Pension Fund Restoration Policy in General Equilibrium
Tinbergen Institute MPhil Thesis

Pim Kastelein*

1st September 2016

Abstract

This paper quantifies the business cycle effects and distributional implications of pension fund restoration policy after the economy has been hit by a financial shock. We extend a canonical New-Keynesian dynamic general equilibrium model with a tractable demographic structure and a pension fund. Numerical simulations show that economies with pension funds that primarily write off accumulated pension wealth to restore financial adequacy behave similarly to an economy without a pension fund. Significant deviations from laissez-faire arise when the pension fund increases the pension fund contribution rate to close the funding gap or postpones the closure of the funding gap. At a cost of significantly distorting aggregate labour supply and output, the pension fund can shelter the group of retirees from unanticipated shocks by guaranteeing the value of their accumulated pension wealth. A defined benefit pension fund can be welfare improving to the group of agents that is already born in the period the financial shock hits. However, since pension funding gaps are typically closed over an extended period of time, a part of the welfare gains to currently alive agents comes at the expense of future, currently unborn, generations.

JEL classification: J32, E32, D91, E21

Keywords: Pension Fund, Regulation, Business Cycles, Life-cycle, New Keynesian model

*Supervised by R.M.W.J. Beetsma (University of Amsterdam, CEPR, CESifo, Netspar, Tinbergen Institute) and W.E. Romp (University of Amsterdam, Netspar, Tinbergen Institute)

1 Introduction

With asset values plummeting due to the financial crisis of 2008 and the ensuing sovereign debt crisis of 2009, many pension funds are left with a funding deficit where the present discounted value of existing pension promises to fund participants far exceed the value of managed assets. Federal Reserve Flow of Funds data indicate that U.S. retirement fund assets were virtually cut in half between 2007 and 2009 as a result of the 2008 financial crisis (Treasury, 2012). Estimations by Novy-Marx and Rauh (2009) imply that the funding gap of U.S. state-sponsored pension plans in 2008 was as large as 3.23 trillion dollars. The experience in other countries has been similar. Natali (2011) shows that the weighted-average real investment rate of return of OECD pension funds was -20.93% in 2008. Additionally, an analysis by Laboul (2010) highlights that the estimated pension fund liabilities of 2100 exchange-listed companies from OECD countries were on average roughly 25% larger than their assets in 2007, 2008, and 2009. Even today many pension funds are in a situation of financial distress, as exemplified by the current Puerto Rican situation where pension funds need to cover 45 billion dollars worth of pension promises to fund participants with a measly 2 billion dollars worth of assets (Timiraos, 2016).

These funding deficits obviously need to be covered if pension funds are to avoid exhausting their assets. Often times, regulations stipulate pension funds to restore their buffers in due time in order to avoid shifting the costs to future generations. In the lingo of the pension literature: regulations require the pension funds to conduct restoration policy such that the funding gap gets closed. However, there are various ways in which this can be done. On the one hand, the value of pension promises to fund participants can be written off in order to bring the liabilities of pension funds closer to their assets. On the other hand, pension funds can increase the required contribution payments by the current group of workers to bring the value of their assets closer to their liabilities. The 2013 Pensions at a Glance report of the OECD shows that there is little consensus amongst pension funds and regulators with regards to the 'right' way of restoring the financial adequacy of pension funds. From 2009 till 2013, all OECD countries have reformed their pension systems, but the taken measures differ widely.

On top of this, pension fund restoration policy is not simply a matter of bringing the assets and liabilities of pension funds closer together. On the contrary, different policy measures have different distributional consequences and have different implications for macroeconomic aggregates such as output, consumption, and investment. This is especially relevant when the economy is in a state of crisis. From a theoretical perspective, cutting the pension entitlements of the current group of retirees can have severe impacts for aggregate consumption since retirees tend to have a larger marginal propensity to consume than the young. However, hiking pension fund contribution rates will affect the current group of workers and distort its optimal labour supply decisions. In short: pension fund restoration policy is likely to have an effect on the business cycle. Unfortunately, most of the pension economics literature has studied pension funds only from a long-term perspective which inherently abstracts from business cycle considerations (see for instance Gollier (2008) and Beetsma and Bovenberg (2009)). With the ongoing process of population ageing (which has motivated many countries to replace their pay-as-you-go pension systems with funded systems) and the recently experienced sensitivity of pension funds to financial crises, insights about sound pension fund policy at a business cycle frequency are now required more than ever.

This paper aims to fill this gap and thus aims to provide an assessment of the business cycle effects and distributional consequences of pension fund restoration policy after the economy has been hit by a financial shock. To do so, we extend a canonical New-Keynesian, closed economy, dynamic general equilibrium model with a tractable demographic structure and a pension fund. We build on the overlapping generations framework of Gertler (1999) who introduces life-cycle behaviour in a model calibrated at business cycle frequency. However, we start out with the New-Keynesian version of this model, engineered by Kara and von Thadden (2016) and Fujiwara and Teranishi (2008), in order to generate empirically realistic responses of the economy to exogenous shocks. The supply-side of the model incorporates capital adjustment costs, imperfect competititon in the intermediate goods sector, and nominal Calvo (1983)-pricing rigidities. Furthermore, the flow utility function of agents is augmented to be of the 'money-in-the-utility-function'-type. As a novelty, we embed the pension fund framework of Romp (2013) into our model. This framework can flexibly mimick various types of pension funds observed in reality, depending on the specific parametrisation.[1] Our contribution to the literature can be viewed from two angles. On the one hand, we extend the literature on Gertler (1999) models such that a wider variety of pension arrangements can be studied.[2] On the other hand, we significantly enrich the demand-side of Romp (2013) in which the optimisation problems of workers and retirees is highly stylised.

At each point in time, the economy of our model is populated by two distinct groups of agents: workers and retirees. Workers face a probability of becoming retired and retirees face a probability of passing away. Agents take into account their expected finiteness of life when deciding upon their actions. The necessary assumptions of risk neutral preferences and constant transition probabilities into retirement and death ensure that closed-form aggregate consumption and savings relations can be derived despite the heterogeneity of agents at the micro-level.[3] In order to avoid agents having empirically unrealistic preferences for smoothing income over time (a direct consequence of the risk neutrality assumption in classical models), we invoke a special case of RINCE (Risk Neutral Constant Elasticity) preferences that restricts individuals to be risk neutral with respect to income risk, but allows them to have any arbitrary intertemporal elasticity of substitution.[4] As documented by Farmer (1990), this class of preferences yields that workers and retirees consume a fraction of their total wealth in each period. The marginal propensity to consume out of wealth for workers and retirees are two important variables in our analysis.

The pension fund is embedded into the decision problem of agents as follows: when earning labour income, agents pay a mandatory contribution to the pension fund. In return, they accumulate pension benefits which they receive upon retirement. The pension fund invests the contributions in the capital stock of the economy and is in control of three policy instruments: it sets the contribution rate on labour income, it sets the accrual rate (which determines how much additional pension benefits a retiree accumulates for one additional unit of wage income), and it controls the indexation instrument with which it can mark up or write down the accumulated stock of pension benefits of the workers and retirees. Pension fund policy is

[1] Examples of nested pension fund arrangements includes individual defined contribution, collective defined contribution, defined benefit, and hybrid systems.

[2] Thus far this literature has only considered pay-as-you-go pension arrangements.

[3] These two assumptions allow us to not have to keep track of the period in which agents are born and in which period agents become retired. We can instead consider the groups of workers and retirees as stand-alone entities rather than comprised of a range of agents born in different periods. As such, the state-space of the model remains small and this guarantees that the model can be solved with ease.

[4] See Epstein and Zin (1989).

determined on the basis of the funding gap, which is the difference between the assets of the pension fund (i.e. its holdings of capital) and its liabilities (i.e. the value of all promised pension benefits to currently alive retirees and workers). When the pension fund faces a funding deficit (which can occur after, for instance, a financial crisis hits), it has to restore the balance between its assets and liabilities by using the three policy instruments. Agents take into account that when they supply additional labour, they accumulate additional pension benefits. Depending on the specific parametrisation, the accumulation of pension benefits acts as an effective subsidy or tax on labour. It is important to appreciate that the pension fund has a noticable effect on the rest of the economy. A well-known consequence of the life-cycle of agents in the Gertler (1999) model is that government policy becomes non-Ricardian, which stems from the fact that the implied finiteness of life drives a wedge between the market interest rate and the effective discount rate that workers apply. Pension fund policy is non-neutral in our model for the same reason. The pension fund directly influences the current income of agents by using its policy instruments and therefore influences the spending patterns of workers and retirees. Additionally, the pension fund effectively creates a new asset in the economy with its policy.

Our analysis has a similar narrative as Shimer (2012) who evaluates the transition path back to the stable equilibrium after starting out with a capital stock below the steady state. An unanticipated financial crisis which evaporates a certain fraction of the capital stock necessitates the pension fund to conduct restoration policy. The pension fund has a range of policy instruments at its disposal to close the funding gap, but the timing and the use of specific instruments has profound effects on the rest of the economy. For instance, if the pension fund immediately writes off accumulated pension benefits, aggregate consumption is depressed since retirees are primarily affected and have relatively higher marginal propensities to consume. However, if the pension fund increases the pension fund contribution rate, it distorts the labour supply decision of agents and in turn aggregate supply is depressed. Ultimately, the pension fund restoration policy is a matter of allocating the pension fund losses to different groups of agents (workers and retirees) from different generations (current and future). Immediately cutting the existing accrued pension benefits of workers and retirees primarily affects the current group of retirees as they are most reliant on their accumulated pension wealth. Quickly restoring pension fund assets by increasing contribution rates primarily affects the current group of workers as they supply the most labour. Postponing the closure of the funding gap entails that future generations become responsible for closing the pension funding gap.

Numerical simulations show that economies with pension funds that primarily write off accumulated pension wealth in order to restore financial adequacy (such as collective defined contribution funds) respond similarly to unanticipated capital stock shocks as a laissez-faire economy in which no pension fund is present. Significant deviations from the laissez-faire economy arise when the pension fund increases the pension fund contribution rate to close the funding gap or postpones the closure of the funding gap. At a cost of significantly distorting aggregate labour supply and output, the pension fund can shelter the group of retirees from unanticipated financial shocks by guaranteeing the value of their accumulated pension wealth. Since retirees have few means to re-accumulate assets if they were to lose their pension wealth, a defined benefit pension fund can be welfare improving to the group of agents that is already born in the period the capital stock shock hits. This finding is in line with policy recipes brought forth by the intergenerational risk-sharing literature (such as Gollier (2008) and Beetsma and Bovenberg (2009)). However, since pension funding gaps

are typically closed over an extended period of time, a part of the welfare gains to currently alive agents comes at the expense of future, currently unborn, generations.

Sensitivity analyses indicate that a defined benefit pension system is most likely to be welfare improving if retirees have few means of re-accumulating wealth after a capital stock shock (e.g. if the productivity of retirees is low). Furthermore, if the pension fund manages a greater portion of the economy's capital stock, it can isolate retirees from unexpected financial shocks to a greater extent (but at the cost of imposing more distortions on the economy). An increased life expectancy of retirees makes them more capable of re-accumulating assets after an unexpected capital stock shock, and therefore limits the welfare improving scope of a defined benefit pension fund. Lastly, we find a trade-off between slow and speedy closure of the pension funding gap. A quick recovery distorts the economy more severely in starting periods, but in later periods distortions are smaller compared to slower recovery. Unfortunately, our model does not allow us to compute the welfare of future, unborn generations. This ultimately hinders the extent to which we can draw firm conclusions about the desirability of slow versus speedy recovery.

In the related literature on overlapping generations models that are calibrated at a business cycle frequency, two main strands can be identified. First, there are papers that enrich the Gertler (1999) model along various dimensions.[5] Second, there are papers that construct large-scale overlapping generations models with detailed, but certain, life-cycle dynamics. With respect to the first strand of literature, recent extensions of Gertler (1999) have primarily focussed on the long-term impact of demographic ageing on the interest rate and the transition towards new steady states. The implications for monetary policy are considered by Carvalho et al. (2016) and Kara and von Thadden (2016), while Katagiri (2012) focuses on output, deflation, and unemployment. Fujiwara and Teranishi (2008) pay attention to the asymmetric effects of monetary policy on workers and retirees. Grafenhofer et al. (2006) build a probabilistic ageing model that essentially is a generalised version of Gertler (1999) with richer life-cycle dynamics. The probabilistic ageing framework is applied to similar research questions such as the economic impact of demographic change (Grafenhofer et al., 2007) and pension reforms (Keuschnigg and Keuschnigg, 2004). The aforementioned studies strictly consider perfect foresight models. As Fujiwara and Teranishi (2008) explain, the literature on extensions of Gertler (1999) has not found a way to deal with the asset valuation complications deriving from the combination of heterogeneous agents and uncertainty. While most papers consider unanticipated changes in long-term processes (such as demographic ageing), Fujiwara and Teranishi (2008) consider the short-run effects of productivity shocks and Kilponen and Ripatti (2006) study the short-run effects of reforms in labour and product markets. Similarly, we solve a perfect foresight model and evaluate the short-run implications of unanticipated capital stock shocks.

As an alternative paradigm and second related strand of literature, papers have built on the large-scale numerical overlapping generations models of Auerbach and Kotlikoff (1987) and DeNardi et al. (1999) to analyse the economic effects of population ageing and pension reform. Examples include Equipe Ingenue (2001), Boersch-Supan et al. (2006), and Krueger and Ludwig (2007). However, the primary focus of these papers is mostly on global imbalances and the international implications of demographic change. Lastly, our model can be linked to recent developments in the macroeconomic literature relating to Heterogeneous Agent New-Keynesian (HANK) models. Kaplan and Violante (2014) show that the effectiveness of fiscal

[5] This paper belongs to this strand of literature.

stimulus payments crucially depends on the marginal propensities to consume of the groups of agents that are targeted. The authors find on the basis of an incomplete-markets model that targeting wealthy 'hand-to-mouth' consumers is the driving force behind the effectiveness of fiscal stimuli. On a related note, Kaplan et al. (2016) find that unexpected cuts in interest rates can have significant effects on aggregate consumption if they implicitly redistribute income between groups of agents with different marginal propensities to consume. In this literature, marginal propensities to consume are determined by the extent to which groups of agents are liquidity constrained. Our model provides an alternative and intuitive explanation for heterogeneous marginal propensities to consume: the life-cycle.

This paper is structured as follows. Section 2 presents and solves the model. More specifically, it describes in turn the decision problems of retirees and workers, aggregation, the workings of the pension fund, the supply-side of the economy, and the actions of fiscal and monetary authorities. Section 3 provides an analysis of the model. To be precise, it discusses the calibration and the resulting steady state of the baseline model. More importantly, it analyses the effects of pension fund restoration policy on the rest of the economy after an unexpected capital stock shock and presents several sensitivity analyses. Section 4 concludes. Technical issues are delegated to Appendix A, while Appendix B gives a summary of all equilibrium conditions and their steady-state equivalents.

2 The model

The model will be discussed and solved in the following order. First we outline the demographic structure. We then solve the decision problems of retirees and workers and aggregate over the groups of retirees and workers. Afterwards, we describe the finances of the pension fund and the way in which it sets its policy. Lastly, the behaviour of firms and monetary and fiscal policymakers is characterised.

2.1 Demographic structure

We consider a unit mass of individuals that is split up in two distinct groups. As in Gertler (1999), individuals have finite lives and flow through two consecutive stages of life: work and retirement. Each individual is born as a worker, and conditional on being a worker in the current period, the probability of remaining one in the next period is ω, while the probability of becoming retired in the next period is $1-\omega$. Upon reaching retirement, the probability of surviving until the next period is γ, while the probability of death is $1-\gamma$. In order to facilitate aggregation within each group, we assume that the probabilities of retirement and death are independent of age (as do Blanchard (1985) and Weil (1989)). Furthermore, we assume that the number of individuals within each cohort is 'large'. The average time spent as a worker for an individual is $\frac{1}{1-\omega}$, while similarly the average time spent in retirement for an individual is $\frac{1}{1-\gamma}$. Denote by N^w the stock of workers and by N^r the stock of retirees. As the lack of time subscripts indicates, we focus on the steady state of the demographics in which the stock of workers and retirees is stable. However, the composition of the groups of workers and retirees is constantly evolving: workers retire and retirees decease. Since each period a share $1-\omega$ of workers retires, we assume that $(1-\omega)N^w$ workers are born each period. In order

to keep the stock of retirees constant, we need that $N^r = (1-\omega)N^w + \gamma N^r$. This holds when we start out with the (old-age) dependency ratio $\psi = \frac{N^r}{N^w} = \frac{1-\omega}{1-\gamma}$.

2.2 Decision problems of workers and retirees

Before moving to an in-depth treatment of the decision problems of workers and retirees, it is required to discuss the types of risk that the individuals in this economy are facing. As mentioned previously, the model does not incorporate aggregate risk. However, individuals do face two types of idiosyncratic risk throughout their life-cycle. Firstly, workers might become retired in the next period, which constitutes an income loss due to the assumed lower productivity of retirees. Secondly, retirees face the uncertainty about their time of death. As in Gertler (1999), we have to make specific assumptions about the insurability of idiosyncratic risk and the risk preferences of individuals so that aggregation of individual decision rules will still be possible.

Similar to Blanchard (1985) and Yaari (1965), we introduce annuity markets that entirely shelter retirees from the risk of the timing of death. Upon retirement, individuals hand over all their private financial savings to a perfectly competitive mutual fund that invests the proceeds in the market and promises a return $\frac{1+r}{\gamma}$ only to those who are lucky enough to survive until the next period. Since the return of the mutual fund dominates the return of the market (which is $1+r$), all retiring individuals fully hand over their private savings. Additionally, it makes sure that there are no accidental bequests that need to be distributed over the surviving individuals, which is attractive since Heijdra et al. (2014) show that the distribution of accidental bequests can be crucial in determining the outcomes of the model.

While in principle it is possible to introduce an insurance market that mitigates the risk of income loss as a result of retirement, doing so would allow individuals to smooth their income over their life-cycle and in turn would kill the life-cycle structure that we are looking to impose. Instead, we specify that individuals are risk neutral with respect to income risk. Since the income risk in this model flows from the mechanical assumption of a constant transition probability $1-\omega$ into retirement, it appears natural to have risk neutral preferences so as to decrease the impact of income variation in the model.

A convenient utility class to invoke is that of RINCE (Risk Neutral Constant Elasticity) preferences. This has two reasons. Firstly, as shown by Epstein and Zin (1989), RINCE preferences restrict individuals to be risk neutral with respect to income risk, but allow them to have any arbitrary intertemporal elasticity of substitution. Since we motivate the presence of income risk on the mechanical grounds of generating meaningful life-cycle behaviour, it is favourable that this class of preferences allows for meaningful preferences with respect to smoothing income over time. Secondly, the specification of RINCE preferences allows us to aggregate the behaviour of workers and retirees. Farmer (1990) shows that this special class of nonexpected utility functions yields closed-form solutions to many dynamic choice problems, which includes the one at hand here. More specifically, our specification of individual's preferences yields that all individuals consume a certain fraction of their total lifetime wealth, irrespective of their age or the amount of wealth they possess.

Let $V_t^{z,i}(\cdot)$ be the value function of a particular individual i at period t, where $z = w, r$ indicates whether the individual is a worker (w) or a retiree (r) in that period. Preferences are given by:

$$
\begin{aligned}
V_t^{z,i}(\cdot) &= \left[[(c_t^{z,i})^{v_1}(1-l_t^{z,i})^{v_2}(m_t^{z,i})^{v_3}]^{\rho} + \beta^z E_t[V_{t+1}^i(\cdot)|z]^{\rho}\right]^{\frac{1}{\rho}} \\
\beta^w &= \beta \\
\beta^r &= \gamma\beta \\
E_t[V_{t+1}^i(\cdot)|w] &= \omega V_{t+1}^{w,i}(\cdot) + (1-\omega)V_{t+1}^{r,i}(\cdot) \\
E_t[V_{t+1}^i(\cdot)|r] &= V_{t+1}^{r,i}(\cdot),
\end{aligned}
$$

where $E_t[V_{t+1}^i(\cdot)|z]$ is the expectation of the value function next period, conditional on being in life-cycle state z in period t and alive in period $t+1$. Additionally, $c_t^{z,i}$, $m_t^{z,i}$, and $l_t^{z,i}$ denote consumption, real balances, and labour supply, respectively. Each individual has one unit of time and enjoys $1 - l_t^{z,i}$ units of leisure. The curvature parameter ρ implies that individuals have a desire to smooth consumption over time. As shown by Farmer (1990), $\sigma = \frac{1}{1-\rho}$ is the familiar intertemporal elasticity of substitution. We assume that the Cobb-Douglas flow utility of individuals exhibits constant returns to scale, i.e. that $v_1 + v_2 + v_3 = 1$.

We are now equipped to analyse the decision problems of workers and retirees. It will be most convenient to firstly solve the decision problem of retirees, and then the decision problem of workers. Afterwards, we will consider aggregation. Most technical details will be treated in Appendix A.

2.2.1 Retiree decision problem

A retiree has to decide in each period how much to consume, how much to save, how much labour to supply, and how many real balances to acquire. In doing so, he takes into account the finiteness of his life. While the structure of this decision problem shares many similarities with other papers in the literature, we incorporate a pension fund framework as a novelty. This framework can flexibly mimick various types of pension arrangements observed in reality, depending on the specific parametrisation. The pension fund is embedded into the retiree decision problem as follows: when earning labour income, the retiree pays a mandatory contribution to the pension fund. In return, he accumulates per-period pension benefits which will be received from next period onwards until death. The pension transfers that the retiree receives are therefore like an annuity.

The pension fund is in control of three policy instruments: it sets the contribution rate on labour income, it sets the accrual rate (which determines how much additional per-period pension benefits a retiree accumulates for one additional unit of wage income), and it controls the indexation instrument with which it can mark up or write down the accumulated per-period pension benefits of the retiree. While a full description of the pension fund is deferred to section 2.3, it is useful at this stage to have a general understanding of the way in which the pension fund uses its three policy instruments. Pension fund policy is determined on the basis of the funding gap, which is the difference between the assets of the pension fund (i.e. its holdings of capital) and its liabilities (i.e. the value of all promised pension benefits to currently alive retirees and workers). When the pension fund faces a funding deficit (which can occur after, for instance, a financial crisis hits),

it has to restore the balance between its assets and liabilities by using the three policy instruments. It can influence the amount of assets it has by changing the labour income contribution rate and it can influence the amount of liabilities it has by changing the accrual rate or by writing off the value of existing pensions with the indexation instrument. The retiree, when deciding on his optimal amount of labour to supply, takes into account the future path of pension fund policy and takes into account that when he works, he accumulates additional pension benefits.

As mentioned earlier, the method of solving the retiree decision problem is still the same as in other papers in the literature (such as Gertler (1999), Kara and von Thadden (2016), and Grafenhofer et al. (2006)), except that the optimal labour supply decision is more involved due to the presence of the pension fund. Otherwise, solving the decision problem of the retiree constitutes performing the following three familiar steps. First, we derive a consumption function which specifies that the retiree consumes in period t a fraction $\epsilon_t \pi_t$ of his total lifetime wealth (which in our model constitutes the present discounted value of private financial wealth, human capital, and pension entitlements). The marginal propensity to consume out of wealth for a retiree (henceforth abbreviated as MPCW) is thus denoted by $\epsilon_t \pi_t$, where the MPCW of a worker is given by π_t and ϵ_t therefore denotes the ratio between the MPCW of a retiree and the MPCW of a worker. Second, we derive an implicit definition of $\epsilon_t \pi_t$. Third, we derive an analytical expression for the value of a retiree $V_t^{r,i}$.

A retiree, who is indexed by i, maximises the following objective in period t:

$$V_t^{r,i}(\frac{1+r_t}{\gamma}a_{t-1}^{r,i}, \mu_t P_t^{r,i}) = \max_{c_t^{r,i}, a_t^{r,i}, l_t^{r,i}, m_t^{r,i}} \left[[(c_t^{r,i})^{v_1}(1-l_t^{r,i})^{v_2}(m_t^{r,i})^{v_3}]^{\rho} + \beta\gamma[V_{t+1}^{r,i}(\frac{1+r_{t+1}}{\gamma}a_t^{r,i}, \mu_{t+1}P_{t+1}^{r,i})]^{\rho} \right]^{\frac{1}{\rho}}$$

subject to:

$$c_t^{r,i} + a_t^{r,i} + \frac{i_t}{1+i_t}m_t^{r,i} = \frac{1+r_t}{\gamma}a_{t-1}^{r,i} + (1-\tau_t)\xi w_t l_t^{r,i} + \mu_t P_t^{r,i} - \tau_t^g,$$

where $a_t^{r,i}$ is the number of consumption goods saved by the retiree at period t, yielding a return of $\frac{1+r_{t+1}}{\gamma}a_t^{r,i}$ in period $t+1$, r_t is the real interest rate on savings from period $t-1$ till period t, and i_t is the nominal interest rate from period t till period $t+1$. Note that we thus index endogenous variables by the time subscript t if the endogenous variable is chosen or determined in period t. The private financial wealth of the retiree is given by $\frac{1+r_t}{\gamma}a_{t-1}^{r,i}$. $P_t^{r,i}$ are the accumulated per-period pension benefits of the retiree, which is adjusted by μ_t, the indexation instrument of the pension fund. τ_t is the contribution rate to the pension fund on labour income, ξw_t is the effective wage of the retiree, and $1-\xi \in (0,1)$ is the productivity loss of retirees relative to workers. Indeed, we allow retirees to continue to supply labour and to accumulate additional pension benefits when retired, making the term 'retiree' a bit of a poor descriptor. However, allowing retirees to continue to be active on the labour market makes the analysis of the decision problem of retirees conveniently similar to the decision problem of workers. In any case, it turns out that in equilibrium the labour supply of retirees lies close to zero. Lastly, τ_t^g is a per capita lump-sum tax levied by the government in order to offset any changes in the money supply.[6]

[6]The decision problem of the retiree is written entirely in real terms. To understand where the term $\frac{i_t}{1+i_t}m_t^{r,i}$ comes from, it is insightful to start from a simplified nominal budget constraint of the retiree and to proceed with writing it in real terms. The nominal budget constraint is written as (with capital letters denoting nominal variables and lowercase letters denoting their real counterparts):

$$C_t^{r,i} + M_t^{r,i} + \tilde{A}_t^{r,i} = (1+i_{t-1})\tilde{A}_{t-1}^{r,i} + M_{t-1}^{r,i},$$

This specification of the budget constraint assumes that the retiree was retired already in the previous period. Kara and von Thadden (2016) show that this characterisation is sufficient to derive the aggregate behaviour of retirees and workers. For a complete description of the cohort-specific behaviour of all individuals the decision problem would have to be conditioned on the birth period and age at which retirement takes place. However, for our purposes this is not necessary.

Accumulated per-period pension benefits evolve according to (where ν_t is the accrual rate of per-period pension benefits):

$$P_{t+1}^{r,i} = \mu_t P_t^{r,i} + \nu_t \xi w_t l_t^{r,i}$$

This shows that the stock of per-period pension benefits of the retiree in the next period is equal to the sum of his current stock of per-period pension benefits (corrected for indexation) and his accrual of new per-period pension benefits based on his labour income. Before moving on, recall our previously made assumptions that participation in the pension scheme is mandatory for those who work and that the number of individuals within each cohort is 'large'. This implies that retirees (and also workers) take the policy of the pension fund as exogenously given, which simplifies the analysis of the retiree (and worker) decision problem.

As shown in Appendix A.1.1 and A.1.2, the decision problem of the retiree gives rise to the following three first-order conditions:

$$c_{t+1}^{r,i} = \left[\beta(1+r_{t+1})(\frac{(1-\tau_t^r)w_t}{(1-\tau_{t+1}^r)w_{t+1}})^{v_2\rho}(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t})^{v_3\rho}\right]^{\sigma} c_t^{r,i} \tag{1}$$

$$1 - l_t^{r,i} = \frac{v_2}{v_1}\frac{c_t^{r,i}}{(1-\tau_t^r)\xi w_t} \tag{2}$$

$$m_t^{r,i} = \frac{v_3}{v_1}\frac{1+i_t}{i_t}c_t^{r,i}, \tag{3}$$

where (1) is the intertemporal Euler equation, (2) the optimal labour supply decision, and (3) the optimal real balances decision. The term $\tau_t^r = \tau_t - (R_t^r - 1)\nu_t$ is the effective labour income tax rate that the retiree faces. We recursively define the annuity factor $R_t^r = 1 + \mu_{t+1}\frac{\gamma}{1+r_{t+1}}R_{t+1}^r$, which denotes the present discounted value to a retiree of receiving one consumption good each period until death, continuously corrected for indexation μ. In other words, the annuity factor R_t^r denotes the amount of period t consumption goods

where $\tilde{A}_t^{r,i}$ are the nominal savings of the retiree excluding money holdings. Dividing through by P_t:

$$c_t^{r,i} + m_t^{r,i} + \tilde{a}_t^{r,i} = \frac{(1+i_{t-1})P_{t-1}\tilde{a}_{t-1}^{r,i}}{P_t} + \frac{P_{t-1}m_{t-1}^{r,i}}{P_t}$$

$$c_t^{r,i} + m_t^{r,i} + \tilde{a}_t^{r,i} = (1+r_t)(\tilde{a}_{t-1}^{r,i} + \frac{1}{1+i_{t-1}}m_{t-1}^{r,i})$$

$$c_t^{r,i} + \frac{i_t}{1+i_t}m_t^{r,i} + a_t^{r,i} = (1+r_t)a_{t-1}^{r,i},$$

where in the second line we use the Fisher relation $1 + r_t = (1+i_{t-1})\frac{P_{t-1}}{P_t}$ and define $a_t^{r,i} = \tilde{a}_t^{r,i} + \frac{1}{1+i_t}m_t^{r,i}$ to be the real savings of the worker including money holdings. The benefits of writing the budget constraint like this is that the first-order condition of the retiree decision problem with respect to $m_t^{r,i}$ will be considerably simplified. Additionally, we can denote with $c_t^{r,i} + \frac{i_t}{1+i_t}m_t^{r,i}$ the 'total' consumption of a worker in period t, which allows us to derive the consumption and value functions in similar fashion as Gertler (1999).

the retiree would judge equivalent to receiving one consumption good each period from period t until death (corrected for indexation). If the retiree earns an additional unit of labour income in period t, he is forced to pay a contribution τ_t to the pension fund, but he also accumulates ν_t additional per-period pension benefits, which he will receive from period $t+1$ onwards. The retiree takes into account that the pension fund can use its indexation instrument μ to adjust the number of consumption goods it transfers to the retiree for each 'earned' unit of per-period pension benefits. If $\mu > 1$ the accumulated per-period pension benefits are marked up, while in the opposite case they are written off. Depending on the pension fund policy (characterised by accrual ν, indexation μ, and contribution τ) the effective labour income tax rate τ^r can be positive or negative.

Let $\epsilon_t \pi_t$ denote the marginal propensity to consume out of wealth of a retiree, where consumption includes the 'consumption' of real balances $\frac{i_t}{1+i_t} m_t^{r,i}$. Using (3), 'total' consumption is given by $c_t^{r,i} + \frac{i_t}{1+i_t} m_t^{r,i} = c_t^{r,i}(1 + \frac{v_3}{v_1})$. Additionally, let retiree disposable income $d_t^{r,i}$ and retiree human capital $h_t^{r,i}$ be defined recursively as:

$$d_t^{r,i} = (1-\tau_t)\xi w_t l_t^{r,i} + \mu_t P_t^{r,i} - \tau_t^g \tag{4}$$

$$h_t^{r,i} = d_t^{r,i} + \frac{\gamma}{1+r_{t+1}} h_{t+1}^{r,i} \tag{5}$$

Appendix A.1.3 shows that one can then establish that the consumption function and MPCW of a retiree satisfy the following two conditions:

$$c_t^{r,i}(1+\frac{v_3}{v_1}) = \epsilon_t \pi_t (\frac{1+r_t}{\gamma} a_{t-1}^{r,i} + h_t^{r,i}) \tag{6}$$

$$\epsilon_t \pi_t = 1 - \frac{\epsilon_t \pi_t}{\epsilon_{t+1}\pi_{t+1}} \beta^\sigma (1+r_{t+1})^{\sigma-1} \gamma \left(\frac{(1-\tau_t^r) w_t}{(1-\tau_{t+1}^r) w_{t+1}} \right)^{v_2 \rho \sigma} \left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t} \right)^{v_3 \rho \sigma} \tag{7}$$

As mentioned above, our model set-up yields that the retirees spend a fraction $\epsilon_t \pi_t$ of their total wealth on consumption goods and real balances. Since the MPCW of a retiree is the same for all retirees, irrespective of age and total wealth, and since the choice of labour supply and real balances are proportional to consumption, aggregation over retirees will be possible.

Appendix A.1.4 shows that (6) and (7) are used to derive an analytical expression for the retiree value function:

$$V_t^{r,i} = (\epsilon_t \pi_t)^{-\frac{1}{\rho}} c_t^{r,i} (\frac{v_2}{v_1} \frac{1}{(1-\tau_t^r)\xi w_t})^{v_2} (\frac{v_3}{v_1} \frac{1+i_t}{i_t})^{v_3},$$

which is needed to solve the decision problem of a worker. Note that, using (2) and (3), $c_t^{r,i}(\frac{v_2}{v_1} \frac{1}{(1-\tau_t^r)\xi w_t})^{v_2} (\frac{v_3}{v_1} \frac{1+i_t}{i_t})^{v_3}$ is simply the optimised period t flow utility of the retiree. The value function therefore states that the value of the retiree is his period t flow utility multiplied by a factor $(\epsilon_t \pi_t)^{-\frac{1}{\rho}}$ which depends on his MPCW and his intertemporal elasticity of substitution.

2.2.2 Worker decision problem

The decision problem of a worker is similar to the decision problem of a retiree. A worker has to decide in each period how much to consume, how much to save, how much labour to supply, and how many real balances to acquire. In doing so, he takes into account that he might retire in future periods. The pension

fund is incorporated into the worker decision problem in identical fashion as in the retiree decision problem. When earning labour income, the worker pays a mandatory contribution to the pension fund. In return, he accumulates per-period pension benefits which will be received once he retires and until death (such that the per-period pension benefits are like an annuity). Again, the pension fund policy is characterised by paths for the pension contribution rate τ, accrual rate ν, and indexation μ. The worker, when deciding on his optimal amount of labour to supply, takes into account the future path of pension fund policy and takes into account that when he works more, he accumulates additional pension benefits. The method of solving the worker decision problem is still the same as in other papers in the literature (such as Gertler (1999), Kara and von Thadden (2016), and Grafenhofer et al. (2006)), except that the optimal labour supply decision is more involved due to the presence of the pension fund. Otherwise, solving the decision problem of the worker constitutes executing the following three familiar steps. First, we derive a consumption function which specifies that the worker consumes a fraction π_t of his total lifetime wealth in every period. Second, we derive an implicit definition of π_t. Third, we derive an analytical expression for the value of a worker $V_t^{w,j}$.

A worker, who is indexed by j, maximises the following objective in period t:

$$V_t^{w,j}((1+r_t)a_{t-1}^{w,j}, \mu_t P_t^{w,j}) = \max_{c_t^{w,j}, a_t^{w,j}, l_t^{w,j}, m_t^{w,j}} \Bigg[[(c_t^{w,j})^{v_1}(1-l_t^{w,j})^{v_2}(m_t^{w,j})^{v_3}]^{\rho} + \beta \Big[\omega V_{t+1}^{w,j}((1+r_{t+1})a_t^{w,j}, \mu_{t+1}P_{t+1}^{w,j}) + (1-\omega)V_{t+1}^{r,j}((1+r_{t+1})a_t^{r,j}, \mu_{t+1}P_{t+1}^{r,j}) \Big]^{\rho} \Bigg]^{\frac{1}{\rho}}$$

subject to the constraints that become operative once he retires and subject to:

$$c_t^{w,j} + a_t^{w,j} + \frac{i_t}{1+i_t} m_t^{w,j} = (1+r_t)a_{t-1}^{w,j} + (1-\tau_t)w_t l_t^{w,j} + f_t - \tau_t^g,$$

where again the budget constraint is written in real terms, the worker takes pension fund policy as exogenously given, and we assume that the worker was already alive in period $t-1$. Newly born workers start out with no private financial wealth. Note that the worker reaps a return $1+r_t$ (rather than $\frac{1+r_t}{\gamma}$) on his savings from the previous period. The worker faces the full wage rate w_t, receives profits f_t from the intermediate goods producing firms, and pays the per capita lump-sum tax τ_t^g. Accumulated per-period pension benefits evolve according to:

$$P_{t+1}^{w,j} = \mu_t P_t^{w,j} + \nu_t w_t l_t^{w,j}$$

This shows that the stock of per-period pension benefits of the worker in the next period is equal to the sum of his current stock of per-period pension benefits (corrected for indexation) and his accrual of new per-period pension benefits based on his labour income.

As shown in Appendix A.2.1, A.2.2, and A.2.3, the decision problem of the worker gives rise to the following three first-order conditions:

$$\omega c_{t+1}^{w,j} + (1-\omega) c_{t+1}^{r,j} \Lambda_{t+1} \chi_{t+1} = c_t^{w,j} \left[\beta (1+r_{t+1}) \Omega_{t+1} \left(\frac{(1-\tau_t^w) w_t}{(1-\tau_{t+1}^w) w_{t+1}} \right)^{v_2 \rho} \left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t} \right)^{v_3 \rho} \right]^{\sigma} \tag{8}$$

$$1 - l_t^{w,j} = \frac{v_2}{v_1} \frac{c_t^{w,j}}{(1-\tau_t^w) w_t} \tag{9}$$

$$m_t^{r,j} = \frac{v_3}{v_1} \frac{1+i_t}{i_t} c_t^{w,j}, \tag{10}$$

where we define the following:

$$\begin{aligned} \Lambda_{t+1} &= (\epsilon_{t+1})^{\frac{\sigma}{1-\sigma}} \\ \chi_{t+1} &= \left(\frac{1-\tau_{t+1}^w}{1-\tau_{t+1}^r} \frac{1}{\xi} \right)^{v_2} \\ \Omega_{t+1} &= \omega + (1-\omega) \chi_{t+1} (\epsilon_{t+1})^{\frac{1}{1-\sigma}} \end{aligned} \tag{11}$$

The worker Euler equation (8) requires some elaboration. In determining how much to save in period t, the worker takes into account that he might become retired in period $t+1$. The subjective reweighting of transition probabilities term Ω arises endogenously from the optimisation procedure and reflects that a worker, when switching into retirement, reaches the next (and irreversible) stage in his life-cycle. The retirement stage is characterised by a different effective wage rate (captured by ξ), a different MPCW (captured by ϵ), and by a different effective labour income tax rate (captured by τ^r and τ^w, which is discussed in more detail below). For this reason, the worker discounts the future at a different rate than the market interest rate.[7]

Conditions (9) and (10) are virtually identical to those of retirees. The only difference is that a worker faces an effective labour income tax rate $\tau_t^w = \tau_t - R_t^w \nu_t$. We recursively define the annuity factor $R_t^w = \frac{\mu_{t+1}}{1+r_{t+1}} \left(\frac{\omega}{\Omega_{t+1}} R_{t+1}^w + (1 - \frac{\omega}{\Omega_{t+1}}) R_{t+1}^r \right)$, which denotes the present discounted value to a worker of receiving one consumption good each period when retired until death, continuously corrected for indexation μ. In other words, the annuity factor R_t^w denotes the amount of period t consumption goods the worker would judge equivalent to receiving one consumption good each period when retired until death (corrected for indexation). Notice that the subjective reweighting of transition probabilities Ω shows up in determining R_t^w as well. Depending on the pension fund policy the effective labour income tax rate τ^w can be positive or negative.

Recall that π_t denotes the marginal propensity to consume out of wealth of a worker, where consumption includes the 'consumption' of real balances $\frac{i_t}{1+i_t} m_t^{w,j}$. Using (10), 'total' consumption is given by $c_t^{w,j} + \frac{i_t}{1+i_t} m_t^{w,j} = c_t^{w,j} (1 + \frac{v_3}{v_1})$. Additionally, let worker disposable income $d_t^{w,j}$ and worker human capital $h_t^{w,j}$ be

[7] In our numerical simulations, it will hold that $\epsilon > 1$ and that $\Omega > 1$ as well. This means that the worker attaches more value to having income when retired in period $t+1$ (to which he applies a subjective transition probability of $\frac{\omega}{\Omega_{t+1}}$) than to remaining a worker in period $t+1$ (to which he applies a subjective transition probability of $1 - \frac{\omega}{\Omega_{t+1}}$) compared to the transition probabilities ω and $1-\omega$.

defined recursively as:

$$d_t^{w,j} = (1-\tau_t)w_t l_t^{w,j} + f_t - \tau_t^g \tag{12}$$

$$h_t^{w,j} = d_t^{w,j} + \frac{\omega}{\Omega_{t+1}}\frac{1}{1+r_{t+1}}h_{t+1}^{w,j} + (1-\frac{\omega}{\Omega_{t+1}})\frac{1}{1+r_{t+1}}h_{t+1}^{r,j}, \tag{13}$$

with $h_{t+1}^{r,j}$ defined similarly as in (5). Appendix A.2.4 shows that one can then establish that the consumption function and MPCW of a worker satisfy the following two conditions:

$$c_t^{w,j}(1+\frac{v_3}{v_1}) = \pi_t\left((1+r_t)a_{t-1}^{w,j} + h_t^{w,j}\right) \tag{14}$$

$$\pi_t = 1 - \frac{\pi_t}{\pi_{t+1}}\beta^\sigma\left((1+r_{t+1})\Omega_{t+1}\right)^{\sigma-1}\left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}}\right)^{v_2\rho\sigma}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho\sigma} \tag{15}$$

As mentioned above, our model set-up yields that the workers spend a fraction π_t of their total wealth on consumption goods and real balances. Since the MPCW of a worker is the same for all workers, irrespective of age and total wealth, and since the choice of labour supply and real balances are proportional to consumption, aggregation over workers will be possible.

Appendix A.2.3, A.2.4, and A.2.5 show that (14) and (15) are consistent with the following analytical expression for the worker value function:

$$V_t^{w,j} = (\pi_t)^{-\frac{1}{\rho}}c_t^{w,j}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^w)w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}$$

Note that, using (9) and (10), $c_t^{w,j}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^w)w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}$ is simply the optimised period t flow utility of the worker. The value function therefore states that the value of the worker is his period t flow utility multiplied by a factor $(\pi_t)^{-\frac{1}{\rho}}$ which depends on his MPCW and his intertemporal elasticity of substitution.

Before moving to the aggregation over retirees and workers, it is important to appreciate the implications that Ω and ϵ have for the model. As seen above, the subjective reweighting of transition probabilities drives a wedge between the market interest rate and the effective discount rate that workers apply to their human wealth. As a result, fiscal policy will be non-Ricardian and therefore the path of government debt and deficits will influence consumption. Additionally, social security that redistributes income between retirees and workers will influence consumption due to the different MPCW of retirees and workers. Gertler (1999) uses precisely this wedge to assess the effect of social security and government debt in a life-cycle economy. In this model, the presence of Ω and ϵ have the added consequence that the pension fund will be non-Ricardian as well, as the pension fund collects labour income contributions from workers and pays out the accumulated pension benefits only once workers are retired. Furthermore, the pension fund can directly influence the current income of workers and retirees by adjusting τ, ν, and μ, and therefore influence their spending patterns.

2.2.3 Aggregation over retirees and workers

In order to characterise aggregate variables, we will sum the previously derived equations over the respective groups of workers and retirees. Aggregate variables will be identified by the lack of a superscript i and j. Recall that the number of alive workers and retirees is N^w and N^r, respectively.

Aggregate labour supply relations satisfy (using (2) and (9)):

$$l_t^r = \sum^i (1 - \frac{v_2}{v_1} \frac{c_t^{r,i}}{(1-\tau_t^r)\xi w_t}) = N^r - \frac{v_2}{v_1} \frac{c_t^r}{(1-\tau_t^r)\xi w_t} \tag{16}$$

$$l_t^w = \sum^j (1 - \frac{v_2}{v_1} \frac{c_t^{w,j}}{(1-\tau_t^w) w_t}) = N^w - \frac{v_2}{v_1} \frac{c_t^w}{(1-\tau_t^w) w_t} \tag{17}$$

$$l_t = l_t^w + \xi l_t^r, \tag{18}$$

where c_t^r and c_t^w denote the aggregate consumption of the group of retirees and workers, to be characterised below.

Aggregate disposable income relations satisfy (using (4) and (12)):

$$d_t^r = \sum^i \left((1-\tau_t)\xi w_t l_t^{r,i} + \mu_t P_t^{r,i} - \tau_t^g \right) = (1-\tau_t)\xi w_t l_t^r + \mu_t P_t^{r,f} - \tau_t^g N^r \tag{19}$$

$$d_t^w = \sum^j \left((1-\tau_t) w_t l_t^{w,j} + f_t - \tau_t^g \right) = (1-\tau_t) w_t l_t^w + f_t N^w - \tau_t^g N^w, \tag{20}$$

where $P_t^{r,f}$ denotes the aggregate per-period pension benefits of the group of retirees, to be characterised below.

Aggregate human wealth relations satisfy (using (5) and (13)):

$$h_t^r = \sum^i (d_t^{r,i} + \frac{\gamma}{1+r_{t+1}} h_{t+1}^{r,i}) = d_t^r + \frac{\gamma}{1+r_{t+1}} h_{t+1}^r \tag{21}$$

$$h_t^w = \sum^j \left(d_t^{w,j} + \frac{1}{1+r_{t+1}} \left(\frac{\omega}{\Omega_{t+1}} h_{t+1}^{w,j} + (1 - \frac{\omega}{\Omega_{t+1}}) h_{t+1}^{r,j} \right) \right) = d_t^w + \frac{1}{1+r_{t+1}} \left(\frac{\omega}{\Omega_{t+1}} h_{t+1}^w + (1 - \frac{\omega}{\Omega_{t+1}}) \frac{1}{\psi} h_{t+1}^r \right), \tag{22}$$

where it is necessary to reweigh h_{t+1}^r in (22) with the term $\frac{1}{\psi} = \frac{N^w}{N^r}$ in order to be consistent with the definition of h_t^r in (21).

Aggregate private financial wealth relations satisfy the following recursive formulations:

$$a_t^r = (1+r_t) a_{t-1}^r + d_t^r - c_t^r - \frac{i_t}{1+i_t} m_t^r + (1-\omega) \left((1+r_t) a_{t-1}^w + d_t^w - c_t^w - \frac{i_t}{1+i_t} m_t^w \right) \tag{23}$$

$$a_t^w = \omega \left((1+r_t) a_{t-1}^w + d_t^w - c_t^w - \frac{i_t}{1+i_t} m_t^w \right) \tag{24}$$

$$a_t = a_t^w + a_t^r,$$

where (23) shows that the aggregate private savings brought into period $t+1$ by those who are retired in period $t+1$ consiststs of two parts. Firstly, it consists of the sum of income that was not spent by retirees in period t. The lack of a multiplication by γ reflects that all savings by retirees in period t are transferred to surviving retirees in period $t+1$. Secondly, it consists of the sum of income not spent in period t by those workers who become retired in period $t+1$. The remainder of the sum of income not spent in period t by workers is given by (24), since newly born workers start out without private financial wealth.

Aggregate consumption relations satisfy (using (6) and (14)):

$$c_t^r = \sum^i \left(\epsilon_t \pi_t \left(\frac{(1+r_t)}{\gamma} a_{t-1}^{r,i} + h_t^{r,i} \right) \right) = \epsilon_t \pi_t \left((1+r_t) a_{t-1}^r + h_t^r \right) \tag{25}$$

$$c_t^w = \sum^j \left(\pi_t \left((1+r_t) a_{t-1}^{w,j} + h_t^{w,j} \right) \right) = \pi_t \left((1+r_t) a_{t-1}^w + h_t^w \right) \tag{26}$$

$$c_t = c_t^r + c_t^w,$$

where the lack of a multiplication by $\frac{1}{\gamma}$ in (25) is a reflection of the perfectly competitive mutual fund: retirees earn a return $\frac{1+r_t}{\gamma}$ on private financial wealth, but only a fraction γ of them survives.

Aggregate real balance relations satisfy (using (3) and (10)):

$$m_t^r = \sum^i \left(\frac{v_2}{v_1} \frac{1+i_t}{i_t} c_t^{r,i} \right) = \frac{v_2}{v_1} \frac{1+i_t}{i_t} c_t^r \tag{27}$$

$$m_t^w = \sum^j \left(\frac{v_2}{v_1} \frac{1+i_t}{i_t} c_t^{w,j} \right) = \frac{v_2}{v_1} \frac{1+i_t}{i_t} c_t^w \tag{28}$$

$$m_t = m_t^w + m_t^r$$

We now have derived aggregate relations for all variables related to the demand-side of the economy. We next turn to a treatment of the pension fund finances and the way it determines its policy.

2.3 Pension fund

This section describes the measurement of the assets and liabilities of the pension fund. Furthermore, it specifies how the pension fund determines its policy. We will show that the model set-up is flexible enough to nest a wide range of different types of pension funds, depending on the specific parametrisation, and discuss the (general equilibrium) impacts that each specific type of pension fund restoration policy will have.

2.3.1 Pension fund finances

The assets of the pension fund are the paid contributions by workers and retirees, which consequently are invested in the capital stock of the economy. Each period, the pension fund receives the pension contributions $\tau_t w_t l_t$ and pays out $\mu_t P_t^{r,f}$ to the currently retired (where $P_t^{r,f}$ are the aggregate per-period pension benefits of the current group of retirees, to be defined below). Additionally, the pension fund starts out in period t with K_t^f worth of assets. This gives the following recursive formulation for the pension fund capital:

$$K_{t+1}^f = (1 + r_{t+1})(K_t^f + \tau_t w_t l_t - \mu_t P_t^{r,f}) \tag{29}$$

At the start of period t, the liabilities of the pension fund are the present discounted value of all the promised per-period pension benefits to currently alive workers and retirees. Note that current and future adjustments to indexation (i.e. $\mu_t \neq 1$ for some t) and accrual of pension benefits do not yet belong to the current liabilities of the pension fund. In order to compute the liabilities of the pension fund, we first define the pension fund annuity factor equivalents of R_t^r and R_t^w:

$$R_t^{r,f} = 1 + \frac{\gamma}{1 + r_{t+1}} R_{t+1}^{r,f} \tag{30}$$

$$R_t^{w,f} = \frac{1}{1 + r_{t+1}} (\omega R_{t+1}^{w,f} + (1 - \omega) R_{t+1}^{r,f}) \tag{31}$$

In words, $R_t^{r,f}$ denotes the present value from the perspective of the pension fund of the expected cash flow to a retiree in period t per unit of accumulated pension benefits (similarly for $R_t^{w,f}$). It can be seen that, in computing $R_t^{r,f}$ and $R_t^{w,f}$, the pension fund sets $\mu_t = 1$, $\forall t$. We can therefore understand $R_t^{r,f}$ and $R_t^{w,f}$ as the 'no policy'-equivalents of R_t^r and R_t^w, reflecting a 'normal' course of future action in which the pension fund fully covers extended promises to retirees and workers. This is in accordance with Novy-Marx and Rauh (2011) who recognise the Accumulated Benefit Obligation (ABO) as a proper definition of the liabilities of a pension fund. Even if the pension fund would be completely frozen, the ABO would denote the current value of accrued pension benefits still contractually owed to pension fund participants. Furthermore, note that $R_t^{w,f}$ fundamentally differs from R_t^w due to the omission of the term Ω_{t+1}, coming from the fact that the pension fund is an ongoing concern which does not have a life-cycle motive like workers do. The pension fund simply discounts its liabilities at the market interest rate given the actual transition probabilities ω and $1 - \omega$. This again is a manifestation of the non-Ricardian nature of the pension fund, similar to the government in Gertler (1999).

In order to compute the liabilities of the pension fund, we also need to define the aggregate per-period pension benefits of the group of retirees and workers at the beginning of period t, $P_t^{r,f}$ and $P_t^{w,f}$ respectively:

$$P_t^{r,f} = \gamma \left(\mu_{t-1} P_{t-1}^{r,f} + \nu_{t-1} \xi w_{t-1} l_{t-1}^r \right) + (1 - \omega) \left(\mu_{t-1} P_{t-1}^{w,f} + \nu_{t-1} w_{t-1} l_{t-1}^w \right) \tag{32}$$

$$P_t^{w,f} = \omega \left(\mu_{t-1} P_{t-1}^{w,f} + \nu_{t-1} w_{t-1} l_{t-1}^w \right) \tag{33}$$

For our purposes, we only need to know the aggregate per-period pension benefits of the group of retirees and workers to determine the liabilities of the pension fund. Our previously made assumptions ensure that we do not need to keep track of the promises the pension fund made to specific workers or retirees. All that matters for the management of the pension fund (and the determination of economic aggregates) is the aggregate stock of extended per-period pension promises. This again follows from the fact that within the groups of workers and retirees, all individuals have the same marginal propensity to consume (irrespective of age and wealth).

We can now define[8] the liabilities of the pension fund at the beginning of period t as:

$$L_t^f = R_t^{r,f} P_t^{r,f} + R_t^{w,f} P_t^{w,f} \tag{34}$$

For the next section it will be useful to write (34) recursively. We can do so by rolling (34) one period forward and plugging in the identities (30), (31), (32), and (33):

$$L_{t+1}^f = (1 + r_{t+1}) \left(\mu_t L_t^f + (R_t^{r,f} - 1)\nu_t \xi w_t l_t^r + R_t^{w,f} \nu_t w_t l_t^w - \mu_t P_t^{r,f} \right), \tag{35}$$

which states that the pension fund liabilities at the start of period $t + 1$ are equal to the future value of this period's liabilities (corrected for current indexation), plus newly issued pension entitlements to working retirees and workers, minus this period's fulfilled pension promises to retirees.

2.3.2 Pension fund policy

As is typically the case in reality, the policy of the pension fund will be determined on the basis of the financial position of the pension fund rather than on the basis of the maximisation of the utility of the pension fund participants. Pension fund regulations generally stipulate that any funding surplus or deficit should be reduced over time. To replicate such regulations in our model, we suppose that the policy of the pension fund is set to reduce the next period's funding gap (assets minus liabilities) to a fraction υ of the current funding gap[9]:

$$K_{t+1}^f - L_{t+1}^f = \upsilon(K_t^f - L_t^f), \tag{36}$$

where if $\upsilon = 0$ the funding gap should be closed within one period and if $0 < \upsilon < 1$ the funding gap is slowly closed over time.

Before moving on, we should be explicit about the timing of the pension fund policy and the ways in which workers and retirees can influence the policy decisions of the pension fund. With respect to timing, we assume that at the start of the period an unanticipated shock can occur, which the pension fund observes and bases its policy upon. Thus, the pension fund announces its policy after the shock materialises. The other agents in the model (workers, retirees, producers, and the government) then make their decisions. In determining its policy, the pension fund therefore takes into account the way it will influence the decisions of others.

Our previously made assumptions guarantee that the other agents in the model will not be able to influence the policy of the pension fund. They simply take the pension fund policy as exogenously given. This stems from the fact that participation in the pension scheme is mandatory for the working retirees and workers and that the number of individuals within each cohort is 'large'. More specifically, the assumption of a 'large' number of retirees and workers ensures that the contributions of a single agent have negligible effects on the financial position of the pension fund (and therefore its policies). Furthermore, mandatory participation ensures that the pension fund does not collapse in case of underfunding or overfunding. As highlighted

[8] Note that this definition of pension fund liabilities holds exactly (and not only in expectation) due to the large group assumption of workers and retirees, and the lack of aggregate uncertainty in the model.

[9] Note that this specification holds exactly since we consider a certainty economy.

by Beetsma et al. (2013), newly born workers would not want to participate in case the pension fund is underfunded as they would have to help restore funding adequacy. Additionally, van Bommel and Penalva (2012) highlight that older agents have an incentive to exclude newly born workers from participating in case the pension fund is overfunded so as to capture the funding surplus for themselves. Our assumptions prohibit such behaviour.

We can substitute (29) and (35) into (36) to obtain:

$$\frac{1+r_{t+1}-\upsilon}{1+r_{t+1}}(K_t^f - L_t^f) = (\mu_t - 1)L_t^f + w_t\nu_t\left((R_t^{r,f}-1)\xi l_t^r + R_t^{w,f} l_t^w\right) - w_t\tau_t l_t, \tag{37}$$

where the left-hand side denotes the 'gap to be filled' and the right-hand side specifies the ways in which the pension fund can do so. For instance, if $K_t^f < L_t^f$ the pension fund can reduce the funding gap by reducing indexation ($\mu_t < 1$), lowering accumulation of pension benefits (decrease ν_t), or hiking labour income pension fund contributions (increase τ_t).[10] Again, the pension fund internalises that its policy has general equilibrium effects on factor prices and labour supply.

Since ν and τ have similar effects in the model, we assume that the accrual rate is constant across all periods and exogenously determined, i.e. $\nu_t = \nu$, $\forall t$. We can then introduce the parameter υ_μ which dictates the share of the funding gap closure stemming from adjustments to indexation:

$$(\mu_t - 1)L_t^f = \upsilon_\mu\left(\frac{1+r_{t+1}-\upsilon}{1+r_{t+1}}(K_t^f - L_t^f)\right) \tag{38}$$

In our analysis, υ, υ_μ, and ν are exogenously determined parameters that resemble the regulations of various types of pension systems. Equations (37) and (38) then jointly determine the pension fund policy τ_t and μ_t. As a result, the pension fund does not internalise that its policy can have asymmetric effects on the utility of workers and retirees. This simply is a manifestation of the fact that pension fund policy is based on a funding gap closure rule rather than welfarian considerations. However, even though the pension fund does not consider the utilities of workers and retirees, the pension fund policy will affect them. For instance, if the pension fund restores a funding gap primarily by adjusting indexation it will largely affect retirees as they are most dependent on their pension benefits. On the contrary, restoring a funding gap by increasing labour income contributions hits workers the hardest as they supply most labour.

In case of underfunding, the pension fund thus has to allocate the burden of funding restoration over the groups of workers and retirees, and doing so will have different general equilibrium implications. On the one hand, adjusting indexation will depress aggregate consumption since retirees have a higher marginal propensity to consume out of wealth than workers. On the other hand, increasing the effective labour income tax rate through increased pension fund contributions will distort labour supply. The following section will explore this argument in more detail.

Before moving to the consideration of specific pension arrangements, we should consider one final aspect of the non-Ricardian nature of the pension fund in our model. As mentioned above, due to the presence of the subjective reweighting of transition probabilities term Ω in the worker decision problem and the different

[10] Note that $\frac{1+r_{t+1}-\upsilon}{1+r_{t+1}} > 0$.

marginal propensities to consume out of wealth for retirees and workers (captured by ϵ) the pension fund is non-Ricardian. Another manifestation of this lies in the fact that the pension fund, with its policy, effectively creates a new asset in the economy. Workers reap a return of $1 + r$ on their private financial savings, while retirees reap a return of $\frac{1+r}{\gamma}$ on their private financial savings through the perfectly competitive mutual fund that provides shelter from the risk of the timing of death. Since the pension fund invests the paid contributions in the same capital stock as workers and retirees, it also reaps a return of $1 + r$ on its assets. However, the design of the pension fund creates a disconnect between the return on assets the pension fund faces and the returns it distributes to workers and retirees. More specifically, the mandatory contributions to the pension fund do not earn an explicit return such as $1 + r$ and $\frac{1+r}{\gamma}$. Rather, the effective returns on pension contributions depend on the pension fund policy τ, μ, and ν. Since these pension fund policy parameters are the same for all retirees and workers, the pension funds in this model are of the 'collective' type, meaning that workers and retirees reap similar effective returns on their mandatory pension savings. Heuristically, the pension fund therefore forces working retirees and workers to purchase an asset with a return that is different from $1 + r$ and $\frac{1+r}{\gamma}$, in turn introducing a new asset in the economy.

2.3.3 Various types of pension systems

The model accomodates a range of different pension systems by setting specific values for the parameters υ and υ_μ. We consider the following four types of pension systems: laissez-faire, defined contribution, defined benefit, and hybrid systems.

- **Laissez-faire (also known as individual defined contribution):** In this pension arrangement there effectively is no pension system. All agents save for themselves. The pension fund does not levy contributions ($\tau = 0$), and does not allow agents to build up pension benefits ($\nu = 0$). As such, $K^f = L^f = 0$. The pension fund does not impact the economy and never needs to restore a funding gap. This laissez-faire system could alternatively be identified as an individual defined contribution pension system. Agents reap a private return on the capital market (contrary to the collective return that would be reaped through the pension fund), and are maximally exposed to any unanticipated changes to the capital stock and their savings. This pension arrangement presents itself as a useful benchmark for the upcoming numerical simulations.

- **Defined contribution (also known as collective defined contribution):** In this pension arrangement, we set $\upsilon_\mu = 1$ and can arbitrarily set $\upsilon \in [0, 1)$. This entails that any funding gap must be closed entirely through the indexation instrument μ. If $\upsilon = 0$, the funding gap is closed in similar fashion as in the laissez-faire case: pension entitlements are cut immediately in order to completely restore funding adequacy. Since retirees are most reliant on pension benefits, their consumption will likely be depressed the most in case of an adverse shock to the capital stock. Due to the higher MPCW of retirees, their downfall in consumption could have substantial consequences for aggregate demand. However, if $\upsilon \in (0, 1)$ the closure of the funding gap is smoothed out over time, and therefore the immediate drop in retiree consumption will likely be smaller.

- **Defined benefit:** In this pension arrangement, we set the other extreme in terms of pension fund policy: $\upsilon_\mu = 0$. This means that any funding gap must be closed entirely through the contributions

instrument τ. Correspondingly, the per-period pension benefits are guaranteed with $\mu = 1$ in all periods. Rather than writing down accumulated pension benefits, instead the labour market supply is distorted. As a result, workers are mostly affected in case of an adverse shock to the capital stock since they supply more labour compared to retirees. Additionally, the implied labour market distortions could have substantial consequences for aggregate supply.

- **Hybrid:** In this pension arrangement, $\upsilon_\mu \in (0, 1)$, meaning that both the indexation and contribution instrument are used to restore the pension funding gap. Such a hybrid system can be compared to nominal defined benefit pension systems, which are common in The Netherlands. Since only nominal pension benefits are guaranteed, inflation implies that the real value of accumulated pension benefits is decreasing over time (i.e. $\mu < 1$ in our model setting). In case of an adverse shock to the stock of capital, pension funds generally aim to divide the burden of financial adequacy restoration over both workers and retirees. As such, both the inflation and contribution instrument are used to fill the funding gap.

2.4 Firms

The supply-side of the economy has a familiar New-Keynesian structure as described in Woodford (2003). More specifically, there is a perfectly competitive final goods producing sector which transforms the output from imperfectly competitive intermediate goods producers. Intermediate goods producers (in the spirit of Dixit and Stiglitz (1977)) face Calvo (1983)-type pricing frictions. Additionally, there are perfectly competitive capital goods producing firms who face capital adjustment costs à la Fernandez-Villaverde (2006) and Christiano et al. (2005). This specification of the supply-side has been chosen so that the model generates realistic short-term dynamics and so that investment dynamics exhibit persistent reactions to shocks hitting the economy. Since the derivations relating to the supply-side of our economy have already been documented elsewhere, we omit most intermediate steps.

2.4.1 Final goods sector

There is a continuum of intermediate good producers, indexed by $z \in [0, 1]$. The perfectly competitive final goods sector assembles intermediate goods according to:

$$y_t = \left[\int_0^1 (y_{z,t})^{\frac{\theta-1}{\theta}} \, dz \right]^{\frac{\theta}{\theta-1}},$$

where $\theta > 1$ is the elasticity of demand for intermediate goods. Each intermediate good z is produced by one firm (which is also indexed by z) and sold at price $P_{z,t}$. The final goods producing sector maximises profits taking all prices (P_t, the price of the final good, and $P_{z,t}$, $\forall z \in [0, 1]$) as given. This gives rise to the following demand function for the output of a particular intermediate good z producing firm:

$$y_{z,t} = y_t \left[\frac{P_{z,t}}{P_t} \right]^{-\theta}$$

Imposing zero profits in the final goods sector yields that the price of the final good can be understood as an average of the intermediate good z prices:

$$P_t = \left[\int_0^1 (P_{z,t})^{1-\theta} \, dz \right]^{\frac{1}{1-\theta}}$$

2.4.2 Intermediate goods sector

The intermediate good z is produced according to:

$$y_{z,t} = (A_t^{lap} l_{z,t})^{\alpha} (k_{z,t})^{1-\alpha},$$

where A_t^{lap} is the exogenously given labour augmenting productivity and $l_{z,t}$ and $k_{z,t}$ denote the employed labour and capital by the intermediate good z producing firm. Since the intermediate goods producing firms rent their employed capital from the capital goods producing sector, capital used in production in period t is indexed by t as well. The markets for labour and capital are perfectly competitive, and so the intermediate good z producing firm takes their prices as given. Cost minimisation by the intermediate good z producing firm implies:

$$w_t = mc_t [\alpha \left(\frac{k_{z,t}}{A_t^{lap} l_{z,t}} \right)^{1-\alpha} A_t^{lap}] \rightarrow mc_t = \frac{w_t l_{z,t}}{\alpha y_{z,t}} \tag{39}$$

$$r_t^k = mc_t [(1-\alpha) \left(\frac{A_t^{lap} l_{z,t}}{k_{z,t}} \right)^{\alpha}] \rightarrow mc_t = \frac{r_t^k k_{z,t}}{(1-\alpha) y_{z,t}}, \tag{40}$$

where mc_t is the real marginal cost (and Lagrangian multiplier on the cost minimisation constraint), w_t the real wage, and r_t^k the real rental rate. This implies for the profits of the intermediate good z producing firm:

$$\begin{aligned} f_{z,t} &= \frac{P_{z,t}}{P_t} y_{z,t} - w_t l_{z,t} - r_t^k k_{z,t} \\ &= y_{z,t} \left(\frac{P_{z,t}}{P_t} - mc_t \right) \end{aligned}$$

Furthermore, from (39), (40), and the fact that all intermediate goods producing firms face the same input prices it follows that each intermediate good z producing firm employs the same capital-labour ratio:

$$\frac{k_{z,t}}{l_{z,t}} = \frac{1-\alpha}{\alpha} \frac{w_t}{r_t^k}$$

This in turn entails that we can write mc_t as:

$$mc_t = \left(\frac{w_t}{\alpha A_t^{lap}} \right)^{\alpha} \left(\frac{r_t^k}{1-\alpha} \right)^{1-\alpha}$$

The intermediate goods producing firms are subject to Calvo (1983) pricing. Each period a fraction $1-\zeta$ of firms can reset its price (and it will do so in an optimal fashion, taking into account the probability that

it cannot change prices in future periods). A fraction ζ of firms cannot adjust its price. Denote with $P^*_{z,t}$ the optimal reset price in period t by the intermediate good z producing firm that can change its price. Since for simplicity the workers are assumed to receive the profits of intermediate goods producing firms, the appropriate pricing kernel used to value profits received in i periods, as shown by Fujiwara and Teranishi (2008), is $\beta^i \Delta_{t+i} = \beta^i \frac{\partial V^w_{t+i}}{\partial c^w_{t+i}} = \beta^i (\pi_{t+i})^{-\frac{1}{\rho}} (\frac{v_2}{v_1} \frac{1}{(1-\tau^w_{t+i})w_{t+i}})^{v_2} (\frac{v_3}{v_1} \frac{1+i_{t+i}}{i_{t+i}})^{v_3}$.[11]

Profit maximisation by an intermediate good z producing firm that can change its price in period t yields the following condition characterising the optimal reset price:

$$\frac{P^*_t}{P_t} = \frac{\theta}{\theta - 1} \frac{\sum_{i=0}^{\infty} (\zeta\beta)^i \Delta_{t+i} \left(\frac{1}{P_{t+i}}\right)^{1-\theta} y_{t+i} mc_{t+i} \frac{P_{t+i}}{P_t}}{\sum_{i=0}^{\infty} (\zeta\beta)^i \Delta_{t+i} \left(\frac{1}{P_{t+i}}\right)^{1-\theta} y_{t+i}}, \tag{41}$$

where we used that the symmetric nature of the economic environment implies that all adjusting firms will choose the same price, i.e. $P^*_t = P^*_{z,t}$, $\forall z$. Contrary to typical New-Keynesian models, we are studying a certainty case such that we do not have an expectation operator in (41).

The evolution of the price level can be written as:

$$P_t = \left[\zeta (P_{t-1})^{1-\theta} + (1-\zeta)(P^*_t)^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

In anticipation of the upcoming numerical simulations, we need to write (41) recursively so that it does not contain infinite sums. Fernandez-Villaverde (2006) shows that we can formulate the optimal reset price condition as $\theta g^1_t = (\theta - 1) g^2_t$ with $g^1_t = \Delta_t mc_t y_t + \beta\zeta (\frac{P_{t+1}}{P_t})^{\theta} g^1_{t+1}$ and $g^2_t = \Delta_t \frac{P^*_t}{P_t} y_t + \beta\zeta (\frac{P_{t+1}}{P_t})^{\theta-1} (\frac{P_{t+1} P^*_t}{P_t P^*_{t+1}}) g^2_{t+1}$.

2.4.3 Capital goods sector

The perfectly competitive capital goods producing sector is responsible for accumulating fresh capital and renting out capital to the intermediate goods producing firms. After the production of intermediate and final goods is completed, the capital goods producing sector purchases i^k_t units of output in order to produce new capital goods according to:

$$k_t = (1-\delta) k_{t-1} + \left(1 - S[\frac{i^k_t}{i^k_{t-1}}]\right) i^k_t,$$

with $S[\frac{i^k_t}{i^k_{t-1}}] = \frac{\chi}{2} (\frac{i^k_t}{i^k_{t-1}} - 1)^2$. This capital evolution specification contains capital adjustment costs in the sense that investing i^k_t consumption goods in period t will only increase tomorrow's capital stock by $\left(1 - S[\frac{i^k_t}{i^k_{t-1}}]\right) i^k_t$. This specification is similar to Fernandez-Villaverde (2006) and Christiano et al. (2005) and χ (the second derivative of $S[\frac{i^k_t}{i^k_{t-1}}]$) represents the severity of the capital adjustment costs. The addition of capital adjustment costs will give rise to persistent investment reactions to shocks hitting the economy and, as Dotsey (1999) argues, will make the impulse response functions of our model smoother.

[11] Note that the pricing kernel does not need to be divided by $\frac{\partial V^w_t}{\partial c^w_t}$ as this term will drop out in (41), which is the condition characterising the optimal reset price.

In maximising its profits, the capital goods producing sector uses the pricing kernel $\frac{1}{1+r}$ to value its profits over time. Profit maximisation with respect to k_t gives rise to the following arbitrage condition:

$$1 + r_t = \frac{P_t^k(1-\delta) + r_t^k}{P_{t-1}^k},$$

where P_t^k is the real price of capital, also known as Tobin's Q (i.e. the Lagrangian multiplier on the capital accumulation constraint). Profit maximisation with respect to i_t^k gives rise to the following difference equation for i_t^k:

$$1 = P_t^k \left(1 - S[\frac{i_t^k}{i_{t-1}^k}] - S'[\frac{i_t^k}{i_{t-1}^k}]\frac{i_t^k}{i_{t-1}^k}\right) + \frac{P_{t+1}^k}{1+r_{t+1}} S'[\frac{i_{t+1}^k}{i_t^k}](\frac{i_{t+1}^k}{i_t^k})^2$$

2.4.4 Aggregate supply-side relations and resource constraints

The analysis of the supply-side of the economy is concluded by the following aggregate supply-side relations and resource constraints:

$$l_t = \int_0^1 l_{z,t} dz$$

$$k_{t-1} = \int_0^1 k_{z,t} dz \tag{42}$$

$$y_t = \left[\int_0^1 (y_{z,t})^{\frac{\theta-1}{\theta}} dz\right]^{\frac{\theta}{\theta-1}}, \text{ with } y_{z,t} = (A_t^{lap} l_{z,t})^{\alpha}(k_{z,t})^{1-\alpha} \tag{43}$$

$$f_t N^w = \int_0^1 f_{z,t} dz = \int_0^1 \left(\frac{P_{z,t}}{P_t} - mc_t\right) y_{z,t} dz, \text{ with } y_{z,t} = (A_t^{lap} l_{z,t})^{\alpha}(k_{z,t})^{1-\alpha} \tag{44}$$

$$a_t + K_t^f + \tau_t w_t l_t - \mu_t P_t^{r,f} = P_t^k k_t + \frac{m_t}{1+i_t} \tag{45}$$

$$y_t = c_t + i_t^k$$

Note that condition (42) reflects that, at the aggregate level, capital is a predetermined variable. Condition (45) is the market clearing condition for savings. It ensures that the total value of savings (which is the sum of private financial wealth of workers and retirees and the assets of the pension fund brought into the next period) equates the total value of assets ($P_t^k k_t + \frac{m_t}{1+i_t}$) in the economy.

In anticipation of the upcoming numerical simulations, we need to rewrite (43) and (44) so that they do not contain integrals. Fernandez-Villaverde (2006) shows that, when exploiting that all intermediate goods producing firms employ the same capital-labour ratio, we can write aggregate supply as $y_t = \frac{(k_{t-1})^{1-\alpha}(A_t^{lap} l_t)^{\alpha}}{v_t^p}$, where $v_t^p = \int_0^1 \left(\frac{P_t}{P_{z,t}}\right)^{\theta} dz$ is a measure of price dispersion. In a flexible price equilibrium (and thus in the steady state), we have that $v_t^p = 1$. Otherwise, $v_t^p > 1$. It therefore reflects the 'loss' in output due to the nominal rigidities of Calvo (1983)-type pricing. The term is written recursively as $v_t^p = \zeta(\frac{P_t}{P_{t-1}})^{\theta} v_{t-1}^p + (1-\zeta)(\frac{P_t}{P_t^*})^{\theta}$. Given this, aggregate profits are simply $f_t N^w = y_t - w_t l_t - r_t^k k_{t-1}$.[12]

[12] Note that we multiply f_t with N^w in order to consistent with the notation used in the worker decision problem above.

2.5 Central bank and government

As in Gertler and Karadi (2011) we suppose that the central bank follows a simple Taylor rule with interest rate smoothing. We follow the specification of Kara and von Thadden (2016). The monetary authority responds to deviations of inflation from the target inflation rate of zero and the output gap $\tilde{y}_t = \ln(\frac{y_t}{\bar{y}})$, where $\bar{y}$ is the steady-state level of output:

$$i_t = \eta i_{t-1} + (1-\eta)[r_{t+1} + \gamma_\pi \pi_t^p + \gamma_y \tilde{y}_t],$$

where $\eta \in (0,1)$ is the interest rate smoothing parameter, $\pi_t^p = \frac{P_t - P_{t-1}}{P_{t-1}}$ inflation, γ_π the weight given to inflation, and γ_y the weight given to the output gap. Lastly, we have the Fisher relation $1+i_t = (1+r_{t+1})\frac{P_{t+1}}{P_t}$.

Since the government is non-Ricardian in our model, we elect to minimise the role of the fiscal authority so as to not potentially distort our research findings with respect to the effects of restoration policy on the economy. We simply assume that the only role of the government is to levy a per-capita lump sum tax τ_t^g in order to offset any changes in the money supply:[13]

$$\tau_t^g = m_{t-1}\frac{P_{t-1}}{P_t} - m_t$$

2.6 Equilibrium definition

For our economy to be in equilibrium, we require that the actions of the government, central bank, pension fund and optimising workers, retirees, and firms are mutually consistent at the aggregate level. Formally, an equilibrium is a set of sequences of the endogenous variables $\{\epsilon_t, \pi_t, \Omega_t, R_t^w, R_t^r, \tau_t^w, \tau_t^r, l_t^w, l_t^r, l_t, d_t^w, d_t^r, h_t^w, h_t^r, c_t^w, c_t^r, c_t, a_t^w, a_t^r, a_t, m_t^w, m_t^r, m_t, R_t^{w,f}, R_t^{r,f}, P_t^{w,f}, P_t^{r,f}, L_t^f, K_t^f, y_t, k_t, f_t, mc_t, w_t, r_t^k, i_t^k, P_t^k, P_t, P_t^*, g_t^1, g_t^2, \Delta_t, v_t^p, i_t, r_t, \tau_t^g\}_{t=0}^\infty$ which satisfies the system of equilibrium conditions archived in Appendix B, taking as given $\{A_t^{lap}, \tau_t, \nu_t, \mu_t\}_{t=0}^\infty$ and appropriate initial conditions for the state variables. In the steady state, we set $A^{lap} = \mu = 1$.

3 Model analysis

3.1 Baseline calibration

We elect to calibrate the model at a yearly frequency. While our model is intended for short-term analysis, numerical simulations by Shimer (2012), who also considers the transitional dynamics after a shock to the capital stock, show that the convergence back to the steady state can take a multitude of years even in a frictionless model. Furthermore, the choice to calibrate the model at a yearly frequency is in accordance

[13] A derivation can be found in Walsh (2010). Starting from the nominal government budget constraint:

$$T_t^g + M_t + B_t = M_{t-1} + (1+i_{t-1})B_{t-1} + G_t,$$

where T^g is nominal tax income, M the aggregate nominal money holdings of workers and retirees, B nominal government bonds, and G nominal government spending. In our setting, we have that $B_t = B_{t-1} = G_t = 0$. Therefore, we need that $T_t^g = M_{t-1} - M_t$ and therefore that $\tau_t^g = \frac{M_{t-1} - M_t}{P_t} = m_{t-1}\frac{P_{t-1}}{P_t} - m_t$.

with other writings related to our line of research, such as Kara and von Thadden (2016), Carvalho et al. (2016), and Gertler (1999).

We set the demographic parameters of the model to match the empirical estimates for the Euro area in 2008 reported in the statistical annex of the 2009 Ageing Report by the European Commission. Assuming that workers enter the labour force at age 20 and retire at age 65, we set $\omega = 0.978$ such that the implied average working period is 45 years. Since the life expectancy at birth is roughly 80 years and retirement starts at age 65, we set $\gamma = 0.933$ such that the implied average retirement period is 15 years. The resulting old-age dependency ratio is 0.33, which matches the estimates of the Ageing Report. Table 1 provides an overview of our chosen demographic parameters. Below we will perform sensitivity analyses with respect to the implied average retirement period.

Table 1: Demographic parameters

Retirement probability of workers	$1-\omega$	0.022
Implied average working period	$\frac{1}{1-\omega}$	45
Death probability of retirees	$1-\gamma$	0.067
Implied average retirement period	$\frac{1}{1-\gamma}$	15
Implied old-age dependency ratio	$\frac{1-\omega}{1-\gamma}$	0.333

Table 2 provides an overview of the selected structural parameters of the model. The combination of parameters β, σ and δ has been set in accordance with many other general equilibrium models calibrated at a yearly frequency and gives rise to a long-run real interest rate $r = 4.2\%$. The relative productivity of retirees ξ is set to approximate the age-profile productivity data in the large scale overlapping generations model of Auerbach and Kotlikoff (1987). Below we will perform sensitivity analyses with respect to the relative productivity of retirees. As in Kara and von Thadden (2016), we fix $\theta = 10$ in order to ensure that the profits of the intermediate goods producing firms (which accrue to workers as a simplifying assumption) are not too sizable. The preference parameters v_1, v_2, and v_3 are also matched by those of Kara and von Thadden (2016).

Table 2: Structural parameters

Intertemporal elasticity of substitution	σ	0.333
Implied ρ	ρ	1.5
Discount factor	β	0.96
Cobb-Douglas share of labour	α	0.333
Relative productivity of retirees	ξ	0.6
Depreciation rate of capital	δ	0.05
Elasticity of demand for intermediate goods	θ	10
Consumption preference parameter	v_1	0.64
Leisure preference parameter	v_2	0.358
Real balances preference parameter	v_3	0.02

The calibration of the pension fund is a sensitive issue. Due to the closed economy-nature of our model, setting realistic parameters for the accrual rate ν and the corresponding contribution rate τ results in the unrealistic calibration that the pension fund virtually completely manages the capital stock of the economy in the steady state. A study by PwC, commissioned by The Association of the Luxembourg Fund Industry,

finds that pension funds in OECD countries in 2014 invested on average one third of their total assets abroad, with positive outliers such as The Netherlands investing roughly three quarters of its total assets internationally. Much of the invested assets are allocated to developing regions that have a less developed pension system. Since our model considers a closed economy, it cannot simultaneously match the empirically observed size of the balance sheet of European pension funds and their accrual and contribution parameters. As such, we need an alternative calibration strategy. We elect to match the size of pension funds in terms of managed capital and accordingly set relatively low values for the accrual and contribution rate. In the baseline scenario, we target the 2009 OECD average pension fund assets as a share of GDP, as documented in the OECD Pensions at a Glance 2011 report.[14] Additionally, we verify whether the implied steady-state per-period transfers to the retirees as a share of total output matches the OECD 2007 average public and private pension benefit spending as a share of GDP (documented in the same report).[15] Table 3 indicates that we have set $\nu = 0.0055$, which implies $\tau = 0.022$.[16] As a consequence, the pension fund manages $\frac{K^f}{y} = 0.89$, which comes reasonably close to the reported 0.872 in the aforementioned OECD report. Furthermore, the steady-state transfers of the pension fund to the group of retirees constitutes roughly 5% of total output, lying below the documented 8.6% in the OECD report (which also includes expenditures on pay-as-you-go pension systems that are not present in our model). The resulting contribution rate of 2.2% is considerably smaller than the OECD 2009 average of 8.4%[17], but this is simply an unfortunate consequence of our closed-economy model. Below we will perform sensitivity analyses with respect to the size of the pension fund and corresponding contribution and accrual rates.

Table 3: Pension fund parameters and targeted pension fund variables

Accrual rate	ν	0.0055
Implied contribution rate	τ	0.022
Implied pension fund capital to output ratio	$\frac{K^f}{y}$	0.89
Implied retiree pension transfers to output ratio	$\frac{P^{r,f}}{y}$	0.049
Implied pension fund capital to aggregate capital ratio	$\frac{K^f}{k}$	0.274

Lastly, we need to fix the parameters that are responsible for the adjustment dynamics of the model. Table 4 presents an overview of the chosen values. For the calibration of the Taylor rule we follow the empirical estimates of Hofmann and Bogdanova (2012). The Calvo survival probability of contracts is chosen such that firms adjust their prices on average every five quarters, which is consistent with the Euro area empirical evidence summarised in Altissimo et al. (2006). We set the capital adjustment costs according to estimates of Christiano et al. (2005), but we are aware that estimates vary from 0.25 (Dotsey, 1999) to 9.81 (Fernandez-Villaverde, 2010) and will therefore perform sensitivity analyses with respect to this parameter.

[14] Reported on page 179 of OECD (2011).
[15] Reported on page 157 of OECD (2011).
[16] Recall that $\mu = 1$ in the steady state.
[17] Reported on page 153 of OECD (2011).

Table 4: Adjustment dynamics parameters

Inertial parameter in Taylor rule	η	0.7
Inflation coefficient in Taylor rule	γ_π	1.5
Output gap coefficient in Taylor rule	γ_y	0.5
Calvo survival probability of contracts	ζ	0.2
Capital adjustment costs parameter	χ	2.5

3.2 Steady state

To verify whether our calibration is sensible, we compare the steady-state values of important endogenous variables with other papers in the literature. Table 5 provides an overview. As can be seen, the marginal propensity to consume out of wealth is considerably higher for retirees than for workers, confirming the findings of Gertler (1999) and Kara and von Thadden (2016). The subjective reweighting of transition probabilities $\Omega > 1$ drives a substantial wedge between the worker annuity factor $R^w = 5.4$ and the annuity factor applied by the pension fund $R^{w,f} = 3.3$. Since the worker becomes less productive upon reaching retirement, he places a higher value on accumulated pension benefits than the pension fund. Because we set $\mu = 1$ in the steady state, we have that the steady-state effective labour income tax rates are negative for both workers and retirees. In effect, the presence of the pension fund acts as an implicit subsidy on labour (albeit a relatively small one). It is not surprising that the effective labour income tax rate for retirees is smaller than the one for workers because the additionally accumulated pension benefits are distributed to retirees sooner.

As a consequence of the effective subsidy on labour income, the labour supply of workers and retirees is slightly skewed compared to the findings of Gertler (1999) and Kara and von Thadden (2016). The labour force participation of workers is 0.6, which still comes reasonably close to the estimated 0.7 in the 2009 Ageing Report of the European Commission.[18] The labour force participation of retirees is rather high at 0.4 (compared to the 0.2 in Gertler (1999)), which to an extent is caused by the presence of the pension fund. Nonetheless, estimates by D'Addio and Whitehouse (2010) highlight that retiree labour force participation lies between 0.25 and 0.75 in many Western countries. In any case, retirees supply less than ten percent of the effective labour supply in the steady state. We will perform sensitivity analyses with respect to the retiree productivity parameter ξ to investigate the extreme case in which retirees do not supply any labour (akin to the simulations of Kara and von Thadden (2016)). Since our model does not contain government spending, the consumption to output ratios are relatively high. However, the estimated share of private financial wealth held by retirees and the capital to output ratio follow Kara and von Thadden (2016) and Gertler (1999) closely.

3.3 Restoring pension funding adequacy after a financial crisis

Our narrative is similar to Shimer (2012), who evaluates the transition path back to the stable equilibrium after starting out with a capital stock below the steady state. Obviously, Shimer (2012) has in mind the global financial crisis of 2008 which destroyed a significant portion of the economy's capital stock. We have

[18] Reported on page 171 of the statistical annex, table 'Main demographic and macroeconomic assumptions'.

Table 5: Steady-state values of selected endogenous variables

MPCW for workers	π	0.062
MPCW for retirees	$\epsilon\pi$	0.104
Subjective reweighting of transition probabilities	Ω	1.035
Effective labour income tax rate for workers	τ^w	-0.008
Effective labour income tax rate for retirees	τ^r	-0.025
Participation rate of workers	$\frac{l^w}{N^w}$	0.600
Participation rate of retirees	$\frac{l^r}{N^r}$	0.418
Capital to output ratio	$\frac{k}{y}$	3.250
Share of private financial wealth held by retirees	$\frac{a^r}{a}$	0.255
Worker consumption to output ratio	$\frac{c^w}{y}$	0.648
Retiree consumption to output ratio	$\frac{c^r}{y}$	0.189
Investment to output ratio	$\frac{i^k}{y}$	0.163

the same scenario in mind, but with a specific interest in the financial situation of pension funds and the effects that the pension fund's restoration policies have on the rest of the economy. Suppose an unexpected (financial) shock hits the economy which evaporates a certain fraction of the capital stock. Not only the private financial wealth of workers and retirees has been damaged, but also the assets of pension funds have taken a hit. As an example, Federal Reserve Flow of Funds data indicate that U.S. retirement fund assets were virtually cut in half between 2007 and 2009 as a result of the 2008 financial crisis (Treasury, 2012). Since the existing liabilities of pension funds are relatively insensitive to the capital stock shock, many pension funds are now faced with a funding deficit that needs to be filled in one way or another. As an example, recall the previously mentioned severe underfunding reflected in the financial statements of 2100 OECD exchange-listed companies (Laboul, 2010).

Generally, pension fund legislation requires funding adequacy to be restored to a certain extent and within a specific time window. Pension funds have a range of policy instruments at their disposal to fulfill legislative requirements, but the timing and the use of specific instruments could have profound effects on the rest of the economy. For instance, if the pension fund immediately cuts the accumulated pension benefits of workers and retirees, consumption could take a nosedive. On the other hand, if the pension fund increases the pension fund contribution rate on labour income, it distorts the labour supply decision of workers and retirees, and in turn aggregate supply could be severely affected. The pension fund restoration policy effectively is a matter of allocating the pension fund losses to different groups of agents (workers and retirees) from different generations (current and future). Immediately cutting the existing accrued pension benefits of workers and retirees primarily affects the current group of retirees as they are most reliant on their accumulated pension wealth. Quickly restoring pension fund assets by increasing contribution rates primarily affects the current group of workers as they supply the most labour. Postponing the closure of the funding gap entails that future generations become responsible for closing the pension funding gap.

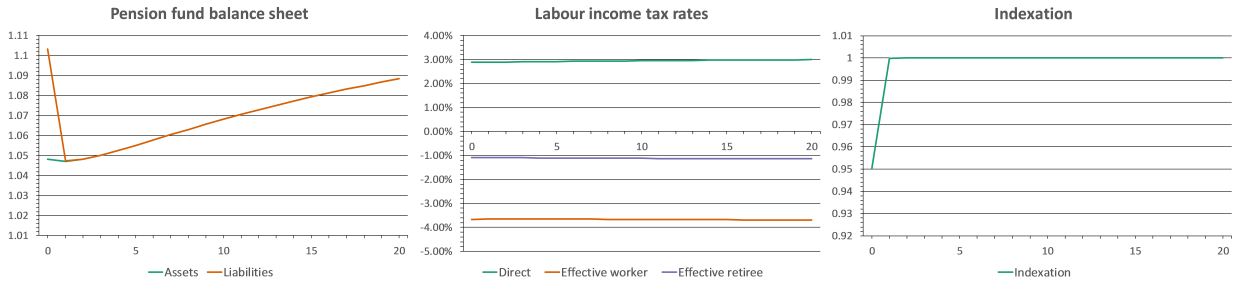
The model that we have built in this writing is sufficiently rich to evaluate the effects of pension fund restoration policy on the rest of the economy in case the pension fund is faced with a funding deficit that needs to be covered. Additionally, since we have analytical expressions for the value of workers and retirees, our model gives a crude indication of the resulting welfare losses (or gains) accruing to the aggregate group

of retirees and workers. In order to replicate a financial crisis, we introduce an unexpected shock which evaporates 10% of the economy's capital stock in the first simulation period. We estimate the model using the Dynare toolbox. Since we consider a perfect foresight model, the solution does not require linearisation. Instead, Dynare simply numerically simulates in order to find the exact paths of the endogenous variables that meet the equilibrium conditions and the paths of exogenously specified shocks. Dynare makes use of the Newton method of simultaneously solving all the equations for every period and makes the simplifying assumption that our system of equations is back in equilibrium at the end of the simulation period. As such, it is required to consider a lengthy simulation period. We will now move to a mapping between the existing types of pension funds in reality and the model pension fund parameters υ and υ_μ to investigate the pension fund restoration policy they imply.

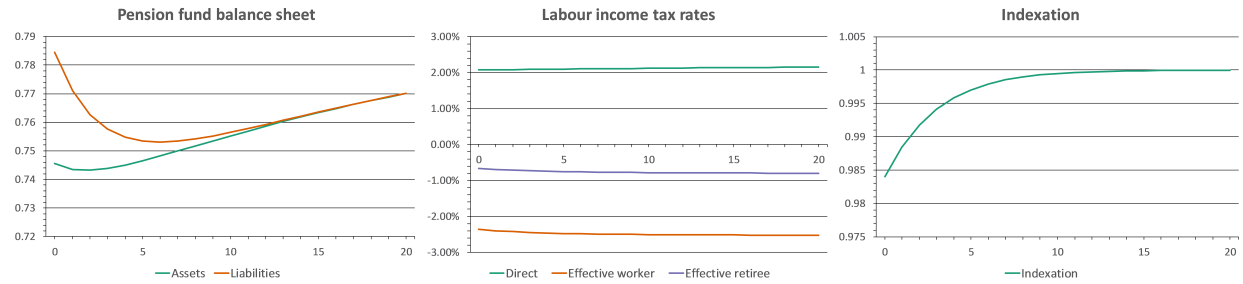
3.3.1 Different types of pension fund restoration policy

The unexpected capital stock shock described in the previous section implies that the pension fund assets K_t^f are smaller than its liabilities L_t^f. Condition (36) then postulates that next period's funding gap should be reduced to a fraction υ of the current funding gap, i.e. that the pension fund should conduct restoration policy in order to move closer towards funding adequacy. As discussed in section 2.3, there are various ways in which the pension fund can impact its assets and liabilities in order to close its funding gap. Recall that the accrual parameter ν has been fixed, which leaves the pension fund with control over the contribution rate τ and indexation μ. Essentially, the contribution rate is a means of building up assets, while indexation is a means of writing down liabilities. Subsection 2.3.3 highlighted the various types of pension funds that our model nests. We consider four pension fund arrangements in our simulations: 'Defined Contribution', 'Indexation', 'Hybrid' and 'Defined Benefit'. Figure 1 presents a graphical representation for each of them with the implied paths for the contribution rate τ and indexation μ along with the financial situation of the pension fund.

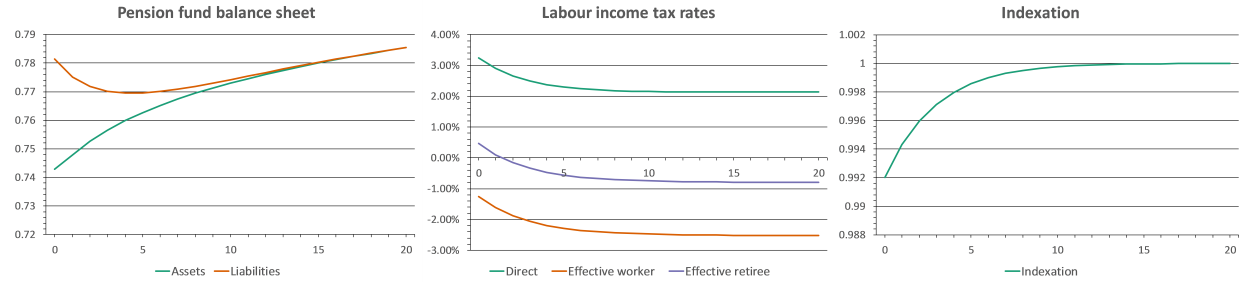
In the Defined Contribution scenario (corresponding to figure 1a) the indexation instrument μ is immediately used to write down the accumulated pension benefits of workers and retirees such that next period the pension fund does not face a funding gap. As can be seen, roughly 5% of all built-up pension wealth should be cut in order to restore the funding adequacy immediately. The contribution rate τ is left unaffected, only changing slightly over time in order to re-accumulate pension fund assets. The Indexation scenario (corresponding to figure 1b) is similar to the Defined Contribution scenario in the sense that only the indexation instrument μ is used to fill the funding gap. However, the speed of recovery is such that the half-life of the funding gap is two years. As shown in the figure, only 1.5% of accumulated pension wealth should be cut in the period the financial shock hits, but in subsequent periods pension wealth has to be written down further. The Hybrid scenario (corresponding to figure 1c) applies a mix of both pension fund instruments in order to close the funding gap. Again, the half-life of the gap is set to two years. About 0.8% of pension wealth is written down in the first period, along with an increase of the contribution rate τ of a little over 1 percentage point. After roughly five periods the contribution rate is back to its steady-state level, implying that labour supply decisions of workers and retirees will be distorted for an extended period of time. In the Defined Benefit scenario (corresponding to figure 1d) solely the contribution rate is used to make up for the losses of the pension fund. Again, the half-life of the gap is set to two years. All existing promises to workers and retirees



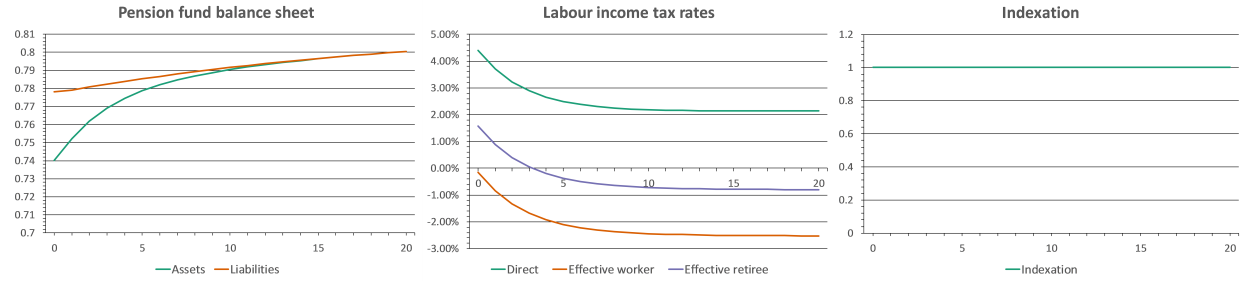
(a) Defined Contribution



(b) Indexation



(c) Hybrid



(d) Defined Benefit

Figure 1: Pension fund policy dynamics for the four different pension system scenarios after a 10% capital stock shock, with a pension funding gap half-life of two years (except in the Defined Contribution case, in which the half-life is zero years).

are guaranteed. It is therefore understandable that the contribution rate should be increased by more than 2 percentage points in the first period, causing significant distortions of optimal labour supply decisions.

3.3.2 The impact of pension fund restoration policy on the rest of the economy

Having discussed pension fund restoration policy, we can now compare the effects of the different pension arrangements on the rest of the economy. More specifically, we evaluate the effects on output, re-accumulation of capital, labour supply, consumption, and welfare. In addition to the previously described pension fund types, we consider a laissez-faire economy in which there is no pension fund and in which all agents accumulate pension savings by themselves. Figure 2 presents impulse response diagrams for the aforementioned variables for the different pension arrangements.

Several conclusions can be drawn from the impulse response diagrams. Generally, the responses of the Defined Contribution and Indexation scenarios lie close to the Laissez-faire case, with the Hybrid and Defined Benefit scenarios significantly deviating. This is expected since in the Laissez-faire case the private financial wealth of retirees and workers evaporates in similar fashion to the writing down of accumulated pension wealth in the Defined Contribution and Indexation scenarios. There are, however, some subtle differences between the Defined Contribution, Indexation, and Laissez-faire cases which we will explore below. Most striking, however, is the distortion of output in the Hybrid and Defined Benefit case. While the re-accumulation of capital is similar across all pension arrangements, labour supply and consumption are significantly affected under the pension arrangements that distort labour supply decisions when conducting restoration policy. It can be seen that worker labour supply decreases as much as 0.6 percentage points and retiree labour supply decreases as much as 2.3 percentage points under a Defined Benefit scheme compared to the Laissez-faire economy. As a result, output is distorted by as much as 0.7 percentage points (and for an extended period of time). The hump-shaped response of labour supplies partially derives from the hump-shaped investment response (as implied by the capital adjustment frictions). Initially, labour productivity and in turn wages are significantly lowered due to the capital stock shock. As capital is re-accumulated labour productivity increases and with it labour supplies.

In the Laissez-faire and Defined Contribution-type scenarios retirees have more of an incentive to increase their labour supply in order to make up for their loss in financial wealth. Workers have a similar motivation but much less so as they have a longer life to live and therefore have more time to make up their financial losses. The significant drop in labour supplies under Defined Benefit-type scenarios stems from the distortions caused by the higher effective labour income tax rates. When the value of pension wealth is guaranteed, retirees would rather enjoy more leisure than supply additional labour. Furthermore, workers would rather supply less labour since they reap less of the benefits. The impulse response diagrams of consumption highlight that the drop in output is mostly mirrored by a drop in worker demand, which can be as large as 0.7 percentage points. Nevertheless, guaranteeing pension wealth entails that retiree consumption can be as much as 0.5 percentage points higher for many periods. However, since retirees are outnumbered by workers, aggregate demand still falls short compared to Laissez-faire-type scenarios.

Let us now highlight some of the subtle differences between the Laissez-faire, Defined Contribution, and Indexation scenarios. The recovery of output and the re-accumulation is virtually identical. However,

there are some noticable differences in terms of labour supply and consumption. Retiree labour supply is generally higher under the presence of a pension fund that solely uses its indexation instrument μ. This stems from the fact that retiree labour supply is most heavily subsidised under a Defined Contribution or Indexation arrangement. As a result, retiree consumption is also slightly higher in these systems. There are, however, small differences also between the Defined Contribution and Indexation case. Since in the Indexation arrangement the accumulated pension wealth is written down slowly over time, the retirees receive a higher pension benefit in the first couple of periods after the shock compared to the Defined Contribution case. As a result, retiree labour supply sparks less, while consumption remains slightly higher. Nevertheless, it should be stressed that the differences between the Laissez-faire, Defined Contribution, and Indexation scenarios are quantitatively insignificant.

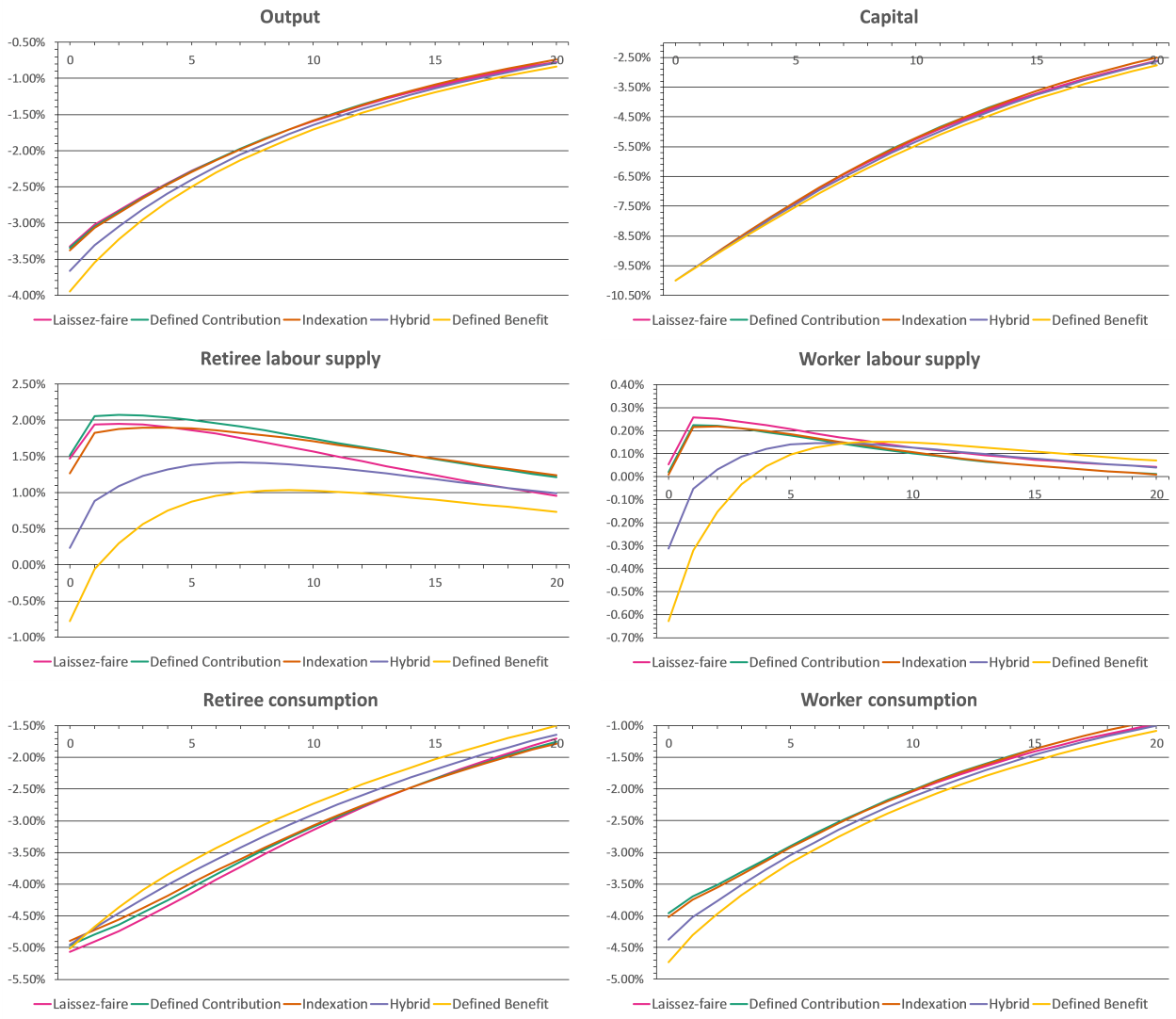


Figure 2: Impulse response diagrams for various variables after a 10% capital stock shock, compared over different pension systems. Values are in percentual deviation from the steady state.

We now turn to a crude assessment of the welfare effects of the various forms of pension fund restoration policy. Since we have analytical expressions for the values of workers and retirees, it is relatively simple to determine the welfare implications for the current groups of workers and retirees. Unfortunately, however, our model does not allow us to determine the welfare effects for specific generations of workers and retirees. This stems from the fact that income effects influence optimal labour supply decisions. Otherwise, a welfare assessment would have been possible in which the welfare of the current groups of workers and retirees is contrasted against the welfare of future generations. Such welfare analysis is performed by Keuschnigg and Keuschnigg (2004) and Jaag et al. (2010), who construct a probabilistic ageing model (essentially a generalised version of Gertler (1999) with more stages in the life-cycle of agents). Since in their model set-up all workers supply the same amount of labour (irrespective of wealth), it is possible to keep track of the wealth, labour supply, and consumption of newborn workers and thus the welfare of current and future generations. Due to the presence of income effects in the optimal labour supply decisions in our model, we can only consider the groups of workers and retirees as individual entities, rather than an amalgamation of agents born in different periods. This ultimately hinders the analysis of pension fund restoration policy with respect to the speed of adjustment parameter υ, since it is not possible to contrast the potential welfare gains from postponing the closure of the funding gap accruing to current generations against the welfare losses of future generations. Future versions of this writing will therefore adjust the worker and retiree decision problem such that no wealth effects are present in the optimal labour supply decisions. Another option would be to consider a deterministic transition through the different stages of the life-cycle, as do large-scale overlapping generation models such as Auerbach and Kotlikoff (1987).

At this stage, however, we can only perform welfare analysis for the groups of workers and retirees in the first simulation period. Ignoring the welfare of future generations has the side-effect that pension fund arrangements that shift the burden of adjustment to future generations look more favourable. Table 6 reports the percentual deviation from the steady state of the worker, retiree, and aggregate values. The displayed figures confirm the patterns observed above. The more retirees are sheltered from financial shocks through their pension wealth, the higher their life-time utility. The more workers are liable for making up the financial losses that without a pension fund would accrue to retirees as well, the lower their life-time utility. Since in our baseline simulations the group of retirees is three times smaller than the group of workers, workers lose less than retirees gain. As a result, aggregate welfare decreases less under a Defined Benefit scheme compared to the other pension arrangements.[19] Essentially, workers, who have a longer remaining lifetime, could in principle partially compensate the losses of retirees, who have less time to recover from the capital stock shock, such that aggregate welfare decreases less compared to the Laissez-faire, Defined Contribution, or Indexation cases. However, this does not mean that the distortions imposed by a Defined Benefit pension system are unambiguously justified. Since part of the welfare gains to the current population derives from the slower closure of the pension funding gap, a portion of the aggregate welfare gains comes at the expense of the welfare of future generations. Without explicitly measuring the welfare of future generations it is impossible to draw welfarian conclusions about the favourability of pension funds that close their funding gap over an

[19] Note that the steady state is the same for the four pension systems. Only the laissez-faire steady state differs from the four pension systems.

Table 6: First period value comparisons after a 10% capital stock shock across different pension arrangements. Values are in percentual deviation from the steady state.

	Retiree value	Worker value	Aggregate welfare
Laissez-faire	−3.19%	−1.83%	−2.23%
Defined Contribution	−3.08%	−1.83%	−2.22%
Indexation	−2.97%	−1.85%	−2.20%
Hybrid	−2.70%	−1.87%	−2.13%
Defined Benefit	−2.44%	−1.88%	−2.06%
Difference between Defined Benefit and Laissez-faire	0.62%	−0.05%	0.17%

extended period of time in the wake of unexpected capital stock shocks. However, if the contribution rate is set such that the pension funding gap is closed in virtually one period (so that future generations are virtually unaffected), we still find that the Defined Benefit pension scheme can be welfare improving.

3.3.3 Sensitivity analyses

As promised in section 3.1, we will now conduct sensitivity analyses in order to further explore the effects of pension fund restoration policy after a capital stock shock. More specifically, we will consider three different parameter configurations in turn: low retiree productivity, increased pension fund size, and increased life expectancy. Additionally, we will consider the interaction between the speed of recovery parameter υ and the capital adjustment cost parameter χ. Table 7 depicts the first period percentual deviation from the steady state for various endogenous variables across different parameter configurations and pension arrangements.[20]

The first sensitivity analysis is with respect to the retiree productivity parameter ξ. As in Kara and von Thadden (2016), we set $\xi = 0.2$ such that the labour supply of retirees is effectively zero. This configuration represents a worst-case scenario in which retirees have no real physical means to restore their lost assets, and as such leaves retirees maximally exposed to unexpected shocks evaporating their financial wealth. In turn, this allows us to gauge the maximal impact a Defined Benefit pension system can have on the welfare of the current group of retirees. While the results are qualitatively similar to the baseline scenario, some interesting quantitative aspects can be highlighted. The retiree labour supply response is amplified as we compare the change in labour supply to a steady state in which labour supply is virtually zero. Since retirees have difficulties with restoring their financial wealth, their drop in consumption and value is considerable. In this case, a Defined Benefit pension fund would allow retirees to experience 2 percentage points less of a drop in life-time utility, at a worker value cost that is slightly larger than in the baseline case. A Defined Benefit pension system is thus most likely to be welfare improving if retirees have few means of re-accumulating wealth after a capital stock shock.

The second sensitivity analysis is with respect to the pension fund accrual rate ν, which was calibrated such that the pension fund manages assets worth roughly 90% of output. We now set $\nu = 0.008$ so that the pension fund manages assets worth roughly 130% of output, which is currently the case in countries such as The Netherlands, Ireland, and Switzerland (OECD, 2011). This entails that a larger fraction of retiree financial wealth is managed by the pension fund. A Defined Benefit pension fund can therefore isolate retiree

[20] Since impulse response diagrams are qualitatively similar to those depicted in figure 2, we elect to simply report the first period values for brevity.

financial wealth from unexpected capital stock shocks to a greater degree. However, this requires a first-period increase in the contribution rate of 3.3 percentage points (compared to the 2 percentage points in the baseline scenario) since the pension funding gap is now also larger. As such, the distortions on output and labour supplies are considerably greater as well.

The third sensitivity analysis is with respect to the life expectancy of agents. More specifically, rather than assuming that agents are retired for fifteen years on average, we now assume that agents are retired for twenty years on average. This corresponds to the demographic projections for the coming fifty years reported in the statistical annex of the 2009 Ageing Report by the European Commission. This scenario is a mirror image of the first sensitivity analysis in the sense that now retirees are better equipped to deal with their loss of financial wealth, as they have more time to re-accumulate assets. The changes in the life-time utilities of the group of retirees and workers are therefore smaller compared to the baseline scenario. However, since retirees pay pension fund contributions for a longer period during retirement, the size of the pension fund is larger as well. This allows a Defined Benefit pension fund to shelter retiree savings from unexpected capital stock shocks to a greater extent as in the second sensitivity analysis, but again with a more distortionary impact on the economy.

Table 7: First period percentual deviation from the steady state of various endogenous variables after a 10% capital stock shock across different pension arrangements and parameter configurations. Values are in percentual deviation from the steady state.

		y	l^r	l^w	c^r	c^w	V^r	V^w
Baseline	Laissez-faire	−3.32%	1.47%	0.05%	−5.07%	−3.96%	−3.19%	−1.83%
	Def. Contrib.	−3.35%	1.52%	0.02%	−4.97%	−3.96%	−3.08%	−1.83%
	Indexation	−3.37%	1.27%	0.01%	−4.90%	−4.02%	−2.97%	−1.85%
	Hybrid	−3.66%	0.24%	−0.31%	−4.96%	−4.37	−2.70%	−1.87%
	Def. Benefit	−3.94%	−0.78%	−0.63%	−5.02%	−4.73%	−2.44%	−1.88%
Low retiree productivity ($\xi = 0.2$)	Laissez-faire	−3.26%	27.6%	0.00%	−6.86%	−4.08%	−5.24%	−2.29%
	Def. Contrib.	−3.28%	117.12%	0.05%	−6.33%	−4.08%	−4.68%	−2.23%
	Indexation	−3.32%	103.34%	0.02%	−6.17%	−4.15%	−4.46%	−2.25%
	Hybrid	−3.76%	34.69%	−0.47%	−6.05%	−4.71%	−3.81%	−2.28%
	Def. Benefit	−4.21%	−33.43%	−0.94%	−5.96%	−5.28%	−3.17%	−2.31%
Large pension fund ($\nu = 0.008$)	Laissez-faire	−3.32%	1.47%	0.05%	−5.07%	−3.96%	−3.19%	−1.83%
	Def. Contrib.	−3.36%	1.57%	0.00%	−4.94%	−3.96%	−3.05%	−1.82%
	Indexation	−3.40%	1.19%	0.00%	−4.84%	−4.04%	−2.89%	−1.85%
	Hybrid	−3.80%	−0.39%	−0.46%	−4.92%	−4.55%	−2.50%	−1.88%
	Def. Benefit	−4.21%	−1.93%	−0.88%	−5.00%	−5.05%	−2.11%	−1.90%
Increased life expectancy ($\gamma = \frac{19}{20}$)	Laissez-faire	−3.32%	1.08%	0.00%	−4.84%	−3.95%	−2.81%	−1.70%
	Def. Contrib.	−3.34%	1.10%	0.05%	−4.76%	−3.96%	−2.72%	−1.69%
	Indexation	−3.38%	0.88%	0.04%	−4.72%	−4.02%	−2.62%	−1.71%
	Hybrid	−3.77%	−0.27%	−0.41%	−4.87%	−4.51%	−2.34%	−1.76%
	Def. Benefit	−4.17%	−1.40%	−0.84%	−5.03%	−5.02%	−2.05%	−1.81%

Lastly, we will study the interaction between the speed of recovery parameter υ and the capital adjustment cost parameter χ. The results above indicate that a Defined Benefit pension system could be welfare improving, despite imposed distortions on the economy. However, this crucially depends on the speed at which the pension funding gap should be closed and the ease with which capital can be re-accumulated. To highlight this mechanism, figure 3 presents impulse response diagrams for output, capital, contribution rate, aggregate labour supply, retiree consumption, and worker consumption for four scenarios. We contrast a case of lower adjustment cost and slower recovery ($\upsilon = 0.70$ and $\chi = 2.5$, as in the baseline scenario) against a case of either higher adjustment costs or speedier recovery, or both (with $\upsilon = 0.40$, implying a funding gap half-life of three quarters, and $\chi = 10$, as estimated in Fernandez-Villaverde (2010)).

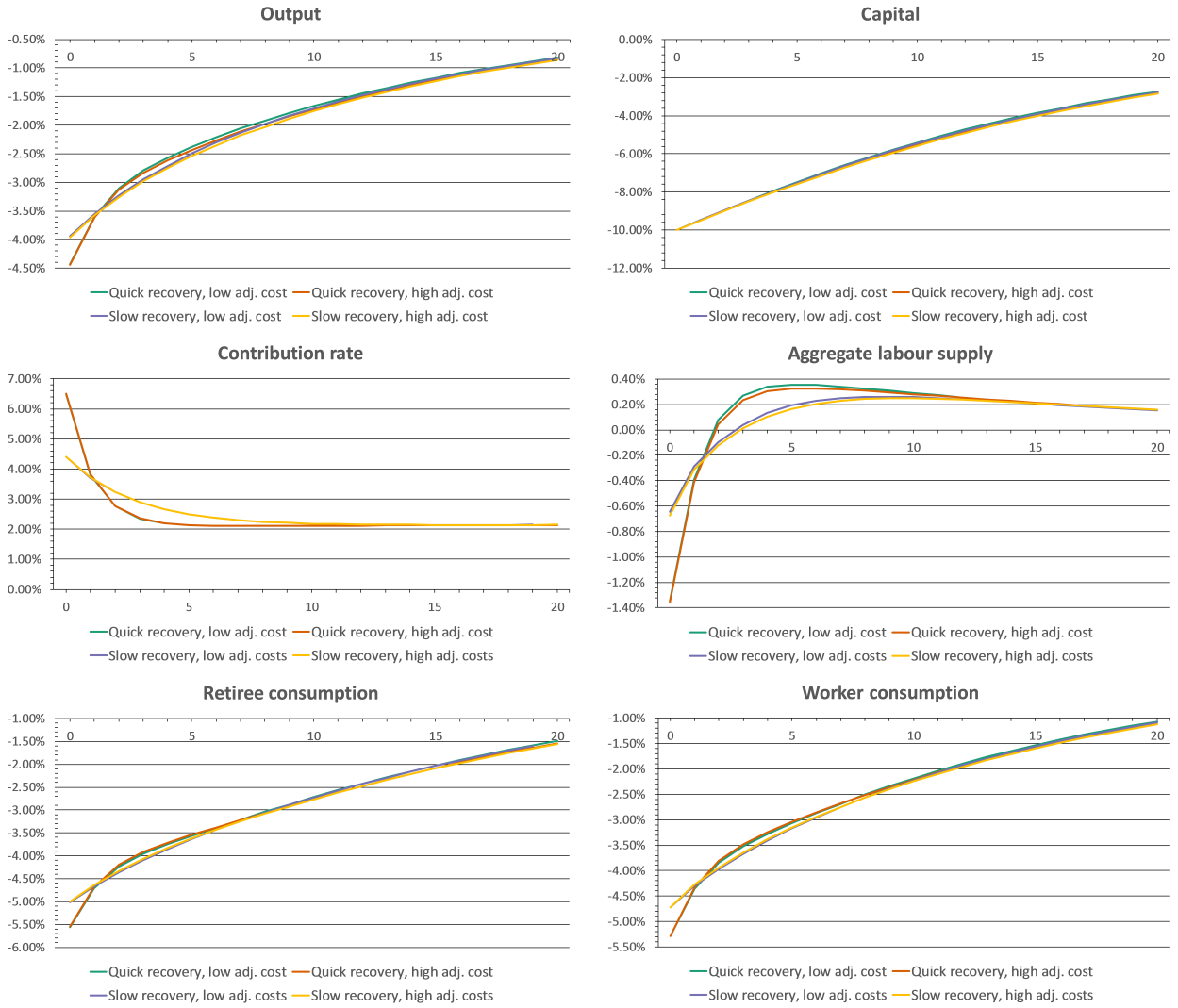


Figure 3: Impulse response diagrams for various variables after a 10% capital stock shock, compared over different combinations of pension funding gap closure speeds and capital adjustment costs. Values are in percentual deviation from the steady state.

It can be observed that the speed of recovery parameter υ has a far greater impact on the impulse response diagrams than the severity of the capital adjustment costs χ. Furthermore, the diagrams highlight the trade-off between slow and speedy closure of the pension funding gap. A quick recovery distorts the economy more severely in the starting periods, but in later periods distortions are smaller compared to slower recovery. Additionally, output and capital recover slightly faster. However, it should be noted that the currently alive population prefers slow recovery to quick recovery. Again, it is impossible to conclude which closure speed of the pension funding gap is welfare maximising, since it is not possible to contrast the potential welfare gains from postponing the closure of the funding gap accruing to current generations against the welfare losses of future generations.

4 Conclusions

This paper assesses the business cycle effects and distributional consequences of pension fund restoration policy and finds that economies with pension funds that primarily write off accumulated pension wealth to restore financial adequacy behave similarly to an economy without a pension fund. Significant deviations from laissez-faire arise when the pension fund increases the pension fund contribution rate to close the funding gap or postpones the closure of the funding gap. At a cost of significantly distorting aggregate labour supply and output, the pension fund can shelter the group of retirees from unanticipated shocks by guaranteeing the value of their accumulated pension wealth. Defined benefit pension schemes can be welfare improving, provided that the pension funding gap is closed fast enough (which ensures that the welfare of future generations is not sacrificed).

Since our model is relatively stylised, there are still various tempting avenues for future research. Most importantly, our assessment needs to be performed with a labour supply structure that is not affected by income effects so that the welfare of future generations can be computed. This will allow conclusions to be drawn about the desirability of slow versus speedy pension funding gap recovery. Other extensions include the study of pension fund restoration policy in an open economy and in a probabilistic ageing framework (which allows for richer life-cycle dynamics). Furthermore, while standard in the Gertler (1999) literature, the consideration of shocks in a perfect foresight economy remains rather unconvincing. To add to the credibility of our results, it is instrumental to perform the analysis in a stochastic environment (despite the fact that the literature has not found a way to incorporate this). Lastly, our quantitative results should not be taken too literally given the aforementioned troubles of pension fund calibration. Bayesian methods or an open economy-specification could improve our calibration and in turn strenghten our research findings. Despite these objections, our qualitative results point towards the direction that pension fund restoration policy has a noticable effect on the business cycle and that the recovery from a financial crisis crucially depends on the conducted policy of pension funds.

References

Altissimo, F., Ehrmann, M., and Smets, F. (2006). Inflation persistence and price-setting behaviour in the Euro area: a summary of the IPN evidence. *ECB Occasional Paper Series No. 46.*

Auerbach, A. and Kotlikoff, L. (1987). *Dynamic Fiscal Policy*. Cambridge University Press.

Beetsma, R. and Bovenberg, A. L. (2009). Pensions and intergenerational risk sharing in general equilibrium. *Economica*, 76:364–386.

Beetsma, R., Romp, W., and Vos, S. (2013). Intergenerational risk sharing, pensions, and endogenous labour supply in general equilibrium. *Scandinavian Journal of Economics*, 115(1):141–154.

Blanchard, O. (1985). Debt, deficits, and finite horizons. *Journal of Political Economy*, 93(2):223–247.

Boersch-Supan, A., Ludwig, A., and Winter, J. (2006). Aging, pension reform and capital flows: a multi-country model. *Eonomica*, 73:625–658.

Calvo, G. (1983). Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics*, 12:983–998.

Carvalho, C., Ferrero, A., and Nechio, F. (2016). Demographics and real interest rates: inspecting the mechanism. *Federal Reserve Bank of San Francisco Working Paper 2016-05*.

Christiano, L., Eichenbaum, M., and Evans, C. (2005). Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of Political Economy*, 113(1):1–45.

D'Addio, A. and Whitehouse, E. (2010). Population ageing and labour markets. *Oxford Review of Economic Policy*, 26(4):613–635.

DeNardi, M., Imrohoroglu, S., and Sargent, T. (1999). Projected u.s. demographics and social security. *Review of Economic Dynamics*, 2:575–615.

Dixit, A. and Stiglitz, J. (1977). Monopolistic competition and optimum product diversity. *American Economic Review*, 67:297–308.

Dotsey, M. (1999). Structure from shocks. *Federal Reserve Bank of Richmond Working Paper Series 99-06*.

Epstein, L. and Zin, S. (1989). Substitution, risk aversion and the temporal behavior of consumption and asset returns: a theoretical framework. *Econometrica*, 57:937–968.

Equipe Ingenue (2001). Macroeconomic consequences of pension reforms in europe: an investigation with the ingenue world model. *CEPII Working Paper 17*.

European Commission (DG ECFIN) and the Economic Policy Committee (AWG) (2009). *The 2009 Ageing Report: Economic and budgetary projections for the EU-27 Member States (2008-2060), Statistical Annex*. European Commission. Available at `http://ec.europa.eu/economy_finance/publications/publication14992_en.pdf`.

Farmer, R. (1990). Rince preferences. *Quarterly Journal of Economics*, 105:43–60.

Fernandez-Villaverde, J. (2006). A baseline dsge model. Available at `http://economics.sas.upenn.edu/~jesusfv/benchmark_DSGE.pdf`.

Fernandez-Villaverde, J. (2010). The econometrics of dsge models. *SERIEs*, 1:3–49.

Fujiwara, I. and Teranishi, Y. (2008). A dynamic new keynesian life-cycle model: Societal aging, demographics, and monetary policy. *Journal of Economic Dynamics and Control*, 32:2398–2427.

Gali, J. (2008). *Monetary Policy, Inflation, and the Business Cycle. An Introduction to the New Keynesian Framework*. Princeton University Press.

Gertler, M. (1999). Government debt and social security in a life-cycle economy. *Carnegie Rochester Conference Series on Public Policy*, pages 61–110.

Gertler, M. and Karadi, P. (2011). A model of unconventional monetary policy. *Journal of Monetary Economics*, 58:17–34.

Gollier, C. (2008). Intergenerational risk-sharing and risk-taking of a pension fund. *Journal of Public Economics*, 92(5-6):1463–1485.

Grafenhofer, D., Jaag, C., Keuschnigg, C., and Keuschnigg, M. (2006). Probabilistic aging. *CESifo Working Paper No. 1680*.

Grafenhofer, D., Jaag, C., Keuschnigg, C., and Keuschnigg, M. (2007). Economic aging and demographic change. *Universitat St. Gallen Discussion Paper No, 2007-35*.

Heijdra, B., Mierau, J., and Reijnders, L. (2014). A tragedy of annuitization? longevity insurance in general equilibrium. *Macroeconomic Dynamics*, 18(7):1607–1634.

Hofmann, B. and Bogdanova, B. (2012). Taylor rules and monetary policy: a global "great deviation"? *BIS Quarterly Review*, pages 37–49.

Jaag, C., Keuschnigg, C., and Keuschnigg, M. (2010). Pension reform, retirement and life-cycle unemployment. *International Tax and Public Finance*, 17(5):556–585.

Kaplan, G., Moll, B., and Violante, G. (2016). Monetary policy according to hank. Available at `http://www.econ.nyu.edu/user/violante/Workingpapers/HANK_FINAL.pdf`.

Kaplan, G. and Violante, G. (2014). A model of the consumption response to fiscal stimulus payments. *Econometrica*, 82(4):1199–1239.

Kara, E. and von Thadden, L. (2016). Interest rate effects of demographic change in a new-keynesian life-cycle framework. *Macroeconomic Dynamics*, 20:120–164.

Katagiri, M. (2012). Economic consequences of population aging in japan: effects through changes in demand structure. *Institute for Monetary and Economic Studies Discussion Paper 2012-E-3*.

Keuschnigg, C. and Keuschnigg, M. (2004). Aging, labor markets, and pension reform in austria. *FinanzArchiv*, 60(3):359–392.

Kilponen, J., Kinnunen, H., and Ripatti, A. (2006). Demographic uncertainty and labor market imperfections in a small open economy. *Bank of Finland, mimeo*.

Kilponen, J. and Ripatti, A. (2006). Labor and product market reforms in general equilibrium - simulation results using a dge model of finnish economy. *Bank of Finland Research Discussion Papers 5/2006*.

Krueger, D. and Ludwig, A. (2007). On the consequences of demographic change for returns to capital, and the distribution of wealth and welfare. *Journal of Monetary Economics*, 54:49–87.

Laboul, A. (2010). Pension fund assets struggle to return to pre-crisis levels. *OECD Pension Markets in Focus*, 7.

Natali, D. (2011). Pensions after the financial and economic crisis: a comparative analysis of recent reforms in europe. *European Trade Union Institute Working Paper 07*.

Novy-Marx, R. and Rauh, J. (2009). The liabilities and risks of state-sponsored pension plans. *Journal of Economic Perspectives*, 23(4):191–210.

Novy-Marx, R. and Rauh, J. (2011). Public pension promises: How big are they and what are they worth? *Journal of Finance*, 66(4):1211–1249.

OECD (2011). *Pensions at a Glance 2011: Retirement-income Systems in OECD and G20 Countries*. OECD Publishing. Available at `http://www.oecd-ilibrary.org/finance-and-investment/pensions-at-a-glance-2011_pension_glance-2011-en`.

PwC Luxembourg (2015). *Beyond their borders, evolution of foreign investment by pension funds*. PwC Luxembourg. Available at `http://www.pwc.lu/en/asset-management/docs/pwc-am-beyond-their-borders.pdf`.

Romp, W. (2013). Procyclicality of pension fund regulation and behaviour. *Netspar Discussion Paper 11/2013*.

Shimer, R. (2012). Wage rigidities and jobless recoveries. *Journal of Monetary Economics*, 59S:S65–S77.

Timiraos, N. (2016). Puerto rico's pensions: 2 billion dollars in assets, 45 billion dollars in liabilities. The Wall Street Journal. Available at `http://www.wsj.com/articles/puerto-ricos-pensions-2-billion-in-assets-45-billion-in-liabilities-1472156434?mod=e2fb`.

Treasury (2012). The financial crisis response in charts. Available at `https://www.treasury.gov/resource-center/data-chart-center/Documents/20120413_FinancialCrisisResponse.pdf`.

van Bommel, J. and Penalva, J. (2012). The governance of perpatual financial intermediaries. *LSF Research Working Paper Series 12-10*.

Walsh, C. (2010). *Monetary Theory and Policy*. The MIT Press, 3rd edition.

Weil, P. (1989). Overlapping families of infinitely-lived agents. *Journal of Public Economics*, 38:183–198.

Woodford, M. (2003). *Interest and Prices. Foundations of a Theory of Monetary Policy*. Princeton University Press.

Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies*, 32:137–150.

A Decision problems of retirees and workers

At this point, some notation is introduced in order to make the derivations more readable. Since the main body has explicitly stated the state variables of the value functions, we will mostly continue without writing them down from here on out. $V_t^{r,i}$ thus simply denotes the value function of retiree i at period t with the correct state variables in that period. Further, denote with $V_{2,t+1}^{r,i}$ the derivative of the value function of retiree i in period $t+1$ with respect to per-period pension benefits $\mu_{t+1}P_{t+1}^{r,i}$ (i.e. the second state variable).

A.1 Retiree decision problem

The solution approach is as follows: Take first-order conditions and derive the Euler equation. Solve for the optimal levels of real balances and leisure in terms of $c_t^{r,i}$ in order to write the Euler equation solely in terms of consumption. Conjecture the form of the consumption function (i.e. spending a share $\epsilon_t\pi_t$ of total wealth) and derive a difference equation for the marginal propensity to consume out of wealth. Conjecture the form of the value function and verify that it gives the same difference equation for the marginal propensity to consume out of wealth as before.

We reiterate the decision problem of retiree i:

$$V_t^{r,i}(\frac{1+r_t}{\gamma}a_{t-1}^{r,i},\mu_t P_t^{r,i}) = \max_{c_t^{r,i},a_t^{r,i},l_t^{r,i},m_t^{r,i}} \left[[(c_t^{r,i})^{v_1}(1-l_t^{r,i})^{v_2}(m_t^{r,i})^{v_3}]^\rho + \beta\gamma[V_{t+1}^{r,i}(\frac{1+r_{t+1}}{\gamma}a_t^{r,i},\mu_{t+1}P_{t+1}^{r,i})]^\rho\right]^{\frac{1}{\rho}} \tag{46}$$

subject to:

$$\begin{aligned} a_t^{r,i} &= \frac{1+r_t}{\gamma}a_{t-1}^{r,i} + (1-\tau_t)\xi w_t l_t^{r,i} + \mu_t P_t^{r,i} - \tau_t^g - c_t^{r,i} - \frac{i_t}{1+i_t}m_t^{r,i} \\ P_{t+1}^{r,i} &= \mu_t P_t^{r,i} + \nu_t \xi w_t l_t^{r,i} \end{aligned}$$

A.1.1 First-order conditions

- **The first-order condition with respect to $c_t^{r,i}$, i.e. $\frac{\partial V_t^{r,i}}{\partial c_t^{r,i}} = 0$**

$$\frac{1}{\rho}\left[\left((c_t^{r,i})^{v_1}(1-l_t^{r,i})^{v_2}(m_t^{r,i})^{v_3}\right)^\rho + \beta\gamma\left(V_{t+1}^{r,i}\right)^\rho\right]^{\frac{1-\rho}{\rho}}$$
$$\left(\rho v_1(c_t^{r,i})^{v_1\rho-1}(1-l_t^{r,i})^{v_2\rho}(m_t^{r,i})^{v_3\rho} - \beta\gamma\rho(V_{t+1}^{r,i})^{\rho-1}V_{1,t+1}^{r,i}\frac{\partial\left(\frac{1+r_{t+1}}{\gamma}a_t^{r,i}\right)}{\partial a_t^{r,i}}\frac{\partial a_t^{r,i}}{\partial c_t^{r,i}}\right) = 0$$

Realise that $\frac{\partial\left(\frac{1+r_{t+1}}{\gamma}a_t^{r,i}\right)}{\partial a_t^{r,i}} = \frac{1+r_{t+1}}{\gamma}$, that $\frac{\partial a_t^{r,i}}{\partial c_t^{r,i}} = -1$, and that $\left[\left((c_t^{r,i})^{v_1}(1-l_t^{r,i})^{v_2}(m_t^{r,i})^{v_3}\right)^\rho + \beta\gamma\left(V_{t+1}^{r,i}\right)^\rho\right]^{\frac{1-\rho}{\rho}} = \left(V_t^{r,i}\right)^{1-\rho}$ drops out. This gives:

$$v_1(c_t^{r,i})^{v_1\rho-1}(1-l_t^{r,i})^{v_2\rho}(m_t^{r,i})^{v_3\rho} = \beta(1+r_{t+1})(V_{t+1}^{r,i})^{\rho-1}V_{1,t+1}^{r,i} \tag{47}$$

We can obtain $V_{1,t+1}^{r,i}$ by applying the Envelope Theorem for constrained optimisation:

$$V_{1,t+1}^{r,i} = (V_{t+1}^{r,i})^{1-\rho} v_1 (c_{t+1}^{r,i})^{v_1\rho-1} (1 - l_{t+1}^{r,i})^{v_2\rho} (m_{t+1}^{r,i})^{v_3\rho} \tag{48}$$

Combining (47) and (48) gives the Euler equation:

$$(c_t^{r,i})^{v_1\rho-1}(1 - l_t^{r,i})^{v_2\rho}(m_t^{r,i})^{v_3\rho} = \beta(1 + r_{t+1})(c_{t+1}^{r,i})^{v_1\rho-1}(1 - l_{t+1}^{r,i})^{v_2\rho}(m_{t+1}^{r,i})^{v_3\rho} \tag{49}$$

- **The first-order condition with respect to $m_t^{r,i}$, i.e. $\frac{\partial V_t^{r,i}}{\partial m_t^{r,i}} = 0$**

Realising that $\frac{\partial a_t^{r,i}}{\partial m_t^{r,i}} = -\frac{i_t}{1+i_t}$:

$$v_3(c_t^{r,i})^{v_1\rho}(1 - l_t^{r,i})^{v_2\rho}(m_t^{r,i})^{v_3\rho-1} = \beta(1 + r_{t+1})(V_{t+1}^{r,i})^{\rho-1} V_{1,t+1}^{r,i} \frac{i_t}{1 + i_t} \tag{50}$$

Combining (47) and (50):

$$m_t^{r,i} = \frac{v_3}{v_1} \frac{1 + i_t}{i_t} c_t^{r,i} \tag{51}$$

- **The first-order condition with respect to $l_t^{r,i}$, i.e. $\frac{\partial V_t^{r,i}}{\partial l_t^{r,i}} = 0$**

Realising that $\frac{\partial a_t^{r,i}}{\partial l_t^{r,i}} = (1 - \tau_t)\xi w_t$ and that $\frac{\partial(\mu_{t+1} P_{t+1}^{r,i})}{\partial l_t^{r,i}} = \mu_{t+1}\nu_t\xi w_t$:

$$v_2(c_t^{r,i})^{v_1\rho}(1 - l_t^{r,i})^{v_2\rho-1}(m_t^{r,i})^{v_3\rho} = \beta(1 + r_{t+1})(V_{t+1}^{r,i})^{\rho-1} V_{1,t+1}^{r,i}(1 - \tau_t)\xi w_t + \beta\gamma V_{2,t+1}^{r,i}\mu_{t+1}\nu_t\xi w_t, \tag{52}$$

where the left-hand side denotes the marginal utility of a unit of leisure. The first term on the right-hand side is the familiar marginal utility of a unit of supplied work which yields $(1 - \tau_t)\xi w_t$ consumption units. The second term on the right-hand side is a newly introduced term stemming from the addition of the pension fund to the model. Working one extra hour in period t gives $\mu_{t+1}\nu_t\xi w_t$ additional per-period pension benefits from period $t+1$ onwards. In calculating $V_{2,t+1}^{r,i}$ we seek to determine the proper valuation of one additionally accrued unit of per-period pension benefits. We define the annuity factor $R_{t+1}^r = 1 + \mu_{t+2}\frac{\gamma}{1+r_{t+1}} R_{t+2}^r$ to be the present discounted value to a retiree at period $t+1$ of receiving one consumption good each period from period $t+1$ until death, continuously corrected for future indexation. In other words, the annuity factor R_{t+1}^r denotes the amount of period $t+1$ consumption goods the retiree would judge equivalent to receiving one consumption good each period from period $t+1$ until death (corrected for future indexation).

With the insight that the newly accrued per-period pension benefits are valued the same by the retiree as receiving R^r_{t+1} consumption goods in period $t+1$, we can write $V^{r,i}_{2,t+1} = R^r_{t+1} V^{r,i}_{1,t+1}$[21] and condense (52) into:

$$v_2 (c^{r,i}_t)^{v_1\rho} (1 - l^{r,i}_t)^{v_2\rho - 1} (m^{r,i}_t)^{v_3\rho} = \beta(1 + r_{t+1})(V^{r,i}_{t+1})^{\rho-1} V^{r,i}_{1,t+1} (1 - \tau_t + \nu_t \mu_{t+1} \frac{\gamma}{1 + r_{t+1}} R^r_{t+1}) \xi w_t \tag{53}$$

Rolling back our definition of R^r_{t+1} one period, we can write (53) as:

$$v_2 (c^{r,i}_t)^{v_1\rho} (1 - l^{r,i}_t)^{v_2\rho - 1} (m^{r,i}_t)^{v_3\rho} = \beta(1 + r_{t+1})(V^{r,i}_{t+1})^{\rho-1} V^{r,i}_{1,t+1} (1 - \tau^r_t) \xi w_t, \tag{54}$$

where $\tau^r_t = \tau_t - (R^r_t - 1)\nu_t$ is the effective labour income tax a retiree faces. Combining (47) and (54):

$$1 - l^{r,i}_t = \frac{v_2}{v_1} \frac{c^{r,i}_t}{(1 - \tau^r_t)\xi w_t} \tag{55}$$

A.1.2 Writing the Euler equation solely in terms of consumption

Plugging (51) and (55) in (49):

$$c^{r,i}_{t+1} = c^{r,i}_t \left[\beta(1 + r_t) (\frac{(1 - \tau^r_t) w_t}{(1 - \tau^r_{t+1}) w_{t+1}})^{v_2\rho} (\frac{1 + i_{t+1}}{i_{t+1}} \frac{i_t}{1 + i_t})^{v_3\rho} \right]^{\sigma}, \tag{56}$$

where we have used that $v_1 + v_2 + v_3 = 1$ and that $\sigma = \frac{1}{1-\rho}$.

A.1.3 Conjecturing the consumption function

In the spirit of Kara and von Thadden (2016) we denote with 'total consumption' $c^{r,i}_t + \frac{i_t}{1+i_t} m^{r,i}_t$, which allows us to verify our conjectures for the consumption and value function in similar fashion as Gertler (1999). We conjecture that the retiree spends a fraction $\epsilon_t \pi_t$ of his total wealth in period t on consumption goods and real balances, where π_t is the marginal propensity to consume out of wealth (MPCW) of workers and ϵ_t the relative MPCW of retirees to workers. To fluently verify our conjectures, we exploit the necessary proportionality between $c^{r,i}_t$ and $m^{r,i}_t$: total consumption is given by $c^{r,i}_t + \frac{i_t}{1+i_t} m^{r,i}_t = c^{r,i}_t (1 + \frac{v_3}{v_1})$.

[21] To further strenghten this argument, consider the following. We will find below that $V^{r,i}_t = (\epsilon_t \pi_t)^{-\frac{1}{\rho}} c^{r,i}_t (\frac{v_2}{v_1} \frac{1}{(1-\tau^r_t)\xi w_t})^{v_2} (\frac{v_3}{v_1} \frac{1+i_t}{i_t})^{v_3}$, that $c^{r,i}_t = \frac{1}{1+\frac{v_3}{v_1}} \epsilon_t \pi_t (\frac{1+r_t}{\gamma} a^{r,i}_{t-1} + h^{r,i}_t)$, and that $h^{r,i}_t = (1 - \tau_t)\xi w_t l^{r,i}_t + \mu_t P^{r,i}_t - \tau^g_t + \frac{\gamma}{1+r_{t+1}} h^{r,i}_{t+1}$. It is then easily observed that one additional unit of wealth in period $t+1$, i.e. one additional unit of $\frac{1+r_{t+1}}{\gamma} a^{r,i}_t$, gives the same marginal value as one additional unit of pension transfers in period $t+1$, i.e. one additional unit of $\mu_{t+1} P^{r,i}_{t+1}$. This ensures that, conditional on calculating the appropriate annuity factor of the retiree, we can write that $V^{r,i}_{2,t+1} = R^r_{t+1} V^{r,i}_{1,t+1}$.

Recall the definitions of the disposable income and human wealth of a retiree:

$$d_t^{r,i} = (1-\tau_t)\xi w_t l_t^{r,i} + \mu_t P_t^{r,i} - \tau_t^g \tag{57}$$

$$h_t^{r,i} = d_t^{r,i} + \frac{\gamma}{1+r_{t+1}} h_{t+1}^{r,i} \tag{58}$$

We conjecture:

$$c_t^{r,i} + \frac{i_t}{1+i_t} m_t^{r,i} = c_t^{r,i}(1+\frac{v_3}{v_1}) = \epsilon_t \pi_t (\frac{1+r_t}{\gamma} a_{t-1}^{r,i} + h_t^{r,i}) \tag{59}$$

We combine (59) in turn with (56) and with the retiree budget constraint, (57), and (58) in order to obtain a difference equation for $\epsilon_t \pi_t$.

Combining (59) with (56):

$$a_t^{r,i} + \frac{\gamma}{1+r_{t+1}} h_{t+1}^{r,i} = \frac{\epsilon_t \pi_t}{\epsilon_{t+1}\pi_{t+1}} \gamma (1+r_{t+1})^{\sigma-1} (\frac{1+r_t}{\gamma} a_{t-1}^{r,i} + h_t^{r,i}) \left(\beta (\frac{(1-\tau_t^r) w_t}{(1-\tau_{t+1}^r) w_{t+1}})^{v_2 \rho} (\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t})^{v_3 \rho} \right)^{\sigma} \tag{60}$$

Combining (59) with the retiree budget constraint, (57), and (58):

$$\epsilon_t \pi_t (\frac{1+r_t}{\gamma} a_{t-1}^{r,i} + h_t^{r,i}) + a_t^r + \frac{\gamma}{1+r_{t+1}} h_{t+1}^{r,i} = \frac{1+r_t}{\gamma} a_{t-1}^{r,i} + d_t^{r,i} + h_t^{r,i} - d_t^{r,i} \tag{61}$$

$$a_t^{r,i} + \frac{\gamma}{1+r_{t+1}} h_{t+1}^{r,i} = (1-\epsilon_t \pi_t)(\frac{1+r_t}{\gamma} a_{t-1}^{r,i} + h_t^{r,i}), \tag{62}$$

where in (61) we have added $\frac{\gamma}{1+r_{t+1}} h_{t+1}^{r,i} = h_t^{r,i} - d_t^{r,i}$ on both sides.

Putting the right-hand sides of (60) and (62) together gives the desired result:

$$\epsilon_t \pi_t = 1 - \frac{\epsilon_t \pi_t}{\epsilon_{t+1}\pi_{t+1}} \beta^{\sigma} (1+r_{t+1})^{\sigma-1} \gamma \left(\frac{(1-\tau_t^r) w_t}{(1-\tau_{t+1}^r) w_{t+1}} \right)^{v_2 \rho \sigma} \left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t} \right)^{v_3 \rho \sigma} \tag{63}$$

A.1.4 Conjecturing the value function

Gertler (1999) conjectures $V_t^{r,i} = \Delta_t^r c_t^{r,i} (\frac{v_2}{v_1} \frac{1}{\xi w_t})^{v_2}$ and Kara and von Thadden (2016) conjecture $V_t^{r,i} = \Delta_t^r c_t^{r,i} (\frac{v_2}{v_1} \frac{1}{\xi w_t})^{v_2} (\frac{v_3}{v_1} \frac{1+i_t}{i_t})^{v_3}$. Similarly, we conjecture:

$$V_t^{r,i} = \Delta_t^r c_t^{r,i} (\frac{v_2}{v_1} \frac{1}{(1-\tau_t^r)\xi w_t})^{v_2} (\frac{v_3}{v_1} \frac{1+i_t}{i_t})^{v_3} \tag{64}$$

Realise that, after exploiting the constant returns to scale property of the Cobb-Douglas flow utility function, $c_t^{r,i} (\frac{v_2}{v_1} \frac{1}{(1-\tau_t^r)\xi w_t})^{v_2} (\frac{v_3}{v_1} \frac{1+i_t}{i_t})^{v_3}$ is simply the optimised period t flow utility of the retiree. The value function therefore states that the value of the retiree is its period t flow utility multiplied by a factor Δ_t^r, to be derived below.

Plugging (64) in (46):

$$\Delta_t^r c_t^{r,i}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^r)\xi w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3} =$$
$$\left(\left[c_t^{r,i}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^r)\xi w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}\right]^{\rho} + \beta\gamma\left[\Delta_{t+1}^r c_{t+1}^{r,i}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^r)\xi w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3}\right]^{\rho}\right)^{\frac{1}{\rho}}$$

Using (56):

$$\left[\Delta_t^r c_t^{r,i}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^r)\xi w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}\right]^{\rho} = \left[c_t^{r,i}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^r)\xi w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}\right]^{\rho} +$$
$$\beta\gamma\left[\Delta_{t+1}^r c_t^{r,i}\left[\beta(1+r_t)(\frac{(1-\tau_t^r)w_t}{(1-\tau_{t+1}^r)w_{t+1}})^{v_2\rho}(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t})^{v_3\rho}\right]^{\sigma}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^r)\xi w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3}\right]^{\rho}$$

Since we defined $\sigma = \frac{1}{1-\rho}$, it holds that $\rho = 1 - \frac{1}{\sigma} \rightarrow \sigma\rho = \sigma - 1$. Using this and some tedious algebra, we arrive at:

$$(\Delta_t^r)^{\rho} = 1 + \gamma(\Delta_{t+1}^r)^{\rho}(1+r_{t+1})^{\sigma-1}\left[\beta(\frac{(1-\tau_t^r)w_t}{(1-\tau_{t+1}^r)w_{t+1}})^{v_2\rho}(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t})^{v_3\rho}\right]^{\sigma}$$

The conjectures for the consumption and value function of the retiree are only mutually consistent in case $(\Delta_t^r)^{\rho} = \frac{1}{\epsilon_t\pi_t}$, as we can verify:

$$(\frac{1}{\epsilon_t\pi_t})^{\rho} = 1 + \gamma(\frac{1}{\epsilon_{t+1}\pi_{t+1}})^{\rho}(1+r_{t+1})^{\sigma-1}\left[\beta(\frac{(1-\tau_t^r)w_t}{(1-\tau_{t+1}^r)w_{t+1}})^{v_2\rho}(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t})^{v_3\rho}\right]^{\sigma}$$
$$\epsilon_t\pi_t = 1 - \frac{\epsilon_t\pi_t}{\epsilon_{t+1}\pi_{t+1}}\beta^{\sigma}(1+r_{t+1})^{\sigma-1}\gamma\left(\frac{(1-\tau_t^r)w_t}{(1-\tau_{t+1}^r)w_{t+1}}\right)^{v_2\rho\sigma}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho\sigma},$$

which indeed is exactly the same difference equation for $\epsilon_t\pi_t$ as derived in (63). This concludes the derivation of the retiree decision problem. We have obtained a difference equation for $\epsilon_t\pi_t$ (which is the same for all retirees), the retiree consumption function, and the retiree value function.

A.2 Worker decision problem

The solution approach is as follows: Take first-order conditions and derive the Euler equation. With respect to the first-order condition of labour, conjecture that we can 'translate' the additionally accumulated pension benefits to receiving an equivalent amount of consumption goods at period t. Solve for the optimal levels of real balances and leisure in terms of $c_t^{w,i}$ in order to write the Euler equation solely in terms of consumption and the value functions of retirees and workers. Conjecture the form of the value function and write the Euler equation solely in terms of consumption. Conjecture the form of the consumption function (i.e. spending a share π_t of total wealth) and derive a difference equation for the marginal propensity to consume out of wealth. Then, verify that the value function gives the same difference equation for the marginal propensity to consume out of wealth. Lastly, verify that, given the form of the value function, we can indeed 'translate'

the additionally accumulated pension benefits into receiving an equivalent amount of consumption goods at period t.

We reiterate the decision problem of retiree j:

$$V_t^{w,j}((1+r_t)a_{t-1}^{w,j}, \mu_t P_t^{w,j}) = \max_{c_t^{w,j}, a_t^{w,j}, l_t^{w,j}, m_t^{w,j}} \Bigg[[(c_t^{w,j})^{v_1}(1-l_t^{w,j})^{v_2}(m_t^{w,j})^{v_3}]^{\rho} + \beta \Big[\omega V_{t+1}^{w,j}((1+r_{t+1})a_t^{w,j}, \mu_{t+1}P_{t+1}^{w,j}) + (1-\omega)V_{t+1}^{r,j}((1+r_{t+1})a_t^{r,j}, \mu_{t+1}P_{t+1}^{r,j}) \Big]^{\rho} \Bigg]^{\frac{1}{\rho}} \tag{65}$$

subject to the constraints that become operative once he retires and subject to:

$$a_t^{w,j} = (1+r_t)a_{t-1}^{w,j} + (1-\tau_t)w_t l_t^{w,j} + f_t - \tau_t^g - c_t^{w,j} - \frac{i_t}{1+i_t} m_t^{w,j}$$
$$P_{t+1}^{w,j} = \mu_t P_t^{w,j} + \nu_t w_t l_t^{w,j}$$

A.2.1 First-order conditions

- **The first-order condition with respect to $c_t^{w,j}$, i.e. $\frac{\partial V_t^{w,j}}{\partial c_t^{w,j}} = 0$**

Similarly to the decision problem of the retiree:

$$v_1(c_t^{w,j})^{v_1\rho-1}(1-l_t^{w,j})^{v_2\rho}(m_t^{w,j})^{v_3\rho} = \beta(1+r_{t+1})[\omega V_{t+1}^{w,j} + (1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega V_{1,t+1}^{w,j} + (1-\omega)V_{1,t+1}^{r,j}], \tag{66}$$

where we can find $V_{1,t+1}^{w,j}$ and $V_{1,t+1}^{r,j}$ using the Envelope Theorem for constrained optimisation:

$$V_{1,t+1}^{w,j} = (V_{t+1}^{w,j})^{1-\rho} v_1 (c_{t+1}^{w,j})^{v_1\rho-1}(1-l_{t+1}^{w,j})^{v_2\rho}(m_{t+1}^{w,j})^{v_3\rho} \tag{67}$$
$$V_{1,t+1}^{r,j} = (V_{t+1}^{r,j})^{1-\rho} v_1 (c_{t+1}^{r,j})^{v_1\rho-1}(1-l_{t+1}^{r,j})^{v_2\rho}(m_{t+1}^{r,j})^{v_3\rho} \tag{68}$$

- **The first-order condition with respect to $m_t^{w,j}$, i.e. $\frac{\partial V_t^{w,j}}{\partial m_t^{w,j}} = 0$**

$$v_3(c_t^{w,j})^{v_1\rho}(1-l_t^{w,j})^{v_2\rho}(m_t^{w,j})^{v_3\rho-1} = \beta(1+r_{t+1})[\omega V_{t+1}^{w,j} + (1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega V_{1,t+1}^{w,j} + (1-\omega)V_{1,t+1}^{r,j}]\frac{i_t}{1+i_t} \tag{69}$$

Combining (66) and (69):

$$m_t^{w,j} = \frac{v_3}{v_1}\frac{1+i_t}{i_t} c_t^{w,j} \tag{70}$$

- **The first-order condition with respect to $l_t^{w,j}$, i.e. $\frac{\partial V_t^{w,j}}{\partial l_t^{w,j}} = 0$**

$$v_2(c_t^{w,j})^{v_1\rho}(1-l_t^{w,j})^{v_2\rho-1}(m_t^{w,j})^{v_3\rho} = \beta(1+r_{t+1})(1-\tau_t)w_t[\omega V_{t+1}^{w,j} + (1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega V_{1,t+1}^{w,j} + (1-\omega)V_{1,t+1}^{r,j}] + \beta\mu_{t+1}\nu_t w_t[\omega V_{t+1}^{w,j} + (1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega V_{2,t+1}^{w,j} + (1-\omega)V_{2,t+1}^{r,j}], \tag{71}$$

where the left-hand side denotes the marginal utility of a unit of leisure. The first term on the right-hand side is the familiar marginal utility of a unit of supplied work which yields $(1-\tau_t)w_t$ consumption units. The second term on the right-hand side is a newly introduced term stemming from the addition of the pension fund to the model. Supplying an extra unit of labour in period t adds $\mu_{t+1}\nu_t w_t$ units to one's stock of per-period pension benefits. As with the retiree decision problem, our aim is to translate these additional per-period benefits into an equivalent, one-period transfer of consumption goods in period t. We again apply the annuity factor approach.

Recall our definition for the retiree annuity factor $R_t^r = 1+\mu_{t+1}\frac{\gamma}{1+r_{t+1}}R_{t+1}^r$. Similarly, we can identify R_t^w as the amount of consumption units at period t a working person judges equivalent to receiving one consumption good once he retires (until death, including future indexation). We define $R_t^w = \frac{\mu_{t+1}}{1+r_{t+1}}\left(\frac{\omega}{\Omega_{t+1}}R_{t+1}^w + (1-\frac{\omega}{\Omega_{t+1}})R_{t+1}^r\right)$. The term Ω_{t+1} is left unspecified at this stage, but will be derived endogenously below. As can be seen, the term is a subjective reweighting of the transition probabilities into retirement by the worker. Intuitively, the worker takes into account that receiving one unit of consumption tomorrow if he becomes retired is different from receiving one unit of consumption tomorrow if he remains a worker. In Gertler (1999), Kara and von Thadden (2016), and Fujiwara and Teranishi (2008) the term Ω does not show up yet in the labour first-order condition of the worker, since there the labour supply decision is purely intratemporal. Our addition of the pension fund, however, makes the labour supply decision of the worker an intertemporal one. In calculating the net present value of the extended pension annuity, the appropriate worker discount factor must be applied, which includes taking into account that the worker reweights the transition probabilities ω and $1-\omega$.

As in the retiree decision problem, we can write $V_{2,t+1}^r = R_{t+1}^r V_{1,t+1}^r$. This states that, for a retiree in period $t+1$, having one additional unit of per-period pension benefits from period $t+1$ onwards is equivalent to receiving R_{t+1}^r consumption goods in period $t+1$. Similarly, we conjecture that we can write $V_{2,t+1}^w = R_{t+1}^w V_{1,t+1}^w$. This states that, for a worker in period $t+1$, having one additional unit of per-period pension benefits once one retires is equivalent to receiving R_{t+1}^w consumption goods in period $t+1$. Updating (71):

$$\begin{aligned} v_2(c_t^{w,j})^{v_1\rho}(1-l_t^{w,j})^{v_2\rho-1}(m_t^{w,j})^{v_3\rho} =& \beta(1+r_{t+1})(1-\tau_t)w_t[\omega V_{t+1}^{w,j}+(1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega V_{1,t+1}^{w,j}+(1-\omega)V_{1,t+1}^{r,j}]+ \\ & \beta\mu_{t+1}\nu_t w_t[\omega V_{t+1}^{w,j}+(1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega R_{t+1}^w V_{1,t+1}^{w,j}+(1-\omega)R_{t+1}^r V_{1,t+1}^{r,j}] \end{aligned} \tag{72}$$

Our conjecture that we can write $V_{2,t+1}^w = R_{t+1}^w V_{1,t+1}^w$ also implies that we can 'translate' the additionally accumulated pension benefits in period $t+1$ into receiving an equivalent amount R_t^w of consumption goods at period t, i.e. implies that we can write $\beta\mu_{t+1}\nu_t w_t[\omega V_{t+1}^{w,j}+(1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega R_{t+1}^w V_{1,t+1}^{w,j}+(1-\omega)R_{t+1}^r V_{1,t+1}^{r,j}]$ as $\beta(1+r_{t+1})R_t^w\nu_t w_t[\omega V_{t+1}^{w,j}+(1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega V_{1,t+1}^{w,j}+(1-\omega)V_{1,t+1}^{r,j}]$. Then, using (66) we can write (72) as:

$$\begin{aligned} v_2(c_t^{w,j})^{v_1\rho}(1-l_t^{w,j})^{v_2\rho-1}(m_t^{w,j})^{v_3\rho} =& (1-\tau_t^w)w_t\left(v_1(c_t^{w,j})^{v_1\rho-1}(1-l_t^{w,j})^{v_2\rho}(m_t^{w,j})^{v_3\rho}\right) \\ 1-l_t^{w,j} =& \frac{v_2}{v_1}\frac{c_t^{w,j}}{(1-\tau_t^w)w_t}, \end{aligned} \tag{73}$$

where $\tau_t^w = \tau_t - R_t^w \nu_t$ is the 'effective' tax rate for a worker. The reason why we make the above conjecture stems from the fact that we do not know what Ω looks like at this stage. As we will see below, Ω depends on τ^w, and τ^w in turn depends on Ω through R^w. After we have verified our conjectures for the consumption and value function of the worker, we will return to this conjecture and verify that we can indeed write $\beta\mu_{t+1}\nu_t w_t[\omega V_{t+1}^{w,j} + (1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega R_{t+1}^w V_{1,t+1}^{w,j} + (1-\omega)R_{t+1}^r V_{1,t+1}^{r,j}]$ as $\beta(1+r_{t+1})R_t^w \nu_t w_t[\omega V_{t+1}^{w,j} + (1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega V_{1,t+1}^{w,j} + (1-\omega)V_{1,t+1}^{r,j}]$ (which in turn implies that $V_{2,t+1}^w = R_{t+1}^w V_{1,t+1}^w$). This will ensure that all conjectures add up to consistent solutions across all equations characterising optimal decisions of retirees and workers.

A.2.2 Writing the Euler equation solely in terms of consumption and the value functions of retirees and workers

Recall from the retiree problem that $1 - l_{t+1}^{r,j} = \frac{v_2}{v_1}\frac{c_{t+1}^{r,j}}{(1-\tau_{t+1}^r)\xi w_{t+1}}$, where $\tau_t^r = \tau_t - (R_t^r - 1)\nu_t$. We come back to (66) and use (70) and (73) in order to substitute out for $1 - l_t^{w,j}$, $1 - l_{t+1}^{w,j}$, $m_t^{w,j}$, $m_{t+1}^{w,j}$:

$$
\begin{aligned}
v_1(c_t^{w,j})^{v_1\rho-1}(1-l_t^{w,j})^{v_2\rho}(m_t^{w.j})^{v_3\rho} &= v_1(c_t^{w,j})^{\rho-1}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^w)w_t})^{v_2\rho}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3\rho} \\
V_{1,t+1}^{w,j} &= (V_{t+1}^{w,j})^{1-\rho}v_1(c_{t+1}^{w,j})^{\rho-1}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^w)w_{t+1}})^{v_2\rho}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3\rho} \\
V_{1,t+1}^{r,j} &= (V_{t+1}^{r,j})^{1-\rho}v_1(c_{t+1}^{r,j})^{\rho-1}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^r)\xi w_{t+1}})^{v_2\rho}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3\rho}
\end{aligned}
$$

This yields in (66):

$$
\begin{aligned}
v_1(c_t^{w,j})^{\rho-1}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^w)w_t})^{v_2\rho}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3\rho} =& v_1\beta(1+r_{t+1})[\omega V_{t+1}^{w,j} + (1-\omega)V_{t+1}^{r,j}]^{\rho-1} \\
& \Bigg[\omega(V_{t+1}^{w,j})^{1-\rho}(c_{t+1}^{w,j})^{\rho-1}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^w)w_{t+1}})^{v_2\rho}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3\rho} \\
& + (1-\omega)(V_{t+1}^{r,j})^{1-\rho}(c_{t+1}^{r,j})^{\rho-1}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^r)\xi w_{t+1}})^{v_2\rho}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3\rho}\Bigg]
\end{aligned}
$$

Rearranging gives the Euler equation written solely in terms of consumption and the value functions of retirees and workers:

$$
\begin{aligned}
(c_t^{w,j})^{\rho-1} =& \beta(1+r_{t+1})[\omega V_{t+1}^{w,j} + (1-\omega)V_{t+1}^{r,j}]^{\rho-1}\left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}}\right)^{v_2\rho}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{+i_t}\right)^{v_3\rho} \\
& \left[\omega(V_{t+1}^{w,j})^{1-\rho}(c_{t+1}^{w,j})^{\rho-1} + (1-\omega)(V_{t+1}^{r,j})^{1-\rho}(c_{t+1}^{r,j})^{\rho-1}\left(\frac{1-\tau_{t+1}^w}{1-\tau_{t+1}^r}\frac{1}{\xi}\right)^{v_2\rho}\right] \qquad (74)
\end{aligned}
$$

A.2.3 Conjecturing the value function

In the retiree problem, we conjectured and verified that $V_t^{r,i} = \Delta_t^r c_t^{r,i} (\frac{v_2}{v_1} \frac{1}{(1-\tau_t^r)\xi w_t})^{v_2} (\frac{v_3}{v_1} \frac{1+i_t}{i_t})^{v_3}$, $\Delta_t^r = (\epsilon_t \pi_t)^{-\frac{1}{\rho}}$. Similarly, we conjecture that $V_t^{w,j} = \Delta_t^w c_t^{w,j} (\frac{v_2}{v_1} \frac{1}{(1-\tau_t^w)w_t})^{v_2} (\frac{v_3}{v_1} \frac{1+i_t}{i_t})^{v_3}$, $\Delta_t^w = (\pi_t)^{-\frac{1}{\rho}}$. Plugging these conjectures into (74):

$$
\begin{aligned}
(c_t^{w,j})^{\rho-1} =& \beta(1+r_{t+1}) \left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}} \right)^{v_2\rho} \left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t} \right)^{v_3\rho} \\
& \Big[\omega(\pi_{t+1})^{-\frac{1}{\rho}} c_{t+1}^{w,j} (\frac{v_2}{v_1} \frac{1}{(1-\tau_{t+1}^w)w_{t+1}})^{v_2} (\frac{v_3}{v_1} \frac{1+i_{t+1}}{i_{t+1}})^{v_3} \\
& + (1-\omega)(\epsilon_{t+1}\pi_{t+1})^{-\frac{1}{\rho}} c_{t+1}^{r,j} (\frac{v_2}{v_1} \frac{1}{(1-\tau_{t+1}^r)\xi w_{t+1}})^{v_2} (\frac{v_3}{v_1} \frac{1+i_{t+1}}{i_{t+1}})^{v_3} \Big]^{\rho-1} \\
& \Big[\omega(\pi_{t+1})^{-\frac{1-\rho}{\rho}} (c_{t+1}^{w,j})^{1-\rho} (\frac{v_2}{v_1} \frac{1}{(1-\tau_{t+1}^w)w_{t+1}})^{v_2(1-\rho)} (\frac{v_3}{v_1} \frac{1+i_{t+1}}{i_{t+1}})^{v_3(1-\rho)} (c_{t+1}^{w,j})^{\rho-1} \\
& + (1-\omega)(\epsilon_{t+1}\pi_{t+1})^{-\frac{1-\rho}{\rho}} (c_{t+1}^{r,j})^{1-\rho} (\frac{v_2}{v_1} \frac{1}{(1-\tau_{t+1}^r)\xi w_{t+1}})^{v_2(1-\rho)} (\frac{v_3}{v_1} \frac{1+i_{t+1}}{i_{t+1}})^{v_3(1-\rho)} (c_{t+1}^{r,j})^{\rho-1} \Big]
\end{aligned}
\tag{75}
$$

Realise that the second and third line of (75) simplify into:

$$
(\pi_{t+1})^{-\frac{\rho-1}{\rho}} (\frac{v_2}{v_1} \frac{1}{(1-\tau_{t+1}^w)w_{t+1}})^{v_2(\rho-1)} (\frac{v_3}{v_1} \frac{1+i_{t+1}}{i_{t+1}})^{v_3(\rho-1)} [\omega c_{t+1}^{w,j} + (1-\omega) c_{t+1}^{r,j} (\epsilon_{t+1})^{-\frac{1}{\rho}} \left(\frac{1-\tau_{t+1}^w}{1-\tau_{t+1}^r} \frac{1}{\xi} \right)^{v_2}]^{\rho-1}
$$

Realise that the fourth and fifth line of (75) simplify into:

$$
(\pi_{t+1})^{\frac{\rho-1}{\rho}} (\frac{v_2}{v_1} \frac{1}{(1-\tau_{t+1}^w)w_{t+1}})^{v_2(1-\rho)} (\frac{v_3}{v_1} \frac{1+i_{t+1}}{i_{t+1}})^{v_3(1-\rho)} [\omega + (1-\omega)(\epsilon_{t+1})^{\frac{\rho-1}{\rho}} \left(\frac{1-\tau_{t+1}^w}{1-\tau_{t+1}^r} \frac{1}{\xi} \right)^{v_2}]
$$

We can therefore condense (75):

$$
\begin{aligned}
(c_t^{w,j})^{\rho-1} =& \beta(1+r_{t+1}) \left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}} \right)^{v_2\rho} \left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t} \right)^{v_3\rho} [\omega c_{t+1}^{w,j} + (1-\omega) c_{t+1}^{r,j} (\epsilon_{t+1})^{-\frac{1}{\rho}} \left(\frac{1-\tau_{t+1}^w}{1-\tau_{t+1}^r} \frac{1}{\xi} \right)^{v_2}]^{\rho-1} \\
& [\omega + (1-\omega)(\epsilon_{t+1})^{-\frac{1-\rho}{\rho}} \left(\frac{1-\tau_{t+1}^w}{1-\tau_{t+1}^r} \frac{1}{\xi} \right)^{v_2}]
\end{aligned}
\tag{76}
$$

Let us define the following, where we use that $\sigma = \frac{1}{1-\rho} \Leftrightarrow \rho = 1 - \frac{1}{\sigma} = \frac{\sigma-1}{\sigma}$ and that $\rho\sigma = \sigma - 1$:

$$
\Lambda_{t+1} = \frac{\Delta_{t+1}^r}{\Delta_{t+1}^w} = (\epsilon_{t+1})^{\frac{\sigma}{1-\sigma}} \tag{77}
$$

$$
\chi_{t+1} = \left(\frac{1-\tau_{t+1}^w}{1-\tau_{t+1}^r} \frac{1}{\xi} \right)^{v_2} \tag{78}
$$

$$
\Omega_{t+1} = \omega + (1-\omega)\chi_{t+1}(\epsilon_{t+1})^{\frac{1}{1-\sigma}} \tag{79}
$$

As mentioned above, Ω depends on τ^w, and τ^w in turn depends on Ω through R^w. The subjective reweighting of transition probabilities term Ω arises endogenously from the optimisation procedure and reflects that a

worker, when switching into retirement, reaches the next (and irreversible) stage in his life-cycle. The retirement stage is characterised by a different effective wage rate (captured by ξ), a different MPCW (captcured by ϵ), and by a different effective labour income tax rate (captured by τ^r and τ^w). For this reason, the worker discounts the future at a different rate than the market interest rate.

We can now neatly write (76) as an Euler equation that solely depends on consumption:

$$\omega c_{t+1}^{w,j} + (1-\omega)c_{t+1}^{r,j}\Lambda_{t+1}\chi_{t+1} = c_t^{w,j}\left[\beta(1+r_{t+1})\Omega_{t+1}\left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}}\right)^{v_2\rho}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho}\right]^{\sigma} \tag{80}$$

A.2.4 Conjecturing the consumption function

By conjecturing the consumption function of the worker and combining it with (80), we will be able to derive a difference equation for π_t. We conjecture that $c_t^{w,j} + \frac{i_t}{1+i_t}m_t^{w,j} = c_t^{w,j}(1+\frac{v_3}{v_1}) = \pi_t\left((1+r_t)a_{t-1}^{w,j} + h_t^{w,j}\right)$. Recall from the retiree problem that the 'just retired agent' consumes according to $c_t^{r,j} + \frac{i_t}{1+i_t}m_t^{r,j} = c_t^{r,j}(1+\frac{v_3}{v_1}) = \epsilon_t\pi_t\left((1+r_t)a_{t-1}^{w,j} + h_t^{r,j}\right)$. Define worker human wealth $h_t^{w,j}$ and worker disposable income $d_t^{w,j}$ as:

$$d_t^{w,j} = (1-\tau_t)w_t l_t^{w,j} + f_t - \tau_t^g \tag{81}$$

$$h_t^{w,j} = d_t^{w,j} + \frac{\omega}{\Omega_{t+1}}\frac{1}{(1+r_{t+1})}h_{t+1}^{w,j} + (1-\frac{\omega}{\Omega_{t+1}})\frac{1}{(1+r_{t+1})}h_{t+1}^{r,j} \tag{82}$$

Note that in calculating the human wealth of a worker we apply the same subjective reweighting of transition probabilities as in calculating the annuity factor of a worker.

Plugging the conjectures for consumption into (80):

$$\pi_t\left((1+r_t)a_{t-1}^{w,j} + h_t^{w,j}\right)\left[\beta(1+r_{t+1})\Omega_{t+1}\left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}}\right)^{v_2\rho}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho}\right]^{\sigma} =$$
$$\omega\pi_{t+1}\left((1+r_{t+1})a_t^{w,j} + h_{t+1}^{w,j}\right) + (1-\omega)\epsilon_{t+1}\pi_{t+1}\left((1+r_{t+1})a_t^{w,j} + h_{t+1}^{r,j}\right)\Lambda_{t+1}\chi_{t+1}$$

Divide by $\pi_{t+1}(1+r_{t+1})[(1+r_t)a_{t-1}^{w,j} + h_t^{w,j}]$, and realise that $\epsilon_{t+1}\Lambda_{t+1} = (\epsilon_{t+1})^{\frac{1}{1-\sigma}}$:

$$\frac{\pi_t}{\pi_{t+1}}\beta^{\sigma}(1+r_{t+1})^{\sigma-1}\Omega_{t+1}^{\sigma}\left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}}\right)^{v_2\rho\sigma}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho\sigma} =$$
$$\frac{1}{(1+r_t)a_{t-1}^{w,j} + h_t^{w,j}}\left[\omega(a_t^{w,j} + \frac{h_{t+1}^{w,j}}{1+r_{t+1}}) + (1-\omega)(\epsilon_{t+1})^{\frac{1}{1-\sigma}}(a_t^{w,j} + \frac{h_{t+1}^{r,j}}{1+r_{t+1}})\chi_{t+1}\right]$$

Dividing by Ω_{t+1}, realising that $(1-\omega)(\epsilon_{t+1})^{\frac{1}{1-\sigma}}\chi_{t+1} = \Omega_{t+1} - \omega$, and seeing that $\frac{\omega}{\Omega_{t+1}}\frac{h^{w,j}_{t+1}}{1+r_{t+1}} + (1 - \frac{\omega}{\Omega_{t+1}})\frac{h^{r,j}_{t+1}}{1+r_{t+1}} = h^{w,j}_t - d^{w,j}_t$:

$$\frac{\pi_t}{\pi_{t+1}}\beta^\sigma\left((1+r_{t+1})\Omega_{t+1}\right)^{\sigma-1}\left(\frac{(1-\tau^w_t)w_t}{(1-\tau^w_{t+1})w_{t+1}}\right)^{v_2\rho\sigma}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho\sigma} = \frac{a^{w,j}_t + h^{w,j}_t - d^{w,j}_t}{(1+r_t)a^{w,j}_{t-1} + h^{w,j}_t} \tag{83}$$

Now, let us rewrite the worker budget constraint so that we can simplify the right-hand side of (83):

$$\begin{aligned}
c^{w,j}_t + \frac{i_t}{1+i_t}m^{w,j}_t + a^{w,j}_t &= (1+r_t)a^{w,j}_{t-1} + (1-\tau_t)w_t l^{w,j}_t + f_t - \tau^g_t \\
\pi_t\left((1+r_t)a^{w,j}_{t-1} + h^{w,j}_t\right) + a^{w,j}_t &= (1+r_t)a^{w,j}_{t-1} + d^{w,j}_t \\
a^{w,j}_t + h^{w,j}_t - d^{w,j}_t &= (1+r_t)a^{w,j}_{t-1} + h^{w,j}_t - \pi_t\left((1+r_t)a^{w,j}_{t-1} + h^{w,j}_t\right) \\
\frac{a^{w,j}_t + h^{w,j}_t - d^{w,j}_t}{(1+r_t)a^{w,j}_{t-1} + h^{w,j}_t} &= 1 - \pi_t
\end{aligned} \tag{84}$$

Plugging (84) into (83) gives the desired difference equation for π_t:

$$\pi_t = 1 - \frac{\pi_t}{\pi_{t+1}}\beta^\sigma\left((1+r_{t+1})\Omega_{t+1}\right)^{\sigma-1}\left(\frac{(1-\tau^w_t)w_t}{(1-\tau^w_{t+1})w_{t+1}}\right)^{v_2\rho\sigma}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho\sigma} \tag{85}$$

A.2.5 Verifying the conjectures for V^w_t and c^w_t

In order to verify the conjectures for the value and consumption function of the worker, we have to plug our conjectures into (65) and arrive at exactly the same difference equation as (85):

$$\begin{aligned}
\left((\pi_t)^{-\frac{1}{\rho}}c^{w,j}_t(\frac{v_2}{v_1}\frac{1}{(1-\tau^w_t)w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}\right)^\rho = &\left(c^{w,j}_t(\frac{v_2}{v_1}\frac{1}{(1-\tau^w_t)w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}\right)^\rho + \\
&\beta\Bigg[\omega(\pi_{t+1})^{-\frac{1}{\rho}}c^{w,j}_{t+1}(\frac{v_2}{v_1}\frac{1}{(1-\tau^w_{t+1})w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3} + \\
&(1-\omega)(\epsilon_{t+1}\pi_{t+1})^{-\frac{1}{\rho}}c^{r,j}_{t+1}(\frac{v_2}{v_1}\frac{1}{(1-\tau^r_{t+1})\xi w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3}\Bigg]^\rho,
\end{aligned}$$

which we can write as:

$$\begin{aligned}
\frac{1-\pi_t}{\pi_t}\left(c^{w,j}_t(\frac{v_2}{v_1}\frac{1}{(1-\tau^w_t)w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}\right)^\rho = &\beta\Bigg[\omega(\pi_{t+1})^{-\frac{1}{\rho}}c^{w,j}_{t+1}(\frac{v_2}{v_1}\frac{1}{(1-\tau^w_{t+1})w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3} + \\
&(1-\omega)(\epsilon_{t+1}\pi_{t+1})^{-\frac{1}{\rho}}c^{r,j}_{t+1}(\frac{v_2}{v_1}\frac{1}{(1-\tau^r_{t+1})\xi w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3}\Bigg]^\rho
\end{aligned} \tag{86}$$

When verifying the conjectures for the value and consumption function of the retiree, we made use of the Euler equation. Similarly, we use (80) and substitute $\omega c_{t+1}^{w,j} = c_t^{w,j}\left[\beta(1+r_{t+1})\Omega_{t+1}\left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}}\right)^{v_2\rho}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho}\right]^{\sigma} - (1-\omega)c_{t+1}^{r,j}\Lambda_{t+1}\chi_{t+1}$ in (86), yielding:

$$\begin{aligned}\frac{1-\pi_t}{\pi_t}\left(c_t^{w,j}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^w)w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}\right)^{\rho} =&\beta\Bigg[c_t^{w,j}\left[\beta(1+r_{t+1})\Omega_{t+1}\left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}}\right)^{v_2\rho}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho}\right]^{\sigma}\\ &(\pi_{t+1})^{-\frac{1}{\rho}}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^w)w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3}-\\ &(1-\omega)c_{t+1}^{r,j}\Lambda_{t+1}\chi_{t+1}(\pi_{t+1})^{-\frac{1}{\rho}}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^w)w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3}+\\ &(1-\omega)(\epsilon_{t+1}\pi_{t+1})^{-\frac{1}{\rho}}c_{t+1}^{r,j}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^r)\xi w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3}\Bigg]^{\rho}\end{aligned} \tag{87}$$

We will now show that the third and fourth line of (87) cancel each other out. Realise that $\Lambda_{t+1}(\pi_{t+1})^{-\frac{1}{\rho}} = (\epsilon_{t+1}\pi_{t+1})^{-\frac{1}{\rho}}$ and $\chi_{t+1} = \left(\frac{1-\tau_{t+1}^w}{1-\tau_{t+1}^r}\frac{1}{\xi}\right)^{v_2}$, so that we can write for the third and fourth line of (87):

$$(1-\omega)(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3}(\epsilon_{t+1}\pi_{t+1})^{-\frac{1}{\rho}}c_{t+1}^{r,j}\left[(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^r)\xi w_{t+1}})^{v_2}-(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^r)\xi w_{t+1}})^{v_2}\right]=0$$

This leaves in (87):

$$\begin{aligned}\frac{1-\pi_t}{\pi_t}\left(c_t^{w,j}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^w)w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}\right)^{\rho} =&\beta\Bigg[c_t^{w,j}\left[\beta(1+r_{t+1})\Omega_{t+1}\left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}}\right)^{v_2\rho}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho}\right]^{\sigma}\\ &(\pi_{t+1})^{-\frac{1}{\rho}}(\frac{v_2}{v_1}\frac{1}{(1-\tau_{t+1}^w)w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3}\Bigg]^{\rho}\end{aligned}$$

Use that $\rho\sigma = \sigma - 1$ and rearrange:

$$\pi_t = 1 - \frac{\pi_t}{\pi_{t+1}}\beta^{\sigma}\left((1+r_{t+1})\Omega_{t+1}\right)^{\sigma-1}\left(\frac{(1-\tau_t^w)w_t}{(1-\tau_{t+1}^w)w_{t+1}}\right)^{v_2\rho\sigma}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_3\rho\sigma}$$

Since we arrive at the same difference equation for π_t as in (85), we have succesfully verified our conjectures for the value and consumption function of the worker.

A.2.6 Coming back to the worker first-order condition for labour

As mentioned above, after the conjectures of the value and consumption function of the worker are verified, we need to return to the worker first-order condition for labour. Above we simply conjectured that $V_{2,t+1}^w = R_{t+1}^w V_{1,t+1}^w$, which is akin to conjecturing that $\beta\mu_{t+1}\nu_t w_t[\omega V_{t+1}^{w,j} + (1-\omega)V_{t+1}^{r,j}]^{\rho-1}[\omega R_{t+1}^w V_{1,t+1}^{w,j} +$

$(1-\omega)R^r_{t+1}V^{r,j}_{1,t+1}]$ can be written as $\beta(1+r_{t+1})R^w_t\nu_t w_t[\omega V^{w,j}_{t+1}+(1-\omega)V^{r,j}_{t+1}]^{\rho-1}[\omega V^{w,j}_{1,t+1}+(1-\omega)V^{r,j}_{1,t+1}]$. Now that we have a clue of what Ω and $V^{w,j}$ look like, it is time to show that we can indeed do this.

Notice that we can rewrite (72) by 'isolating' R^r_{t+1}:

$$\begin{aligned} v_2(c^{w,j}_t)^{v_1\rho}(1-l^{w,j}_t)^{v_2\rho-1}(m^{w,j}_t)^{v_3\rho} =& \beta(1+r_{t+1})(1-\tau_t+\nu_t\frac{\mu_{t+1}}{1+r_{t+1}}R^r_{t+1})w_t[\omega V^{w,j}_{t+1}+(1-\omega)V^{r,j}_{t+1}]^{\rho-1} \\ & [\omega V^{w,j}_{1,t+1}+(1-\omega)V^{r,j}_{1,t+1}]+ \\ & \beta\mu_{t+1}\nu_t w_t[\omega V^{w,j}_{t+1}+(1-\omega)V^{r,j}_{t+1}]^{\rho-1}[\omega(R^w_{t+1}-R^r_{t+1})V^{w,j}_{1,t+1}] \end{aligned}$$

We can write $R^w_t=\frac{\mu_{t+1}}{1+r_{t+1}}\left(\frac{\omega}{\Omega_{t+1}}R^w_{t+1}+(1-\frac{\omega}{\Omega_{t+1}})R^r_{t+1}\right)=\frac{\mu_{t+1}}{1+r_{t+1}}\left(R^r_{t+1}+\frac{\omega}{\Omega_{t+1}}(R^w_{t+1}-R^r_{t+1})\right)$. Therefore, since we already isolated $\frac{\mu_{t+1}}{1+r_{t+1}}R^r_{t+1}$, we simply need to isolate the remaining $\frac{\mu_{t+1}}{1+r_{t+1}}\frac{\omega}{\Omega_{t+1}}(R^w_{t+1}-R^r_{t+1})$, which we will be able to do once we show that the following condition holds:

$$\frac{\omega}{\Omega_{t+1}}(R^w_{t+1}-R^r_{t+1})[\omega V^{w,j}_{1,t+1}+(1-\omega)V^{r,j}_{1,t+1}]=\omega(R^w_{t+1}-R^r_{t+1})V^{w,j}_{1,t+1}$$

Or, equivalently, that:

$$\Omega_{t+1}=\omega+(1-\omega)\frac{V^{r,j}_{1,t+1}}{V^{w,j}_{1,t+1}} \tag{88}$$

Let us think about $V^{r,j}_{1,t+1}$ and $V^{w,j}_{1,t+1}$ for a moment. These two expressions show the marginal value from one additional unit of wealth for a retiree and worker in period $t+1$, respectively. Here we can use our verified conjectures for the value and consumption functions to compute $V^{r,j}_{1,t+1}$ and $V^{w,j}_{1,t+1}$. We know that $V^{r,j}_t=(\epsilon_t\pi_t)^{\frac{\rho-1}{\rho}}\frac{1}{1+\frac{v_3}{v_1}}\left(\frac{1+r_t}{\gamma}a^{w,j}_{t-1}+h^{r,j}_t\right)(\frac{v_2}{v_1}\frac{1}{(1-\tau^r_t)\xi w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}$. Similarly, we know that $V^{w,j}_t=(\pi_t)^{\frac{\rho-1}{\rho}}\frac{1}{1+\frac{v_3}{v_1}}\left((1+r_t)a^{w,j}_{t-1}+h^{w,j}_t\right)(\frac{v_2}{v_1}\frac{1}{(1-\tau^w_t)w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}$. We can therefore compute:

$$\begin{aligned} V^{r,j}_{1,t+1} &= (\epsilon_{t+1}\pi_{t+1})^{\frac{\rho-1}{\rho}}\frac{1}{1+\frac{v_3}{v_1}}(\frac{v_2}{v_1}\frac{1}{(1-\tau^r_{t+1})\xi w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3} \\ V^{w,j}_{1.t+1} &= (\pi_{t+1})^{\frac{\rho-1}{\rho}}\frac{1}{1+\frac{v_3}{v_1}}(\frac{v_2}{v_1}\frac{1}{(1-\tau^w_{t+1})w_{t+1}})^{v_2}(\frac{v_3}{v_1}\frac{1+i_{t+1}}{i_{t+1}})^{v_3} \end{aligned}$$

We thus derive that $\frac{V^{r,j}_{1,t+1}}{V^{w,j}_{1,t+1}}=(\epsilon_{t+1})^{\frac{\rho-1}{\rho}}\left(\frac{1-\tau^w_{t+1}}{1-\tau^r_{t+1}}\frac{1}{\xi}\right)^{v_2}=(\epsilon_{t+1})^{\frac{1}{1-\sigma}}\chi_{t+1}$ (since $\frac{\rho-1}{\rho}=\frac{1}{1-\sigma}$). Plugging this into (88):

$$\Omega_{t+1}=\omega+(1-\omega)(\epsilon_{t+1})^{\frac{1}{1-\sigma}}\chi_{t+1}$$

This is exactly the same as (79). Therefore, our conjecture that we could write $\beta\mu_{t+1}\nu_t w_t[\omega V^{w,j}_{t+1}+(1-\omega)V^{r,j}_{t+1}]^{\rho-1}[\omega R^w_{t+1}V^{w,j}_{1,t+1}+(1-\omega)R^r_{t+1}V^{r,j}_{1,t+1}]$ as $\beta(1+r_{t+1})R^w_t\nu_t w_t[\omega V^{w,j}_{t+1}+(1-\omega)V^{r,j}_{t+1}]^{\rho-1}[\omega V^{w,j}_{1,t+1}+(1-\omega)V^{r,j}_{1,t+1}]$ has been verified. Additionally, this means that, by a similar argument rolled forward by one period, $V^w_{2,t+1}=R^w_{t+1}V^w_{1,t+1}$. It is now ascertained that all conjectures add up to mutually consistent solutions across all equations characterising optimal decisions of retirees and workers. This concludes the derivation of the

decision problem of the retiree. We have obtained a difference equation for π_t (which is the same for all workers), the worker consumption function, and the worker value function.

B Equilibrium and steady-state conditions

This section presents an overview of all equilibrium conditions and their steady-state counterparts. In the steady state, we set $\mu = A^{lap} = 1$ and, in order to ensure that the pension funding gap is zero, $\tau = \frac{\nu(R^{w,f} l^w + (R^{r,f}-1)\xi l^r)}{l}$, with $\nu_t = \nu$ exogenously determined.

Marginal propensity to consume out of wealth for retirees:

$$\epsilon_t \pi_t = 1 - \frac{\epsilon_t \pi_t}{\epsilon_{t+1}\pi_{t+1}} \beta^\sigma (1+r_{t+1})^{\sigma-1} \gamma \left(\frac{(1-\tau_t^r) w_t}{(1-\tau_{t+1}^r) w_{t+1}} \right)^{v_2 \rho \sigma} \left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t} \right)^{v_3 \rho \sigma}$$
$$\epsilon \pi = 1 - \beta^\sigma (1+r)^{\sigma-1} \gamma$$

Marginal propensity to consume out of wealth for workers:

$$\pi_t = 1 - \frac{\pi_t}{\pi_{t+1}} \beta^\sigma \left((1+r_{t+1}) \Omega_{t+1} \right)^{\sigma-1} \left(\frac{(1-\tau_t^w) w_t}{(1-\tau_{t+1}^w) w_{t+1}} \right)^{v_2 \rho \sigma} \left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t} \right)^{v_3 \rho \sigma}$$
$$\pi = 1 - \beta^\sigma \left((1+r) \Omega \right)^{\sigma-1}$$

Subjective reweighting of transition probabilities:

$$\Omega_t = \omega + (1-\omega) \left(\frac{1-\tau_t^w}{1-\tau_t^r} \frac{1}{\xi} \right)^{v_2} (\epsilon_t)^{\frac{1}{1-\sigma}}$$
$$\Omega = \omega + (1-\omega) \left(\frac{1-\tau^w}{1-\tau^r} \frac{1}{\xi} \right)^{v_2} (\epsilon)^{\frac{1}{1-\sigma}}$$

Private annuity factors of retirees and workers:

$$R_t^r = 1 + \mu_{t+1} \frac{\gamma}{1+r_{t+1}} R_{t+1}^r$$
$$R^r = \frac{1+r}{1+r-\gamma}$$
$$R_t^w = \frac{\mu_{t+1}}{1+r_{t+1}} \left(\frac{\omega}{\Omega_{t+1}} R_{t+1}^w + (1 - \frac{\omega}{\Omega_{t+1}}) R_{t+1}^r \right)$$
$$R^r = \frac{\omega}{\Omega} \frac{R^w}{1+r} + (1 - \frac{\omega}{\Omega}) \frac{R^r}{1+r}$$

Effective tax rates on labour:

$$\tau_t^r = \tau_t - (R_t^r - 1)\nu_t$$
$$\tau^r = \tau - (R^r - 1)\nu$$
$$\tau_t^w = \tau_t - R_t^w \nu_t$$
$$\tau^w = \tau - R^w \nu$$

Aggregate labour supply of retirees, workers, and total labour force:

$$l_t^w = N^w - \frac{v_2}{v_1}\frac{c_t^w}{(1-\tau_t^w)w_t}$$
$$l^w = N^w - \frac{v_2}{v_1}\frac{c^w}{(1-\tau^w)w}$$
$$l_t^r = N^r - \frac{v_2}{v_1}\frac{c_t^r}{(1-\tau_t^r)\xi w_t}$$
$$l^r = N^r - \frac{v_2}{v_1}\frac{c^r}{(1-\tau^r)\xi w}$$
$$l_t = l_t^w + \xi l_t^r$$
$$l = l^w + \xi l^r$$

Aggregate disposable income of retirees and workers:

$$d_t^r = (1-\tau_t)\xi w_t l_t^r + \mu_t P_t^{r,f} - \tau_t^g N^r$$
$$d^r = (1-\tau)\xi w l^r + P^{r,f}$$
$$d_t^w = (1-\tau_t) w_t l_t^w + f_t N^w - \tau_t^g N^w$$
$$d^w = (1-\tau) w l^w + f N^w$$

Aggregate human capital of retirees and workers:

$$h_t^r = d_t^r + \frac{\gamma}{1+r_{t+1}} h_{t+1}^r$$
$$h^r = \frac{1+r}{1+r-\gamma} d^r$$
$$h_t^w = d_t^w + \frac{1}{1+r_{t+1}}\left(\frac{\omega}{\Omega_{t+1}} h_{t+1}^w + (1-\frac{\omega}{\Omega_{t+1}})\frac{1}{\psi} h_{t+1}^r\right)$$
$$h^w = d^w + \frac{1}{1+r}\left(\frac{\omega}{\Omega} h^w + (1-\frac{\omega}{\Omega})\frac{1}{\psi} h^r\right)$$

Aggregate consumption of retirees, workers, and total population:

$$
\begin{aligned}
c_t^r &= \epsilon_t \pi_t \left((1+r_t)a_{t-1}^r + h_t^r\right) \\
c^r &= \epsilon \pi \left((1+r)a^r + h^r\right) \\
c_t^w &= \pi_t \left((1+r_t)a_{t-1}^w + h_t^w\right) \\
c^w &= \pi \left((1+r)a^w + h^w\right) \\
c_t &= c_t^r + c_t^w \\
c &= c^r + c^w
\end{aligned}
$$

Private financial wealth of retirees, workers (which is redundant due to Walras' law), and total population:

$$
\begin{aligned}
a_t^r &= (1+r_t)a_{t-1}^r + d_t^r - c_t^r - \frac{i_t}{1+i_t} m_t^r + (1-\omega)\left((1+r_t)a_{t-1}^w + d_t^w - c_t^w - \frac{i_t}{1+i_t} m_t^w\right) \\
a^r &= (1+r)a^r + d^r - c^r - \frac{i}{1+i} m^r + (1-\omega)\left((1+r)a^w + d^w - c^w - \frac{i}{1+i} m^w\right) \\
a_t^w &= \omega\left((1+r_t)a_{t-1}^w + d_t^w - c_t^w - \frac{i_t}{1+i_t} m_t^w\right) \\
a^w &= \omega\left((1+r)a^w + d^w - c^w - \frac{i}{1+i} m^w\right) \\
a_t &= a_t^w + a_t^r \\
a &= a^w + a^r
\end{aligned}
$$

Aggregate real money balances of retirees, workers, and total population:

$$
\begin{aligned}
m_t^r &= \frac{v_2}{v_1}\frac{1+i_t}{i_t} c_t^r \\
m^r &= \frac{v_2}{v_1}\frac{1+i}{i} c^r \\
m_t^w &= \frac{v_2}{v_1}\frac{1+i_t}{i_t} c_t^w \\
m^w &= \frac{v_2}{v_1}\frac{1+i}{i} c^w \\
m_t &= m_t^w + m_t^r \\
m &= m^w + m^r
\end{aligned}
$$

Pension fund annuity factors:

$$
\begin{aligned}
R_t^{r,f} &= 1 + \frac{\gamma}{1+r_{t+1}} R_{t+1}^{r,f} \\
R^{r.f} &= \frac{1+r}{1+r-\gamma} \\
R_t^{w,f} &= \frac{1}{1+r_{t+1}} (\omega R_{t+1}^{w,f} + (1-\omega) R_{t+1}^{r,f}) \\
R^{w,f} &= \frac{(1-\omega)(1+r)}{(1+r-\gamma)(1+r-\omega)}
\end{aligned}
$$

Aggregate per-period pension benefits of retirees and workers:

$$
\begin{aligned}
P_t^{r,f} &= \gamma \left(\mu_{t-1} P_{t-1}^{r,f} + \nu_{t-1} \xi w_{t-1} l_{t-1}^r \right) + (1-\omega) \left(\mu_{t-1} P_{t-1}^{w,f} + \nu_{t-1} w_{t-1} l_{t-1}^w \right) \\
P^{r,f} &= \gamma \left(P^{r,f} + \nu \xi w l^r \right) + (1-\omega) \left(P^{w,f} + \nu w l^w \right) \\
P_t^{w,f} &= \omega \left(\mu_{t-1} P_{t-1}^{w,f} + \nu_{t-1} w_{t-1} l_{t-1}^w \right) \\
P^{w,f} &= \frac{\omega}{1-\omega} \nu w l^w
\end{aligned}
$$

Pension fund liabilities:

$$
\begin{aligned}
L_t^f &= R_t^{r,f} P_t^{r,f} + R_t^{w,f} P_t^{w,f} \\
L^f &= R^{r,f} P^{r,f} + R^{w,f} P^{w,f}
\end{aligned}
$$

Pension fund assets:

$$
\begin{aligned}
K_{t+1}^f &= (1+r_{t+1})(K_t^f + \tau_t w_t l_t - \mu_t P_t^{r,f}) \\
K^f &= \frac{\tau w l - P^{r,f}}{r}
\end{aligned}
$$

Market clearing for savings:

$$
\begin{aligned}
a_t + K_t^f + \tau_t w_t l_t - \mu_t P_t^R &= P_t^k k_t + \frac{m_t}{1+i_t} \\
a + \frac{(1+r)(\tau w l - P^{r,f})}{r} &= k + \frac{m}{1+i}
\end{aligned}
$$

Aggregate capital to labour ratio:

$$
\begin{aligned}
\frac{k_{t-1}}{l_t} &= \frac{1-\alpha}{\alpha} \frac{w_t}{r_t^k} \\
\frac{k}{l} &= \frac{1-\alpha}{\alpha} \frac{w}{r^k}
\end{aligned}
$$

Marginal cost:

$$mc_t = \left(\frac{w_t}{\alpha}\right)^{\alpha} \left(\frac{r_t^k}{1-\alpha}\right)^{1-\alpha}$$
$$mc = \left(\frac{w}{\alpha}\right)^{\alpha} \left(\frac{r^k}{1-\alpha}\right)^{1-\alpha}$$
$$mc = \frac{\theta - 1}{\theta}$$

Pricing conditions:

$$\theta g_t^1 = (\theta - 1) g_t^2$$
$$\theta g^1 = (\theta - 1) g^2$$
$$g_t^1 = \Delta_t mc_t y_t + \beta\zeta(\frac{P_{t+1}}{P_t})^{\theta} g_{t+1}^1$$
$$g^1 = \Delta mcy + \beta\zeta g^1$$
$$g_t^2 = \Delta_t \frac{P_t^*}{P_t} y_t + \beta\zeta(\frac{P_{t+1}}{P_t})^{\theta-1}(\frac{P_{t+1}P_t^*}{P_t P_{t+1}^*}) g_{t+1}^2$$
$$g^2 = \Delta y + \beta\zeta g^2$$

Pricing kernel of intermediate goods producing firms:

$$\Delta_t = (\pi_t)^{-\frac{1}{\rho}}(\frac{v_2}{v_1}\frac{1}{(1-\tau_t^w)w_t})^{v_2}(\frac{v_3}{v_1}\frac{1+i_t}{i_t})^{v_3}$$
$$\Delta = (\pi)^{-\frac{1}{\rho}}(\frac{v_2}{v_1}\frac{1}{(1-\tau^w)w})^{v_2}(\frac{v_3}{v_1}\frac{1+i}{i})^{v_3}$$

Evolution of aggregate price level:

$$P_t = \left[\zeta(P_{t-1})^{1-\theta} + (1-\zeta)(P_t^*)^{1-\theta}\right]^{\frac{1}{1-\theta}}$$
$$\frac{P^*}{P} = 1$$

Output:

$$y_t = \frac{(k_{t-1})^{1-\alpha}(A_t^{lap} l_t)^{\alpha}}{v_t^p}$$
$$y = y_z = (l)^{\alpha}(k)^{1-\alpha}$$
$$v_t^p = \zeta(\frac{P_t}{P_{t-1}})^{\theta} v_{t-1}^p + (1-\zeta)(\frac{P_t}{P_t^*})^{\theta}$$
$$v^p = 1$$

Profits:

$$f_t = y_t - w_t l_t - r_t^k k_{t-1}$$
$$f = (1 - mc)y$$

Aggregate capital stock dynamics:

$$k_t = (1-\delta)k_{t-1} + \left(1 - S[\frac{i_t^k}{i_{t-1}^k}]\right) i_t^k$$
$$i^k = \delta k$$

Price of capital:

$$1 = P_t^k \left(1 - S[\frac{i_t^k}{i_{t-1}^k}] - S'[\frac{i_t^k}{i_{t-1}^k}]\frac{i_t^k}{i_{t-1}^k}\right) + \frac{P_{t+1}^k}{1+r_{t+1}} S'[\frac{i_{t+1}^k}{i_t^k}](\frac{i_{t+1}^k}{i_t^k})^2$$
$$P^k = 1$$

Aggregate resource constraint:

$$y_t = c_t + i_t^k$$
$$y = c + i^k$$

Fisher relation:

$$1 + i_t = (1 + r_{t+1})\frac{P_{t+1}}{P_t}$$
$$i = r$$

No-arbitrage relationship:

$$1 + r_t = \frac{r_t^k + P_t^k(1-\delta)}{P_{t-1}^k}$$
$$r = r^k - \delta$$

Government budget constraint:

$$\tau_t^g = m_{t-1}\frac{P_{t-1}}{P_t} - m_t$$
$$\tau^g = 0$$

Monetary policy rule:

$$i_t = \eta i_{t-1} + (1-\eta)[r_{t+1} + \gamma_\pi \pi_t^p + \gamma_y \tilde{y}_t]$$
$$i = r$$