Optimal long-term asset allocation with illiquid assets

Kristy A.E. Jansen MSc

Tilburg University

October 7, 2016

Abstract. This paper solves the optimal asset allocation and consumption pattern of a CRRA investor in case of a single liquid and a single illiquid risky asset, where illiquidity results from restrictions on the trading time of the illiquid asset. The model shows the illiquid asset is less attractive for the short-term investor, whereas the effect of illiquidity is negligible for long-term investors. Moreover, if in certain scenarios the total fraction invested in the illiquid asset becomes sufficiently high and the investor is not able to trade the illiquid asset, the investor reduces her allocation towards the liquid risky asset and the fraction of total wealth to consume. Finally, a positive correlation between the return on the illiquid asset and the probability to be able to trade that asset, leads to an additional reduction in the allocation towards the illiquid asset. This result indicates that the investor faces an increased utility loss if it becomes more difficult to trade the illiquid asset during bad market outcomes.

1 Introduction

The share of illiquid assets in investors' portfolios is large and has increased over the last decades. Kaplan and Violante (2010) show that individuals hold the majority of their net wealth in illiquid assets. The main illiquid asset class for households is residential real estate, followed by art objects. Secondly, the Global Pension Asset Study 2015, held by Towers Watson, shows that pension funds worldwide have increased their asset allocations towards alternative, illiquid asset classes such as private equity, hedge funds, real estate and infrastructure from about 5% in 1995 up to 25% in 2015.

In order to sell and buy these illiquid assets, without facing a significant price drop, the investor has to wait a particular time period before the next opportunity to trade arises. The time between trades results from the idea it takes time to find an appropriate counter-party to trade with (Diamond (1982)). Asset classes such as real estate require the counter-parties to have significant capital and particular knowledge about the asset class. The number of counter-parties who satisfy these capital and abilities requirements are often limited, which implies the next opportunity to trade

takes time and the actual amount of time it takes is uncertain. Moreover, asset classes such as private equity and hedge funds often have a lockup period, which means the investor has to stick to the investment for a certain time period. The average period between trades in real estate markets is approximately 4-5 years for residential housing and already increases up to approximately 10 years for institutional real estate and private equity. The time between transactions for infrastructure can be as large as 50-60 years. Less extreme examples of illiquid assets are corporate bonds and over-the-counter (OTC) equity. Corporate bonds trade approximately once a day, whereas OTC equity trades once a week.[1]

Although most investors have a sense of what illiquidity means, there does not exist a single generally accepted definition of illiquidity. The notion of illiquidity used here implies a restriction on the trading time of the illiquid asset, which is conceptually different from two other definitions of illiquidity used in the literature. Any transaction can be characterized by three dimensions: price, quantity, and time and along these dimensions three concepts of illiquidity can be distinguished. The first, and most often used concept, implies a restriction on the price of the illiquid asset. This implies the illiquid asset can always be traded as long as you pay sufficient costs. Constantinides (1986) solves the optimal asset allocation for an investor who is facing proportional transaction costs, and shows that in this case a no-trading region arises. Only when the asset allocation falls outside the no-trading region, the investor will re-balance back to the boundaries of the no-trading region. Lynch and Tan (2010) confirm this idea and show how the shape of the no-trading region changes if the investor faces fixed transaction costs, fixed and proportional transaction costs and predictability and transaction costs. Gârleanu and Pedersen (2010) solve the optimal asset allocation in case of quadratic transaction costs. Quadratic transaction costs, in contrast to fixed and proportional transaction costs, take into account the price impact of trading. They show that the investor always trades, but only in limited quantities to balance-off the costs and benefits of re-balancing. The second concept implies a restriction on the traded quantity of the illiquid asset. Again, the asset can always be traded, but the investor can only buy and sell in limited quantities. Longstaff (2001) shows that the investor's trading strategy consists of trading as much as possible, whenever possible in case the investor can only trade at a limited rate. The third concept of illiquidity, which is in line with the notion of illiquidity used in this thesis, implies a restriction on the trading time of the illiquid asset. In this case, the asset cannot be traded for a particular time period, however when a trading opportunity arises the investor can buy and sell at no-cost and in unlimited quantities. Gârleanu (2009) shows in an equilibrium model with a single risky asset that in case future expected liquidity is high, agents take more extreme positions, as they do not have to hold these positions for long when they become undesirable. As a result, larger trades should be observed in more liquid markets. Mostly related to the current paper, Ang et al. (2014) solve an optimal consumption problem in case of two risky assets and show the investor reduces its allocation towards illiquid assets

[1] Table 1: Holding Periods and Turnover of Various Asset Classes - Ang, Papanikolaou, and Westerfield (2014).

if the expected time between trading opportunities increases.

This thesis solves the optimal asset allocation for a CRRA investor in case of a single liquid risky asset and a single illiquid risky asset. Illiquidity is modeled by assuming a random probability the investor is able to trade the illiquid asset each period. In contrast to Ang et al. (2014), who use a continuous-time model with an infinite horizon, in this thesis a discrete-time model with a finite horizon is used. This approach allows to study how the effect of illiquidity depends on the horizon of the investor, which is a contribution to the current literature. Moreover, an extension of the baseline model studies the long-term asset allocation in case the severity of illiquidity is depended on the realized return of the illiquid asset.

The inability to trade the illiquid asset continuously is a form of market incompleteness and exposes the investor to additional risk not present in the standard Merton problem. The optimal asset allocation incorporating this additional risk is solved for both an intermediate consumption problem and a final wealth problem. I analyze the results for two extreme cases. In the first case, the investor faces a probability $1-p$ illiquid wealth cannot be liquidated at all at the final date, which means the illiquid asset is worth exactly zero if it cannot be liquidated. In the other extreme case, I assume illiquid wealth can be liquidated at the final date for sure, without facing any discount in the value of the illiquid asset. Comparing the two extremes allows to study which determinants of the cost of illiquidity are the strongest.

The optimal asset allocation in case the investor faces the risk her illiquid wealth might not be liquidated at the final date, which implies the investor does not gain utility over illiquid wealth, reduces the allocation towards the illiquid asset to zero in the years approaching the final date. In those years, the probability the investor is not able to liquidate illiquid wealth at the final date increases, which is the major source of the utility loss the investor faces. By reducing the asset allocation towards the illiquid asset to zero near the final date makes sure all wealth can be liquidated. In case the investor is expected to be able to trade once in two years, the investor has exactly zero allocation towards the illiquid asset in the last two periods and the allocation gradually converges to the Merton solution when further away from the final date (Figure 3, Section 3.2.2). This effect is even stronger in a final wealth problem. In a final wealth problem, the investor cares about optimizing his wealth at the final date only. The utility loss in case the investor will not be able to liquidate his illiquid wealth is therefore larger than for an investor optimizing intermediate consumption, who also gains utility over intermediate dates.

In case the illiquid asset can be liquidated at the final date with probability one, illiquidity is less severe and affects the allocation towards the illiquid asset in a consumption problem only. The allocation towards the illiquid asset reduces especially in the years approaching the final date. The investor smooths consumption, which implies consumption as fraction of total wealth is especially large in the years

approaching the final date. As the investor can only consume out of liquid wealth, illiquid wealth grows at a relatively fast rate compared to liquid wealth in those years. Therefore, the risk illiquid wealth grows too fast and the investor might not be able to finance her desired consumption level increases, leading to a reduction in the allocation towards the illiquid asset. In case the investor is expected to trade once in two years, the reduction in the allocation towards the illiquid risky asset compared to the Merton solution is approximately 4%, and slightly reduces to zero when further away from the final date (Figure 8, Section 3.2.5). In a final wealth problem, the investor only faces utility loss as a result of fluctuations in the actual allocation towards the illiquid asset in case the investor is not able to trade the asset. However, as the utility function of a CRRA investor is a relatively flat function of the fraction invested in the illiquid asset at the optimum, the utility loss is small and the optimal allocation towards the illiquid asset is not significantly different from the Merton solution.

Besides a reduction in the allocation towards the illiquid risky asset, consumption and the allocation towards the liquid risky asset are also affected by illiquidity. Optimal consumption and the allocation towards the liquid risky asset are decreasing in the actual fraction of total wealth invested in the illiquid asset, so the higher the actual fraction invested in illiquid wealth compared to the optimal fraction, the lower consumption and the allocation towards the liquid risky asset. In this way, the investor partly compensates the increased risk exposure as a result of a higher actual allocation towards the illiquid asset. In case the liquid and illiquid risky asset are correlated, the reduction in the allocation towards the liquid risky asset, when the fraction invested in illiquid wealth increases, is even stronger. The total volatility of the investor's portfolio can be reduced to a higher extent by decreasing the allocation towards the liquid risky asset in case of positive correlation between the liquid and illiquid risky asset.

Although our results are not exactly comparable to Ang et al. (2014), as we use a finite-horizon model whereas Ang et al. (2014) use an infinite horizon, the effects of illiquidity found in case illiquid wealth might not be liquidated at the final date in a consumption problem are comparable to the results found in Ang et al. (2014). However, in case liquid wealth can be liquidated at the final date for sure, the effects found in this thesis are significantly smaller in case of a consumption problem. Moreover, Ang et al. (2014) shows that the effect of illiquidity in a final wealth problem is negligible if illiquid wealth can be liquidated at some point. We however find that in case illiquid wealth might not be liquidated at the final date, the effect of illiquidity on the reduction in the allocation towards the illiquid asset is significant.

The baseline model is extended by allowing for correlation between the return on the illiquid asset and the probability to trade this asset. The results show the illiquid asset becomes less attractive in case of a positive correlation compared to the baseline model. A positive correlation leads to a reduction in the allocation towards the illiquid asset of approximately 3% compared to the baseline model (Figure 15, Section 5). This implies that the investor faces a larger utility loss in case she is not

able to trade the illiquid asset during bad market outcomes. The extension sheds a first light on the flight-to-liquidity phenomenon, where investors try to sell more illiquid assets during times of financial stress, which makes it more difficult to trade those assets.

In Section 2 the baseline model is presented and the numerical procedure to solve the model is discussed. In Section 3, the results of the baseline model are presented and the implications for the asset allocation and consumption patterns are discussed. In Section 4 a sensitivity analysis is performed to better understand the main driver behind the costs of illiquidity. In Section 5, the model is solved in case the probability to trade and the realized returns on the illiquid asset are correlated. The last section concludes.

2 Baseline model

In this section, the baseline model is presented and the dynamic programming technique in order to solve the model is described. In the next section, the main implications of the model on the investor's optimal asset allocation and consumption pattern are discussed.

2.1 Baseline problem

The investor maximizes utility over sequences of consumption with a finite horizon T and preferences are represented by a standard constant relative risk aversion (CRRA) expected utility function:

$$\mathbb{E}_0\Big[\sum_{t=0}^{T}\sum_{j=0}^{1/h} e^{-(t+jh)\beta}\frac{C_{t+jh}^{1-\gamma}}{1-\gamma}\Big] \tag{1}$$

where β equals the time preference discount factor and γ the risk-aversion parameter ($\gamma > 1$). The subscript t is defined on a yearly basis and h indicates the length of the time-period between two points in time the portfolio and consumption choice is evaluated within one year. So the investor maximizes consumption over a length of T years with in total $(1/h)T$ evaluation periods. In order to shorten notion, $s = t + jh$ is used in the remainder of this thesis.

The investor has access to three assets: a risk-free asset B, a liquid risky asset S and an illiquid risky asset X. The risk-free asset has a constant rate of return r^f in each time period of length h. The liquid risky asset earns a continuously compounded nominal return R_s^S in the period $[s-h, s]$ and the illiquid risky asset earns a continuously compounded nominal return R_s^X in the period $[s-h, s]$. The prices of risk are denoted by λ_S and λ_X for the liquid and illiquid risky asset respectively. The volatilities of the liquid and illiquid risky assets are denoted by σ_S and σ_X respectively.

The two risky assets have a multivariate normal distribution $[R_s^S; R_s^X] \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = [r^f + \lambda_S \sigma_S - 0.5\sigma_S^2; \quad r^f + \lambda_X \sigma_X - 0.5\sigma_X^2]$ and the variance-covariance matrix equals $\boldsymbol{\Sigma} = [\sigma_S^2 \quad \rho\sigma_S\sigma_X; \quad \rho\sigma_S\sigma_X \quad \sigma_X^2]$.[2]

The investor's wealth consists of two parts, the liquid wealth is denoted by W and the illiquid wealth equals the total amount invested in the illiquid asset and therefore equals X. Illiquid wealth can only be consumed or converted into liquid wealth if a trading opportunity arises. Therefore, the optimization problem of the investor is subject to the following two budget constraints:

$$W_s = (W_{s-h} - \triangle X_{s-h} - C_{s-h})(\exp(r^f) + \theta_{s-h}(\exp(R_s^S) - \exp(r^f))) \tag{2}$$

$$X_s = (X_{s-h} + \triangle X_{s-h}) \exp(R_s^X) \tag{3}$$

where θ_{s-h} is the fraction of liquid wealth W invested in the liquid risky asset S, the remainder $1 - \theta_{s-h}$ is invested in the risk-free asset B. $\triangle X_{s-h}$ is the total amount transferred from liquid to illiquid wealth in case the investor is able to trade at no cost.

The possibility of trading the illiquid asset is defined by the indicator function I_s, which equals 1 in case a trading opportunity arises at time s and equals 0 otherwise. The investor is endowed with initial wealth W_0 and X_0, where $W_0 > 0$ and $X_0 \geq 0$. I assume the investor cannot short the illiquid asset, so $X_s \geq 0$ for all s. Moreover, without loss of generality, I restrict the set of admissible trading strategies to those that imply $W_s > 0$. As the investor can only consume out of her liquid wealth, $W_s \leq 0$ implies zero consumption with positive probability if the illiquid asset cannot be traded at time s ($I_s = 0$), leading to a utility of $-\infty$. The two assumptions together imply $W_s + X_s > 0$.

Problem 2.1. *The optimal consumption problem of the investor equals:*

$$\max_{\{\theta_s, \triangle X_s, C_s\}_{s=0}^T} \mathbb{E}_0 \Big[\sum_{s=0}^{T} e^{-s\beta} \frac{C_s^{1-\gamma}}{1-\gamma} \Big]$$

subject to the two budget constraints (2) and (3), where the transfer from or to the illiquid risky asset can only be non-zero ($\triangle X_s \neq 0$) if $I_s = 1$.

The indicator function I_s is a random variable and the probability a trading opportunity arises at time s is constant over time in the baseline model and denoted by $p = \mathbb{P}\{I_s = 1\}$. Moreover, the exogenous variables are the returns earned on the two risky assets, R_s^S and R_s^X. The endogenous state variables are W_s and X_s and the decision variables are θ_s, $\triangle X_s$ and C_s. The dynamics of the endogenous variables

[2] The means are adjusted for their variances as exponentials over the returns are taken in the model to ensure wealth does not turn negative.

$\{X_s(\omega), W_s(\omega)\}_{s=0}^T$ and the decision variables $\{\theta_s(\omega), \triangle X_s(\omega), C_s(\omega)\}$ are adapted to the filtration $\mathbb{F} = \{\mathcal{F}_s\}_{s=0}^T$ over a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\{\mathcal{F}_s\}$ is the filtration generated by $[R_s^S; R_s^X]$. Moreover, I_s is adapted with respect to filtration $\{\mathcal{F}_s\}$ and I_s is independent of $\mathcal{F}_{s-h}$.

Prices of risk are not time-varying in the baseline model and therefore the value function of the investor is equal to the discounted present value of the sum of all future expected utilities over consumption given the optimal decisions taken by the investor, which depends on the two endogenous variables. This is equivalent to parameterizing the value function s to depend on the value of total wealth $W_s + X_s$ and the fraction invested in the illiquid asset $\xi_s = \frac{X_s}{W_s+X_s}$:

$$V_s(W_s + X_s, \xi_s) = \max_{\{\theta_s, \triangle X_s, C_s\}_{s=0}^T} \mathbb{E}_0\Big[\sum_{s=0}^T e^{-s\beta}\frac{C_s^{1-\gamma}}{1-\gamma}\Big] \tag{4}$$

By applying the principle of dynamic programming I may write the Hamilton-Jacobi-Bellman (HJB) equation:

$$V_s(W_s + X_s, \xi_s) = \max_{\theta_s, \triangle X_s, C_s} e^{-s\beta}\frac{C_s^{1-\gamma}}{1-\gamma} + \mathbb{E}_s V_{s+h}(W_{s+h} + X_{s+h}, \xi_{s+h}) \tag{5}$$

with boundary condition:

$$V_T(W_T + X_T, \xi_T) = pe^{-T\beta}\frac{(W_T + X_T)^{1-\gamma}}{1-\gamma} + (1-p)e^{-T\beta}\frac{((1-d\xi_T)(W_T + X_T))^{1-\gamma}}{1-\gamma}$$

where d is a discount, $0 \leq d \leq 1$.

The boundary condition implies that with probability $p = \mathbb{P}\{I_T = 1\}$ the investor can liquidate both liquid and illiquid wealth at time T without any costs and with probability $1-p$ the investor can only liquidate her illiquid wealth at a discount d. A high d implies the investor can convert the illiquid asset into cash only at high costs, whereas a low d implies the investor can liquidate illiquid wealth at small cost. If a trading opportunity in the illiquid asset arises at time T, the investor is able to convert the illiquid asset into liquid wealth at no cost to finance consumption needs at the final date T. This form of the boundary condition allows to study the implications of a finite horizon, as only in case of a finite horizon the investor faces the risk part of his wealth cannot be fully liquidated at the final date.

The value function in the standard Merton two asset problem is of the form $V_s(W_s + X_s, \xi_s) = e^{-s\beta}\frac{(W_s+X_s)^{1-\gamma}}{1-\gamma}H_s$, where H_s is a deterministic function of time (Merton (1971)). Theorem 2.2 below shows that in this case the form of the value function satisfies:

$$V_s(W_s + X_s, \xi_s) = e^{-s\beta}\frac{(W_s + X_s)^{1-\gamma}}{1-\gamma}\Big(pH_s(\xi_s^*) + (1-p)H_s(\xi_s)\Big)$$

where $H_s(\xi_s^*) = \text{argmin}_{\xi_s} H_s(\xi_s)$.

This means that in the current model H_s is no longer a function of time s only, but also depends on the fraction of total wealth invested in the illiquid asset at time s, ξ_s. In other words, the value function can be represented as a power function of total wealth, times a function depending on the fraction of the portfolio invested in the illiquid asset, and time.

Theorem 2.2. *There exist time-dependent functions α_s, θ_s, H_s and ξ_s such that the optimal solution to Problem 2.1 can be written as:*

$$C_s^* = \alpha_s(\xi_s)(W_s + X_s) \tag{6}$$

$$\theta_s^* = \theta_s(\xi_s) \tag{7}$$

$$\xi_s^* = \text{argmin}_{\xi_s} H_s(\xi_s) \tag{8}$$

$$V_s(W_s + X_s, \xi_s) = e^{-s\beta} \frac{(W_s + X_s)^{1-\gamma}}{1-\gamma} \Big(pH_s(\xi_s^*) + (1-p)H_s(\xi_s) \Big) \tag{9}$$

Proof. We prove the theorem by backward induction and start by establishing (9). At the final date $s = T$, the value function can, due to (5), be written in (9) with $H_T(\xi_T) = (1 - d\xi_T)^{1-\gamma}$ and $H_T(\xi_T^*) = \text{argmin}_{\xi_T} H_T(\xi_T) = 1$ as follows:

$$V_T(W_T + X_T, \xi_T) = e^{-\beta T} \frac{(W_T + X_T)^{1-\gamma}}{1-\gamma} (pH_T(\xi_T^*) + (1-p)H_T(\xi_T)) \tag{10}$$

Now assume that (9) holds at time s and show that (9) also holds at time $s - h$. Indeed, note that the value function at time $s - h$ equals:

$$\begin{aligned}
V_{s-h}(W_{s-h} + X_{s-h}, \xi_{s-h}) &= \max_{\theta_{s-h}, \triangle X_{s-h}, C_{s-h}} e^{-(s-h)\beta} \frac{C_{s-h}^{1-\gamma}}{1-\gamma} + \mathbb{E}_{s-h} V_s(W_s + X_s, \xi_s) \\
&= e^{-(s-h)\beta} \frac{C_{s-h}^{*1-\gamma}}{1-\gamma} + \mathbb{E}_{s-h} V_s(W_s + X_s, \xi_s) \\
&= e^{-(s-h)\beta} \frac{\alpha_{s-h}(\xi_{s-h})^{1-\gamma}(W_{s-h} + X_{s-h})^{1-\gamma}}{1-\gamma} \\
&+ \mathbb{E}_{s-h}[e^{-s\beta} \frac{(W_s + X_s)^{1-\gamma}}{1-\gamma} \Big(pH_s(\xi_s^*) + (1-p)H_s(\xi_s) \Big)] \\
&= e^{-(s-h)\beta} \frac{(W_{s-h} + X_{s-h})^{1-\gamma}}{1-\gamma} \Big(\alpha_{s-h}^{1-\gamma}(\xi_{s-h}) \\
&+ e^{-h\beta} \mathbb{E}_{s-h}[\{(1 - \xi_{s-h} - \alpha_{s-h}(\xi_{T-h}))(\exp(r^f) + \theta_{s-h}(\xi_{s-h})(\exp(R_s^S) \\
&- \exp(r^f))) + \xi_{s-h} \exp(R_s^X)\}^{1-\gamma} \Big(pH_s(\xi_s^*) + (1-p)H_s(\xi_s) \Big)] \Big)
\end{aligned}$$

Notice that in the last step, I have used the dynamics of total wealth $W_s + X_s$:

$$W_s + X_s = (W_{s-h} + X_{s-h})\{(1 - \xi_{s-h} - \alpha_{s-h}(\xi_{s-h}))(\exp(r^f) + \theta_{s-h}(\xi_{s-h})(\exp(R_s^S) - \exp(r^f))) + \xi_{s-h}\exp(R_s^X)\} \tag{11}$$

Therefore at each time $s-h$, the penalty function $H_{s-h}(\xi_{s-h})$ equals:

$$H_{s-h}(\xi_{s-h}) = \alpha_{s-h}^{1-\gamma}(\xi_{s-h}) + e^{-h\beta}\mathbb{E}_{s-h}[\{(1 - \xi_{s-h} - \alpha_{s-h}(\xi_{s-h}))(\exp(r^f) + \theta_{s-h}(\xi_{s-h})(\exp(R_s^S) - \exp(r^f))) + \xi_{s-h}\exp(R_s^X)\}^{1-\gamma}\Big(pH_s(\xi_s^*) + (1-p)H_s(\xi_s)\Big)] \tag{12}$$

where $H_{s-h}(\xi_{s-h}^*) = \min_{\xi_{s-h}} H_{s-h}(\xi_{s-h})$.

Notice that ξ_s can be written as a function of ξ_{s-h}:

$$\xi_s = \frac{X_s}{X_s + W_s} = \frac{\xi_{s-h}\exp(R_s^X)}{(1 - \xi_{s-h} - \alpha_{s-h}(\xi_{s-h}))\{\exp(r^f) + \theta_{s-h}(\xi_{s-h})(\exp(R_s^S) - \exp(r^f))\} + \xi_{s-h}\exp(R_s^X)} \tag{13}$$

Therefore, the penalty function $H_{s-h}(\xi_{s-h})$ is a function of $s-h$ and ξ_{s-h} only and hence (9) holds for all s.

Now that we have proved the functional form of the value function, (8) can be shown as follows:

$$\xi_s^* = \text{argmax}_{\xi_s} V_s(W_s + X_s, \xi_s) = \text{argmax}_{\xi_s} e^{-s\beta}\frac{(W_s + X_s)^{1-\gamma}}{1-\gamma}\Big(pH_s(\xi_s^*) + (1-p)H_s(\xi_s)\Big) = \text{argmin}_{\xi_s} H_s(\xi_s)$$

The functional forms of the decision variables C_{s-h}^* and θ_{s-h}^* can now easily be derived:

$$C_{s-h}^* = \text{argmax}_{C_{s-h}} \frac{C_{s-h}^{1-\gamma}}{1-\gamma} + \mathbb{E}_{s-h}V_s(W_s + X_s, \xi_s)$$

$$\theta^*_{s-h} = \text{argmax}_{\theta_{s-h}} \frac{C^{1-\gamma}_{s-h}}{1-\gamma} + \mathbb{E}_{s-h} V_s(W_s + X_s, \xi_s)$$
$$\Longleftrightarrow$$

$$\frac{\partial V_{s-h}}{\partial C_{s-h}} = \beta^{s-h} C^{-\gamma}_{s-h} - \mathbb{E}_{s-h}[\frac{\partial V_s}{\partial W_s + X_s}\Big(\exp(r^f) + \theta_{s-h}(\exp(R^S_s) - \exp(r^f))\Big)]$$
$$+ \mathbb{E}_{s-h}[\frac{\partial V_s}{\partial \xi_s}\xi_s \frac{1}{W_s + X_s}\Big(\exp(r^f) + \theta_{s-h}(\exp(R^S_s) - \exp(r^f))\Big)] = 0$$

$$\frac{\partial V_{s-h}}{\partial \theta_{s-h}} = \mathbb{E}_{s-h}[\frac{\partial V_s}{\partial W_s + X_s}\Big(\exp(R^S_s) - \exp(r^f)\Big)(W_{s-h} - \triangle X_{s-h} - C_{s-h})]$$
$$- \mathbb{E}_{s-h}[\frac{\partial V_s}{\partial \xi_s}\xi_s \frac{1}{W_s + X_s}\Big(\exp(R^S_s) - \exp(r^f)\Big)(W_{s-h} - \triangle X_{s-h} - C_{s-h})] = 0$$

By using (9) and the dynamics of total wealth (11) again, we get[3]:

$$\frac{\partial V_{s-h}}{\partial C_{s-h}} = \alpha_{s-h}(\xi_{s-h})^{-\gamma} + \beta^h \mathbb{E}_s[\{(1 - \xi_{s-h} - \alpha_{s-h}(\xi_{s-h}))(\exp(r^f) + \theta_{s-h}(\xi_{s-h})(\exp(R^S_s) - \exp(r^f)))$$
$$+ \xi_{s-h}\exp(R^X_s)\}^{-\gamma}\Big((1-p)\frac{H'_s(\xi_s)}{1-\gamma}\xi_s - (pH_s(\xi^*_s) + (1-p)H_s(\xi_s))\Big)\exp(r^f)] = 0 \tag{14}$$

$$\frac{\partial V_{s-h}}{\partial \theta_{s-h}} = \mathbb{E}_s[\{(1 - \xi_{s-h} - \alpha_{s-h}(\xi_{s-h}))(\exp(r^f) + \theta_{s-h}(\xi_{s-h})(\exp(R^S_s) - \exp(r^f)))$$
$$+ \xi_{s-h}(\exp(R^X_s))\}^{-\gamma}\Big((pH_s(\xi^*_s) + (1-p)H_s(\xi_s)) - (1-p)\frac{H'_s(\xi_s)}{1-\gamma}\xi_s\Big)\Big(\exp(R^S_s) - \exp(r^f)\Big)] = 0 \tag{15}$$

Equation (13) shows that $\xi_s = f(\xi_{s-h})$ and hence the first order conditions with respect to consumption, C_{s-h}, and the allocation towards the liquid risky, θ_{s-h}, depend only on time $s-h$ and the fraction invested in the illiquid asset ξ_{s-h} and therefore (6) and (7) are satisfied.

□

[3]Notice the assumption $W_{s-h} + X_{s-h} \geq 0$ implies $(W_{s-h} - \triangle X_{s-h} - C_{s-h}) \geq 0$, as otherwise liquid wealth would turn negative.

Intuitively, the function $H_s(\xi_s)$ can be viewed as a penalty function, which is minimized at the optimal fraction of total wealth invested in the illiquid asset, $\xi_s^* = \text{argmin}_{\xi_s} H_s(\xi_s) = H_s(\xi_s^*)$, at time s. As $1-\gamma < 0$, the value function is therefore optimized at ξ_s^*. In case the investor is able to trade the illiquid asset at time s, she will re-balance her portfolio towards the optimal ratio of illiquid wealth ξ_s^* to total wealth as there are no costs involved when the investor trades and the return processes have constant prices of risk. Moreover, the optimal consumption choice and optimal investment strategy in the liquid risky asset depend on the fraction of total wealth invested in the illiquid asset, ξ_s. If the investor cannot trade the illiquid asset, she will choose her consumption and investment in the illiquid asset in such a way that given ξ_s, consumption and the allocation towards the liquid risky asset is optimized. Notice that in the original problem, the investor decides the optimal transfer from or to the illiquid asset in case a trading opportunity with probability p arises, $\triangle X_s$, at each s, which can now simply be found by solving:

$$\triangle X_s^* = (\xi_s^* - \xi_s)(W_s + X_s) \tag{16}$$

2.2 Solution technique

In this section the numerical method to solve the baseline model is described. Before explaining the numerical method, I will clearly define the sequence of decision making and the evolution of the endogenous variables.

2.2.1 Timing

In Figure 1, the sequence of decision making is depicted.

The endogenous variables, liquid wealth W_{s-h} and illiquid wealth X_{s-h}, are defined as total wealth before consumption and returns earned in period $[s-h, s]$. Based on the value of liquid and illiquid wealth at time $s-h$, the investor first decides the fraction of total wealth to be invested in the illiquid asset in case a trading opportunity with probability p arises, ξ_{s-h}^*. Subsequently, based on the actual fraction invested in illiquid wealth, ξ_{s-h}, the investor chooses the optimal fraction of total wealth to be consumed in period $[s-h, s]$, $\alpha_{s-h}(\xi_{s-h})$, and the optimal allocation towards the liquid risky asset, $\theta_{s-h}(\xi_{s-h})$. This implies ξ_{s-h} is a decision variable only when a trading opportunity in the illiquid risky asset arises. In case the investor is not able to trade the illiquid asset, ξ_{s-h} is endogenous and depends on the chosen fraction of total wealth to be consumed at $s-2h$, the fraction of liquid wealth invested in the liquid risky asset at time $s-2h$, the allocation towards the illiquid asset at time $s-2h$, and the returns earned on both the illiquid and liquid risky asset in period

$[s-2h, s-h]$. So in sum, if the investor cannot trade the illiquid asset the decision variables are θ_{s-h} and α_{s-h} and if the investor can trade the decision variables are θ_{s-h}, α_{s-h} and ξ_{s-h}. The decision frequency is $1/h$ each year. Notice that by the assumption $W_{s-h} > 0$ and $X_{s-h} \geq 0$, the possible values for ξ_{s-h} are restricted to the interval $[0, 1)$.

2.2.2 Numerical solution technique

The baseline model is solved by means of backward induction, where I start solving the problem at the final date $s = T$ and solve the model backwards for each period until arriving at time $s = 0$. To solve the optimal consumption, the optimal fraction invested in the illiquid asset and the optimal fraction invested in the liquid risky asset at time $T - h$, I construct a grid for the fraction invested in the illiquid asset, $\xi_{T-h} \in [0, 1)$. Moreover, I simulate $M = 100000$ trajectories for the exogenous state variables, the returns on the liquid and illiquid risky asset in the period $[T-h, T]$, R_T^S and R_T^X, from a multi-normal distribution with means and variance-covariance matrix as described in Section 2.1.

For each grid point, I solve the first order conditions with respect to consumption and the allocation towards the liquid risky asset, (14) and (15), by using $H_T(\xi_T)$ as defined in (10), R_T^S, and R_T^X to find the optimal fraction of total wealth the investor consumes, $\alpha_{T-h}(\xi_{T-h})$ and the optimal fraction invested in the liquid risky asset, $\theta_{T-h}(\xi_{T-h})$ for each $\xi_{T-h} \in [0, 1)$. Using $\alpha_{T-h}(\xi_{T-h})$, $\theta_{T-h}(\xi_{T-h})$, R_T^S, R_T^X, and the penalty function at time T, $H_T(\xi_T)$, the penalty function $H_{T-h}(\xi_{T-h})$ can be computed for each ξ_{T-h} as in (12). Finally, ξ_{T-h}^* can be solved by $\xi_{T-h}^* = \text{argmin}_{\xi_{T-h}} H_{T-h}(\xi_{T-h})$.

Again, I draw M trajectories from a multi-normal distribution to obtain R_{T-h}^S and R_{T-h}^X. Using R_{T-h}^S, R_{T-h}^X and $H_{T-h}(\xi_{T-h})$ again allows for solving $\alpha_{T-2h}(\xi_{T-2h})$ and $\theta_{T-2h}(\xi_{T-2h})$ by solving (14) and (15). Then the penalty function at time $T-2h$ can be solved and the optimal allocation towards the illiquid risky asset, ξ_{T-2h}^* can be derived. This approach can be continued until we arrive at the starting date $s = 0$.

Figure 1 shows the functional form of $H_s(\xi_s)$ in case $T = 6$. The graphs clearly show the further away ξ_s from ξ_s^*, the higher the value of the penalty function, which implies the higher ξ_s is deviated from the optimum, the lower the corresponding value function of the investor.

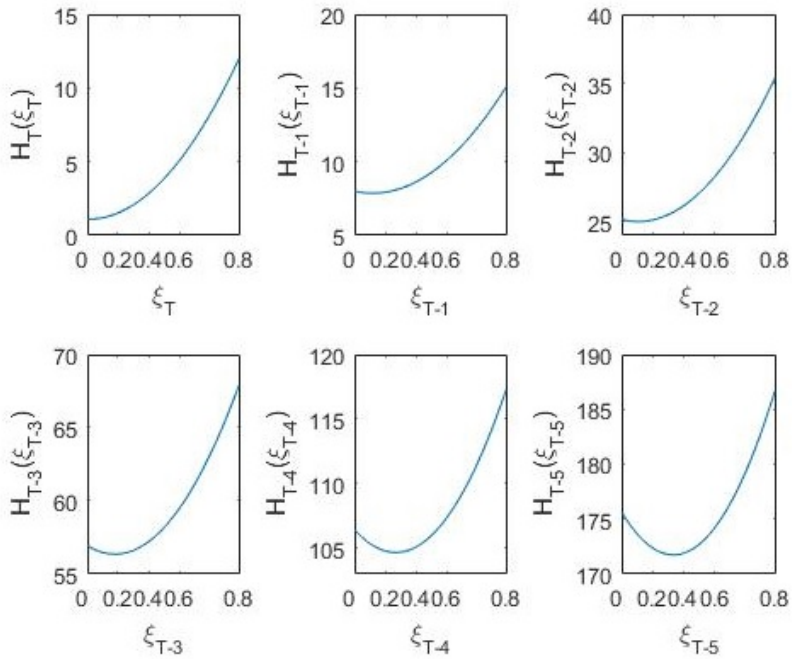


Figure 1. **Penalty function** This figure shows the functional form of $H_s(\xi_s)$ for each s, $s = 0, 1, ..., T$, $T = 6$. The graph is constructed by using the following parameter values: $\gamma = 3, \beta = 0.02, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, $\rho = 0$, $d = 1$ and $p = 0.5$.

3 Results baseline model

In this section, the main implications of the baseline model are shown. Moreover, this section disentangles two aspects of illiquidity that lead to a reduction in the allocation towards the illiquid asset.

3.1 Parameter values

The chosen parameter values are in line with empirical implications for the financial market on an annually basis. The liquid asset has a price of risk $\lambda_S = 0.25$ and $\sigma_S = 0.2$ and the risk-free rate is set equal to $r^f = 0.02$. The risk-aversion parameter is set equal to $\gamma = 3$, which results in an optimal equity allocation of approximately 40% and a risk-free bond allocation of 60%. The time-preference discount factor equals $\beta = 0.02$ and the discount to transfer the illiquid wealth into cash at the final date T equals $d = 1$[4].

[4]In Section 3.2.5 the results are discussed in case $d = 0$ at time T

The parameter values of the illiquid asset are set equal to the parameter values of the liquid risky asset; $\lambda_X = 0.25$ and $\sigma_X = 0.2$. In this way, I isolate the effect of illiquidity, instead of getting results that are due to higher Sharpe ratio of the illiquid asset. In line with this reasoning, I also assume no correlation between the liquid and illiquid risky asset; $\rho = 0$. In the sensitivity analysis I will clearly show how the results change when these assumptions are relaxed.

The final date T is set equal to 20 and evaluation period h equals 1. Regarding the severity of the illiquidity, I assume there is a probability $p = 0.5$ the investor is able to trade each period. This implies the expected time between trades in the illiquid asset equals two years.

3.2 Optimal portfolio implications

In this section I characterize the investor's optimal asset allocation and consumption patterns. Moreover, I compare all portfolio and consumption implications of the model with the Merton two-risky asset case.

3.2.1 The fully liquid case

First, I will discuss the results in case the illiquid risky asset can always be traded: the Merton two-asset problem. The allocation towards the liquid and illiquid risky asset are the same and constant over time in the fully liquid case and satisfy the Merton two-asset solution (Merton (1971)):

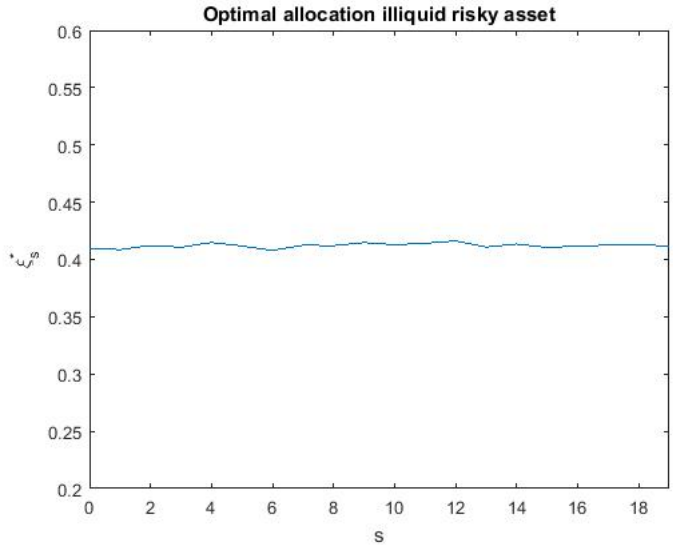
$$\begin{pmatrix} \theta_s^{Merton} \\ \xi_s^{Merton} \end{pmatrix} = \frac{1}{\gamma}\left(\sum\right)^{-1} \begin{pmatrix} \lambda_S \sigma_S \\ \lambda_X \sigma_X \end{pmatrix}$$

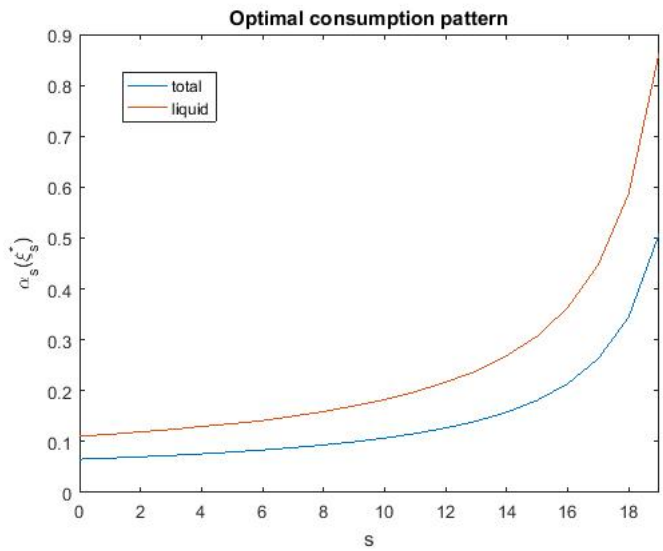
The optimal consumption pattern satisfies the optimal consumption pattern derived in the Merton Model (Merton (1971)):

$$\alpha_s^{Merton} = (e^{-s\beta} H_s)^{-\frac{1}{\gamma}}$$

where H_s is a deterministic function of time s. Notice that in this model in case the investor is able to trade each evaluation period, we use $H_s^*(\xi_s^*)$ each s. $H_s^*(\xi_s^*)$ has the same value for each ξ_s and therefore depends on time s only.

Figure 2 confirms the output of the model is consistent with the Merton two-risky asset problem described in this section in case $p = 1$. Consumption as a fraction of total wealth exponentially increases when approaching T and the allocation towards the liquid and illiquid risky asset remain constant over time and are both equal to the Merton solution, which implies the optimal consumption and the asset allocation decision are independent of each other.


Optimal allocation illiquid risky asset
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
ξ_s^*
0
2
4
6
8
10
12
14
16
18
s


Optimal consumption pattern
total
liquid
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$\alpha_s(\xi_s^*)$
0
2
4
6
8
10
12
14
16
18
s

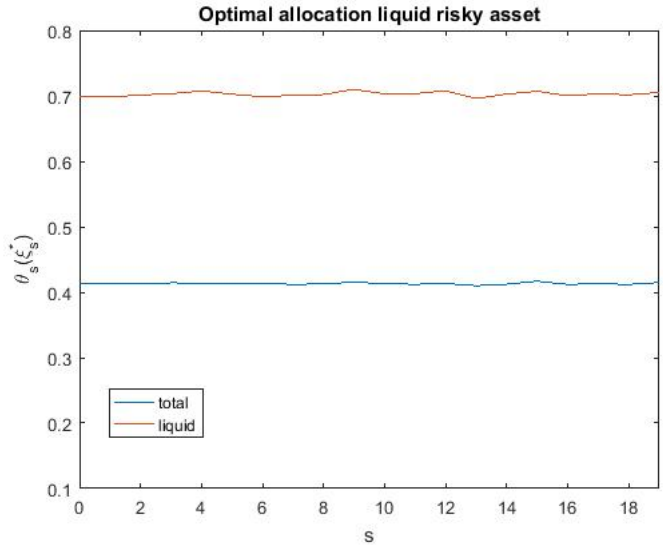


Figure 2. **Optimal policy implications fully liquid case** The first graph shows the optimal allocation towards the illiquid asset in the fully liquid case. The second graph shows the optimal fraction consumed of total and liquid wealth, where consumption as fraction of liquid wealth is computed as $\alpha_s/(1-\xi_s)$. The third graph shows the optimal allocation towards the liquid risky asset as a fraction of total and liquid wealth, where the allocation towards the liquid risky asset as fraction of total wealth is computed as $\theta_s(1-\xi_s)$. The graph is constructed by using the following parameter values: $\gamma = 3, \beta = 0.02, d = 0, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

3.2.2 Allocation towards the illiquid risky asset

Figure 3 shows the optimal allocation towards the illiquid risky asset for different levels of illiquidity, measured by the trading probability p. First, the asset allocation implication towards the illiquid asset is discussed in case $p = 0.5$. In the starting years, the optimal allocation towards the illiquid asset approximately satisfies the Merton solution. However, in the years approaching the final date T, the allocation towards the illiquid asset significantly reduces and even equals zero in the last two time periods. The explanation for the reduction in the allocation towards the illiquid asset is twofold. In the years approaching the final date T, illiquid wealth grows at a relatively faster rate compared to liquid wealth as the investor smooths consumption. Consumption as fraction of total wealth, as seen in the Section 3.2.1, increases exponentially in the years approaching T. As the investor can only consume out of liquid wealth, in some scenarios illiquid wealth might grow too fast, which makes her unable to finance the desired consumption level in those years. Moreover, at the final date T the investor faces the risk not being able to liquidate illiquid wealth. Therefore, as the final date becomes more important during the years, the allocation towards the illiquid asset reduces as a result of having a higher risk not being able to liquidate final

illiquid wealth. Obviously, the lower p, the more severe illiquidity and the stronger the reduction in the allocation towards the illiquid asset. In case $p = 0.1$, the investor even allocates less than the Merton solution in the more recent years. Our results are not exactly comparable to the results found in Ang et al. (2014), as we use a finite horizon, whereas Ang et al. (2014) use an infinite model. However, the effect of illiquidity found in the baseline model is in line with the results found in Ang et al. (2014).

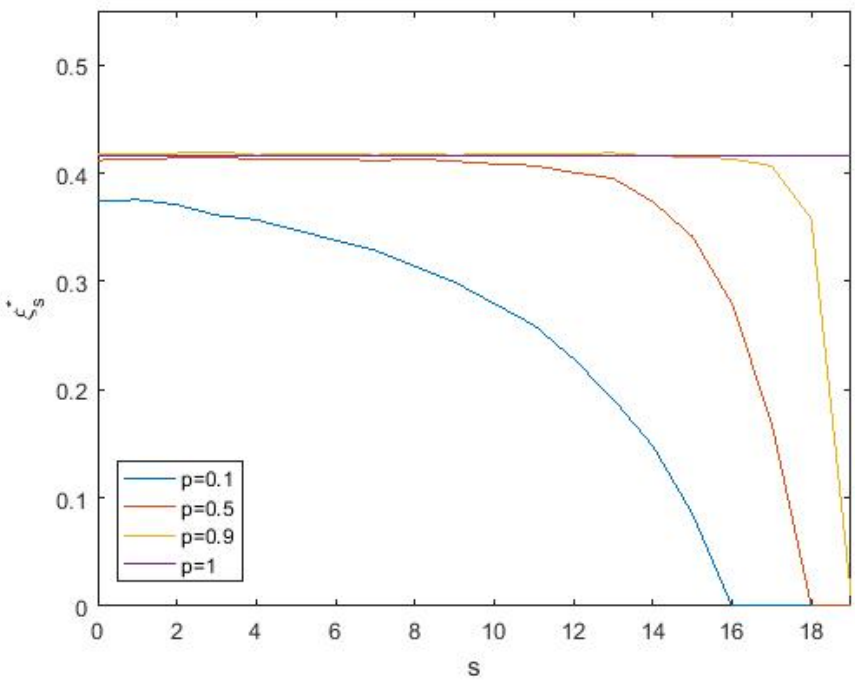


Figure 3. **Optimal allocation towards the illiquid risky asset in a consumption problem** This graph plots the optimal asset allocation towards the illiquid asset, ξ_s^*, at each point in time $s = 0, 1, ..., T$ in case the expected time between trades is once in two years ($p = 0.5$), once in ten years ($p = 0.1$) and once in 1.1 year ($p = 0.9$). The horizontal line shows the optimal allocation in case the investor can trade each evaluation period (Merton solution). The graph is constructed by assuming there is a probability $1 - p$ illiquid wealth cannot be liquidated at the final date and the following parameter values are used: $\gamma = 3, \beta = 0.02, d = 1, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

3.2.3 Allocation towards the liquid risky asset

In the baseline model I assume the liquid and illiquid risky asset are uncorrelated, $\rho = 0$, however the allocation towards the liquid risky asset is still affected by

illiquidity. In scenarios where the actual allocation to the illiquid asset is significantly above the optimum, the investor becomes more risk averse to take investments in the liquid risky asset compared to a Merton investor. As long as the actual allocation towards the illiquid asset is below its optimum, $\xi_s \leq \xi_s^*$, the optimal allocation towards the liquid risky asset satisfies the Merton solution. However, as the investor can only consume out of his liquid wealth, the illiquid risky asset is no longer simply a substitute for the liquid risky asset if the actual allocation towards the illiquid asset is above its optimum, $\xi_s > \xi_s^*$, and therefore the investor reduces the allocation towards the liquid risky asset. This result is in line with the findings in Ang et al. (2014).

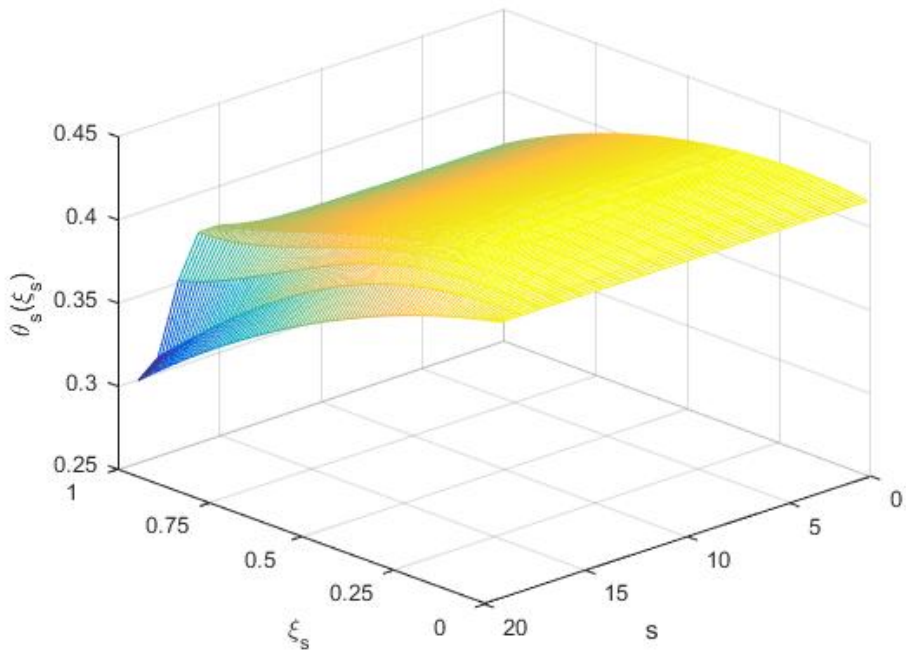


Figure 4. **Optimal allocation towards the liquid risky asset as a fraction of total wealth in a consumption problem** This graph plots the optimal asset allocation towards the liquid asset as a function of time $s = 0, 1, ..., T$ and the fraction invested in the illiquid asset, ξ_s, in case the expected time between trades is once in two years ($p = 0.5$). The graph is constructed by using the following parameter values: $\gamma = 3, \beta = 0.02, d = 1, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

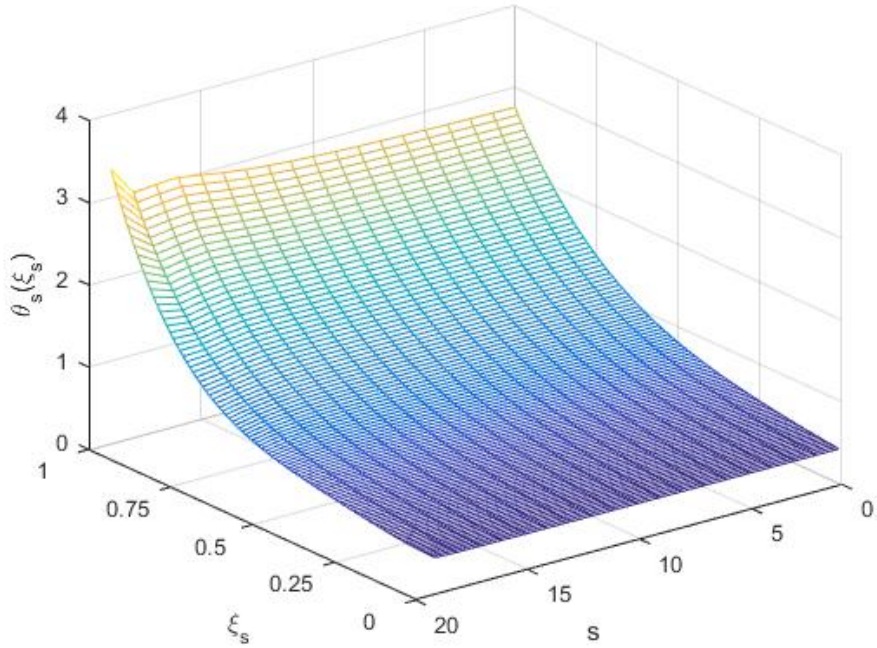


Figure 5. **Optimal allocation towards the liquid risky asset as a fraction of liquid wealth in a consumption problem** This graph plots the optimal asset allocation towards the liquid asset as a function of time $s = 0, 1, ..., T$ and the fraction invested in the illiquid asset, ξ_s, in case the expected time between trades is once in two years ($p = 0.5$). The graph is constructed by using the following parameter values: $\gamma = 3, d = 1, \beta = 0.02, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

Figure 4 shows the allocation towards the liquid risky asset reduces faster in the years approaching the final date T, from the Merton solution, 42%, down to $30\% - 35\%$ when $\xi_s \to 1$. As ξ_s^* is significantly below the Merton solution in the years approaching T, $\xi_s \to 1$ gives the investor a higher utility loss in those years compared to earlier years and therefore the reduction in the allocation towards the liquid risky asset is stronger. In the years further away from the final date T, the allocation towards the liquid risky asset reduces at approximately the same speed, from about 42% to 37% when $\xi_s \to 1$. Moreover, a lower p implies the optimal allocation towards the illiquid asset, ξ_s^*, is lower compared to the baseline model, so therefore the reduction in the allocation towards the liquid risky asset is also stronger as the deviation when $\xi_s \to 1$ from the optimum ξ_s^* is larger.

Figure 5 shows that the fraction of liquid wealth invested in the liquid risky asset is increasing in ξ_s^*. If only a small fraction is allocated to liquid wealth, the investor has to invest a larger fraction of her liquid wealth in the liquid risky asset to

still obtain a significant exposure of total wealth towards the liquid risky asset.

3.2.4 Consumption path

Consumption is also affected by illiquidity. Figure 6 shows, in line with the findings in Ang et al. (2014), that the investor consumes a lower fraction of her total wealth compared to the two-asset Merton problem, irrespective of time s or ξ_s.[5] Consumption must be funded by liquid wealth, therefore the larger the allocation towards the illiquid asset, ξ_s, the lower optimal consumption as a fraction of total wealth. Again, this is to partly compensate the increased total risk exposure by holding a higher allocation towards the illiquid asset. Consumption as fraction of liquid wealth increases when ξ_s increases. If $\xi_s \to 1$, the investor has to consume a large part of liquid wealth in order to fulfill the desired consumption level.

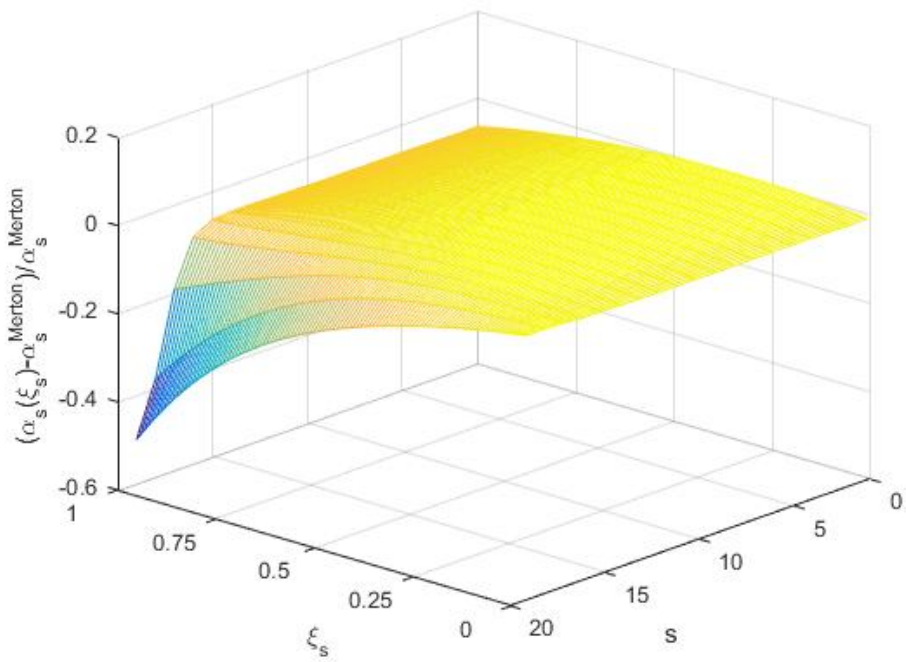


Figure 6. **Optimal consumption pattern as a fraction of total wealth.** This graph plots the optimal consumption pattern as a function of time $s = 0, 1, ..., T$ and the fraction invested in the illiquid asset, ξ_s, relative to the optimal fraction of total wealth consumed in a two-asset Merton problem in case the expected time between trades is once in two years ($p = 0.5$). The graph is constructed by using the following parameter values:

[5] The numerical solutions show that for all s, $\alpha_s(\xi_s)$ is strictly below the Merton solution, α_s^{Merton}.

$\gamma = 3, \beta = 0.02, d = 1, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

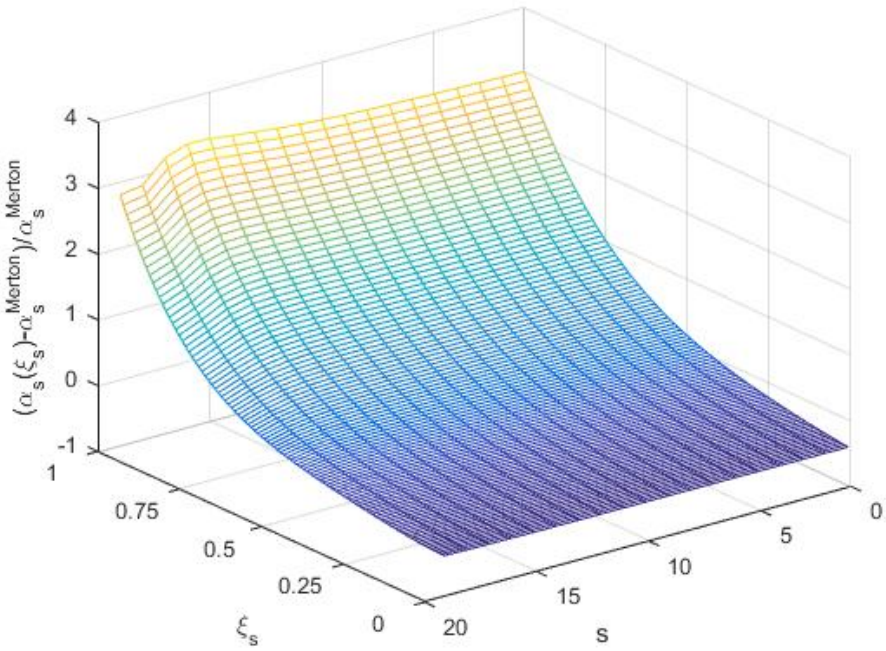


Figure 7. **Optimal consumption pattern as a fraction of liquid wealth.** This graph plots the optimal consumption pattern as a function of time $s = 0, 1, ..., T$ and the fraction invested in the illiquid asset, ξ_s, relative to the optimal fraction of liquid wealth consumed in a two-asset Merton problem in case the expected time between trades is once in two years ($p = 0.5$). The graph is constructed by using the following parameter values: $\gamma = 3, \beta = 0.02, d = 1, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

Figure 6 shows the investor reduces the optimal fraction consumed relative to the optimal Merton consumption to a higher extent in the years approaching the final date. Again, in those years the effect of illiquidity is stronger and therefore also the reduction on the optimal fraction consumed is larger.

3.2.5 Boundary condition value function

In this section, I describe in which way the results of the baseline model change under the assumption both liquid and illiquid wealth can be liquidated at the final date T at no cost, i.e. when $d = 0$. In case illiquid wealth can be liquidated with probability one at the final date T, the risk the investor might not be able to liquidate illiquid

wealth vanishes compared to the baseline model. Therefore, under the assumption liquid wealth can be liquidated for sure, only the consumption smoothing explanation for the reduction in the allocation towards the illiquid asset is left. In this way, I can disentangle the effects of consumption smoothing and the risk not being able to liquidate illiquid wealth on the allocation towards the illiquid asset.

Obviously, under this assumption the effect of illiquidity is less severe, as the risk the investor might not be able to liquidate illiquid wealth completely disappears. In the graph below, the optimal asset allocation towards the illiquid asset is shown in case $p = 0.1$, $p = 0.5$, $p = 0.9$ and $p = 1$.

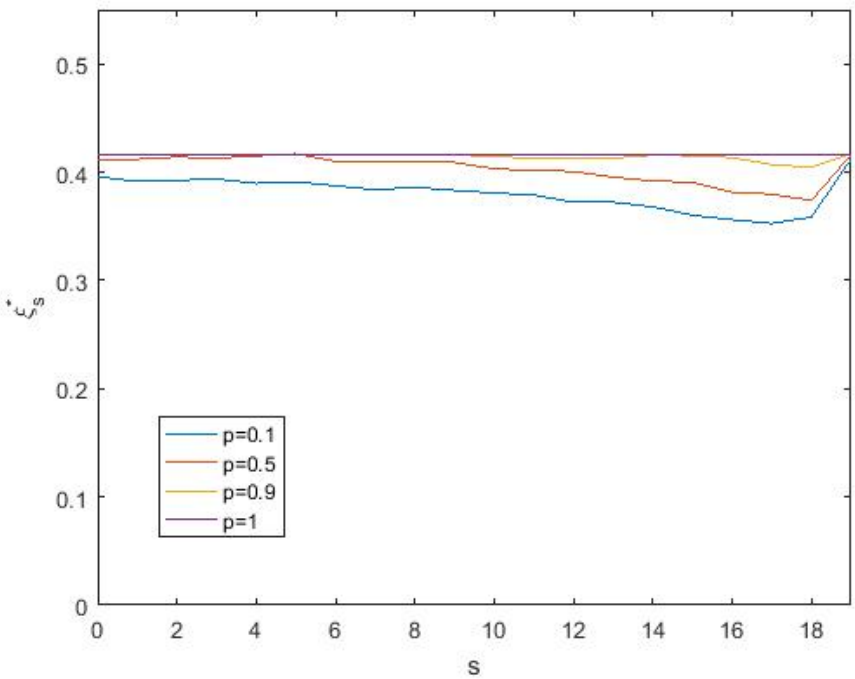


Figure 8. **Optimal allocation towards the illiquid risky asset in a consumption problem** This graph plots the optimal asset allocation towards the illiquid asset, ξ_s^*, at each point in time $s = 0, 1, ..., T$ in case the expected time between trades is once in two years ($p = 0.5$), once in ten years ($p = 0.1$) and once in 1.1 year ($p = 0.9$). The horizontal line shows the optimal allocation in case the investor can trade each evaluation period (Merton solution). The graph is constructed by assuming illiquid wealth can be liquidated at the final date and the following parameter values are used: $\gamma = 3, \beta = 0.02, d = 0, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

Figure 8 shows that absence of the risk illiquid wealth might not be liquidated at

the final date significantly reduces the effect of illiquidity. At date $T-h$, the optimal allocation towards the illiquid asset is again the Merton solution, irrespective of the ability to trade, as at time T the investor can consume both his liquid and illiquid wealth with probability one, so there is no reason to take into account illiquidity. In the years approaching $T-h$, the allocation towards the illiquid asset decreases as a result of consumption smoothing, explained in Section 3.2.2. If the expected time between trading opportunities equals two years, the investor reduces its optimal asset allocation towards the illiquid asset in the years approaching the final date T by approximately 4%, which slowly converges to 0% when further away from the final date T. Figure 3 and Figure 8 therefore confirm that the main cost of illiquidity is the fact the investor might not be able to liquidate his illiquid wealth at the final date T. Obviously, when $0 < r < 1$, the results shown in Figure 3 and 8 will be in between, for large r, the graph looks similar to Figure 3, whereas for small r the graph looks similar to Figure 8.

4 Sensitivity analysis

In this section a sensitivity analysis is performed to show how the outcome of the model depends on the underlying assumptions and the assumed parameter values. First, the results in case of a final wealth problem are discussed. Then its shown how the results of the model change in case of different time-preference discount factors and risk-aversion parameters. Third, the effect of stochastic trading is disentangled from deterministic trading opportunities. Finally, the results are discussed in case the liquid and illiquid risky asset are correlated.

4.1 Preferences

4.1.1 Final wealth problem

In this section, the problem is solved in case the investor optimizes terminal wealth only. At each point in time the investor chooses the optimal allocation towards the illiquid risky asset, ξ_s^*, if a trading opportunity arises and the trading strategy towards the liquid risky asset, θ_s, such that his terminal wealth is optimized. The budget constraints of the investor are as follows:

$$W_s = (W_{s-h} - \triangle X_{s-h})(\exp(r^f) + \theta_{s-h}(\exp(R_s^S) - \exp(r^f))) \tag{17}$$

$$X_s = (X_{s-h} + \triangle X_{s-h}) \exp(R_s^X) \tag{18}$$

The investor maximizes CRRA utility over terminal wealth $W_T + X_T$ at the terminal date $s = T$:

Problem 4.1. *The optimal final wealth problem of the investor equals:*

$$\max_{\{\theta_s, \triangle X_s\}_{s=0}^{T}} \mathbb{E}_0\Big[e^{-\beta T}\Big(p\frac{(W_T + X_T)^{1-\gamma}}{1-\gamma} + (1-p)\frac{((1-d\xi_T)(W_T + X_T))^{1-\gamma}}{1-\gamma}\Big)\Big]$$

subjected to the two budget constraints (17) and (18), where the transfer from or to the illiquid risky asset can only be non-zero ($\triangle X_s \neq 0$) if $I_s = 1$.

The solution technique to solve the final wealth problem is the same as for the optimal consumption problem, except that liquid wealth is not corrected for the fraction consumed.

Figure 9 shows the optimal asset allocation towards the illiquid risky asset is not significantly affected by illiquidity in the final wealth problem if liquid wealth can be liquidated for sure, i.e. when $d = 0$. Only in case the investor is expected to trade once in ten years, i.e. when $p = 0.1$, the optimal fraction invested in the illiquid asset is slightly below the optimal Merton solution. If the investor has no incentive to smooth consumption, the effect of illiquidity is negligible. In this case liquid wealth does not decrease with consumption $\alpha_s(\xi_s)$, so illiquid wealth grows at a relatively slower rate compared to the baseline model. Therefore, the actual allocation towards the illiquid risky asset will deviate less from the optimal one compared to the baseline model. Moreover, the utility function of a CRRA investor is rather flat at the optimum, so small differences in the actual allocation towards the illiquid asset being different from the optimal allocation only has a second order effect on the utility loss. The results for a final wealth problem under the assumption illiquid wealth can be liquidated at the final date are in line with the findings in Ang et al. (2014).

The effect of illiquidity on the allocation towards the liquid risky asset has the same shape as in the consumption problem when illiquid wealth can be liquidated for sure at T (Appendix B). In the final wealth problem, in line with the consumption problem, the allocation towards the liquid risky asset reduces if the actual allocation towards the illiquid asset increases. This implies the investor adapts the allocation towards the liquid risky asset to compensate for the total increased risk exposure as a result of higher holdings in the illiquid risky asset. As in this case the actual fraction invested in the illiquid asset will deviate less from the optimal allocation towards the illiquid asset, ξ_s^*, the reduction is smaller when $\xi_s \to 1$, from approximately 42% to 37%.

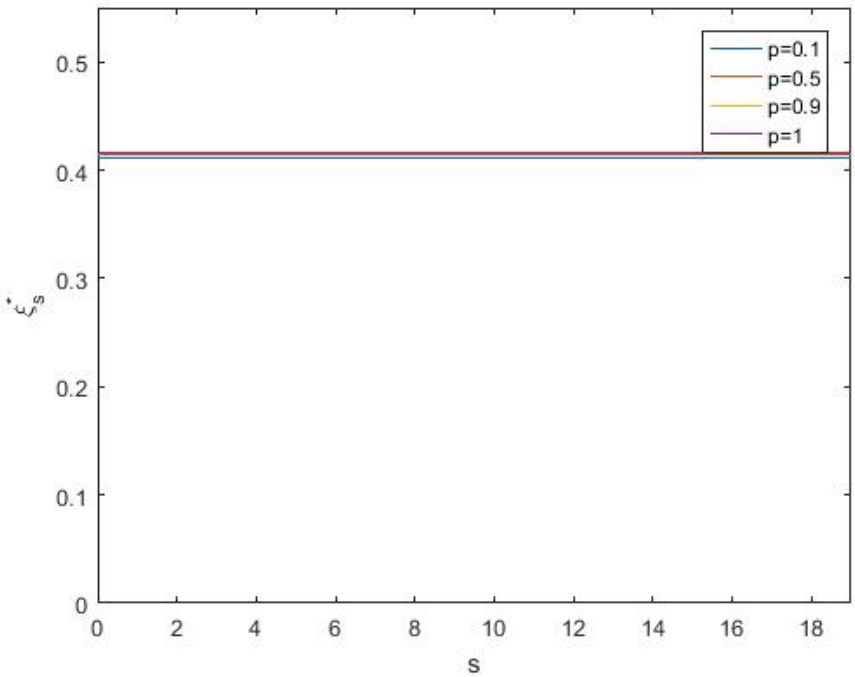


Figure 9. **Optimal allocation towards the illiquid risky asset in a final wealth problem** This graph plots the optimal asset allocation towards the illiquid asset ξ_s^* at each point in time $s = 0, 1, ..., T$ in case the expected time between trades is once in two years ($p = 0.5$), once in ten years ($p = 0.1$) and once in 1.1 year ($p = 0.9$). The horizontal line shows the optimal allocation in case the investor can trade each evaluation period (Merton solution) in case of a final wealth problem. The graph is constructed by assuming illiquid wealth can be liquidated at the final date and the following parameter values are used: $\gamma = 3, \beta = 0.02, r^f = 0.02, d = 0, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

In case the investor might face the risk not being able to liquidate illiquid wealth, i.e. when $d = 1$, the effect of illiquidity is even severe compared to the consumption problem. In a consumption problem, the investor obtains utility over intermediate dates and final wealth, whereas in a final wealth problem, the investor is interested in optimizing final wealth only. Therefore the utility loss in a final wealth problem is larger in case illiquid wealth cannot be liquidated at the final date T. This can also be seen in the stronger reduction in the allocation towards the liquid risky asset when $\xi_s \to 1$ compared to the consumption problem (Appendix B).

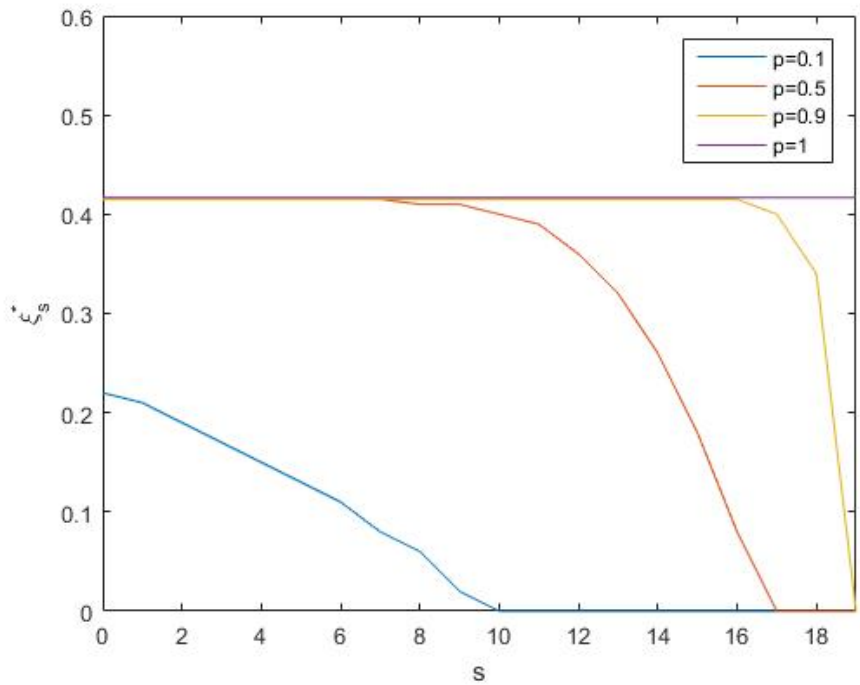


Figure 10. **Optimal allocation towards illiquid risky asset in a final wealth problem** This graph plots the optimal asset allocation towards the illiquid asset ξ_s^* at each point in time $s = 0, 1, ..., T$ in case the expected time between trades is once in two years ($p = 0.5$), once in ten years ($p = 0.1$) and once in 1.1 year ($p = 0.9$). The horizontal line shows the optimal allocation in case the investor can trade each evaluation period (Merton solution) in case of a final wealth problem. The graph is constructed by assuming illiquid wealth is not liquidated with probability $1 - p$ and using the following parameter values: $\gamma = 3, \beta = 0.02, r^f = 0.02, d = 1, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

In sum, illiquidity is more severe in case the investor faces the risk not being able to liquidate illiquid wealth at the final date T in both the final wealth and consumption problem. Not being able to liquidate final wealth leads to a higher utility loss in a final wealth problem as no intermediate utility is obtained. In case the investor is able to liquidate illiquid wealth at the final date, illiquidity is more severe in a consumption problem in the years before the final date as of consumption smoothing the investor faces the risk illiquid wealth becomes too large compared to liquid wealth, affecting the desired consumption level, which risk is absent in a final wealth problem.

4.1.2 Time preference and risk-aversion

In case the investor values consumption in the future less, which implies a high β, the effect of illiquidity is smaller. Figure 11 shows that an investor with high β

slightly increases the allocation towards the illiquid asset compared to an investor with a low β. A higher β implies the investor cares more about consumption today instead of consumption in the future. Therefore, the effect the investor might not be able to liquidate illiquid wealth is (slightly) smaller for this investor.

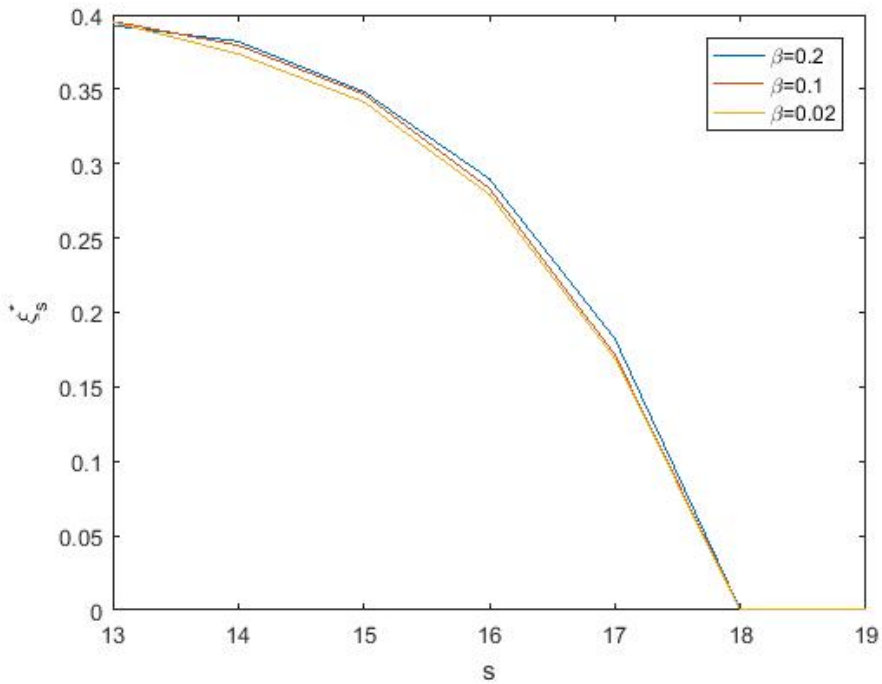


Figure 11. **Optimal allocation towards illiquid risky asset for different time-preference discount factors** This graph plots the optimal asset allocation towards the illiquid asset ξ_s^* at each point in time $s = 0, 1, ..., T$ in case the time-preference discount factor equals $\beta = 0.2$, $\beta = 0.1$ and $\beta = 0.02$. The graph is constructed by assuming illiquid wealth is not liquidated with probability $1 - p$ and using the following parameter values: $\gamma = 3, p = 0.5, r^f = 0.02, d = 1, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

If the risk-aversion parameter γ increases, the effect of illiquidity decreases. An investor with high risk-aversion coefficient γ would allocate only a small amount to the illiquid risky asset even if the investor is continuously able to trade the illiquid asset. The hurdle of not being able to trade for this investor is smaller compared to an investor with a higher risk-aversion coefficient γ, which would allocate a significant part of her wealth to the illiquid asset if she could continuously trade. Figure 12 shows in percentages the deviation from the optimal Merton solution in case $\gamma = 3$, $\gamma = 6$ and $\gamma = 12$.

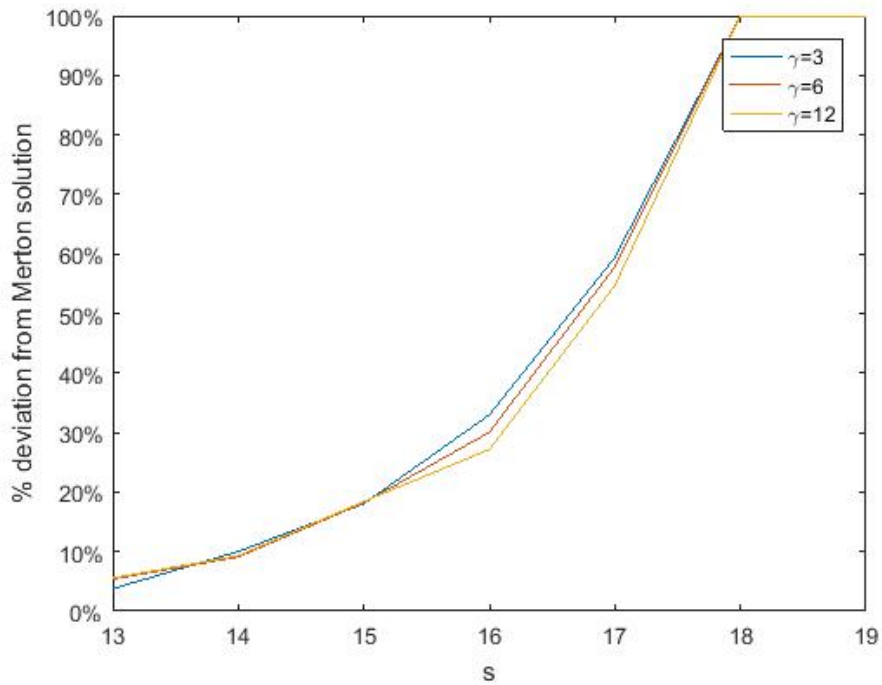


Figure 12. **Optimal allocation towards illiquid risky asset for different risk-aversion parameters** This graph plots the optimal asset allocation towards the illiquid asset ξ_s^* at each point in time $s = 0, 1, ..., T$ in case the risk aversion parameter equals $\gamma = 3$, $\gamma = 6$ and $\gamma = 12$. The graph is constructed by assuming illiquid wealth is not liquidated with probability $1 - p$ and using the following parameter values: $\beta = 0.02, p = 0.5, r^f = 0.02, d = 1, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

4.2 Financial market

4.2.1 Deterministic trading opportunities

In this section the effect of the length of the no-trading period only is studied. This means the investor is able to trade at know deterministic points in time and cannot trade in-between. I denote the intervals between two trades by τ. The problem in case of deterministic trading opportunities remains the same as in baseline model, however in this case we know exactly at which points in time $\triangle X_s \neq 0$[6]. This means the probability to trade equals $p = 1$ at some points and equals $p = 0$ otherwise:

[6]Notice that in case of deterministic trading opportunities, only the case illiquid wealth can be liquidated with probability one at the final date T is discussed. If the investor knows when she is able to trade, she will never invest at $T - h$ in the illiquid asset if she knows illiquid wealth can not be liquidated for sure.

Problem 4.2. *The optimal consumption problem of the investor in case of deterministic trading opportunities equals:*

$$\max_{\{\theta_s, \triangle X_s, C_s\}_{s=0}^{T}} \mathbb{E}_0\Big[\sum_{s=0}^{T} e^{-s\beta} \frac{C_s^{1-\gamma}}{1-\gamma}\Big]$$

subjected to the two budget constraints (2) and (3), where the transfer from or to the illiquid risky asset can only be non-zero ($\triangle X_s \neq 0$) at times $s = 0, \tau, 2\tau, ..., T$.

Figure 13 shows the result in case of deterministic trading opportunities when $\tau = 2$. In the baseline model the expected time between trades equals two years, however in contrast to the case with stochastic trading, Figure 13 shows illiquidity affects the asset allocation only at the last trading opportunity $T - 2$.

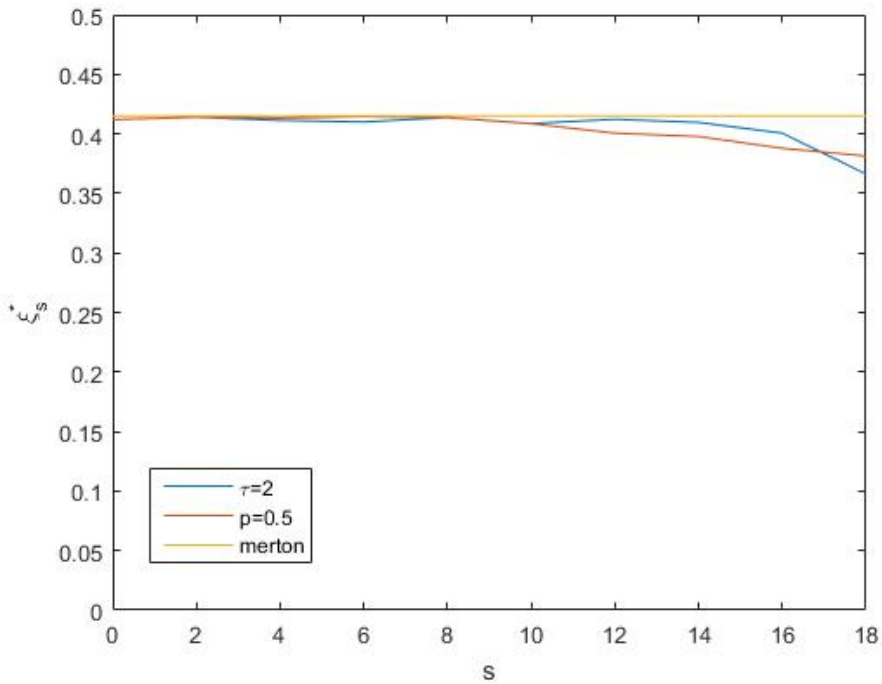


Figure 13. **Optimal allocation towards illiquid risky asset in case of deterministic trading opportunities** This graph plots the optimal asset allocation towards the illiquid asset ξ_s^* at each point in time $s = 0, \tau, ..., T - \tau$ with $\tau = 2$ in case of determinstic trading opportunities and in case of a probability $p = 0.5$ to be able to trade each period. The graph is constructed by assuming liquid wealth is liquidated with probability one and using the following parameter values: $\gamma = 3, \beta = 0.02, d = 0, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

At $T-2$ the investor allocates below the Merton solution as the investor knows he will not be able to trade at time $T-1$ for sure. As the investor wants to avoid states of the world illiquid wealth becomes too large at $T-1$ such that she might not be able to consume the desired fraction of total wealth, she will reduce the allocation towards the illiquid asset. The reduction is even larger compared to the case where there is a probability $p = 0.5$ to be able to trade each period, as in the latter case there is still positive probability the investor is able to re-balance her portfolio at $T-1$. However, the asset allocation converges much faster to the Merton solution when further away from the final date T. If the investor is able to trade at fixed intervals, the fraction of illiquid wealth will stay within certain bounds in contrast to the baseline model. In case of deterministic trading opportunities, the investor knows beforehand the next opportunity to be able to re-balance his asset allocation. In the baseline model there is uncertainty when the next opportunity to trade arises, so the investor has to take into account scenarios where illiquid wealth becomes considerable large compared to liquid wealth, leading to a stronger effect of illiquidity.

4.2.2 Correlation between liquid and illiquid risky asset

In case the liquid and the illiquid risky asset are correlated, the reduction in the allocation towards the illiquid asset relative to the Merton solution does not significantly change. However, the allocation towards the liquid risky asset does. In case the two risky asset are positively correlated, i.e. when $0 < \rho < 1$, the reduction in the allocation towards the liquid risky asset when the total fraction invested in illiquid wealth increases is much stronger in case of a positive correlation.[7]

[7] Figure 14 is constructed using $s = 10$, as before $T-5$, the reduction in the allocation towards the liquid risky asset is at approximately the same speed for each s. This allows to better compare the effect of the two assets being correlated or not.

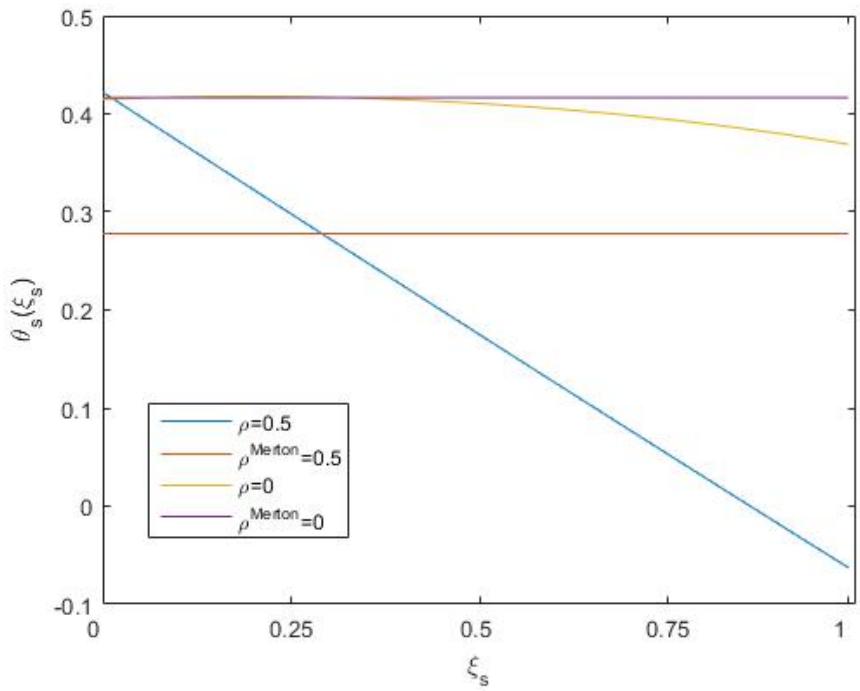


Figure 14. **Optimal allocation towards the liquid risky asset in case of correlation between the liquid and illiquid risky asset** This graph plots the optimal asset allocation towards the liquid risky asset $\theta_s(\xi_s^*)$ in case $\rho = 0.5$ and $\rho = 0$ for $s = 10$. The graph is constructed using the following parameter values: $\gamma = 3, \beta = 0.02, d = 1, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$.

The total volatility of the investor's portfolio can be reduced to a larger extent in case of a positive correlation between the liquid and illiquid risky asset compared to the case where the assets are uncorrelated by decreasing the allocation towards the liquid risky asset.[8] The investor uses the liquid risky asset as a hedge against increases and decreases in the actual fraction invested in the illiquid risky asset. In case the allocation towards the liquid risky asset is below the Merton solution, the investor increases the allocation towards the liquid risky asset in order to increase the total exposure. On the other hand, in case the allocation towards the illiquid asset is above the Merton solution, the investor reduces the allocation towards the liquid risky asset significantly to reduce the total risk exposure.

[8]The variance of the portfolio equals $\theta_s^2\sigma^2 + \xi_s^2\sigma^2 + 2\theta_s\xi_s\rho\sigma^2$, as $\sigma_S = \sigma_X$.

5 Correlation probability to trade and realized returns on the illiquid asset

In times of financial market stress, investors often prefer to hold highly liquid assets (Beber, Brandt and Kavajecz (2009)). As a result of an increased demand for liquid assets, prices of relatively less liquid assets decrease and it becomes more difficult to trade the illiquid assets. For instance, during the 1998 Russian default, Treasury bonds suddenly increased in value relative to less liquid debt instruments. Based on the 'flight-to-liquidity' phenomenon, the baseline model is extended to allow for correlation between the return earned on the illiquid asset and the probability to trade.

Concretely, when the return on the illiquid risky asset R_s^X over period $[s-h, s]$ materializes, conditionally on the realized return, there is a probability to trade $p_s(R_s^X)$ at time s. The problem of the investor remains exactly the same as in the baseline model, however now the probability to trade at time s is a function of the realized return on the liquid risky asset in period $[s-h, s]$. The problem becomes:

Problem 5.1. *The optimal consumption problem of the investor equals:*

$$\max_{\{\theta_s, \triangle X_s, C_s\}_{s=0}^{T}} \mathbb{E}_0\Big[\sum_{s=0}^{T} e^{-s\beta} \frac{C_s^{1-\gamma}}{1-\gamma}\Big]$$

subjected to the two budget constraints (2) and (3), where the transfer from or to the illiquid risky asset can only be non-zero ($\triangle X_s \neq 0$) if a trading opportunity with probability $p_s(R_s^X)$ arises.

The first order Euler conditions remain the same as in the case trading is random, however in this case the value function does not only depend on time s, total wealth $W_s + X_s$, and ξ_s, but also on p_s.

I assume negative returns on the illiquid asset indicate a bad market outcome of the illiquid asset market and in line with the flight-to-liquidity phenomenon, trading the illiquid asset becomes more difficult. To start with, I assume if the realized return R_s^X is below the risk-free rate, the probability to trade equals $p_s(R_s^X) = 0.1$, whereas in case of a realized return above the risk-free rate, the probability to trade equals $p_s(R_s^X) = 0.9$. The benchmark in this case is the risk-free rate, where I assume the asset performed well in case the realized return is above the risk-free rate, and performs poorly if its below the risk-free rate. In order to see whether and to which extent the results are affected in case the correlation between realized return and probability to trade is positive, I compare the results with the unconditional probability the investor is able to trade in this market, which equals $p = 0.548$.[9]

[9] 56% of the trajectories drawn from the multi-normal distribution are above the risk-free rate, therefore the unconditional probability equals $p = 0.1 * 0.44 + 0.9 * 0.56 = 0.548$.

Moreover, I also show the results in case of a negative correlation, so in that case the probabilities are exactly the opposite and the unconditional probability equals $p = 0.452$.[10]

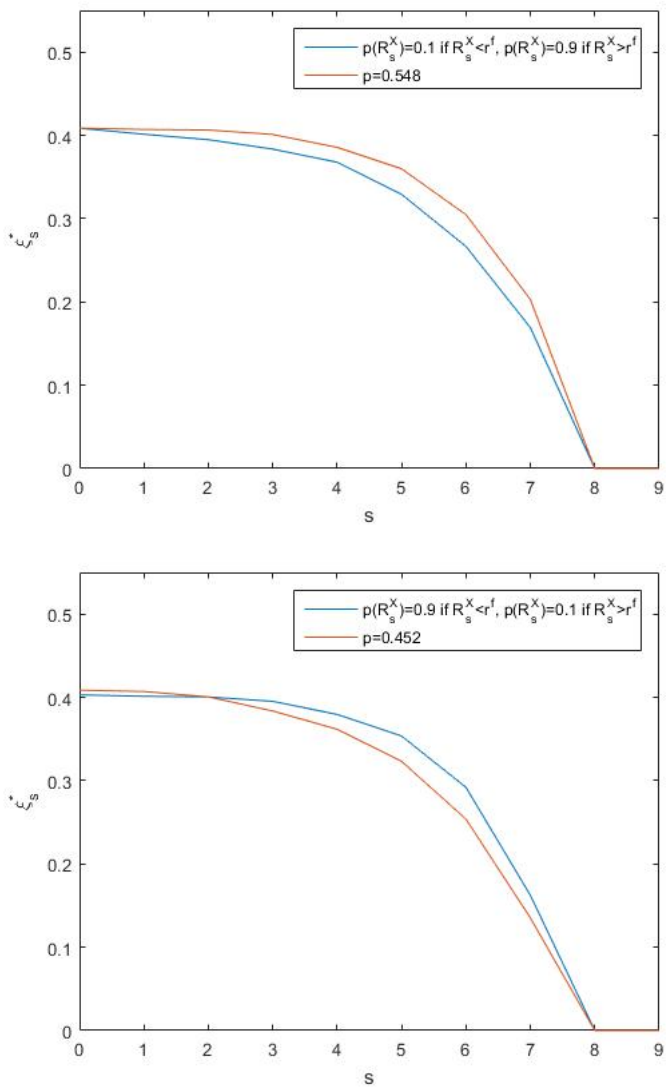


Figure 15. **Optimal allocation towards illiquid risky asset in case of correlation between the returns on the illiquid asset and the probability to trade** This graph plots the optimal asset allocation towards the illiquid asset ξ_s^* at

[10]In the analysis, $T = 10$, as the years before $T = 10$ will lead in any case to an allocation approximately equal to the Merton solution.

each point in time $s = 0, 1..., T$ in case $p_s(R_s^X) = 0.1$ if $R_s^X < \exp(r^f)(> \exp(r^f))$ and $p_s(R_s^X) = 0.9$ if $R_s^X > \exp(r^f)(< \exp(r^f))$. The graph is constructed by assuming illiquid wealth is not liquidated with probability $1 - p$ and using the following parameter values: $\gamma = 3, \beta = 0.02, r^f = 0.02, d = 1, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$.

Figure 15 shows in case the probability to trade, p_s and the return on the illiquid asset, R_s^X, are positively correlated, the illiquid asset is less attractive compared to the unconditional case, i.e. $p = 0.548$. The investor allocates approximately 3% less to the illiquid asset compared to the baseline model with $p = 0.548$ in the years approaching the final date T and the two lines converge to the Merton solution when going further ahead from the final date T. In case of a negative correlation, exactly the reverse is observed, so the investor allocates a higher fraction of his total wealth to the illiquid asset compared to the unconditional case. A positive correlation implies the investor increases its allocation towards the illiquid asset by approximately 3% compared to the baseline model with $p = 0.452$. These results imply investors value states of the world where the probability to trade increases during bad market times, whereas they dislike states of the world where the probability to trade decreases during bad market outcomes.

In this section, I have shown that a positive correlation between the return on the illiquid asset and the probability to trade this asset makes the asset less attractive to the CRRA investor. This is consistent with the risk-averse investor who values trading more during bad market outcomes. Therefore, a lower probability to trade in case of bad market outcomes, makes the investor more risk-averse against this asset, leading to a lower optimal allocation towards this asset. In future research, beyond the scope of this thesis, it would be interesting to model an environment that is more consistent with real implications of the flight to liquidity phenomenon to study more precisely the magnitude on the asset allocation in case of a positive correlation between returns and the probability to trade.

6 Conclusion

In this thesis, the long-term optimal asset allocation for a CRRA investor is derived in case of one liquid risky and one illiquid risky asset, where illiquidity results from the restriction on the trading time of the illiquid asset. A finite, discrete time model is used to understand how the effect of illiquidity depends on time and the horizon of the investor. The problem is solved for both a final wealth problem and a consumption problem. Moreover, the baseline model is extended by allowing for correlation between the returns on the illiquid asset and the probability to trade.

In case of a consumption problem, the investor significantly reduces her allocation towards the illiquid asset in the years approaching the final date. The lower the probability to trade, the sooner the effect of illiquidity is visible. The reason for the reduction in the allocation towards the illiquid asset is twofold. First, in the years

approaching the final date, illiquid wealth grows at a relatively faster rate compared to liquid wealth as a result of consumption smoothing. In order to avoid states of the world where illiquid wealth becomes too large such that the investor might not be able to finance the desired consumption level, the investor reduces the allocation towards the illiquid asset. The second reason for the reduction in the allocation towards the illiquid asset is an increased probability illiquid wealth cannot be liquidated at the final date. Not being able to liquidate illiquid wealth is the main source of the significant reduction in the allocation towards the illiquid asset when approaching the final date.

In case of a final wealth problem, the asset allocation is hardly affected in case the investor can liquidate illiquid wealth at the final date. The only utility loss the investor experiences are fluctuations in the actual allocation towards the illiquid asset in case the investor is not able to trade. As the utility function is relatively flat at the optimum, the effect of a slightly different allocation towards the illiquid asset is negligible. However, if the investor might not be able to liquidate illiquid wealth at the final date, the reduction in the allocation towards the illiquid asset is even larger compared to consumption problem. As the investor gains no utility over intermediate periods, not being able to liquidate illiquid wealth has a stronger effect in a final wealth problem.

This paper also shows that not only the allocation towards the illiquid asset is affected. Both the optimal consumption and the optimal allocation towards the liquid risky asset decrease if the actual allocation towards the illiquid asset becomes significantly large. In this way, the investor partly compensates the increased risk exposure as a result of a higher actual allocation towards the illiquid asset. In case the liquid and illiquid risky asset are correlated the reduction in the allocation towards the liquid risky asset, when the fraction invested in illiquid wealth increases, is even stronger. The total volatility of the investor's portfolio can be reduced to a higher extent by decreasing the allocation towards the liquid risky asset in case of positive correlation between the liquid and illiquid risky asset.

Finally, an extension of the baseline model shows that in case the returns on the illiquid asset and the probability to trade are positively correlated, the investor allocates less to this asset compared to the baseline model. This implies that the investor faces a larger utility loss in case she is not able to trade the illiquid asset during bad market outcomes. These primarily results indicate that the flight-to-liquidity phenomenon often observed in financial markets may have serious implications for the optimal allocation towards illiquid asset classes. In the future, it would be interesting to study a more realistic setting of correlations between the probability to trade and the returns earned on the illiquid asset to measure more accurately the magnitude of the effect of the flight-to-liquidity phenomenon.

This paper assumes the same return on both liquid and illiquid risky asset in

order to obtain only the pure timing effect of illiquidity on the optimal asset allocation and consumption pattern is studied. The next step would be to calculate the illiquidity premium the investor requires such that he obtains the same welfare level as in the fully liquid case. This premium is likely to depend on the horizon and the severity of illiquidity. Moreover, in the current model no dividends are included. Some illiquid asset classes, such as real estate, do in fact pay dividends. Including a dividend paying illiquid asset in the model is likely to decrease the effect of illiquidity, as liquid wealth will decrease at a slower rate when there is cash inflow from dividends.

Long-term investors such as pension funds have besides a relatively long horizon, also relatively small short-term liabilities. The model clearly shows illiquidity significantly reduces the allocation towards the illiquid asset in the years approaching the final date and in the years when total payout that has to be financed by liquid wealth, i.e. consumption, is relatively large. Therefore, a long horizon and relatively small short-term liabilities implies that in this setting, illiquidity has a negligible effect on the optimal asset allocation for those long-term investors.

Appendix A

In this section, the first order Euler conditions are derived in case there is correlation between the realized returns on the illiquid asset and the probability to trade. The value function at the final date T is written as follows:

$$V_T(W_T+X_T,\xi_T,p_T) = e^{-\beta T}\frac{(W_T+X_T)^{1-\gamma}}{1-\gamma}\Big(p_T(R_T^X)H_T(\xi_T^*)+(1-p_T(R_T^X))H_T(\xi_T)\Big)$$

where $H_T(\xi_T) = (1-d\xi_T)^{1-\gamma}$ and $H_T(\xi^*) = 1$.

The first order Euler conditions with respect to consumption and the allocation towards the liquid risky asset are as follows:

$$\begin{aligned}\frac{\partial V_{s-h}}{\partial C_{s-h}} &= \alpha_{s-h}(\xi_{s-h})^{-\gamma} + \beta^h \mathbb{E}_s[\{(1-\xi_{s-h}-\alpha_{s-h}(\xi_{s-h}))(\exp(r^f)+\theta_{s-h}(\xi_{s-h})(\exp(R_s^S)-\exp(r^f)) \\ &+ \xi_{s-h}\exp(R_s^X)\}^{-\gamma}\Big((1-p_s(R_s^X))\frac{H_s'(\xi_s)}{1-\gamma}\xi_s - (p_s(R_s^X)H_s(\xi_s^*)+(1-p_s(R_s^X))H_s(\xi_s))\Big)\exp(r^f)] = 0\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{\partial V_{s-h}}{\partial \theta_{s-h}} &= \mathbb{E}_s[\{(1-\xi_{s-h}-\alpha_{s-h}(\xi_{s-h}))(\exp(r^f)+\theta_{s-h}(\xi_{s-h})(\exp(R_s^S)-\exp(r^f))) \\ &+ \xi_{s-h}(\exp(R_s^X))\}^{-\gamma}\Big((p_s(R_s^X)H_s(\xi_s^*)+(1-p_s(R_s^X))H_s(\xi_s)) - (1-p_s(R_s^X))\frac{H_s'(\xi_s)}{1-\gamma}\xi_s\Big)\Big(\exp(R_s^S) \\ &- \exp(r^f)\Big)] = 0\end{aligned} \tag{20}$$

As in the baseline model, notice that $\xi_s = f(\xi_{s-h})$. Moreover, the penalty function in case the return on the illiquid asset and the probability to trade are correlated is defined as follows:

$$\begin{aligned}H_{s-h}(\xi_{s-h}) &= \alpha_{s-h}^{1-\gamma}(\xi_{s-h}) + e^{-h\beta}\mathbb{E}_{s-h}[\{(1-\xi_{s-h}-\alpha_{s-h}(\xi_{s-h}))(\exp(r^f)+\theta_{s-h}(\xi_{s-h})(\exp(R_s^S)-\exp(r^f))) \\ &+ \xi_{s-h}\exp(R_s^X)\}^{1-\gamma}\Big(p_s(R_s^X)H_s(\xi_s^*)+(1-p_s(R_s^X))H_s(\xi_s)\Big)]\end{aligned} \tag{21}$$

where $H_{s-h}(\xi_{s-h}^*) = \min_{\xi_{s-h}} H_{s-h}(\xi_{s-h})$.

The FOCs as in equation (19) and (20) together with the penalty function (21) allow for solving ξ^*_{s-h}, $\theta_{s-h}(\xi_{s-h})$ and $\alpha_{s-h}(\xi_{s-h})$ in the same way as done in the baseline model.

Appendix B

In this section the allocation towards the liquid risky asset as a function of the fraction invested in the illiquid asset is shown in case of a final wealth problem, both in case illiquid wealth is liquidated at time T for sure and in case there is a probability $1-p$ illiquid wealth cannot be liquidated.

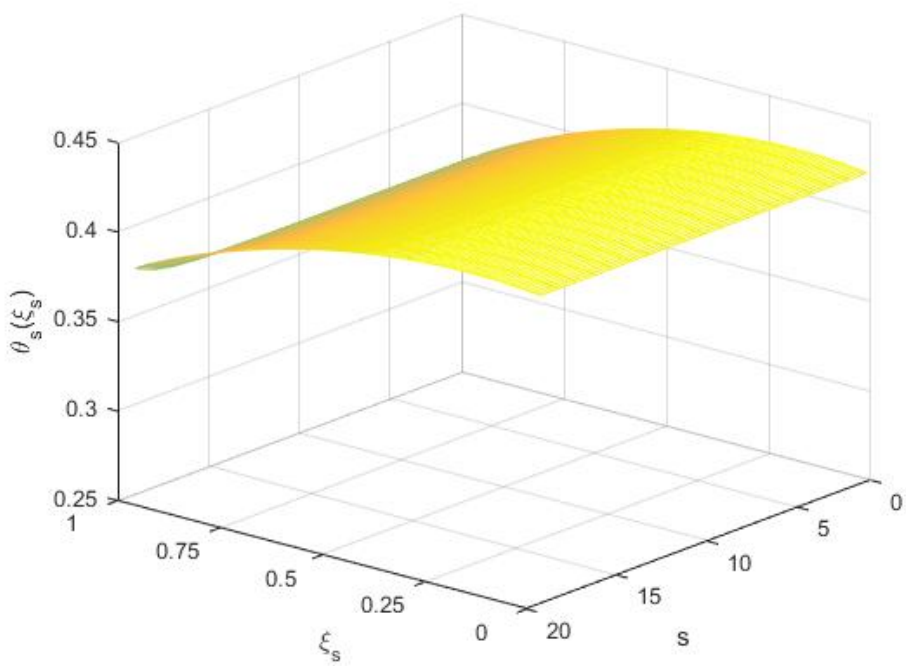


Figure 16. **Optimal allocation towards the liquid risky asset as a fraction of total wealth in a final wealth problem in case illiquid wealth is liquidated at the final date.** This graph plots the optimal asset allocation towards the liquid asset as a function of time $s = 0, 1, ..., T$ and the fraction invested in the illiquid asset, ξ_s, in case the expected time between trades is once in two years ($p = 0.5$) and illiquid wealth is liquidated at the final date T with probability 1. The graph is constructed by using the following parameter values: $\gamma = 3, \beta = 0.02, r^f = 0.02, d = 0, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and

$\rho = 0$.

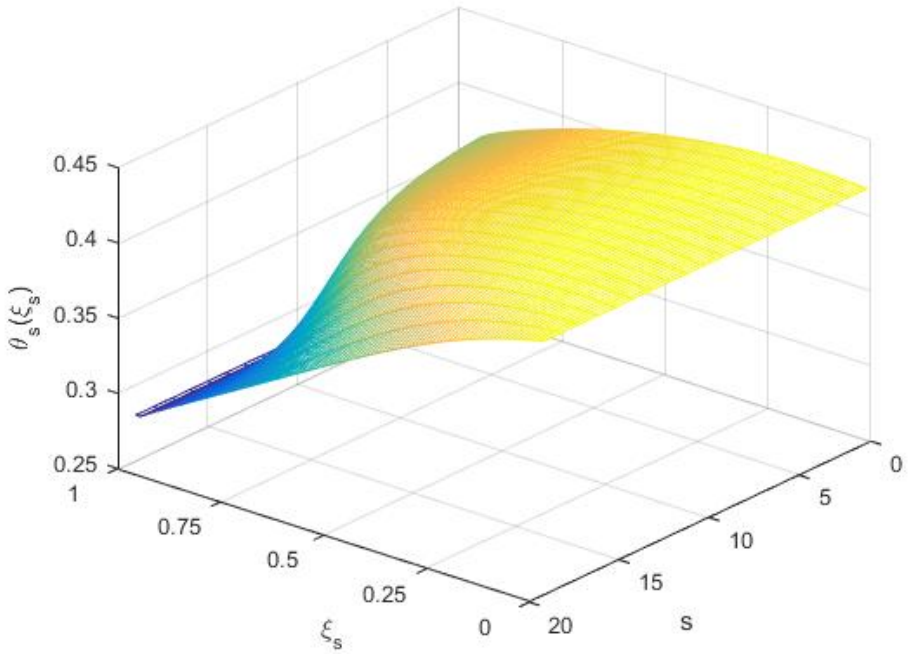


Figure 16. **Optimal allocation towards the liquid risky asset as a fraction of total wealth in a final wealth problem in case illiquid wealth is liquidated with probability p at the final date.** This graph plots the optimal asset allocation towards the liquid asset as a function of time $s = 0, 1, ..., T$ and the fraction invested in the illiquid asset, ξ_s, in case the expected time between trades is once in two years ($p = 0.5$) and illiquid wealth is liquidated at the final date T with probability p. The graph is constructed by using the following parameter values: $\gamma = 3, \beta = 0.02, d = 1, r^f = 0.02, \lambda_X = \lambda_S = 0.25, \sigma_X = \sigma_S = 0.2$, and $\rho = 0$.

References

Ang, Andrew, Papanikolaou, Dimitris and Westerfield, Mark M., 2014, Portfolio choice with illiquid assets, *Management Science*, 60 (11) 27372761.

Beber, Alessandro, Brandt, Michael W. and Kavajecz, Kenneth A., 2009, Flight-to-Quality or Flight-to-Liquidity? Evidence from the Euro-Area Bond Market, *The Review of Financial Studies*, 22(3) 925-957.

Constantinides, George M., 1986, Capital Market Equilibrium with Transaction Costs, *The Journal of Political Economy*, 94 842-862.

Diamond, Peter A., 1982, Aggregate demand management in search equilibrium, *Journal of Political Economy*, 90 (5) 881-894.

Gârleanu, Nicolae, 2009, Portfolio choice and pricing in illiquid markets, *Journal of Economic Theory*, 144 (2009) 532-564.

Gârleanu, Nicolae and Pedersen, Lasse Heje, 2013, Dynamic Trading with Predictable Returns and Transaction Costs, *Journal of Finance*, 68 (6) 2309-2340.

Jansen, Kristy, 2015, Optimal Asset Allocation with Illiquid Assets, Netspar Academic Series.

Longstaff, Francis A., 2001, Optimal Portfolio Choice and the Valuation of Illiquid Securities, *The Review of Financial Studies*, 14 (2) 407-431.

Lynch, Anthony W. and Tan, Sinan, 2011, Explaining the Magnitude of Liquidity Premia: The Roles of Return Predictability, Wealth Shocks, and State-Dependent Transaction Costs, *The Journal of Finance*, 66 13291368.

Kaplan G., Violante, GL, 2010, How much consumption insurance beyond self-insurance?, *American Economic Journal: Macroeconomics*, 2(4) 53-87.

Merton, Robert C.,1971, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory*, 3(4) 373413.

Global Pensions Asset Study - Towers Watson