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A life-cycle overlapping-generations model of the small open economy

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We construct an overlapping generations model for the small open economy incorporating a realistic description of the mortality process. With age-dependent mortality, the typical life-cycle pattern of consumption and saving results from the maximizing behaviour of individual households. Our 'Blanchard-Yaari-Modigliani' model is used to analytically study a number of typical shocks affecting the small open economy, namely a balanced-budget public spending shock, a temporary Ricardian tax cut, and an interest rate shock. The demographic details matter a lot—both the impulse-response functions and the welfare profiles (associated with the different shocks) are critically affected by them. These demographic details furthermore do not wash out in the aggregate. The model is flexible and can be applied to a wide variety of theoretical and policy issues.

JEL classifications: E10, D91, F41, J11.

1. Introduction

> It is possible that death may be the consequence of two generally co-existing causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration or an increased inability to withstand destruction. (Gompertz, 1825)

The opening quotation is a verbal introduction to a phenomenon that is now often called Gompertz' law of mortality. In his path-breaking paper, Benjamin Gompertz[1] (1825) identified two main causes of death, namely one due to pure chance and another depending on the person's age. He pointed out that if only the first cause were relevant, then 'the intensity of mortality' would be constant and the surviving fraction of a given cohort would decline in geometric progression. In contrast, if only the second cause would be relevant, and 'if mankind be continually gaining seeds of indisposition, or in other words, an increased liability

[1] As Hooker (1965) points out, Benjamin Gompertz can be seen as one of the founding fathers of modern demographic and actuarial theory. See also Preston *et al.* (2001, p.192). Blanchard (1985, p.225) and Faruqee (2003, p.301) incorrectly refer to the non-existing 'Gomperty's law'.

to death' then the force of mortality would increase with age. Gompertz' law was subsequently generalized by Makeham (1860) who argued that the instantaneous mortality rate depends both on a constant term (first cause) and on a term that is exponential in the person's age (second cause).

The microeconomic implications for consumption behaviour of lifetime uncertainty—resulting from a positive death probability—were first studied in the seminal paper by Yaari (1965). He showed that, faced with a positive mortality rate, individual agents will discount future felicity more heavily due to the uncertainty of survival. Furthermore, with lifetime uncertainty the consumer faces not only the usual solvency condition but also a constraint prohibiting negative net wealth at any time—the agent is simply not allowed by capital markets to expire indebted. Yaari assumes that the household can purchase (annuity) or sell (life insurance) actuarial notes at an actuarially fair interest rate. In the absence of a bequest motive, the household will use such notes to fully insure against the adverse effect of lifetime uncertainty.

The Yaari insights were embedded in a general equilibrium growth model by Blanchard (1985). In order to allow for exact aggregation of individual decision rules, Blanchard simplified the Yaari model by assuming a constant death probability, i.e. only the first cause of death is introduced into the model and households enjoy a perpetual youth. Because of its flexibility, the Blanchard-Yaari model has achieved workhorse status in the last two decades.[2] As Blanchard himself points out, his modelling approach has the disadvantage that it cannot capture the life-cycle aspects of consumption and saving behaviour—the age-independent mortality rate ensures that the propensity to consume out of total wealth is the same for all households.[3]

Blanchard's modelling dilemma is clear: exact aggregation is 'bought' at the expense of a rather unrealistic description of the demographic process.[4] Of course, in a closed-economy context, the aggregation step is indispensable because equilibrium factor prices are determined in the aggregate factor markets. However, in the context of a small open economy, factor prices are typically determined in world markets so that the aggregation step is not necessary and life-cycle effects can be modelled. The main objective of this paper is to elaborate on exactly this point. As we demonstrate below, provided we restrict attention to

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[2] For the purpose of this paper, the most important extension is due to Buiter (1988) who allows for non-zero population growth by using the insights of Weil (1989). For a textbook treatment of the Blanchard-Yaari model, see Blanchard and Fischer (1989, ch. 3) or Heijdra and van der Ploeg (2002, ch.16).

[3] Blanchard shows that a 'saving-for-retirement' effect can be mimicked by assuming that labour income declines with age. Faruqee and Laxton (2000) use this approach in a calibrated simulation model.

[4] Blanchard suggests that a constant mortality rate may be more reasonable if the model is applied to 'dynastic families' rather than to individual agents (1985, p.225, fn.1). Under this interpretation the mortality rate refers to the probability that the dynasty literally becomes extinct or that the chain of bequests between the generations is broken. Yaari (1965) studies bequests in the presence of lifetime uncertainty in his Case D.

the case of a small open economy, it is quite feasible to construct and analytically analyze a Blanchard-Yaari type overlapping-generations model incorporating a realistic description of demography. In addition we show that such a model gives rise to drastically different impulse-response functions associated with various macroeconomic shocks—the demographic realism matters.

The remainder of this paper is organized as follows. Section 2 sets out the model. Following Calvo and Obstfeld (1988) and Faruqee (2003), we assume that the mortality rate is age-dependent and solve for the optimal decision rules of the individual households.[5] We establish that the propensity to consume out of total wealth is an increasing function of the individual's age provided the mortality rate is non-decreasing in age. Next, we postulate a constant birth rate and characterize both the population composition and the implied aggregate population growth rate associated with the demographic process. Still using the general demographic process we characterize the steady-state age-profiles for consumption, human wealth, and asset holdings.

In Section 3 we employ actual demographic data for the Netherlands to estimate the parameters of the Blanchard and Gompertz-Makeham demographic models. Not surprisingly, the latter model provides by far the superior fit with the data. Interestingly, the estimated Gompertz-Makeham (G-M hereafter) model distinguishes two 'phases' of life, namely youth and old-age. During youth, Gompertz' first cause of death dominates and the mortality rate is virtually constant, but during old-age it rises exponentially with age. In our view, the G-M model is interesting for at least two reasons. First, it presents a continuous-time generalization of the Diamond (1965) model, allowing individuals to differ even within each 'phase' of life. Second, it gives rise to relatively simple analytical expressions for the propensity to consume and the steady-state age profiles for consumption, human wealth, and financial assets. In the remainder of the section we show that the G-M model also gives rise to a bell-shaped age profile for financial assets. Interestingly, assets are decumulated later on in life not because labour income falls (as in Modigliani's life-cycle model) but because of the exponentially increasing mortality rate.

In Section 4 we compute and visualize the effects on the key variables of three typical macroeconomic shocks affecting the small open economy, namely a balanced-budget spending shock, a temporary tax cut (Ricardian equivalence experiment), and an interest rate shock. We compare and contrast the results obtained for the Blanchard and G-M models. In the second part of Section 4 we also present the welfare effects associated with the shocks and demonstrate

[5] The relationship between these papers and ours is as follows. Calvo and Obstfeld (1988) recognize age-dependent mortality but do not solve the decentralized model. Instead, they characterize the dynamically consistent social optimum in the presence of time- and age-dependent lumpsum taxes. Faruqee (2003) models age-specific mortality in a decentralized setting but is ultimately unsuccessful. Indeed, he confuses the cumulative density function with the mortality rate (by requiring the death rate to go to unity in the limit; see Faruqee (2003, p.302)). Furthermore, he is unable to solve the transitional dynamics.

that the G-M model may give rise to non-monotonic welfare effects on existing generations, something which is impossible in the Blanchard case. We conclude Section 4 by showing that the two models also give rise to significantly different impulse-response functions for the aggregate variables (especially for asset holdings)—the heterogeneity does not 'wash out' in the aggregate.

Finally, in Section 5 we mention a number of possible model applications and extensions and we draw some conclusions. The paper concludes with a brief Appendix containing the key derivations and proofs.

2. The model

2.1 Households

2.1.1 *Individual consumption* From the perspective of birth, the expected lifetime utility of an agent is given by:

$$\Lambda(v, v) \equiv \int_v^\infty [1 - \Phi(\tau - v)]\, U[\bar{c}(v, \tau)] e^{\theta(v-\tau)} d\tau, \tag{1}$$

where v is the birth date, $U[\cdot]$ is 'felicity' (or instantaneous utility), $\bar{c}(v, \tau)$ is consumption of a vintage-v agent at time τ $(\geqslant v)$, and θ is the constant pure rate of time preference $(\theta > 0)$. Intuitively, $1 - \Phi(\tau - v)$ is the probability that an agent born at time v is still alive at time τ (at which time the agent's age is $\tau - v$). The instantaneous mortality rate (or death probability) of an agent of age s is given by the hazard rate of the stochastic distribution of the age of death $m(s) \equiv \phi(s)/[1 - \Phi(s)]$, where $\phi(s)$ and $\Phi(s)$ denote, respectively, the density and distribution (or cumulative density) functions. These functions exhibit the usual properties, i.e. $\phi(s) \geqslant 0$ and $0 \leqslant \Phi(s) \leqslant 1$ for $s \geqslant 0$. Since, by definition, $\Phi'(s) = \phi(s)$ and $\Phi(0) = 0$, it follows that the first term on the right-hand side of (1) can be simplified to:[6]

$$1 - \Phi(\tau - v) = e^{-M(\tau - v)}, \tag{2}$$

where $M(\tau - v) \equiv \int_0^{\tau - v} m(s) ds$ is the cumulative mortality factor. By using (2) in (1) we find that the utility function of a newborn agent can be written as:

$$\Lambda(v, v) \equiv \int_v^\infty U[\bar{c}(v, \tau)] e^{-[\theta(\tau - v) + M(\tau - v)]} d\tau. \tag{3}$$

[6] All derivations are documented in a separate Mathematical Appendix (see Heijdra and Romp, 2006). The key results are derived in the Appendix to this paper.

As was pointed out by Yaari (1965), future felicity is discounted both because of pure time preference (as $\theta > 0$) and because of life-time uncertainty (as $M(\tau - v) > 0$).[7]

From the perspective of some later time period t ($> v$), the utility function of the agent born at time v takes the following form:

$$\Lambda(v, t) \equiv e^{M(t-v)} \int_t^{\infty} U[\bar{c}(v, \tau)] e^{-[\theta(\tau - t) + M(\tau - v)]} d\tau, \tag{4}$$

where the discounting factor due to life-time uncertainty ($M(\tau - v)$) depends on the age of the household at time τ.[8] The felicity function is iso-elastic:

$$U[\bar{c}(v, \tau)] = \begin{cases} \dfrac{\bar{c}(v, \tau)^{1-1/\sigma} - 1}{1 - 1/\sigma} & \text{for } \sigma \neq 1 \\ \ln \bar{c}(v, \tau) & \text{for } \sigma = 1 \end{cases}, \tag{5}$$

where σ is the constant intertemporal substitution elasticity ($\sigma \geqslant 0$). The household budget identity is given by:

$$\dot{\bar{a}}(v, \tau) = [r + m(\tau - v)]\bar{a}(v, \tau) + \bar{w}(\tau) - \bar{z}(\tau) - \bar{c}(v, \tau), \tag{6}$$

where $\bar{a}(v, \tau)$ is real financial wealth, r is the exogenously given (constant) world rate of interest, $\bar{w}(\tau)$ is the wage rate, and $\bar{z}(\tau)$ is the lumpsum tax (the latter two variables are assumed to be independent of age and we assume that $\bar{w}(\tau) > \bar{z}(\tau)$). Labour supply is exogenous and each household supplies a single unit of labour. As usual, a dot above a variable denotes that variable's time rate of change, e.g. $\dot{\bar{a}}(v, \tau) \equiv d\bar{a}(v, \tau)/d\tau$. Following Yaari (1965) and Blanchard (1985), we postulate the existence of a perfectly competitive life insurance sector which offers actuarially fair annuity contracts to the households. Since household age is directly observable, the annuity rate of interest faced by a household of age $\tau - v$ is equal to the sum of the world interest rate and the instantaneous mortality rate of that household.

Abstracting from physical capital, financial wealth can be held in the form of domestic government bonds ($\bar{d}(v, \tau)$) or foreign bonds ($\bar{f}(v, \tau)$).

$$\bar{a}(v, \tau) \equiv \bar{d}(v, \tau) + \bar{f}(v, \tau). \tag{7}$$

[7] Yaari (1965, p.143) attributes the latter insight to Fisher (1930, pp. 216–7).

[8] The appearance of the term $e^{M(t-v)}$ in front of the integral is a consequence of the fact that the distribution of expected remaining lifetime is not memoryless in general. Blanchard (1985) uses the memoryless exponential distribution for which $M(s) = \mu_0 s$ (where μ_0 is a constant) and thus $M(t - v) - M(\tau - v) = -M(\tau - t)$. Equation (4) can then be written in a more familiar format as $\Lambda(v, t) \equiv \int_t^{\infty} U[\bar{c}(v, \tau)] e^{-(\theta + \mu_0)(\tau - t)} d\tau$.

The two assets are perfect substitutes in the households' portfolios and thus attract the same rate of return.

In the planning period t, the household chooses paths for consumption and financial assets in order to maximize lifetime utility (4) subject to the flow budget identity (6) and a solvency condition, taking as given its initial level of financial assets $\bar{a}(v,t)$. The household optimum is fully characterized by:

$$\frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} = \sigma[r-\theta], \tag{8}$$

$$\Delta(u,r^*)\bar{c}(v,t) = \bar{a}(v,t) + \bar{h}(v,t), \tag{9}$$

$$\bar{h}(v,t) \equiv e^{ru+M(u)} \int_u^\infty [\bar{w}(s+v) - \bar{z}(s+v)] e^{-[rs+M(s)]} ds, \tag{10}$$

where $u \equiv t-v$ is the age of the household in the planning period, $r^* \equiv r - \sigma[r-\theta]$, and $\Delta(u,\lambda)$ is defined in general terms as:

$$\Delta(u,\lambda) \equiv e^{\lambda u+M(u)} \int_u^\infty e^{-[\lambda s+M(s)]} ds, \qquad (\text{for } u \geqslant 0). \tag{11}$$

Equation (8) is the 'consumption Euler equation', relating the optimal time profile of consumption to the difference between the interest rate and the pure rate of time preference. The instantaneous mortality rate does not feature in this expression because households fully insure against the adverse effects of lifetime uncertainty (Yaari, 1965). In order to avoid having to deal with a taxonomy of different cases, we restrict attention in the remainder of this paper to the case of a nation populated by patient agents, i.e. $r > \theta$.[9] Equation (9) shows that consumption in the planning period is proportional to total wealth, consisting of financial wealth ($\bar{a}(v,t)$) and human wealth ($\bar{h}(v,t)$). The marginal (and average) propensity to consume out of total wealth equals $1/\Delta(u,r^*)$, where r^* can be seen as the 'effective' discount rate facing the consumer. Clearly, $\Delta(u,r^*)$ depends only on the household's age in the planning period and not on time itself. For future reference, Lemma 1 establishes some important properties of the $\Delta(u,\lambda)$ function. Finally, human wealth is defined in (10) and represents the market value of the unit time endowment, i.e. the present value of after-tax wage income, using the annuity rate of interest for discounting purposes. Unless after-tax wage income is time-invariant, human wealth depends on both time and on the household's age in the planning period.

[9] The results for the other cases (with $r < \theta$ or $r = \theta$) are easily deduced from our mathematical expressions.

Lemma 1 Let $\Delta(u,\lambda)$ be defined as in (11) and assume that the mortality rate is non-decreasing, i.e. $m'(s) \geqslant 0$ for all $s \geqslant 0$. Then the following properties can be established for $\Delta(u,\lambda)$: (i) decreasing in λ, $\partial\Delta(u,\lambda)/\partial\lambda < 0$; (ii) non-increasing in household age, $\partial\Delta(u,\lambda)/\partial u \leqslant 0$; (iii) upper bound, $\Delta(u,\lambda) \leqslant 1/[\lambda + m(u)]$ (if $\lambda + m(u) > 0$); (iv) $\Delta(u,\lambda) > 0$; (v) $\lim_{\lambda\to\infty} \Delta(u,\lambda) = 0$; (vi) for $m'(s) > 0$ and $m''(s) \geqslant 0$, the inequalities in (ii)-(iii) are strict and $\lim_{u\to\infty} \Delta(u,\lambda) = 0$.

Proof See Appendix.

2.1.2 *Demography* In order to allow for non-zero population growth, we employ the analytical framework developed by Buiter (1988) which distinguishes the instantaneous mortality rate $m(s)$ and the birth rate b (>0) and thus allows for net population growth or decline. The population size at time t is denoted by $L(t)$ and the size of a newborn generation is assumed to be proportional to the current population:

$$L(v,v) = bL(v). \tag{12}$$

The size of cohort v at some later time τ is:

$$L(v,\tau) = L(v,v)[1 - \Phi(\tau - v)] = bL(v)\mathrm{e}^{-M(\tau-v)}, \tag{13}$$

where we have used (2) and (12). The aggregate mortality rate, $\bar{m}$, is defined as:

$$\bar{m}L(t) = \int_{-\infty}^{t} m(t-v)L(v,t)\mathrm{d}v, \tag{14}$$

and it is assumed that the system is in a 'demographic steady state' so that $\bar{m}$ is constant (see also eq. (16) below). Despite the fact that the expected remaining lifetime of each individual is stochastic, there is no aggregate uncertainty in the economy. In the absence of international migration, the growth rate of the aggregate population, n, is equal to the difference between the birth rate and the aggregate mortality rate, i.e. $n \equiv b - \bar{m}$. It follows that $L(v) = A_0\mathrm{e}^{nv}$, $L(t) = A_0\mathrm{e}^{nt}$ and thus $L(v) = L(t)\mathrm{e}^{-n(t-v)}$. Using this result in (13) we obtain the generational population weights:

$$l(v,t) \equiv \frac{L(v,t)}{L(t)} = b\mathrm{e}^{-[n(t-v)+M(t-v)]}, \qquad (\text{for } t \geqslant v). \tag{15}$$

The key thing to note about (15) is that the population proportion of generation v at time t only depends on the age of that generation and not on time itself.

The growth rate of the population in the demographic steady state is computed by combining (14) and (15) and simplifying:

$$\frac{1}{b} = \Delta(0, n). \tag{16}$$

For a given birth rate b, eq. (16) implicitly defines the coherent solution for n and thus for the aggregate mortality rate, $\bar{m} \equiv b - n$.[10]

2.1.3 *Per capita household sector* *Per capita* variables are calculated as the integral of the generation-specific values weighted by the corresponding generation weights. For example, per capita consumption, $c(t)$, is defined as:

$$c(t) \equiv \int_{-\infty}^{t} l(v, t)\bar{c}(v, t)\mathrm{d}v, \tag{17}$$

where $l(v, t)$ and $\bar{c}(v, t)$ are defined in, respectively, (15) and (9) above. Exact aggregation of (9) is impossible because both $\Delta(u, r^*)$ and the wealth components, $\bar{a}(v, t)$ and $\bar{h}(v, t)$, depend on the generations index v. The 'Euler equation' for *per capita* consumption can nevertheless be obtained by differentiating (17) with respect to time and noting (8) and (15):

$$\dot{c}(t) = b\bar{c}(t, t) + \sigma[r - \theta]c(t) - \int_{-\infty}^{t} [n + m(t - v)]l(v, t)\bar{c}(v, t)\mathrm{d}v. \tag{18}$$

Per capita consumption growth is boosted by the arrival of new generations who start to consume out of human wealth (first term on the right-hand side) and by individual consumption growth (second term). The third term on the right-hand side of (18) corrects for population growth and (age-dependent) mortality.[11]

Per capita financial wealth is defined as $a(t) \equiv \int_{-\infty}^{t} l(v, t)\bar{a}(v, t)\mathrm{d}v$. By differentiating this expression with respect to t we obtain:

$$\dot{a}(t) = (r - n)a(t) + w(t) - z(t) - c(t), \tag{19}$$

where $w(t) = \bar{w}(t)$, $z(t) = \bar{z}(t)$, and we have used eq. (6) and noted the fact that newborns are born without financial assets ($\bar{a}(t, t) = 0$). The interest rate net of population growth is assumed to be positive, i.e. $r > n$. As in the standard Blanchard model, annuity payments drop out of the expression for *per capita*

[10] For a constant mortality rate m, we have $1/\Delta(0, n) = n + m$ so that (16) implies $n = b - m$. Blanchard (1985) sets $b = m$ so that $n = 0$ (constant population).

[11] If the mortality rate were constant, as in Blanchard (1985) and Buiter (1988), then $n \equiv b - m$ and eq. (18) would simplify to $\dot{c}(t) = \sigma[r - \theta]c(t) - b[c(t) - c(t, t)]$.

asset accumulation because they constitute transfers (via the life insurance companies) from those who die to agents who stay alive.

Finally, per capita human wealth is defined as $h(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{h}(v,t)\mathrm{d}v$ so that $\dot{h}(t)$ can be written as:

$$\dot{h}(t) = (r-n)h(t) + b\bar{h}(t,t) - w(t) + z(t). \tag{20}$$

In the standard Buiter model *per capita* human wealth is the same for all generations and accumulates at the constant annuity rate of interest $(r+m)$. In contrast, in the present model the effects of the net interest rate $(r-n)$ and the birth rate (b) are separate, with the former applying to per capita human wealth and the latter applying to the human wealth of newborn generations.

2.2 Firms, government, and foreign sector

Following Buiter (1988) we keep the production side of the model as simple as possible by abstracting from physical capital altogether.[12] Competitive firms face the technology $Y(t) = k_0 L(t)$ where k_0 is an exogenous productivity index and $L(t)$ is the aggregate supply of labour. The real wage rate is then given by $w(t) = k_0$.

The government budget identity is given by:

$$\dot{d}(t) = (r-n)d(t) + g(t) - z(t), \tag{21}$$

where $d(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{d}(v,t)\mathrm{d}v$ is the per capita stock of domestic bonds, and $g(t)$ is per capita government goods consumption. The government solvency condition is $\lim_{\tau\to\infty} d(\tau)\ \mathrm{e}^{(r-n)(t-\tau)} = 0$, so that the intertemporal budget constraint of the government can be written as:

$$d(t) = \int_t^{\infty} \left[z(\tau) - g(\tau)\right]\mathrm{e}^{-(r-n)(\tau-t)}\mathrm{d}\tau. \tag{22}$$

To the extent that there is outstanding debt (positive left-hand side), it must be exactly matched by the present value of current and future primary surpluses (positive right-hand side), using the net interest rate $(r-n)$ for discounting.

Finally, the evolution of the *per capita* stock of net foreign assets is explained by the current account:

$$\dot{f}(t) = (r-n)f(t) + w(t) - c(t) - g(t), \tag{23}$$

[12] In the context of a small open economy with firms facing convex investment adjustment costs, our approach does not entail much loss of generality because the investment and savings systems decouple in that case. See Matsuyama (1987), Bovenberg (1993, 1994), Heijdra and Meijdam (2002), and Heijdra and van der Ploeg (2002, pp. 571–581).

where we have used the fact that $y(t) \equiv Y(t)/L(t) = w(t)$ and where $f(t) \equiv \int_{-\infty}^{t} l(v,t)f(v,t)\mathrm{d}v$ denotes the *per capita* stock of foreign bonds in the hands of domestic households.

2.3 Steady-state equilibrium

It is relatively straightforward to characterize the steady state of the model. The steady-state values for all variables are designated by means of a hat overstrike, e.g. $\hat{c}$ is steady-state per capita consumption. Where no confusion can arise, the time index is also suppressed. Since technology is held constant, the wage rate is time-invariant, i.e. $w(t) = \hat{w} = k_0$. If the government variables are also held constant, so that $z(t) = \hat{z}$, $g(t) = \hat{g}$, and $d(t) = \hat{d} \equiv (\hat{z} - \hat{g})/(r-n)$, then the economy settles into a unique saddle-point stable steady-state equilibrium in which $c(t) = \hat{c}$, $h(t) = \hat{h}$, $a(t) = \hat{a}$, and $f(t) = \hat{f}$.[13]

In the steady-state equilibrium, all variables applying to individuals can be rewritten solely in terms of their age, $u \equiv t - v$ (as is also the case outside the steady state for $\Delta(u, r^*)$—see eq. (11) above). After some straightforward substitutions we find:

$$\hat{\bar{h}}(u) \equiv \hat{\bar{h}}(v,t) = [\hat{w} - \hat{z}]\Delta(u,r), \tag{24}$$

$$\hat{\bar{c}}(u) \equiv \hat{\bar{c}}(v,t) = \frac{\hat{\bar{h}}(0)}{\Delta(0,r^*)} \mathrm{e}^{\sigma[r-\theta]u}, \tag{25}$$

$$\hat{\bar{a}}(u) \equiv \hat{\bar{a}}(v,t) = \Psi(u,r,r^*)\hat{\bar{h}}(0), \tag{26}$$

where $r^* \equiv r - \sigma[r-\theta]$, $\Delta(u,\lambda)$ is defined in eq. (11), and $\Psi(u,r,r^*)$ is given by:

$$\Psi(u,r,r^*) \equiv \mathrm{e}^{ru+M(u)}\left[\frac{\int_u^\infty \mathrm{e}^{-[r^*s+M(s)]}\mathrm{d}s}{\Delta(0,r^*)} - \frac{\int_u^\infty \mathrm{e}^{-[rs+M(s)]}\mathrm{d}s}{\Delta(0,r)}\right]. \tag{27}$$

Deferring the economic intuition behind (24)-(26) to Section 3.2, we simply note that human wealth is positive (since $\hat{w} > \hat{z}$) and proportional to $\Delta(u,r)$, the properties of which are covered in Lemma 1. Human wealth at birth is an important determinant for the age profiles for both consumption and financial assets. In the absence of initial financial wealth (e.g. received bequests), $\hat{\bar{h}}(0)$ is the key 'initial condition' facing agents. Consumption rises monotonically with age but the age profile of financial assets depends critically

[13] Saddle-point stability follows trivially from the fact that all agents in the economy satisfy their respective solvency conditions. Consumption and human wealth are forward-looking variables (able to feature discrete jumps) whilst total financial assets and net foreign assets are predetermined (non-jumping) variables.

on the demographic process, i.e. on the $\Psi(u, r, r^*)$ function. For convenience, the main properties of this function are stated in Lemma 2. With a constant mortality rate, financial wealth rises monotonically with age (Lemma 2(iv)). When the mortality rate increases with age, however, the assets are positive and increasing early on in life, but return to zero at higher ages provided the condition in Lemma 2(iii) is satisfied. For the general case, the asset profile may display multiple peaks though there is only a single peak for the estimated G-M model studied in Section 3.2 below.

Lemma 2 Let $\Psi(u, r, r^*)$ be defined as in (27) and note that $r^* = r \Leftrightarrow \sigma[r - \theta] = 0$. The following properties can be established for $\Psi(u, r, r^*)$: (i) $\Psi(u, r, r) = 0$ for all $u \geqslant 0$; (ii) for $r > r^*$, $\Psi(u, r, r^*) \geqslant 0$ with the equality sign only holding for $u = 0$; (iii) if $r > r^*$ then $\lim_{u\to\infty} \Psi(u, r, r^*) = 0$ if and only if $\lim_{u\to\infty} e^{[r-r^*]u}/(r + m(u)) = 0$; (iv) if $m(u) = m_0$ (Blanchard) then $\Psi(u, r, r^*) \equiv e^{\sigma[r-r^*]u} - 1$ is a strictly increasing function in u.

Proof See Appendix.

Simple expressions for the steady-state *per capita* variables can also be found:

$$\hat{c} = \hat{\bar{c}}(0)\frac{\Delta(0, n^*)}{\Delta(0, n)}, \tag{28}$$

$$\hat{h} = \frac{\hat{w} - \hat{z}}{r - n}\left[1 - \frac{\Delta(0, r)}{\Delta(0, n)}\right] \tag{29}$$

$$\hat{a} \equiv \hat{d} + \hat{f} = \frac{\hat{w} - \hat{z}}{r - n}\left[\frac{\Delta(0, r)}{\Delta(0, r^*)}\frac{\Delta(0, n^*)}{\Delta(0, n)} - 1\right], \tag{30}$$

where $n^* \equiv n - \sigma[r - \theta]$ and the term in square brackets on the right-hand side of (30) is positive. Not surprisingly, per capita consumption exceeds consumption by newborns (because $n > n^*$ so that $\Delta(0, n^*) > \Delta(0, n)$), and both per capita human and financial wealth are positive.

Armed with these expressions it is straightforward to derive the long-run effects of various shocks impacting the economy.[14] A balanced-budget increase in government consumption ($d\hat{z} = d\hat{g} > 0$) leads to a decrease in steady-state human wealth and consumption for all cohorts:

$$\frac{d\hat{\bar{h}}(u)}{d\hat{z}} = -\Delta(u, r) < 0, \tag{31}$$

$$\frac{d\hat{\bar{c}}(u)}{d\hat{z}} = -\frac{\Delta(0, r)}{\Delta(0, r^*)}e^{\sigma[r-\theta]u} < 0. \tag{32}$$

[14] The impact and transitional effects of these shocks are studied in Section 4 of the paper.

Obviously, *per capita* steady-state consumption and human wealth also fall (see eqs (28) and (29)). It follows from (30) that *per capita* steady-state financial assets decline:

$$\frac{\mathrm{d}\hat{a}}{\mathrm{d}\hat{z}} = \frac{1}{r-n}\left[1+\frac{\mathrm{d}\hat{c}}{\mathrm{d}\hat{z}}\right] < 0. \tag{33}$$

This implies that consumption is crowded out more than one for one. Finally, since government debt is unchanged (by design) it follows from the first equality in (30) that $\mathrm{d}\hat{f}/\mathrm{d}\hat{z} = \mathrm{d}\hat{a}/\mathrm{d}\hat{z}$. The balanced-budget increase in government consumption thus leads to a long-run reduction in financial assets and a reduction in net imports, just as in the standard open-economy Blanchard model with $r > \theta$ (1985, p. 230-1).

A long-run tax-financed increase in public debt ($(r-n)\mathrm{d}\hat{d} = \mathrm{d}\hat{z} > 0$) leads to a decrease in generation-specific and per capita steady-state consumption and human wealth (see (31)-(32)). It follows from (30) that:

$$(r-n)\frac{\mathrm{d}\hat{f}}{\mathrm{d}\hat{z}} \equiv -(r-n)\frac{\mathrm{d}\hat{d}}{\mathrm{d}\hat{z}} + \frac{\mathrm{d}\hat{c}}{\mathrm{d}\hat{z}} + 1 = \frac{\mathrm{d}\hat{c}}{\mathrm{d}\hat{z}} < -1. \tag{34}$$

As in the standard Blanchard model (with $r > \theta$), government debt more than displaces foreign assets in the households' portfolios (1985, p. 242).

An increase in the world interest rate leads to higher discounting of after-tax wages and a reduction in both individual and aggregate human wealth:

$$\frac{\mathrm{d}\hat{\bar{h}}(u)}{\mathrm{d}r} = [\hat{w} - \hat{z}]\frac{\partial \Delta(u,r)}{\partial r} < 0, \tag{35}$$

$$\frac{\mathrm{d}\hat{h}}{\mathrm{d}r} = \int_0^{\infty} l(u)\frac{\mathrm{d}\hat{\bar{h}}(u)}{\mathrm{d}r}\,\mathrm{d}u < 0, \tag{36}$$

where we have used Lemma 1(i) to establish the sign in (35). The interest elasticity of individual consumption is given by:

$$\frac{r}{\hat{\bar{c}}(u)}\frac{\mathrm{d}\hat{\bar{c}}(u)}{\mathrm{d}r} = \frac{r}{\hat{\bar{h}}(0)}\frac{\mathrm{d}\hat{\bar{h}}(0)}{\mathrm{d}r} + r\sigma u - (1-\sigma)\frac{r}{\Delta(0,r^*)}\frac{\mathrm{d}\Delta(0,r^*)}{\mathrm{d}r^*}. \tag{37}$$

The effect on individual consumption is ambiguous in general because it results from the interplay of three effects, namely the (initial) human-wealth effect (HWE), the consumption-growth effect (CGE), and the (initial) consumption-propensity effect (CPE). The HWE is represented by the first term on the right-hand side of (37) and is negative as after-tax income is discounted more heavily. The CGE effect (the second term on the right-hand side) is positive and increasing in the household's age. An increase in the interest rate causes agents to adopt a steeper age-profile for consumption. Finally, the third term on the right-hand side represents the CPE, i.e. the effect of the interest rate change on a newborn's propensity to consume, $1/\Delta(0, r^*)$. In the empirically plausible case, with $\sigma < 1$, the CPE is positive, thus partially offsetting the negative HWE. For the case with a logarithmic felicity function, which we focus on from Section 3.2 onward, $\sigma = 1$ and the CPE is zero ($\Delta(0, r^*) = \Delta(0, \theta)$ in that case).

The effect on *per capita* consumption can be written as:

$$\frac{r}{\hat{c}}\frac{d\hat{c}}{dr} = \frac{r}{\hat{\bar{c}}(0)}\frac{d\hat{\bar{c}}(0)}{dr} - \sigma\frac{r}{\Delta(0, n^*)}\frac{d\Delta(0, n^*)}{dn^*}. \tag{38}$$

and is thus also ambiguous in general. The sign of first term on the right-hand side is ambiguous for $\sigma < 1$, because the HWE is negative and the CPE is positive (see (37)). For the logarithmic case ($\sigma = 1$), however, the first term must be negative. Since the second term on the right-hand side is positive, it is nevertheless possible for *per capita* consumption to rise (as is the case in the simulations performed in Section 4). Finally, the effect on individual and *per capita* assets is ambiguous for the general specification of the model.

3. Demography

As was stressed by Blanchard (1985, p.223), exact aggregation of the consumption function is generally impossible because both the propensity to consume (our $1/\Delta(u, r^*)$) and the wealth components (our $\bar{a}(v, t)$ and $\bar{h}(v, t)$) are age dependent. Blanchard cuts this Gordian knot by assuming the mortality rate to be constant, i.e. $m(s) = \mu_0 > 0$ and $M(u) = \mu_0 u$. The advantages of his approach are its simplicity and flexibility—the expected remaining planning horizon is $1/\mu_0$ so, by letting $\mu_0 \to 0$, the infinite-horizon Ramsey model is obtained as a special case. The main disadvantage of the Blanchard approach is that it cannot capture the life-cycle aspect of consumption behaviour. In addition, the perpetual youth assumption is easily refuted empirically as it runs foul of the Gompertz-Makeham law of mortality (see Preston *et al.* (2001) and Section 3.1 below).

In the context of a small open economy, however, it is quite feasible to incorporate a realistic demographic structure because the aggregation step is not necessary. Since both the interest rate and the wage rate are exogenous, the macroeconomic equilibrium can be studied directly at the level of individual households (see Section 3.2 and Section 4).

3.1 Estimates

In this paper we estimate the survival function $(1 - \Phi(\tau - v))$ by using actual demographic data for the Netherlands taken from the Human Mortality Database (2006). The data are annual and apply to the population cohort born in 1920. Actual mortality figures are available up to 2003, implying that demographic projections have only been used to compute the survival probabilities for the age range 84-105.[15] Denoting the actual surviving fraction up until age u_i of the people born in 1920 by $S(u_i)$, we estimate the parameters of a given parametric distribution function by means of non-linear least squares. Denoting the parameter vector by μ, the model to be estimated is thus:

$$S(u_i) = 1 - \Phi(u_i, \mu) + \varepsilon_i = \mathrm{e}^{-M(u_i, \mu)} + \varepsilon_i, \tag{39}$$

where $M(u_i, \mu) = \int_0^{u_i} m(s, \mu)\mathrm{d}s$ and ε_i is the stochastic error term. The estimates are reported in Table 1 for two specifications of the mortality process. In that table, $\hat{\sigma}$ is the estimated standard error of the regression, $\bar{m}$ is the average mortality rate, the t-statistics are given in round brackets below the estimates, and $1 - \widehat{\Phi(100)}$ represents the estimated proportion of centenarians.

We consider two different functional forms for the instantaneous mortality rate and the associated $M(u_i)$ functions. The Blanchard model, which is based on a constant mortality rate, yields an estimated mortality rate of 1.15% per annum and displays a rather poor fit—the estimated standard error is 0.22 which far exceeds the standard error for the second model. Model 2 postulates the instantaneous mortality rate to follow the G-M process:

$$m(u_i) = \mu_0 + \mu_1 \mathrm{e}^{\mu_2 u_i}, \tag{40}$$

with $\mu_i > 0$. As Table 1 confirms, the parameter estimates are all positive and highly significant. The standard deviation is very small and the model features a realistic

Table 1 Estimated survival functions

1. Blanchard demography	$M(u) \equiv \mu_0 u$		
$\hat{\sigma} = 0.2213$	$\hat{\mu}_0$		
$\bar{m} = 1.15\%$	0.1147×10^{-1}		
$1 - \widehat{\Phi(100)} = 31.8\%$	(14.3)		
2. G-M demography	$M(u) \equiv \mu_0 u + (\mu_1/\mu_2)[e^{\mu_2 u} - 1]$		
$\hat{\sigma} = 0.4852 \times 10^{-2}$	$\hat{\mu}_0$	$\hat{\mu}_1$	$\hat{\mu}_2$
$\bar{m} = 1.02\%$	0.2437×10^{-2}	5.52×10^{-5}	0.0964
$1 - \widehat{\Phi(100)} = 0.1\%$	(65.8)	(20.5)	(138.2)

[15] Child mortality was still a real issue in the 1920s—almost 11% of the 1920 cohort died during their first year. Since it is not the phenomenon that we wish to focus on, however, we adjust the mortality figures by assuming the death probability for ages 0-14 to be equal to that of a 15-year-old. This takes out the downward sloping segment of the mortality function at the start of life that the G-M process is ill-equipped to deal with in any case. See also Preston *et al.* (2001, p.194).

prediction for the fraction of centenarians (0.1% rather than the unrealistic prediction of almost 32% for the Blanchard model).

In the top panel of Fig. 1 we illustrate data points at five-year intervals (stars) as well as the estimated survival functions for the two models. The poor fit of the Blanchard model is confirmed–the surviving fraction is underestimated up to about age 73 and overestimated thereafter. In contrast, the G-M model tracks the data quite well. Another way to visualize the difference between the two models makes use of their predicted mortality rates (middle panel) and expected remaining

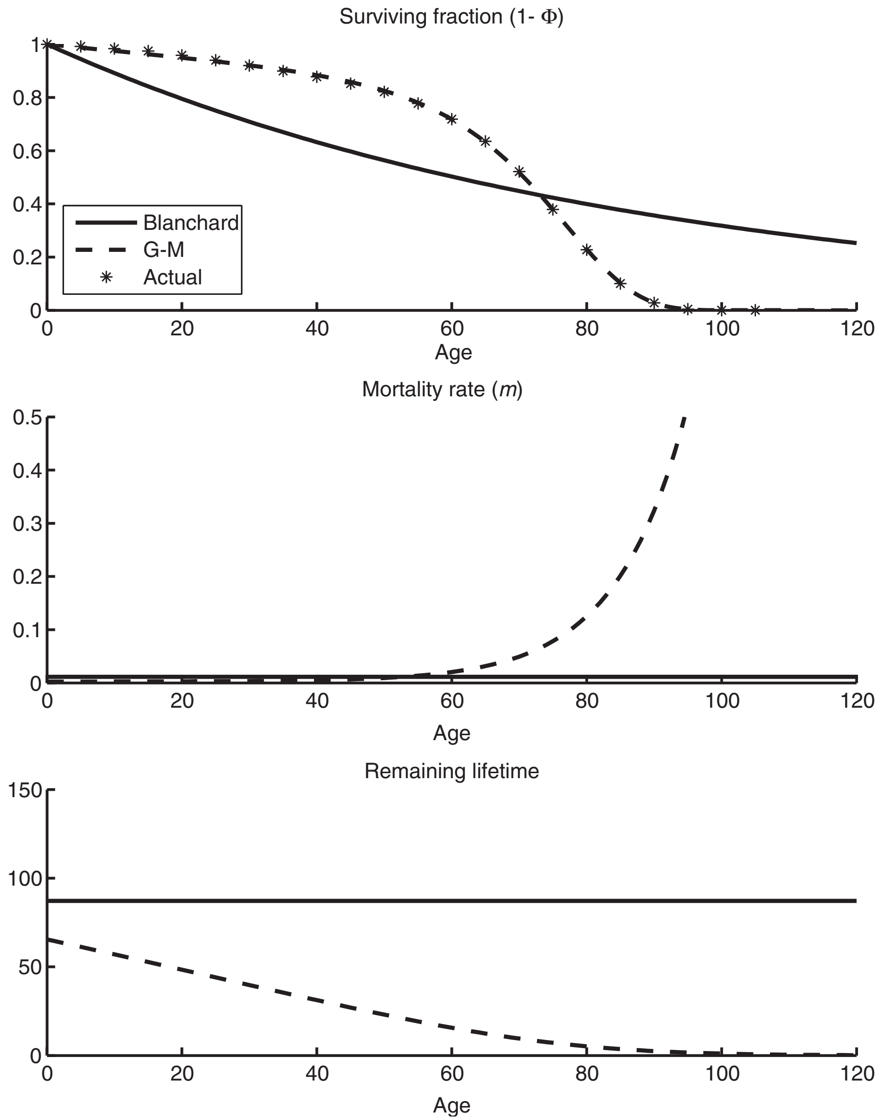

Fig. 1. Actual and estimated survival rates

lifetimes (bottom panel of Fig. 1). After about age 60, the mortality rate of the G-M model rises exponentially with age. The estimated G-M model thus distinguishes two phases of life, namely 'youth', lasting until about age 60, and 'old age' thereafter. Of course, for the Blanchard model expected remaining lifetime is constant (and equal to 87 years) so the agent enjoys a 'perpetual youth.'

3.2 Steady-state profiles

In Fig. 2 we visualize (for both estimated models) the steady-state age profiles for the propensity to consume, human wealth, consumption, and financial assets. The analytical expressions for these variables are given in, respectively, eqs (11), (24), (25), and (26). In order to avoid a taxonomy of cases, we restrict attention to the 'unit-elastic case' in the remainder of the paper, i.e. we set the intertemporal substitution elasticity equal to one ($\sigma = 1$). This implies that $r^* = \theta$ so that the marginal propensity to consume out of total wealth equals $1/\Delta(u, \theta)$ and is thus independent of the interest rate.

Clearly the $\Delta(u, \lambda)$ function (defined in (11)) plays a key role in the model. Fortunately, for both demographic specifications, easily computed closed-form solutions for $\Delta(u, \lambda)$ can be derived. Indeed, for the Blanchard model it reduces to $\Delta(u, \lambda) = 1/(\lambda + \mu_0)$ and is thus independent of the age of the household. We show in the Appendix that the solution for the G-M model can be written in terms of the upper-tailed incomplete gamma function (Kreyszig, 1999, p.A55). Not surprisingly, since the estimated G-M model satisfies the assumptions stated in Lemma 1, it follows that the marginal propensity to consume, $1/\Delta(u, \theta)$, increases with age. This is confirmed in the top left-hand panel of Fig. 2.

In the top right-hand panel of Fig. 2 the age profile for steady-state human wealth is plotted for the two mortality models.[16] For the standard Blanchard model the annuity rate of interest is age-independent because the mortality rate is constant. As a result, human wealth is age-independent also. In stark contrast, for the G-M model the annuity rate of interest rises with age so that discounting of after-tax wage income is heavier the older the household is. Human wealth gradually falls with age as a result. Indeed, it follows from (24) that $\bar{\hat{h}}(u)$ is proportional to $\Delta(u, r)$ which is downward sloping in u for any demography with a non-decreasing mortality rate (see Lemma 1). Exploiting the proportionality

[16] Following Cardia (1991, p.423) we set $r = 0.04$ and $\theta = 0.039$. We interpret the G-M demography as the truth and choose b such that $n = 0.0134$ (the average population growth rate during the period 1920–40). This yields a value of $b = 0.0236$ (which falls in between the observed birth rates for 1920 ($= 0.028$) and 1940 ($= 0.02$)). The estimated G-M model yields an expected remaining lifetime at birth of 65.5 years, which is very close to the value used by Cardia ($= 67$). For the unimportant scaling variables we use $y = w = 5$ and $z = 0$. For the G-M model, it follows that $\hat{h} = 89.42$, $\hat{c} = 5.05$, and $\hat{a} = 1.84$. For the Blanchard model we have $\hat{h} = 97.15$, $\hat{c} = 5.12$, and $\hat{a} = 4.29$. The simulation results are quite robust for different parameter values.

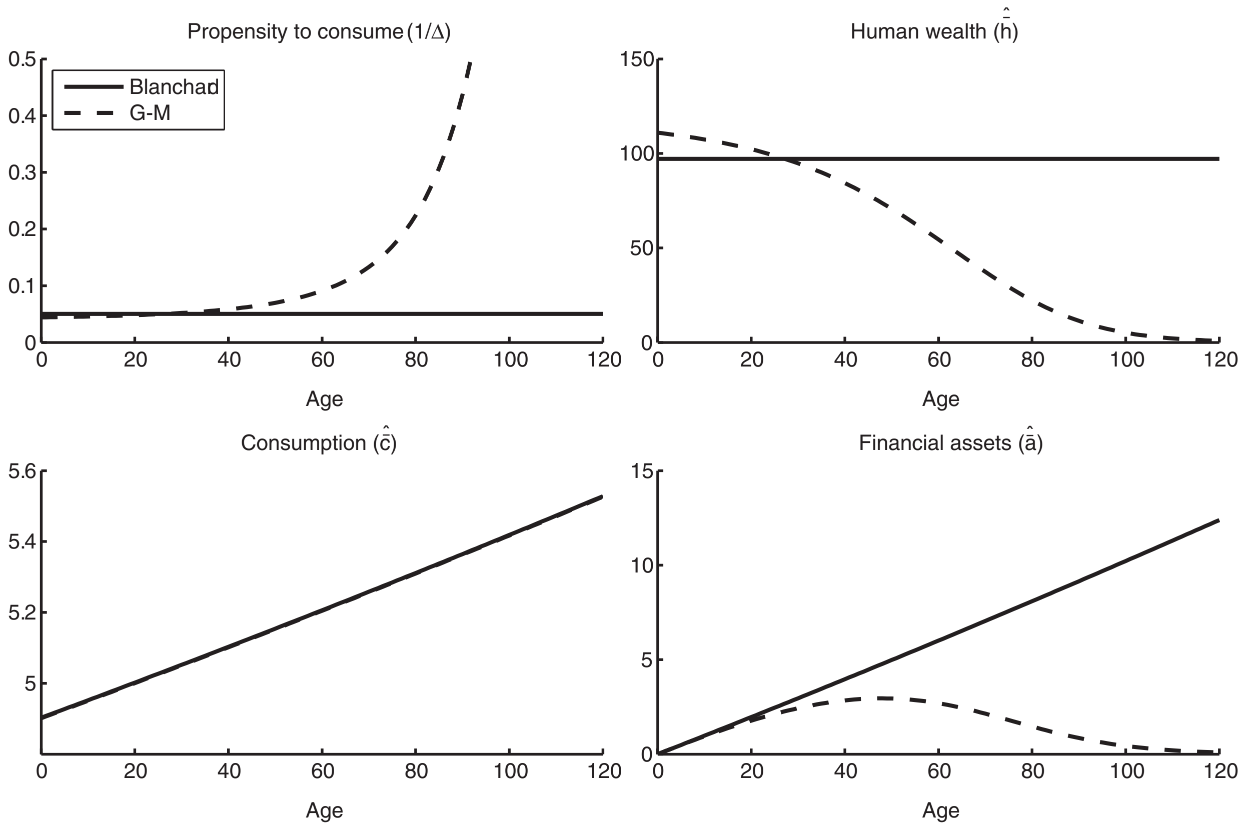

Fig. 2. Steady-state profiles for individuals

between $\hat{\bar{h}}(u)$ and $\Delta(u, r)$, we find that the slope of the human wealth profile is given by:

$$\frac{d\hat{\bar{h}}(u)}{du} = (\hat{w} - \hat{z})\left[(r + m(u))\Delta(u, r) - 1\right] < 0, \tag{41}$$

where the term in square brackets on the right-hand side of (41) is equal to $\partial\Delta(u, r)/\partial u$. During the early phase of life, the annuity rate $r + m(u)$ is relatively low, $\Delta(u, r)$ is relatively high, and human wealth falls only slightly as young agents are still on the flat part of the mortality curve. At high ages, $r + m(u)$ is high, $\Delta(u, r)$ is low, and $d\hat{\bar{h}}(u)/du$ is again relatively low. The G-M model thus gives to an inverse-S-shaped profile for human wealth with a point of inflexion located at the approximate age of 55.

In the bottom left-hand panel of Fig. 2 the age profile of steady-state consumption is visualized. As follows readily from eq. (25), the growth rate of individual consumption is the same for both demographic models. Interestingly, the estimated mortality models both predict very similar steady-state consumption paths (in level terms).

Finally, in the bottom right-hand panel of Fig. 2 the age profile of steady-state financial assets is visualized. For the Blanchard model financial assets rise with age—see Lemma 2(iv). Matters are vastly different for the G-M model. Indeed, for that model financial asset holdings follow the classic life-cycle pattern stressed by Modigliani and co-workers, i.e. households start life with zero assets, then save up until middle age, after which dissaving takes place. Despite the fact that very old agents have hardly any financial assets or human wealth left, the annuity rate of interest is so high for them that a high consumption level can nevertheless be maintained.[17]

The upshot of the discussion so far is as follows. The Blanchard specification tracks the demographic data very poorly and predicts unrealistic age patterns for the consumption propensity, human wealth, and financial wealth. In contrast, the G-M model tracks the data rather well and predicts the relevant life-cycle patterns in these variables. A further theoretical advantage of the G-M model is that it enables a conceptual distinction between youth and old age (just as is possible in the two-period Diamond (1965) model).

4. Visualizing shocks with realistic demography

In this section we compute and visualize the effects on the different variables of a number of prototypical shocks affecting a small open economy

[17] The estimated G-M model satisfies the condition stated in Lemma 2(iii) so that assets go to zero as the agent gets very old. In addition, the model gives rise to a single peak in the asset profile, a result we have been unable to prove analytically in general.

at time $t = 0$.[18] The analytical expressions for the general demographic model are reported in Heijdra and Romp (2006).

4.1 Shocks

4.1.1 *Balanced-budget fiscal policy* The first shock consists of an unanticipated and (believed to be) permanent increase in government consumption which is financed by means of lumpsum taxes (i.e. $d\hat{g} = d\hat{z} > 0$). The effects of this shock on individual human wealth and financial assets are illustrated in Fig. 3. In that figure, the left-hand panels depict the Blanchard case whilst the right-hand panels illustrate the results for the G-M model.

In the Blanchard case, the increase in the lumpsum tax causes a once-off decrease in human wealth which is the same for all existing and future generations. In stark contrast, in the G-M model the fall in human wealth depends both on time and on the generations index. The top right-hand panel of Fig. 3 shows the effects for two existing households (aged, respectively, 40 and 20 at the time of the shock) and two future households (born respectively one second and 40 years after the shock). As a result of the shock there is a once-off change in the age profile of human wealth. This profile itself does not depend on time because there is no transitional dynamics in after-tax wages (see eq. (24) above).

In the bottom two panels of Fig. 3 the paths for financial assets are illustrated. In the Blanchard case these assets rise monotonically over time for each household. The shock induces a slight kink (at time $t = 0$) in the profile for each generation. For the G-M model in the right-hand panel, the crowding-out effect due to the tax increase is much more visible. The peak in financial asset holdings is higher, the older the existing household is (compare, for example, the 40 and 20 year old households). The profiles for the future households born, respectively, in 0 and 40 years time are identical in shape. Again, this is because of the lack of transitional dynamics in after-tax wages, i.e. in terms of eq. (26) the effect operates entirely via steady-state human wealth at birth for post-shock generations.

4.1.2 *Temporary tax cut* The second shock consists of a typical Ricardian equivalence experiment. At impact ($t = 0$), the lumpsum tax is reduced and deficit financing is used to balance the budget. As a result, the stock of government debt gradually increases over time. In order to ensure that government solvency is maintained, the tax is gradually increased over time and ultimately rises to a level higher than in the initial situation. The shock that is administered thus takes the following form:

$$dz(t) = -dz_0 e^{-\chi t} + d\hat{z}[1 - e^{-\chi t}], \qquad (\textit{for } t \geqslant 0), \tag{42}$$

[18] These shocks do not have to be infinitesimal as no linearization techniques have been used. Indeed, in Section 4.1.1 we set $d\hat{z} = 1$ whilst in 4.1.2 we set $dz_0 = 1$. Since production equals $y = 5$, these tax shocks are large. In Section 4.1.2, the interest rate is increased from 4% to 4.2% per annum.

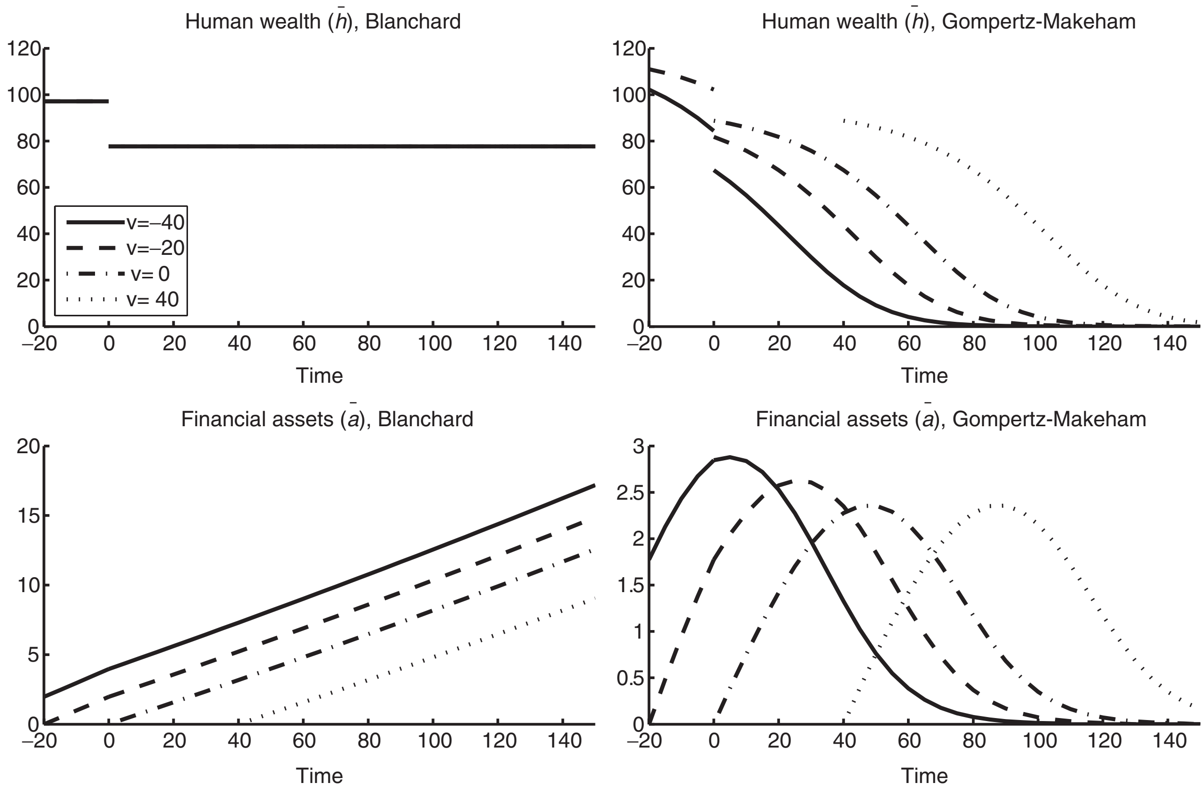

Fig. 3. Balanced-budget fiscal policy

where $0 < \chi \ll \infty$, $dz_0 > 0$, and $d\hat{z} = [(r-n)/\chi]dz_0 > 0$. At impact, the lumpsum tax falls by dz_0 but in the long run it rises by $d\hat{z}$. (The long-run effect on public debt equals $d\hat{d} = dz_0/\chi > 0$.) In the simulations, the persistence parameter is set at $\chi = 0.1$ implying that the tax reaches its pre-shock level after about 15 to 16 years.

The effects on human and financial wealth are illustrated for the two cases in Fig. 4. In the Blanchard case, human wealth is age-independent. It nevertheless features transitional dynamics because the path of lumpsum taxes is time dependent. Human wealth increases at impact (because of the tax cut), but during transition it gradually falls again (because of the gradual tax increase). In the long run, the permanently higher taxes (needed to finance interest payments on accumulated debt) ensure that human wealth is less than before the shock.

In the G-M model, the effect on human wealth is both time- and age-dependent. At impact, all existing households experience an increase in their human wealth because of the tax cut. For each household, human wealth declines during transition both because of ageing (gradual increase in the annuity rate of interest) and because the tax rises over time. For the future household born 40 years after the shock, the human wealth profile is virtually in the new steady state as most of the shock has worn out by then.

In the bottom panels of Fig. 4 the profiles for financial assets are illustrated. In the Blanchard case the tax cut causes an acceleration in asset accumulation at impact. This kink also occurs for the G-M model in the bottom right-hand right panel. The G-M case illustrates quite clearly that the Ricardian equivalence experiment redistributes resources from distant future generations toward near future and existing generations. Especially members of the generation born at the time of the shock react strongly to the tax cut as far as their savings behaviour is concerned. Indeed, their maximum asset holding peaks at a much higher level than that of 40 year old existing generations and generations born 40 years after the shock.

4.1.3 *Interest rate shock* The final shock analysed in this paper consists of an unanticipated and permanent increase in the world interest rate (i.e. $dr > 0$ for $t \geqslant 0$). The effects of this shock on human and financial wealth are illustrated in Fig. 5. In the Blanchard case the shock causes a once-off decrease in age-independent human wealth. The higher annuity rate of interest leads to stronger discounting of future after-tax wages. For the G-M model there is a once-off downward shift in the age profile of human wealth. Like the shock itself, this age profile displays no further transitional dynamics over time.

The bottom panels of Fig. 5 illustrate the effects on financial assets. Whilst the effects for the Blanchard case speak for themselves, those for the G-M model warrant some further comment. For future generations, the age profile of financial assets features a once-off upward shift at impact and displays no further transitional dynamics thereafter. In contrast, for existing generations the time path of assets depends both on their age and on time. This transitional dynamics is caused by

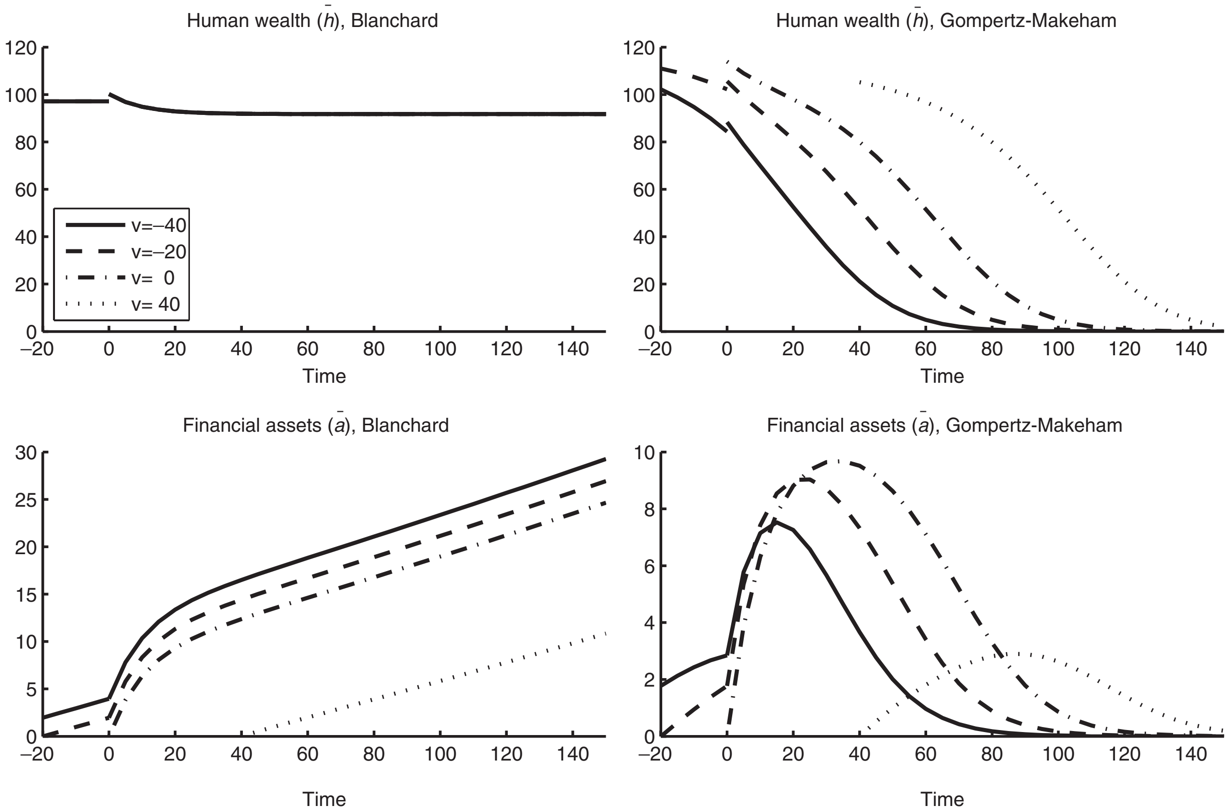

Fig. 4. Ricardian equivalence experiment: temporary tax cut

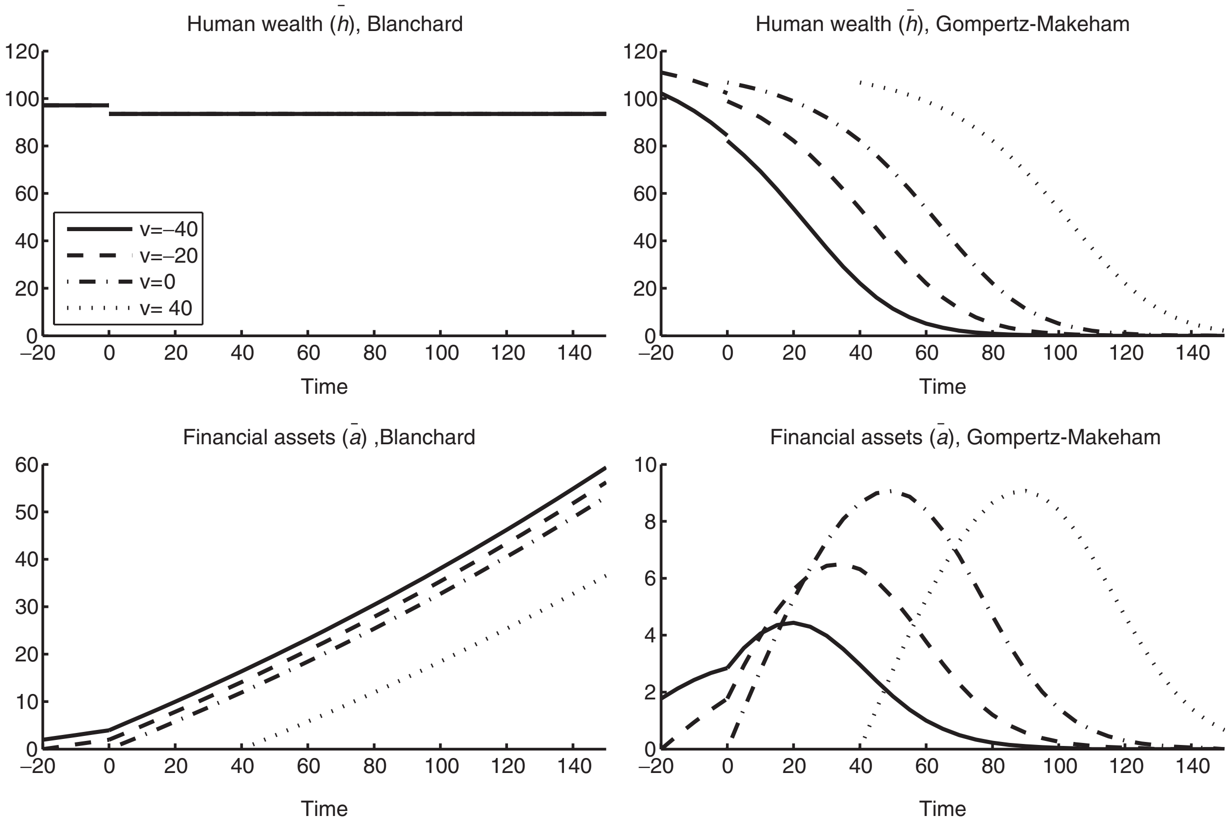

Fig. 5. Increase in the world interest rate

the fact that the consumption path for such generations depends on both t and v separately. Existing generations are affected by the interest rate hike both via their human wealth and via their accumulated financial assets which attract a higher rate of return after the shock.

4.2 Welfare effects

The Blanchard model is often used to investigate the intergenerational welfare effects of various policy measures.[19] In this section we visualize the intergenerational welfare effects associated with the three shocks studied above. For existing households, the change in welfare from the perspective of the shock period $t=0$ is evaluated ($\mathrm{d}\Lambda(v,0)$ for $v\leqslant 0$) whereas for future agents the welfare change from the perspective of their birth date is computed ($\mathrm{d}\Lambda(v,v)$ for $v>0$). The welfare effect for existing agents ($v\leqslant 0$) can be written as:

$$\mathrm{d}\Lambda(v,0)=\mathrm{d}r\int_0^\infty \tau e^{-\theta\tau-M(\tau-v)+M(-v)}\mathrm{d}\tau+\Delta(-v,\theta)\ln\Gamma_E(v),\quad (\text{for } v\leqslant 0),\qquad (43)$$

where $\Delta(-v,\theta)$ is defined in eq. (11) above and where $\Gamma_E(v)$ is defined as:

$$\Gamma_E(v)\equiv\frac{\hat{\bar{a}}(-v)+\bar{h}(v,0)}{\hat{\bar{a}}(-v)+\hat{\bar{h}}(-v)},\quad (\text{for } v\leqslant 0).\qquad (44)$$

Intuitively, $\Gamma_E(v)$ captures the effect of the impact change in human wealth for existing generations. The welfare effect consists of two separate components. The first term on the right-hand side of (43) represents the 'consumption growth effect' and is only relevant for the world interest rate shock (i.e., if $\mathrm{d}r>0$). Individual consumption growth is equal to $r-\theta$ and an increase in r leads to a steeper consumption time profile. The mortality process exerts a non-trivial influence on the consumption growth effect via the utility function. The second term on the right-hand side of (43) summarizes the welfare effect of the change in the level of consumption caused by the impact change in human wealth. This 'human wealth effect' is relevant for all shocks and is equal to the product of $\ln\Gamma_E(v)$ (defined in (44)) and the inverse propensity to consume of a v-year old agent ($\Delta(-v,\theta)$).

The welfare effect for future generations can be written as:

$$\mathrm{d}\Lambda(v,v)=\mathrm{d}r\int_0^\infty s e^{-[\theta s+M(s)]}\mathrm{d}s+\Delta(0,\theta)\ln\Gamma_F(v),\quad (\text{for } v>0),\qquad (45)$$

[19] See, for example, Bovenberg (1993, 1994) on capital taxation and investment subsidies, Bettendorf and Heijdra (2001a,b) on product subsidies and tariffs under monopolistic competition, and Heijdra and Meijdam (2002) on government infrastructure. All these studies are set in the context of a small open economy.

where $\Delta(0,\theta)$ is the inverse propensity to consume of a newborn and $\Gamma_F(v)$ is defined as:

$$\Gamma_F(v) \equiv \frac{\bar{h}(v,v)}{\hat{\bar{h}}(0)}, \quad (\text{for } v > 0). \tag{46}$$

Here, $\Gamma_F(v)$ represents the effect on the human wealth of a future newborn. Just as for existing generations, the welfare effect for future generations consists of a consumption growth effect (first term on the right-hand side of (45)) and a human wealth effect (second term).

The welfare effects of the different shocks are illustrated in Fig. 6. The left-hand panels present the results for the Blanchard case whilst the right-hand panels visualize those for the G-M model. The welfare effects of balanced-budget fiscal policy are illustrated in the top panels. All present and future generations experience a reduction in human wealth and as a result the welfare effect is negative for all generations. The effect is the same for all future generations because there is no transitional dynamics in human wealth (see above). For existing generations the welfare loss declines with the age of the generation. The human wealth effect decreases with age because both the inverse propensity to consume ($\Delta(-v,\theta)$) and the relative importance of human wealth ($\ln \Gamma_E(v)$ in (43) above) decline with age. The Blanchard and G-M models thus give qualitatively similar welfare results for the spending shock. A key difference between the two models concerns the slope of the welfare profile for existing generations. In the G-M model (right-hand panel) the welfare effect is practically zero for all generations older than 100 years. In contrast, for the Blanchard case (left-hand panel) there is still a noticeable welfare effect for 200 year old generations. This low 'generational adjustment speed' of the Blanchard model is also observed for the other shocks. Intuitively, in the Blanchard case, old generations are not killed off rapidly enough (see also the top panel of Fig. 1).

The middle two panels of Fig. 6 illustrate the welfare effects for the Ricardian tax cut experiment. All existing generations as well as future generations born close to the time of shock benefit at the expense of more distant future generations. For future generations the welfare loss is larger the later they are born. For existing generations the welfare profile is monotonically decreasing in age for the Blanchard case but non-monotonic for the G-M model. In the Blanchard case, $\Delta(-v,\theta) = \Delta(0,\theta) = 1/(\theta + \mu_0)$ is constant and $\ln \Gamma_E(v)$ declines monotonically with age. In contrast, for the G-M model, $\Delta(-v,\theta)$ decreases with age but $\ln \Gamma_E(v)$ is non-monotonic. Indeed, $\ln \Gamma_E(v)$ is increasing in age for all generations up to about 90 years and only decreases in age thereafter.[20] As a result, the welfare

[20] Of course, there are virtually no centenarians predicted by the G-M model so the downward sloping part of the $\ln \Gamma_E(v)$ function is practically irrelevant. In contrast, the estimated Blanchard demography predicts that about 32% of newborns will still be alive at age 100. See Table 1 and the top panel of Fig. 1.

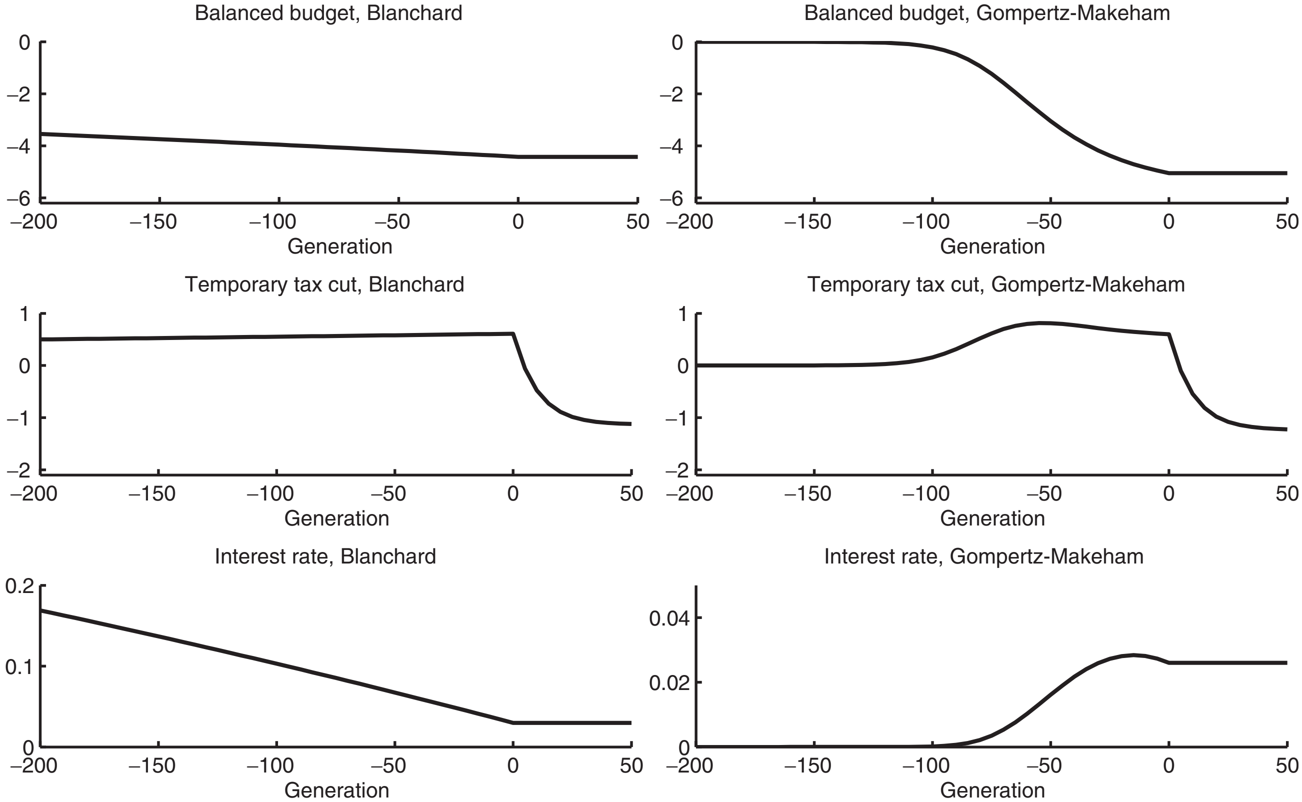

Fig. 6. Welfare effects

profile for existing generations displays a bump around the age of 55 in the middle right-hand panel of Fig. 6. At that point, the drop in $\Delta(-\nu, \theta)$ just matches the increase in $\ln \Gamma_E(\nu)$.

In the bottom two panels of Fig. 6 the welfare effects for the interest rate shock are illustrated. Since the shock induces no transitional dynamics in the age profile of human wealth for future generations, the welfare effect is the same for all future generations in both models. For existing generations the welfare effect increases with age in the Blanchard model, but is non-monotonic for the G-M model. For an interest rate shock both the consumption growth effect and the human wealth effect are relevant. The shock induces a decrease in $\ln \Gamma_E(\nu)$ which falls with age in both models. In the Blanchard case, the consumption growth effect is constant (and positive) for all generations. In contrast, for the G-M model, the consumption growth effect is positive and constant for future generations, but falling in age for existing generations. As a result, the total effect on welfare displays a bump around the age of 15 for the G-M model (see the bottom right-hand panel of Fig. 6).

4.3 Aggregate effects

As was pointed out above, Blanchard (1985) assumes a constant mortality rate in order to allow for exact aggregation of the consumption function. With the more general mortality process considered in this paper, only numerical aggregation is possible. This subsection visualizes the aggregate effects on the key variables of the three shocks considered above. To what extent do the aggregate results predicted by the Blanchard and G-M models differ?

In Fig. 7 we illustrate the effects on human wealth (first row), consumption (second row), and financial assets (third row) for the spending shock (first column), the Ricardian tax cut (second column), and the interest rate shock (third column). To facilitate the comparisons between the two models, we report the percentage deviations from the steady state for all variables, i.e. $(h(t) - \hat{h})/\hat{h}$, $(c(t) - \hat{c})/\hat{c}$, are plotted $(a(t) - \hat{a})/\hat{a}$ in Fig. 7.

For the spending shock, the results for human wealth are identical and those for consumption and financial assets are qualitatively very similar but differ in terms of the speed of adjustment towards the new steady state. The slow speed of convergence is also a feature of the Blanchard results for the other two shocks.

For the Ricardian tax cut, the effects on human wealth are again similar but those on consumption and financial wealth are not. For the G-M model, the impact effect on consumption is much larger, and the slope of the aggregate Euler equation is much steeper during transition, than for the Blanchard model. Similarly, the savings response is much more pronounced for the G-M model. This is to a large extent due to a numerator effect, i.e. $\hat{a}$ is much smaller for the G-M model than for the Blanchard model (see also footnote 16).

Finally, for the interest rate shock the effect on human wealth is qualitatively the same for the two models, though the Blanchard model overestimates the fall in

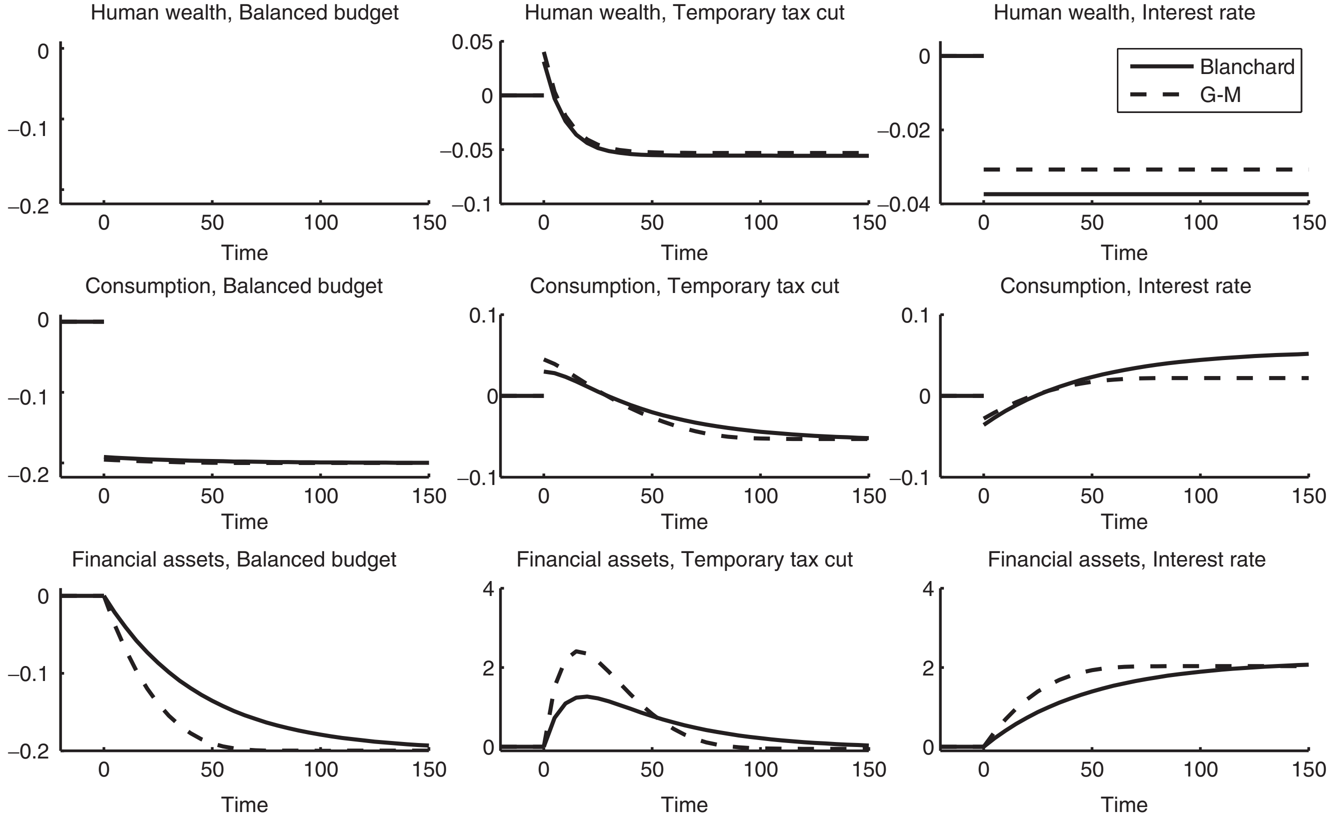

Fig. 7. Aggregate effects of the shocks

human wealth. The impact reduction in consumption is virtually the same for the two models but transition is much faster for the G-M model. Again, the savings response at impact is stronger for the G-M model.

4.4 Discussion

The key findings of this section are as follows. Incorporating a realistic demographic structure is quite feasible in the context of a small open economy facing a constant world interest rate. At the level of individual households, a realistic description of the mortality process reinstates the classic life-cycle saving insights of Modigliani and co-workers.

The welfare effects associated with the different shocks are also potentially affected in a non-trivial manner by the incorporation of a more realistic demography. Two key differences stand out between the Blanchard and G-M models. First, the G-M model predicts a much faster (and in our view more realistic) 'generational convergence speed' of the welfare effects than the Blanchard model. Second, the G-M model incorporates more extensive age-dependency and as a result may give rise to a non-monotonic welfare effect on existing generations—something which is impossible in the Blanchard case (for the shocks studied).

Finally, we have demonstrated that the demographic details do not 'wash out' at the aggregate level. The impulse-response functions for the different shocks are quite different for the Blanchard and G-M models, especially the ones for *per capita* consumption and financial assets.

In some applications of our model, it may be the case that individual behaviour depends in part on aggregate variables so that knowledge of the latter is crucial. For example, if the revenue of a consumption tax (t_C) is recycled in a lumpsum fashion to households (i.e. $\bar{z}(t) = z(t) = -t_C c(t)$) then individual consumption, human wealth, and financial assets will all depend on the aggregate tax revenue. This complication can be easily dealt with by using an iterative procedure in the simulations. In the first step the initial tax revenue and implied lumpsum transfer are guessed and individual and aggregate consumption levels are computed. In subsequent steps, the aggregate information is used to update the guess for transfers until convergence is achieved.

5. Concluding remarks

The framework developed in this paper can be extended in a number of directions. First, in order to investigate the effects of demographic change, it is necessary to generalize the stochastic distribution for expected remaining lifetimes. Two possibilities can be distinguished. 'Embodied' mortality change can be studied by assuming the instantaneous mortality rate to be generation-specific, i.e. by writing it as $m(v, s)$. An example of embodied mortality change could be the ability to extract and store embryonic stem cells to be used for future organ repairs. In contrast, 'disembodied' demographic change can be modelled by writing the mortality rate $m(t, s)$, i.e. by postulating a time-dependent mortality process. An example of disembodied mortality change would be a comprehensive cure for cancer or heart and vascular diseases.

A second extension endogenizes the household's labour supply and retirement decisions. By entering leisure hours into the felicity function, the agent has an additional choice variable with which to determine optimal labour income and lifetime utility. Two approaches can be considered. In the 'divisible labour' case, the agent can freely choose the number of working hours at each instant. In the typical formulation, consumption and leisure are both normal goods so that, as the agent gets older and richer, labour supply gradually declines to it lower bound (of zero). Hence, the agent gradually retires from the labour market. In the 'indivisible labour' case, employment is assumed to be a participation decision, i.e. the agent either works a fixed number of hours (full time) or not at all. In such a setting the retirement decision constitutes a withdrawal from the labour market altogether. In both types of labour supply models, the most interesting shocks that can be studied are ageing shocks and pension reform.

Whereas the first two extensions are relatively straightforward, the third and final one is not. The introduction of a realistic mortality process in a closed economy is complicated by the fact that exact aggregation of the consumption function is impossible (see above). Of course, the steady state can still be characterized analytically quite easily (see Section 2.3 above). The transitional effects of various shocks are, however, much more difficult to compute due to the fact that equilibrium factor prices will generally be time-varying. In the near future we wish to investigate whether approximate aggregation of the key behavioral relationships is feasible for particular shock parameterizations. If that fails, numerical methods will be employed to characterize transitional dynamics.

In conclusion, we hope that the Blanchard-Yaari-Modigliani model constructed in this paper will prove to be a useful addition to the toolbox of both theoretical economists and policy practitioners alike. At least in the context of a small open economy, there is no justification whatsoever to use models based on a blatantly unrealistic description of demography. Had mortality not caught up with him, Benjamin Gompertz would probably support that conclusion.

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Appendix

In this brief Appendix we derive the key results used in the paper. More detailed derivations are presented in Heijdra and Romp (2006).

1. Proof of Lemma 1

By definition, $M(u) \equiv \int_0^u m(s)ds$ so that $M(0) = 0$, $M'(u) = m(u) > 0$, and $M''(u) = m'(u) \geqslant 0$. First consider the standard case, with $\lambda + m(u) > 0$. Since $M(s)$ is a convex function of s we have $M(s) \geqslant M(u) + m(u)[s - u]$ and thus:

$$\Delta(u, \lambda) \leqslant e^{\lambda u + M(u)} \int_u^\infty e^{-[\lambda s + m(u)(s-u) + M(u)]} ds = \frac{1}{\lambda + m(u)}. \tag{A.1}$$

This establishes part (iii). Part (i) follows by straightforward differentiation:

$$\frac{\partial \Delta(u, \lambda)}{\partial \lambda} = -e^{\lambda u + M(u)} \int_u^\infty [s - u] e^{-[\lambda s + M(s)]} ds < 0. \tag{A.2}$$

Similarly, part (ii) is obtained by differentiating $\Delta(u, \lambda)$ with respect to u:

$$\frac{\partial \Delta(u, \lambda)}{\partial u} = [\lambda + m(u)]\Delta(u, \lambda) - 1 \leqslant 0, \tag{A.3}$$

where the sign follows from (A.1). For the alternative case, with $\lambda + m(u) < 0$, (A.1) no longer holds but (A.2)-(A.3) do. For $m'(u) > 0$ the inequalities in (A.2)-(A.3) are strict. Parts (iv)–(vi) are obvious. □

2. Incomplete gamma function

The demographic discount function for the G-M process can be written as:

$$\Delta(u, \lambda) = e^{[\lambda + \mu_0]u + \frac{\mu_1}{\mu_2} e^{\mu_2 u}} \int_u^\infty e^{-[\lambda + \mu_0]s - \frac{\mu_1}{\mu_2} e^{\mu_2 s}} ds. \tag{A.4}$$

We define $\beta(u) \equiv \mu_1/(\mu_2)e^{\mu_2 u}$ and $t = \beta(s)$. Changing the integrand we obtain:

$$\Delta(u, \lambda) = \frac{\mu_2^{\alpha-1}}{\mu_1^{\alpha}} e^{[\lambda+\mu_0]u+\beta(u)} \Gamma(\alpha, \beta(u)), \tag{A.5}$$

where $\alpha \equiv -(\lambda + \mu_0)/\mu_2$ and $\Gamma(\alpha, \beta(u))$ is the upper tailed incomplete gamma function (defined in general terms as $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$; see Kreyszig (1999, p.A55)). The incomplete gamma function is well documented (see every mathematical/statistical handbook) and, more importantly, most software packages have very fast routines to calculate it.

There is one slight complication: the incomplete gamma function is usually only defined for $\alpha \geqslant 0$, whereas we also need to evaluate it for $\alpha < 0$. We can solve this problem by using the 'functional relation of the incomplete gamma function'. Indeed, by integrating the incomplete gamma function by parts we obtain the following recursion formula:

$$\Gamma(\alpha, x) = \frac{1}{\alpha} e^{-t} t^{\alpha} \Big|_{t=x}^{\infty} + \frac{1}{\alpha} \int_x^\infty t^{\alpha} e^{-t} dt = -\frac{1}{\alpha} e^{-x} x^{\alpha} + \frac{1}{\alpha} \Gamma(\alpha + 1, x). \tag{A.6}$$

Repeated application of eq. (A.6) gives for $k = 0, 1, 2, \ldots$:

$$\begin{aligned} \Gamma(\alpha, x) = &- e^{-x} x^{\alpha} \left[\frac{1}{\alpha} + \frac{1}{\alpha} \frac{1}{\alpha+1} x + \frac{1}{\alpha} \frac{1}{\alpha+1} \frac{1}{\alpha+2} x^2 + \cdots + \frac{1}{\alpha} \frac{1}{\alpha+1} \cdots \frac{1}{\alpha+k-1} x^{k-1} \right] \\ &+ \frac{1}{\alpha} \frac{1}{\alpha+1} \cdots \frac{1}{\alpha+k-1} \Gamma(\alpha+k, x). \end{aligned} \tag{A.7}$$

Hence, by choosing the smallest integer k such that $\alpha + k$ is non-negative, the value of $\Gamma(\alpha, x)$ can be computed in a standard fashion.

3. Proof of Lemma 2

We denote the term in square brackets on the right-hand side of (27) by $\Omega(u, r, r^*)$ and note that, $\Omega(0, r, r^*) = \lim_{u \to \infty} \Omega(u, r, r^*) = 0$. Taking the derivative with respect to u we find:

$$\Omega'(u, r, r^*) = \mathrm{e}^{-M(u)} \left[\frac{\mathrm{e}^{-ru}}{\Delta(0, r)} - \frac{\mathrm{e}^{-r^* u}}{\Delta(0, r^*)} \right], \tag{A.8}$$

which clearly has a single root (at $\bar{u} \equiv 1/(r - r^*) \ln(\Delta(0, r)/\Delta(0, r^*)) > 0$) and satisfies $\Omega'(0, r, r^*) > 0$ (for $r > r^*$). This in combination with continuity of $\Omega(u, r, r^*)$ shows that $\Omega(u, r, r^*) > 0$ for $u > 0$ (and $r > r^*$). Since $\Psi(u, r, r^*) \equiv e^{ru+M(u)} \Omega(u, r, r^*)$, this proves part (ii).

To show part (iii), rewrite $\lim_{u\to\infty}\Psi(u,r,r^*)$ and use l'Hopital's rule

$$\lim_{u\to\infty}\Psi(u,r,r^*)=\lim_{u\to\infty}\left\{\frac{1}{\Delta(0,r^*)}\frac{\int_u^\infty e^{-r^*s-M(s)}ds}{e^{-ru-M(u)}}-\frac{1}{\Delta(0,r)}\frac{\int_u^\infty e^{-rs-M(s)}ds}{e^{-ru-M(u)}}\right\}$$

$$\ldots=\lim_{u\to\infty}\left\{\frac{1}{\Delta(0,r^*)}\frac{e^{-r^*u-M(u)}}{[r+m(u)]e^{-ru-M(u)}}-\frac{1}{\Delta(0,r)}\frac{e^{-ru-M(u)}}{[r+m(u)]e^{-ru-M(u)}}\right\}$$

$$\ldots=\lim_{u\to\infty}\left\{\frac{1}{\Delta(0,r^*)}\frac{e^{[r-r^*]u}}{r+m(u)}-\frac{1}{\Delta(0,r)}\frac{1}{r+m(u)}\right\},$$

from which (iii) follows immediately. Part (iv) is obvious. □