The Value and Risk of Intergenerational Risk Sharing

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1 Introduction

In this short note we quantify the *value* and *risk* associated to Intergenerational Risk Sharing (IGR) in typical Dutch[1] collective pension contracts[2]. The goal is to derive *analytical* formulas that can easily be assessed, thus avoiding possible ambiguities that may arise with simulation-based methods. An accompanying Excel sheet gives such an analytical implementation.

There exist multiple definitions of IGR which complicates the general discussion. We formalize IGR as the possibility that pension payments of a certain generation depend on stock market returns that occur before this generation enters the pension system. The existing generations are needed to facilitate this. As a result, generations in the pension system may profit from an additional diversification option, with associated risk premium. We quantify exactly the gains, and risks, of this form of IGR.

A few points are important to stress for proper reading of this document. We quantify the value and risk of IGR in terms of a percentage of (the net present value of) life-time pension contributions. Alternatively some papers express it in terms of a percentage of human capital or life-time consumption. This affects results by a significant factor of about 10[3]. Also, this document does not address any other advantages or disadvantages of IGR other than the purely numerical value and risk in a stylized setting.

Lately, several papers have appeared that discuss and measure the value and risk of IGR. We refer to [1] for an overview (in Dutch).

2 The financial market

The financial market model we use is standard. This means that there is a single risk factor that we identify with stock market risk. Taking exposure to this risk factor induces risk and an expected return that need to be balanced.

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[1]In the Dutch debate, alternative terms for IGR are used like "Solidariteit met Toekomstige Opbouw", “Genoeg is genoeg”, “Doorschuiven van schokken”, “Verlengen van de life cycle”, ...

[2]More precisely, our contract looks much like the so-called SER I-B variant in the discussion on reforming the pension system in the Netherlands.

[3]This factor is based on the idea that about 10% of labor income is paid into 2nd pillar pensions in the Netherlands. Note, however, that this number may depict quite some heterogeneity over individuals.

We thus exclude interest rate risk. Analytical expressions with interest rate risks are likely to be available for standard Vasicek interest rate risk models, but not necessarily insightful when it concerns IGR, see [2]. We also exclude longevity risk and non-traded inflation risk. Generally speaking, these risks are often considered to be much smaller than investment risk, so that the effect on IGR is limited as well.

Let's define the parameters of interest. All returns and interest rates in this paper are geometrically compounded.

- There is a constant interest rate r;
- There is a systematic risk factor Z with price λ;
- There is stock, with exposure to Z, that has volatility σ;
- Agents have CRRA utility with risk aversion γ;
- The (continuously rebalanced) stock exposure will be denoted by w.

If we denote the stock price at time t by S_t, it thus evolves according to

$$\mathrm{d}S_t = (r + \lambda\sigma)\, S_t \mathrm{d}t + S_t \sigma \mathrm{d}Z_t. \quad (1)$$

From this expression follows an expected (arithmetic) return on stocks of approximately (see appendix for details) $\mu = r + \lambda\sigma$ so that λ can be identified with the Sharpe ratio. The risk premium on the stock is $\lambda\sigma$.

3 Modeling IGR

As explained in the introduction, we assume that agents actually have access to stock risk *before* they enter the labor market (at time $t = 0$). Thus, instead of starting to invest one unit of wealth at time $t = 0$, we assume that the agent can invest an amount $\exp(-Br)$ at time $t = -B$.

This paper, and the accompanying Excel sheet, consider three possibilities of *how much* of the pension contributions are exposed to IGR. We mention these here.

First-best exposure The *full* pension contribution over the life-cycle of the agent is *optimally* exposed to risks before entering the labor market.

Full exposure with smoothing The *full* pension contribution over the life-cycle of the agent is exposed to risks before entering the labor market according to the a smoothing mechanism.

Gradual exposure with smooting Only pension contributions at the *start of the life-cycle* of the agent are exposed to risks before entering the labor market and they are exposed according to a smoothing mechanism .

The case with first-best exposures is presented mainly for reasons of exposition and it leads to the often-used $\exp\left(B\lambda^2/\gamma\right)$ formula, e.g., in [3]. This case means that (all future) pension premiums are paid instantaneously upon entering the labor market and are exposed optimally to previous shocks. In particular, no smoothing of shocks is applied.

In the second case ("Full exposure with smoothing") we still assume that all life-time pension premiums are paid at once upon entering the labor market, but that they are exposed to IGR using the SER I-B mechanism that is currently discussed. This situation would occur if pension wealth is transfered from some other pension contract to a new contract ("invaren"). As a result, exposures decrease with the horizon. Note, actually, that such a form af smoothing is suboptimal in the setting of the present note.

In the third and last case ("Gradual exposure with smoothing") we assume that agents pay pension contributions only gradually over their life-time so that contributions early in life have more IGR exposure than contributions later in life. This is the most realistic setting for agents entering the labor market at young age.

In all cases, we do not only derive the *value* of IGR, but also the associated *risk*.

3.1 IGR: First-best exposures

We first derive the first-best exposures to shocks before entering the labor market. This will also lead to the formula's that will be needed later. As the problem is still a Merton problem, we know that it is optimal to have a constant stock exposure w over the investment period. Then, wealth at time t will equal[4]

$$\begin{aligned} W_t &= \exp\left(-Br\right)\exp\left(\left(r+w\lambda\sigma\right)\left(B+t\right)-\frac{1}{2}w^2\sigma^2\left(B+t\right)+w\sigma\left(Z_t-Z_{-B}\right)\right) \\ &= \exp\left(rt+\left[w\lambda\sigma-\frac{1}{2}w^2\sigma^2\right]\left(B+t\right)+w\sigma\left(Z_t-Z_{-B}\right)\right). \qquad (2) \end{aligned}$$

Observe that, for $B=0$, we indeed get the standard expression for the evolution of wealth in a Merton model. Essentially, the above expression is the same, only starting at $t=-B$ with an initial wealth of $\exp\left(-Br\right)$.

Now, assume that the agent wishes to maximize utility of wealth at time $t=T$. With CRRA utility, the agent thus maximizes

$$\begin{aligned} EW_T^{1-\gamma} &= \exp\left(\left(1-\gamma\right)\left(rt+\left[w\lambda\sigma-\frac{1}{2}w^2\sigma^2\right]\left(B+t\right)\right)+\frac{1}{2}\left(1-\gamma\right)^2w^2\sigma^2\left(B+t\right)\right) \\ &= \exp\left(\left(1-\gamma\right)rt+w\left(1-\gamma\right)\lambda\sigma\left(B+t\right)-\frac{1}{2}\left(1-\gamma\right)\gamma w^2\sigma^2\left(B+t\right)\right). \qquad (3) \end{aligned}$$

[4] This easily follows from Itô's lemma. Also note that Z is in this case a two-sided Brownian motion with $Z_0=0$, no drift, and unit variance.

One easily verifies (by solving the first-order condition $\lambda\sigma = \gamma w \sigma^2$), that the optimal stock investment is given by the classical expression

$$w^* = \frac{\lambda}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2}. \tag{4}$$

A well-known consequence of this standard Merton setting is that optimal investments do not depend on the investment horizon, nor on wealth accumulated. This is immediate from the above results.

3.1.1 First-best exposures: The value of IGR

Observe that the utility of wealth at horizon T factorizes in the following way. We have

$$EW_T^{1-\gamma} = \exp\left((1-\gamma)\,B\left[w\lambda\sigma - \frac{1}{2}\gamma w^2\sigma^2\right]\right)\exp\left((1-\gamma)\left[rt + w\lambda\sigma t - \frac{1}{2}\gamma w^2\sigma^2 t\right]\right). \tag{5}$$

The utility gain of IGR thus is given by the proportionality factor

$$\exp\left((1-\gamma)\,B\left[w\lambda\sigma - \frac{1}{2}\gamma w^2\sigma^2\right]\right), \tag{6}$$

which, in terms of certainty equivalents, translates into

$$\exp\left(B\left[w\lambda\sigma - \frac{1}{2}\gamma w^2\sigma^2\right]\right). \tag{7}$$

Under the optimal investment $w^* = \lambda/(\gamma\sigma)$, this expression simplifies to the well-known certainty equivalent wealth formula

$$\exp\left(\frac{B}{2}\frac{\lambda^2}{\gamma}\right). \tag{8}$$

For future reference, note that the *additional* value of IGR of having an (additional) exposure α to shocks at time $t = -B$ equals

$$\exp\left(\left[\alpha\lambda\sigma - \frac{1}{2}\gamma\alpha^2\sigma^2\right]\right). \tag{9}$$

3.1.2 First-best exposures: The risk of IGR

We measure the risk of IGR by the distribution of wealth at time $t = 0$. From (2), we have

$$W_0 = \exp\left(\left[w\lambda\sigma - \frac{1}{2}w^2\sigma^2\right]B - w\sigma Z_{-B}\right). \tag{10}$$

Clearly, for $B = 0$ we have $W_0 = 1$, but, for $B > 0$, W_0 can be either larger or smaller than 1. W_0 follows a log-normal distribution. More precisely,

$$W_0 \sim LN\left(\left[w\lambda\sigma - \frac{1}{2}w^2\sigma^2\right]B; w^2\sigma^2 B\right). \tag{11}$$

Again plugging in the optimal stock exposure $w^* = \lambda/\gamma\sigma$ leads to

$$W_0 \sim LN\left(\left[\frac{1}{\gamma} - \frac{1}{2\gamma^2}\right]\lambda^2 B; \frac{\lambda^2}{\gamma^2}B\right). \tag{12}$$

For future reference, note that the *additional* risk of IGR of having an (additional) exposure α to shocks at time $t = -B$ equals

$$LN\left(\left[\alpha\lambda\sigma - \frac{1}{2}\alpha^2\sigma^2\right]; \alpha^2\sigma^2\right). \tag{13}$$

3.2 IGR: Full exposure with smoothing

In the Dutch pension system, shocks are smoothed over a period of 10 years. We formalize this as such that every year a fraction ρ of shocks is transferred to the funding ratio, and a fraction $1 - \rho$ is transferred to entitlements. This means effectively that the exposure of pension entitlements to a financial market shock at time $t = -B$ is given by $\alpha = w\rho^B$, where w denotes the fund exposure to the financial market. The exposure to shocks before entry thus decays with time. As mentioned above, this is suboptimal in the present Merton setting with CRRA utility.

As a result, each exposure to shocks before labor market entry, i.e., to shocks at $t = -1, -2, -3, \ldots$ have to be addressed separately. Each shock leads to some additional value and some additional risk of IGR. As the exposures are suboptimal, it may actually be that the value is negative.

Substituting the effective exposure $\alpha = w\rho^B$ in (9) we find for the *additional* value of IGR due to exposure to shocks at $t = -B$

$$\exp\left(\left[w\rho^B\lambda\sigma - \frac{1}{2}\gamma(w\rho^B)^2\sigma^2\right]\right), \tag{14}$$

and for the *additional* risk

$$LN\left(\left[w\rho^B\lambda\sigma - \frac{1}{2}(w\rho^B)^2\sigma^2\right]; (w\rho^B)^2\sigma^2\right). \tag{15}$$

It is important to note that these are the *additional* value and risk due to (also) having exposure to shocks at time $t = -B$. Note that, for $\rho = 1$ this leads to the formulas in Section 3.1 that are linear in B. In order to get the total value and risk, the above effects have to be cumulated. More precisely, the total value of IGR is given by

$$\exp\left(\sum_{B=1}^{\infty}\left[w\rho^B\lambda\sigma - \frac{1}{2}\gamma(w\rho^B)^2\sigma^2\right]\right), \tag{16}$$

with a risk of

$$LN\left(\sum_{B=1}^{\infty}\left[w\rho^B\lambda\sigma - \frac{1}{2}(w\rho^B)^2\sigma^2\right]; \sum_{B=1}^{\infty}(w\rho^B)^2\sigma^2\right). \tag{17}$$

These formulas could be simplified, but that does not seem to lead to additional insights. We refer to the accompanying Excel sheet for an implementation. Also note that for small values of $w\rho^B$ the second-order terms in (14) and (15) are negligible with respect to the first-order terms. This allows for even simpler analytical expressions. Finally note that in the present Dutch system, IGR is organized via increases or decreases of entitlements, not accrued value. If the value of entitlements does not coincide with the value of pension premiums paid[5], this would lead to an additional correction factor.

3.3 IGR: Gradual exposure with smoothing

The results in Section 3.2 still assume that the total life-time pension contribution would be fully exposed to IGR shocks. This is generally not the case, with the exception of value transfer into a new pension system.

Suppose pension premiums of size $1/H$ are paid over the course of H years that people participate in the pension system, say $H = 40$. The effective exposure to shocks at time $t = -B$ is now given by

1. the premium payment of $1/H$ at $t = 0$ has an exposure of $w\rho^B$;
2. the premium payment of $1/H$ at $t = 1$ has an exposure of $w\rho^{(B+1)}$;
3. the premium payment of $1/H$ at $t = 2$ has an exposure of $w\rho^{(B+2)}$;
4. ...

As no more premium payments will be made after $t = H$, the total exposure (to shocks at $t = -B$) becomes

$$\alpha = \frac{1}{H} w \sum_{t=0}^{H-B} \rho^{B+t} = \frac{1}{H} w \frac{\rho^B - \rho^{H+1}}{1-\rho}. \tag{18}$$

Again, substituting these exposures into (9) and (13) and calculating the total value and risk can be done analytically. Details can be found in the accompanying Excel sheet.

4 Summary

This brief note discusses the value and risk of Intergenerational Risk Sharing is a very simple setting. It mainly serves the purpose of explaining where value and risk of IGR in current Dutch pension contracts comes from. More complicated settings will lead to different numerical results, but conceptually are the same.

Part of the Dutch discussion also refers to so-called borrowing constraints. At young age, pension participants may actually want to invest more in stocks than the wealth they have at that time. This is very similar to the notion of

[5] For instance due to the Dutch “doorsneesystematiek”.

IGR we discuss here: a specific pension contract may lead to exposure to risky returns that are utility increasing. Within the Dutch setting, choosing these optimal exposures and discussing in what institutional settings these are most easily obtained deserves probably more attention.

This paper is accompanied by an Excel sheet that implements the formula's. The reader is invited to consider several concrete parameter settings and check sensitivity of the risk and value of IGR for various parameter configurations.

A Geometric versus arithmetic returns

Actual calculations of the gains of IGR need parameter estimates, in particular for r, λ, and σ. Under the assumptions imposed, we have

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + S_t \sigma \mathrm{d}Z_t, \tag{19}$$

with $\mu = r + \lambda\sigma$. This SDE implies that (gross) arithmetic asset returns S_{t+1}/S_t are log-normally distributed with parameters $\mu - \sigma^2/2$ and σ^2. More precisely, we have

$$\log \frac{S_{t+1}}{S_t} \sim N\left(\mu - \frac{\sigma^2}{2}; \sigma^2\right). \tag{20}$$

As a result, the geometric returns satisfy

$$E \log \frac{S_{t+1}}{S_t} = \mu - \frac{\sigma^2}{2}, \tag{21}$$

$$V \log \frac{S_{t+1}}{S_t} = \sigma^2. \tag{22}$$

Similarly, (net) arithmetic returns satisfy

$$E\left\{\frac{S_{t+1}}{S_t} - 1\right\} = \exp\left(\mu - \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\right) - 1 \approx \mu, \tag{23}$$

$$V\left\{\frac{S_{t+1}}{S_t} - 1\right\} = \exp\left(2\left[\mu - \frac{\sigma^2}{2}\right] + \sigma^2\right)\left(\exp\left(\sigma^2\right) - 1\right). \tag{24}$$

The Dutch Committee Parameters (2014), estimates

$$\hat{\mu} = 8.5\%, \tag{25}$$

$$\hat{\mu} - \frac{1}{2}\hat{\sigma}^2 = 7.0\%. \tag{26}$$

This means that they estimate, implicitly, $\hat{\sigma}^2 = 2 \times 1.5\% = 3\%$, i.e., $\hat{\sigma} = 17.5\%$. All these parameters are nominal. Taking a $r = 2\%$ real interest rate, a $\pi = 2\%$ inflation level, we would thus, to be consistent with the Dutch Committee Parameters (2014) estimates take

$$\hat{\sigma} = 17.5\%, \tag{27}$$

$$\hat{\mu} = 8.5\%, \tag{28}$$

$$\hat{\lambda}\hat{\sigma} = 8.5\% - 2.0\% - 2.0\% = 4.5\%, \tag{29}$$

$$\hat{\lambda} = 4.5\%/17.5\% = 25.7\%. \tag{30}$$

Note that the [3] used parameters $\lambda = \sigma = 20\%$.

References

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[3] Casper van Ewijk, Marcel Lever, Jan Bonenkamp, Roel Mehlkopf *Pensioen in discussie: Risicodeling moeilijker/keuze binnen grenzen*, Netspar Brief 1, 2014, 1994.