{ZLB}} - \pi$	1	0.93	0.96	0.89	0.41	0.01
nominal sr	R_0^{ZLB}	0.93	1	0.88	0.98	0.73	0.03
real 1 yr	$R_1^{\text{ZLB}} - \pi$	0.96	0.88	1	0.90	0.37	0.01
nom.1 yr	R_1^{ZLB}	0.89	0.98	0.90	1	0.73	0.03
CPI afg	π	0.41	0.73	0.37	0.73	1	0.05
MSCI return	y	0.01	0.03	0.01	0.03	0.05	1
mean (%)		0.05	1.74	0.15	1.84	1.69	5.76
st.dev. (%)		1.29	1.71	1.12	1.54	0.71	17.52

and define

$$\begin{aligned} f_0(x;n) &= \sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x} \\ f_1(x;n) &= \sum_{i=0}^{n-1} i\ x^{i-1} = \frac{1-x^{n-1}}{(1-x)^2} - \frac{(n-1)\,x^{n-1}}{1-x} \\ f_2(x;n) &= \sum_{i=0}^{n-1} i^2 x^{i-1} = \frac{2x(1-x^n)}{(1-x)^3} - \frac{2nx^n - 1 + x^{n-1}}{(1-x)^2} - \frac{\left(1-n^2\right) x^{n-1}}{1-x} \\ g(x,y;n) &= \sum_{i=0}^{n-1} \left(1-x^i\right)\left(1-y^i\right) \\ &= \sum_{i=0}^{n-1} \left[1 - x^i - y^i + (xy)^i\right] \\ &= n - f_0(x;n) - f_0(y;n) + f_0(xy;n)\,. \end{aligned}$$

It can be easily derived that for $n \to \infty$,

$$f_0(x;\infty) = \frac{1}{1-x} \qquad f_1(x;\infty) = \frac{1}{(1-x)^2} \qquad f_2(x;\infty) = \frac{1+x}{(1-x)^3}.$$

A.1 Level of interest rates

In this section, expressions are derived for the following interest rates,

$$E\left[f^{sr}_{n,n+1,t}\right] = a_n + b_n\theta \qquad E\left[R^{sr}_{n,t}\right] = \bar{a}_n + \bar{b}_n\theta$$

with $\theta = (I-\rho)^{-1}\mu$. The ZLB forward rate is given by:

$$f^{\text{ZLB}}_{n,n+1,t} = \underline{r} + \left(f^{sr}_{n,n+1,t} - \underline{r}\right)\Phi\left(\frac{f^{sr}_{n,n+1,t} - \underline{r}}{\sigma^{\mathbb{Q}}_n}\right) + \sigma^{\mathbb{Q}}_n f\left(\frac{f^{sr}_{n,n+1,t} - \underline{r}}{\sigma^{\mathbb{Q}}_n}\right)$$

where $\left(\sigma^{\mathbb{Q}}_n\right)^2 := \text{Var}(sr_{t+n}) = \sum_{j=0}^{n-1} b_j' \Sigma\Sigma' b_j$. This means that an explicit expression is not available for the mean and median of the ZLB interest rates $R^{\text{ZLB}}_{n,t}$ with n finite.

$\underline{E\left[f^{sr}_{n,n+1,t}\right] = a_n + b_n\theta:}$

Note first that

$$b_n = \delta_1' \left(\rho^{\mathbb{Q}}\right)^n = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} \rho_1^n & & \\ & \rho_2^n & n\rho_2^{n-1} \\ & & \rho_2^n \end{pmatrix} = \begin{bmatrix} \rho_1^n & \rho_2^n & n\rho_2^{n-1} \end{bmatrix}.$$

Substituting

$$\begin{aligned}
\sum_{j=0}^{n-1} b_j &= \delta_1' \sum_{j=0}^{n-1} \left(\rho^{\mathbb{Q}}\right)^j \\
&= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \left(I - \left(\rho^{\mathbb{Q}}\right)^n\right) \left(I - \rho^{\mathbb{Q}}\right)^{-1} \\
&= \begin{bmatrix} 1-\rho_1^n & 1-\rho_2^n & -n\rho_2^{n-1} \end{bmatrix} \begin{pmatrix} \frac{1}{1-\rho_1} & & \\ & \frac{1}{1-\rho_2} & \frac{1}{(1-\rho_2)^2} \\ & & \frac{1}{1-\rho_2} \end{pmatrix} \\
&= \begin{bmatrix} \frac{1-\rho_1^n}{1-\rho_1} & \frac{1-\rho_2^n}{1-\rho_2} & \frac{1-\rho_2^n}{(1-\rho_2)^2} - \frac{n\rho_2^{n-1}}{1-\rho_2} \end{bmatrix}
\end{aligned}$$

into the definition of a_n in (6) gives

$$\begin{aligned}
a_n &= \delta_0 - \frac{\Delta}{2} \begin{pmatrix} \frac{1-\rho_1^n}{1-\rho_1} & \frac{1-\rho_2^n}{1-\rho_2} & \frac{1-\rho_2^n}{(1-\rho_2)^2} - \frac{n\rho_2^{n-1}}{1-\rho_2} \end{pmatrix} \Sigma\Sigma' \begin{pmatrix} \frac{1-\rho_1^n}{1-\rho_1} \\ \frac{1-\rho_2^n}{1-\rho_2} \\ \frac{1-\rho_2^n}{(1-\rho_2)^2} - \frac{n\rho_2^{n-1}}{1-\rho_2} \end{pmatrix} \\
&= \delta_0 - \frac{\Delta}{2} \mathrm{vec}\left(\begin{pmatrix} \frac{1-\rho_1^n}{1-\rho_1} \\ \frac{1-\rho_2^n}{1-\rho_2} \\ \frac{1-\rho_2^n}{(1-\rho_2)^2} - \frac{n\rho_2^{n-1}}{1-\rho_2} \end{pmatrix} \begin{pmatrix} \frac{1-\rho_1^n}{1-\rho_1} & \frac{1-\rho_2^n}{1-\rho_2} & \frac{1-\rho_2^n}{(1-\rho_2)^2} - \frac{n\rho_2^{n-1}}{1-\rho_2} \end{pmatrix} \right)' \mathrm{vec}\left(\Sigma\Sigma'\right) \\
&= \delta_0 - \frac{\Delta}{2} \begin{pmatrix} \left(\frac{1-\rho_1^n}{1-\rho_1}\right)^2 \\ \frac{2(1-\rho_1^n)}{1-\rho_1} \frac{1-\rho_2^n}{1-\rho_2} \\ \frac{2(1-\rho_1^n)}{1-\rho_1} \left(\frac{1-\rho_2^n}{(1-\rho_2)^2} - \frac{n\rho_2^{n-1}}{1-\rho_2}\right) \\ \left(\frac{1-\rho_2^n}{1-\rho_2}\right)^2 \\ \frac{2(1-\rho_2^n)}{1-\rho_2} \left(\frac{1-\rho_2^n}{(1-\rho_2)^2} - \frac{n\rho_2^{n-1}}{1-\rho_2}\right) \\ \left(\frac{1-\rho_2^n}{(1-\rho_2)^2} - \frac{n\rho_2^{n-1}}{1-\rho_2}\right)^2 \end{pmatrix}' \begin{pmatrix} s_{11} \\ s_{12} \\ s_{13} \\ s_{22} \\ s_{23} \\ s_{33} \end{pmatrix}.
\end{aligned}$$

$\underline{E\left[R^{sr}_{n,t}\right] = \bar{a}_n + \bar{b}_n\theta:}$

$$\bar{a}_n = \frac{1}{n} \sum_{i=0}^{n-1} a_i$$

$$
= \delta_0 - \frac{\Delta}{2n}\sum_{i=0}^{n-1}\begin{pmatrix} \left(\frac{1-\rho_1^i}{1-\rho_1}\right)^2 \\ \frac{2\left(1-\rho_1^i\right)\left(1-\rho_2^i\right)}{(1-\rho_1)(1-\rho_2)} \\ \frac{2\left(1-\rho_1^i\right)}{1-\rho_1}\left(\frac{1-\rho_2^i}{(1-\rho_2)^2} - \frac{i\rho_2^{i-1}}{1-\rho_2}\right) \\ \left(\frac{1-\rho_2^i}{1-\rho_2}\right)^2 \\ \frac{2\left(1-\rho_2^i\right)}{1-\rho_2}\left(\frac{1-\rho_2^i}{(1-\rho_2)^2} - \frac{i\rho_2^{i-1}}{1-\rho_2}\right) \\ \frac{1}{(1-\rho_2)^2}\left[\left(\frac{1-\rho_2^i}{1-\rho_2}\right)^2 - \frac{2i\ \rho_2^{i-1}\left(1-\rho_2^i\right)}{1-\rho_2} + i^2\ \rho_2^{2i-2}\right] \end{pmatrix}' \begin{pmatrix} s_{11} \\ s_{12} \\ s_{13} \\ s_{22} \\ s_{23} \\ s_{33} \end{pmatrix}
$$

$$
= \delta_0 - \frac{\Delta}{2n}\begin{pmatrix} \frac{g(\rho_1,\rho_1;n)}{(1-\rho_1)^2} \\ \frac{2g(\rho_1,\rho_2;n)}{(1-\rho_1)(1-\rho_2)} \\ \frac{2}{(1-\rho_1)(1-\rho_2)}\left(\frac{g(\rho_1,\rho_2;n)}{1-\rho_2} - f_1(\rho_2;n) + \rho_1 f_1(\rho_1\rho_2;n)\right) \\ \frac{g(\rho_2,\rho_2;n)}{(1-\rho_2)^2} \\ \frac{2}{(1-\rho_2)^2}\left[\frac{g(\rho_2,\rho_2;n)}{1-\rho_2} - f_1(\rho_2;n) + \rho_2 f_1\left(\rho_2^2;n\right)\right] \\ \frac{1}{(1-\rho_2)^2}\left[\frac{g(\rho_2,\rho_2;n)}{(1-\rho_2)^2} - \frac{2f_1(\rho_2;n)-2\rho_2 f_1\left(\rho_2^2;n\right)}{1-\rho_2} + f_2\left(\rho_2^2;n\right)\right] \end{pmatrix}' \begin{pmatrix} s_{11} \\ s_{12} \\ s_{13} \\ s_{22} \\ s_{23} \\ s_{33} \end{pmatrix}
$$

$$
\begin{aligned}
\bar{b}_n &= \frac{1}{n}\sum_{i=0}^{n-1} b_i \\
&= \frac{1}{n}\left[\ \sum_{i=0}^{n-1}\rho_1^i \quad \sum_{i=0}^{n-1}\rho_2^i \quad \sum_{i=0}^{n-1} i\rho_2^{i-1}\ \right] \\
&= \frac{1}{n}\left[\ f_0(\rho_1;n) \quad f_0(\rho_2;n) \quad f_1(\rho_2;n)\ \right]
\end{aligned}
$$

where the functions f_0, f_1, f_2, and g are as defined before in Appendix A.

A.2 Variance of the shadow rate

The covariance matrix of the latent variables is (see Koijen, Nijman, and Werker (2010))

$$
\begin{aligned}
\lim_{t\to\infty}\mathrm{Var}(X_t) &= \sum_{i=0}^{\infty}\rho^i\Sigma\Sigma'{\rho'}^i \\
&= \sum_{i=0}^{\infty} V_\rho D_\rho^i V_\rho^{-1}\Sigma\Sigma' V_\rho^{,-1} D_\rho^i V_\rho' \\
&= V_\rho\left(V_\rho^{-1}\Sigma\Sigma' V_\rho'^{-1}\star M\right)V_\rho'
\end{aligned}
$$

where $\star$ is the matrix multiplication by element, and

$$
\rho = V_\rho D_\rho V_\rho^{-1} \qquad\qquad M_{ij} = \frac{1}{1-d_i d_j}.
$$

The variance of the shadow forward rate is ($B_n = b_n' b_n$)

$$
\begin{aligned}
\mathrm{Var}\left(f_{n,n+1,t}^{sr}\right) &= b_n\ \mathrm{Var}\left(X_t\right)\ b_n' \\
&= \mathbf{1}'\left[V_\rho\left(V_\rho^{-1}\Sigma\Sigma' V_\rho'^{-1}\star M\right)V_\rho'\star B_n\right]\mathbf{1}.
\end{aligned}
$$

The variance of the shadow interest rate is

$$
\mathrm{Var}\left(R_{n,t}^{sr}\right) = \mathrm{Var}\left(\frac{1}{n}\sum_{n=0}^{n-1} f_{n,n+1,t}^{sr}\right)
$$

$$= \frac{1}{n^2} \mathbf{1}' \left(V_\rho \left(V_\rho^{-1} \Sigma \Sigma' V_\rho'^{-1} \star M \right) V_\rho' \star \bar{B}_n \right) \mathbf{1}$$

where

$$\begin{aligned}
\bar{B}_n &= \sum_{j=0}^{n-1} B_j \\
&= \sum_{j=0}^{n-1} \begin{pmatrix} \rho_1^{2j} & \rho_1^j \rho_2^j & j \rho_1^j \rho_2^{j-1} \\ \rho_1^j \rho_2^j & \rho_2^{2j} & j \rho_2^{2j-1} \\ j \rho_1^j \rho_2^{j-1} & j \rho_2^{2j-1} & j^2 \rho_2^{2j-2} \end{pmatrix} \\
&= \begin{pmatrix} f_0\left(\rho_1^2; n\right) & f_0(\rho_1 \rho_2; n) & \rho_1 f_1(\rho_1 \rho_2; n) \\ f_0(\rho_1 \rho_2; n) & f_0\left(\rho_2^2; n\right) & \rho_2 f_1\left(\rho_2^2; n\right) \\ \rho_1 f_1(\rho_1 \rho_2; n) & \rho_2 f_1\left(\rho_2^2; n\right) & f_2\left(\rho_2^2; n\right) \end{pmatrix}.
\end{aligned}$$

It is most convenient to derive another, more explicit, expression for $\mathrm{Var}\left(X_t\right)$ (Muns, 2015a): $\mathrm{vec}(\mathrm{Var}(X_t)) = (I - \rho \otimes \rho)^{-1} \mathrm{vec}\left(\Sigma \Sigma'\right)$

$$= \begin{pmatrix}
\frac{1}{1-\rho_1^2} & & & & & & & & \\
& \frac{1}{1-\rho_1\rho_2} & \frac{\rho_1}{(1-\rho_1\rho_2)^2} & & & & & & \\
& & \frac{1}{1-\rho_1\rho_2} & & & & & & \\
& & & \frac{1}{1-\rho_1\rho_2} & & & \frac{\rho_1}{(1-\rho_1\rho_2)^2} & & \\
& & & & \frac{1}{1-\rho_2^2} & \frac{\rho_2}{\left(1-\rho_2^2\right)^2} & & \frac{\rho_2}{\left(1-\rho_2^2\right)^2} & \frac{1+\rho_2^2}{\left(1-\rho_2^2\right)^3} \\
& & & & & \frac{1}{1-\rho_2^2} & & & \frac{\rho_2}{\left(1-\rho_2^2\right)^2} \\
& & & & & & \frac{1}{1-\rho_1\rho_2} & & \\
& & & & & & & \frac{1}{1-\rho_2^2} & \frac{\rho_2}{\left(1-\rho_2^2\right)^2} \\
& & & & & & & & \frac{1}{1-\rho_2^2}
\end{pmatrix}
\begin{pmatrix} s_{11} \\ s_{12} \\ s_{13} \\ s_{12} \\ s_{22} \\ s_{23} \\ s_{13} \\ s_{23} \\ s_{33} \end{pmatrix}.$$

This gives

$$\mathrm{Var}(X_t) = \begin{pmatrix}
\frac{s_{11}}{1-\rho_1^2} & \frac{s_{12}}{1-\rho_1\rho_2} + \frac{\rho_1 s_{13}}{(1-\rho_1\rho_2)^2} & \frac{s_{13}}{1-\rho_1\rho_2} \\
\frac{s_{12}}{1-\rho_1\rho_2} + \frac{\rho_1 s_{13}}{(1-\rho_1\rho_2)^2} & \frac{s_{22}}{1-\rho_2^2} + \frac{2\rho_2 s_{23}}{\left(1-\rho_2^2\right)^2} + \frac{1+\rho_2^2}{\left(1-\rho_2^2\right)^3} s_{33} & \frac{s_{23}}{1-\rho_2^2} + \frac{\rho_2 s_{33}}{\left(1-\rho_2^2\right)^2} \\
\frac{s_{13}}{1-\rho_1\rho_2} & \frac{s_{23}}{1-\rho_2^2} + \frac{\rho_2 s_{33}}{\left(1-\rho_2^2\right)^2} & \frac{s_{33}}{1-\rho_2^2}
\end{pmatrix}.$$

The variance of the shadow rate is

$$\begin{aligned}
\mathrm{Var}(sr_t) &= \mathrm{Var}\left(f_{0,1,t}^{sr}\right) \\
&= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathrm{Var}(X_t) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \frac{s_{11}}{1-\rho_1^2} + \frac{2 s_{12}}{1-\rho_1\rho_2} + \frac{2\rho_1 s_{13}}{(1-\rho_1\rho_2)^2} + \frac{s_{22}}{1-\rho_2^2} + \frac{2\rho_2 s_{23}}{\left(1-\rho_2^2\right)^2} + \frac{\left(1+\rho_2^2\right) s_{33}}{\left(1-\rho_2^2\right)^3}.
\end{aligned}$$

A.3 Asymptotically longterm forward rate

It follows from $|\rho_1|, |\rho_2| < 1$ that

$$\lim_{n\to\infty} a_n = \delta_0 - \frac{\Delta}{2} \begin{pmatrix} \frac{1}{(1-\rho_1)^2} \\ \frac{2}{(1-\rho_1)(1-\rho_2)} \\ \frac{2}{(1-\rho_1)(1-\rho_2)^2} \\ \frac{1}{(1-\rho_2)^2} \\ \frac{2}{(1-\rho_2)^3} \\ \frac{1}{(1-\rho_2)^4} \end{pmatrix}' \begin{pmatrix} s_{11} \\ s_{12} \\ s_{13} \\ s_{22} \\ s_{23} \\ s_{33} \end{pmatrix}$$

$$\lim_{n\to\infty} b_n = [0\ 0\ 0]$$

such that $\lim_{n\to\infty} f^{sr}_{n,n+1,t} = \lim_{n\to\infty} a_n$ is deterministic. For the variance of the ultimate forward rate,

$$\left(\sigma_n^{\mathbb{Q}}\right)^2 := \mathrm{Var}(sr_{t+n}) = \sum_{j=0}^{n-1} b_j' \Sigma\Sigma' b_j \to \mathbf{1}' \left(\Sigma\Sigma' \star \bar{B}_\infty\right) \mathbf{1}$$

where

$$\bar{B}_\infty = \sum_{j=0}^{\infty} B_j = \begin{pmatrix} \frac{1}{1-\rho_1^2} & \frac{1}{1-\rho_1\rho_2} & \frac{\rho_1}{(1-\rho_1\rho_2)^2} \\ \frac{1}{1-\rho_1\rho_2} & \frac{1}{1-\rho_2^2} & \frac{\rho_2}{\left(1-\rho_2^2\right)^2} \\ \frac{\rho_1}{(1-\rho_1\rho_2)^2} & \frac{\rho_2}{\left(1-\rho_2^2\right)^2} & \frac{1+\rho_2^2}{\left(1-\rho_2^2\right)^3} \end{pmatrix}.$$

The (deterministic) limit of the ZLB term structure follows from

$$\begin{aligned} \lim_{n\to\infty} R^{\mathrm{ZLB}}_{n,t} &= \lim_{n\to\infty} f^{\mathrm{ZLB}}_{n,n+1,t} \\ &= \lim_{n\to\infty} \underline{r} + \left(f^{sr}_{n,n+1,t} - \underline{r}\right) \Phi\left(\frac{f^{sr}_{n,n+1,t} - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right) + \sigma_n^{\mathbb{Q}} f\left(\frac{f^{sr}_{n,n+1,t} - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right) \\ &= \underline{r} + (a_\infty - \underline{r}) \Phi\left(\frac{a_\infty - \underline{r}}{\sigma_\infty^{\mathbb{Q}}}\right) + \sigma_\infty^{\mathbb{Q}} f\left(\frac{a_\infty - \underline{r}}{\sigma_\infty^{\mathbb{Q}}}\right). \end{aligned} \tag{15}$$

B Data

The data sample runs from January 1990 to June 2016 and contains the following monthly series:

- Zero-coupon rates are constructed from swap rates in Bloomberg.[7] The swaps are denominated in Deutsche Marks before 1999 and in euros from 1999 onwards. The maturities are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, 30, 40, and 50 years. The same set of maturities is used by the Dutch Central Bank (DNB) for the euroswap term structure.[8]
- The inflation rate is the end-of-month Dutch Consumer Price Index excluding the effects of policy measures (source Statistics Netherlands, Statline). This is a common inflation measure for indexation of pensions. Due to data availability, the standard CPI measure (Datastream code NLCPANNL) is used between January 1990 and December 1996.
- The stock return is the total return from the MSCI World Index, denominated in euros (Datastream code MSWRLDE (RI)). Before 1999, the total return in dollars is manually converted into euros using the series with codes MSWRLDE$ (RI) and EUDOLLR. Extrapolating the latter method gives the total return series in MSWRLDE for the years since 1999.

[7] Peter Vlaar kindly provided the data with swap rates.

[8] See DNB Statistics, Table 1.3.1. Missing maturities shorter than 50 years are obtained by interpolating forward rates. Maturities longer than 50 years are obtained by extrapolating the forward rate between 40 and 50 years.

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