The Effect of Taxation on Optimal Consumption and Portfolio Decisions Over the Life-cycle

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Abstract
This paper investigates how taxes affect the welfare effects of different accumulation and decumulation strategies over the life-cycle. Using numerical optimization, we compare optimal consumption and investment strategy under different tax regimes in the United States, Australia, Netherlands, and Korea. We also evaluate the welfare effect by measuring the certainty equivalent utility loss from not accumulating wealth through the tax-privileged pension account. Our results show how the optimal consumption and investment decisions can be altered by different pension tax rules. We find that unlike the EET system, the optimal decisions and corresponding welfare effects under the TTE regime result in only small differences due to the same taxation structure between two accounts. Our welfare analyses show substantial difference in welfare effects in the presence/absence of the taxation rules over the life-cycle, suggesting that ignoring taxes as in the academic literature would significantly distort welfare implications. We also find that a strict tax penalty on the lump-sum withdrawal from the pension account is essential to discourage people from non-annuitization, however, current tax rates are not enough to incentivize people to save more in the tax-privileged pension account and to annuitize under our tax parameters.

Keywords: Pension tax treatments, Annuitization, Life-cycle modelling
JEL Codes: D14, G51, H24

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1 Introduction

Tax is one of the most crucial elements to be considered in the financial decision-making process in practice. Individuals and financial advisors strive to be as tax-efficient as possible in their financial planning. This is particularly the case in the occupational pension plans in which the tax treatments are designed to provide tax benefits to retirement savings in order to incentivize individuals to contribute more in their pension account and to annuitize the accumulated wealth upon retirement. In recent decades, preferential pension tax treatments have growing impact on the individuals’ financial decisions such as savings, consumption, and portfolio asset allocations as private pensions are becoming a major part of retirement assets in many countries due to shrinking generosity of the public pension systems in the face of increasing longevity risks. In the United States, to take just one example, the retirement assets from the private sector\footnote{This includes retirement assets from private DB, DC plans, and IRAs.} totaled $27.3 trillion in the third quarter of 2021, which accounts for 73% of the total retirement assets (ICI, 2021 [32]). Therefore, the importance of tax considerations in lifetime financial planning of the individuals cannot be overemphasized.

Due to the complex nature of tax rules, however, taxes are often ignored in the academic literature. Such include, as for example, i) progressive marginal income tax rates depending on the level of income, ii) different timing of taxation for ordinary labor income and pension contribution, iii) different tax treatments on capital gains from investments in ordinary bank account and tax-privileged pension account, and iv) different taxation rules on the type of withdrawal after retirement. The consumption-investment choice problem becomes much complicated when such aspects of the tax rules are incorporated in the model. What determines the actual consumption opportunity of the investor is the after-tax disposable income. Hence, the optimal portfolio and consumption decisions of investors as well as corresponding welfare effects can be substantially miscalculated if the tax rules are ignored. Moreover, the heterogeneity of the pension taxation treatments across different countries such
as EET\(^2\), TTE, TET, etc. implies that the optimal strategies and welfare implications can vary even for the investors with identical characteristics under the same market conditions, depending on the pension tax regime of a specific country.

In this paper, we investigate the consumption-investment problem under different assumptions on pension tax treatments in the life-cycle framework, particularly focusing on three taxation stages where the pension tax incentives can be provided: i) when the individual makes contribution to the pension account, ii) when the pension wealth is invested and capital gains occur, and iii) when the pension withdrawals are made. We consider a CRRA investor who faces uncertain lifetime with borrowing and liquidity constraints and accumulates retirement wealth via a taxable ordinary account (“free wealth account”) or a tax-privileged pension account (“pension account”). At the time of retirement, the investor can either choose to annuitize or plan the investment and consumption strategies herself without longevity insurance, and her wealth is subject to different taxation rules based on the type of the savings account and her withdrawal decision. By contrasting optimal strategies of a taxable ordinary account and a tax-privileged pension account with those of no taxation case, we study how the optimal consumption and investment choices can alter if an individual takes into account taxation rules during her lifetime. Furthermore, we examine the welfare effects of different strategies by measuring certainty equivalent utility loss to understand by how much an individual would be better off by choosing one type of account over the other in various life stages - accumulation phase, decumulation phase, and full horizon. We also quantify the cost of ignoring taxes by comparing the welfare effects of the strategies with and without taxation. To account for the international heterogeneity of the pension tax treatments, we employ tax regimes of four different countries: United States, Australia, Netherlands, and Korea.

The contribution of our study is as follows. First, we contribute to the literature by inves-

\(^2\)Note that “E” stands for “Exempt” and “T” stands for “Taxed”, indicating three stages - contributions, investments, and benefits - of the taxation on pension savings. For instance, EET is a form of pension tax treatment, whereby pension contributions are “Exempt”, capital gains of the pension funds are “Exempt”, and pension withdrawals are “Taxed”.

tigating how the optimal consumption and investment decisions would change depending on different tax regimes and individual preferences under liquidity and short-selling constraints. Second, our analyses show that ignoring tax features over the life-cycle may lead to significant differences in welfare implications. Third, we extensively investigate four different pension tax rules of the US, Australia, Netherlands, and Korea and show that optimal decisions as well as welfare effects strongly depends on tax parameters as well as the taxation structure, suggesting that tax-optimized decisions can be significantly different even within the same taxation system.

The remaining parts of the paper proceeds as follows. First, we review the previous literature on the taxation effect on financial decisions and the welfare effects of annuitization in Section 2. Our life-cycle model with taxation is formulated in Section 3, and we present the empirical analysis using numerical optimization in Section 4. Section 5 concludes.

2 Literature Review

In the early literature on the impact of taxes on the optimal decision of individuals, researchers have focused mainly on the optimal realization of the investment returns to minimize capital gains taxes. A seminal work of Constantinides (1983) [15] has shown that, under the costless short selling opportunity and symmetric taxation of long-term and short-term capital gains, it is optimal to realize losses immediately and defer gains until a forced liquidation. Constantinides (1984) [16] has further suggested that under asymmetric taxation where a higher tax rate is given to the short-term capital gains than the long-term capital gains, investors should optimally realize losses in a short-term while realizing gains in a long-term and repurchase the assets immediately. Dammon et al. (2001) [17] have investigated the optimal dynamic consumption and portfolio choices with capital gains taxes, short-sale constraints, and bequest motives. Their numerical results have shown that investors would find stocks more attractive under the presence of the capital gains tax as it makes the risk-
return trade-off more favorable, which has been also suggested by Marekwica (2012) [44] under tax rebate possibility for capital losses. Dammon et al. (2001) [17] also have found that the US elderly would optimally retain higher equity holdings if the capital gains taxes are forgiven upon death. Extended from the single stock and bond setting of Dammon et al. (2001) [17], Gallmeyer et al. (2006) [23] have studied the consumption-portfolio problem under two risky assets and a risk-free asset with and without costly short-sale availability. They have shown that the optimal strategy would be similar to the case of single risky asset if the short selling is not allowed, however, an investor may optimally short one of the risky assets even if both of them have no embedded capital gains if short selling is available and the trading cost is lower.

Recently, an increased attention has been given to a broader taxation rules in addition to the capital gains taxes in the life-cycle framework. Bruhn (2012) [11] has formulated a life-cycle model with taxes on consumption, capital gains, labor income, retirement benefits, and life insurance. By employing the US and Danish tax rates, he has found that the welfare gain of annuity benefits is 35% compared to tax on lump-sum benefits due to a favorable capital gains taxation in the US. However, the welfare gain is only 12% in Danish tax regime because of a lower optimal savings rate as a result of higher labor income taxation. The results have suggested that incorporating several taxation rules may lead to a significant variation in optimal behavior of individuals. Lachance (2013) [39] have compared different tax regimes of the two representative pension plans in US, namely the Roth 401(k) and Traditional 401(k) with considerations of the tax deductibility of contributions, tax on withdrawals, and tax on social security benefits. She has found that choosing one type of plan over another does not create meaningful welfare implications except for high income individuals. However, her life-cycle model assumes no market risk and also capital gains taxation is not included in the model. Horneff et al. (2022) [28] also have investigated the impact of taxes on savings, retirement decision, and inequality using the tax features of the Roth 401(k). They have

\[^3\text{Dammon and Spatt (2012) [18] provide an extensive review on theoretical framework of portfolio choice under taxation, particularly focusing on the US tax code.}\]
found that the tax treatment of the Roth 401(k) triggers delayed retirement, lower lifetime tax payments, and small reductions in consumption. They have claimed, however, that the overall differences in the optimal results under the EET and TEE regimes do not show any significant gap, providing a weak support for favoring one regime over another. Meanwhile, Fischer et al. (2012) [21] have argued that the welfare cost of ignoring taxation in the optimal portfolio decision is less than 2%, providing a justification for the literature to excluding tax rules. They have claimed that under the US tax code where taxation on the capital gains are exempt upon death, tax-optimized decision becomes important only for investors with strong bequest motives. Lastly, there also have been several attempts to empirically examine the effect of tax incentives under a specific taxation system using administrative data of different countries, for instance, in UK (Sweeting, 2009 [61]), Germany (Sauter et al., 2015 [58]), Switzerland (Bütler and Ramsden, 2022 [12]) and Italy (Jappelli and Pistaferri, 2003 [34]).

Our paper is also related to a strand of literature on the optimal annuitization decision. A seminal work of Yaari (1963) [66] has shown that if a risk-averse individual who faces longevity risk and has no bequest motive is an expected utility maximizer with intertemporally separable utility, it is optimal to fully annuitize her retirement wealth when actuarially fair annuities are available. Since then, a great deal of attention has focused on incorporating plausible factors to explain so-called the annuity puzzle, such as the presence of pre-annuitized wealth (Dushi and Webb (2004) [20]; Pashchenko (2013) [52]), marital status (Kotlikoff and Spivak (1981) [37]; Brown and Poterba (2000) [10]; Brown (2001) [7]), bequest motives (Friedman and Warshawsky (1990) [22]; Bernheim (1991) [4]; Laitner and Juster (1996) [38]; Lockwood (2012) [42]; Lockwood (2018) [43]), health costs and liquidity constraints (Sinclair and Smetters (2004) [60]; Turra and Mitchell (2008) [64]; Pang and Warshawsky (2010) [51]; Peijnenburg et al. (2015) [53]), insufficient asset menu, and behavioral biases (Hu and Scott (2007) [29]; Brown (2007) [8]; Brown et al. (2013) [9]; Salisbury and Nenkov (2016) [57]). As none of these papers have incorporated the features of the pension
tax treatments in the optimal decision of individuals, we investigate the annuity puzzle from the perspective of taxation.

3 A Life-cycle Model Incorporating Taxation

In this section, we explore the optimal consumption-investment problem with consideration of the pension tax regime. We compare different savings and withdrawal strategies over the life-cycle using the taxable free wealth account and the tax-privileged pension account. By numerically solving for the optimal consumption path and the portfolio strategy, we evaluate the discounted expected utility of each strategy measured by certainty equivalent wealth. A main focus in this section is on how the optimal decisions would change depending on different structures of the pension tax treatments.

This section is organized as follows. The basic assumptions of our model are described in Section 3.1. In Sections 3.2 and 3.3 we present the optimization problems in the decumulation and in the accumulation phase, respectively, for different types of accounts. We discuss how to quantify welfare effects drawing on the certainty equivalent utility loss in Section 3.4.

3.1 Two Savings Accounts and Taxation

We consider a discrete time model, in which an individual in each period $t$ during her active life invests part of her labor income to finance consumption after retirement date, which we denote by $t = T$. She starts on date zero with initial wealth at time $t = 0$ equal to zero. At any (future) time $t = 0, 1, \ldots, T - 1$ at which the individual is alive and not yet retired, (s)he receives income $Y_t$ and consumes an amount $C_t$. All remaining wealth after income and consumption is invested in a portfolio consisting of a fraction $f_t$ invested in the risky asset and $1 - f_t$ in the riskless asset. She can make these investments either in a taxable free wealth account or in a tax-privileged pension account. At time $t = T$, the individual retires, and the accrued wealth is used to finance consumption after retirement. For both type of accounts,
she either annuitizes her wealth at retirement date, or determines the optimal consumption and investment plan without longevity insurance. That is, we consider the following four strategies:

- Accumulate retirement wealth via a free wealth account and purchase an optimal variable annuity at retirement;

- Accumulate retirement wealth via a free wealth account and determine the optimal consumption and investment plan at retirement for the remaining lifetime without buying an annuity;

- Accumulate retirement wealth via a pension account and convert the accrued wealth into an optimal variable annuity at retirement;

- Accumulate retirement wealth via a pension account, make a full lump-sum cashout at retirement date and determine the optimal consumption and investment plan for the remaining lifetime without buying an annuity.

Our goal is to investigate the effect of taxes on the optimal consumption and investment decisions in the accumulation and the decumulation phase, as well as on the welfare derived from different savings accounts with various tax treatments. We consider the following tax rates:

- $\tau_F$: income tax on labor income put in the free wealth account;

- $\tau_{PC}$: income tax on labor income put in the pension account;

- $\tau_{CG}$: capital gains tax on investment returns;

- $\tau_{AB}$: income tax on annuity benefits after retirement date;

- $\tau_{WB}$: income tax on any withdrawals other than annuity benefits from the pension account after retirement date.
While the free wealth account holder saves all wealth for current and future consumption in the free wealth account, the pension account holder uses the free wealth account only to finance current consumption; all savings for future consumption take place in the pension account. For both types of accounts, any labor income that is put in the free wealth account is taxed at rate $\tau_F > 0$. Whether the other three income sources are taxed or exempt depends on whether the investor saves for retirement in a free wealth account or a pension account, and, in case she chooses a pension account, on the the specific tax treatment of that account. We will consider two types of pension accounts, referred to as “EET” and “TTE”, respectively. In Table 1, we summarize the four strategies that we consider (free wealth account with and without annuitization and pension account with and without annuitization), where we distinguish the EET and TTE tax systems for the two pension account strategies. If a tax is exempt, the corresponding tax rate is set equal to zero.

<table>
<thead>
<tr>
<th>Tax Parameters</th>
<th>Free Wealth</th>
<th>Free Wealth</th>
<th>Pension account</th>
<th>Pension account</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Annuitization</td>
<td>No Annuitization</td>
<td>Annuitization</td>
<td>No Annuitization</td>
</tr>
<tr>
<td>Labor income$^1$ ($\tau_F$)</td>
<td>Taxed</td>
<td>Taxed</td>
<td>Taxed</td>
<td>Taxed</td>
</tr>
<tr>
<td>Pension contr. ($\tau_{PC}$)</td>
<td>N/A</td>
<td>N/A</td>
<td>Exempt</td>
<td>Taxed</td>
</tr>
<tr>
<td>Cap. gains accum. ($\tau_{CG}$)</td>
<td>Exempt</td>
<td>Taxed</td>
<td>Exempt</td>
<td>Taxed</td>
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<tr>
<td>Cap. gains decum. ($\tau_{CG}$)</td>
<td>N/A</td>
<td>N/A</td>
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<tr>
<td>Annuity benefits ($\tau_{AB}$)</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Withdrawal benefits ($\tau_{WB}$)</td>
<td>N/A</td>
<td>Exempt</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Note: If a tax is “Exempt”, the corresponding tax rate is equal to zero. The top (bottom) panel corresponds to the accumulation (decumulation) phase.

1 The tax rate $\tau_F$ is imposed on labor income that is put in the free wealth account, not on the total labor income.

2 In the Australian pension tax regime, the capital gains are taxed during the accumulation phase only.

In each of the six cases, the individual chooses her consumption and investment strategies so as to maximize the discounted expected utility of her consumption over the accumulation and the decumulation phase. In the next section we formulate the corresponding optimization
problems. We make the following assumptions:

- The investor maximizes the expected present value of the utility derived from consumption over the course of her life. She has a power utility function (CRRA) with relative risk aversion coefficient $\gamma > 1$:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}, \text{ for all } C > 0.$$  (1)

- Both in the free wealth account and in the pension account, all wealth that is not consumed in period $t$ can be invested in a portfolio consisting of a fraction $f_t$ invested in the risky asset $S_t$ and $1 - f_t$ in the riskless asset $B_t$. Short selling is not allowed, i.e., $f_t \in [0, 1]$ for all $t$. The price of the risky asset follows a log-normal distribution. Specifically, the after-tax price evolution of the risky and the riskless asset is as follows:

$$dB_t = \tilde{r} B_t dt,$$  \hspace{1cm} (2)

$$dS_t = (\tilde{r} + \lambda \tilde{\sigma})dt + \tilde{\sigma} S_t dZ_t,$$  \hspace{1cm} (3)

where $Z_t \sim_{iid} N(0,1)$, $\tilde{r}$ denotes the after-tax risk-free rate of return, and $\tilde{\mu}$, $\tilde{\sigma}$, and $\lambda \equiv (\tilde{\mu} - \tilde{r})/\tilde{\sigma}$ denote the after-tax expected return, standard deviation and Sharpe ratio of the risky asset. Because the tax rate imposed on capital gains is denoted $\tau_{CG}$, it holds that:

$$\tilde{r} = [1 - \tau_{CG}] r,$$ \hspace{1cm} (4)

$$\tilde{\mu} = [1 - \tau_{CG}] \mu,$$ \hspace{1cm} (5)

$$\tilde{\sigma} = [1 - \tau_{CG}] \sigma,$$ \hspace{1cm} (6)

where $r$, $\mu$, and $\sigma$ are the corresponding pre-tax values. If capital gains are exempt from taxes (e.g., in an EET pension account), it holds that $\tau_{CG} = 0$, and so after-tax
value equals pre-tax value. We assume that
\[
\frac{\mu - r}{\gamma \sigma^2} \leq 1 - \tau_{CG}. \tag{7}
\]
This ensures that the after-tax Merton fraction is less than or equal to 1.\textsuperscript{4}

- The investor is liquidity constrained, i.e., she cannot borrow. Moreover, wealth in the pension account can only be withdrawn at or after retirement date. Hence, in any given period prior to retirement, she cannot consume more than the wealth available in the free wealth account.

- For any given \( v \) and \( t \leq s \), the probability that the individual is still alive at time \( v + s \) conditional on being alive at time \( v \), denoted by \( sP_v \), satisfies the relationship \( sP_v = tP_v \cdot s-tP_{v+t} \).

In the next sections, we derive the optimal consumption and investment strategies in the accumulation and decumulation phase, for the four strategies displayed in Table 1, i.e., free wealth account with and without annuitization, and pension wealth account with and without annuitization. Because the optimal strategies are available in closed form expression for the decumulation phase, we split the problem in two parts. In Section 3.2 we present the optimization problems that need to be solved to determine the optimal consumption and investment strategies in the accumulation phase for a free wealth account holder and a pension account holder, respectively, taking into account that any available wealth at retirement date will be decumulated according to an optimal decumulation strategy. In Section 3.3 we determine closed form expressions for the optimal consumption and investment strategies in the decumulation phase with and without annuitization, as a function of the level of accrued pension wealth at retirement date.

\textsuperscript{4}The maximum capital gains tax rate that we consider in our empirical analysis in \( \tau_{CG} = 15.4\% \) for Korea. Hence, condition (7) is satisfied as long as the pretax Merton fraction is weakly lower than 0.846. In our numerical analysis, we consider \( \sigma = 0.2 \) and \( \mu - r = 0.04 \). Then, (7) is satisfied for all \( \tau_{CG} \leq 15.4\% \) iff \( \gamma \geq 1.18 \).
3.2 Optimal Strategies in the Accumulation Phase

In this section we present the optimization problems that need to be solved to determine the optimal consumption and investment strategies in the accumulation phase, for a free wealth account holder and a pension account holder, respectively.

3.2.1 Optimal accumulation for free wealth account holder

We first consider the optimization problem of a free wealth account holder. At the beginning of each year $t = 0, 1, \ldots, T - 1$, the individual receives labor income $Y_t$ and consumes $C_t$. The labor income gradually grows every year with a deterministic growth rate of $g_t$. We let $W_t$ denote aggregate after-tax wealth in the free wealth account just after income is received and taxes are paid but before consumption takes place. Then, the optimal investment and consumption strategy in the accumulation phase solves the following optimization problem:

$$\sup_{C_t, f_t} E_0 \left[ \sum_{t=0}^{T-1} e^{-\beta t} \cdot t p_0 \cdot U(C_t) + e^{-\beta T} \cdot T p_0 \cdot u(W_T) \right],$$  \hspace{1cm} (8)$$

subject to:

$$W_0 = (1 - \tau_F)Y_0,$$

$$W_{t+1} = e^{R_t(f_t)} \cdot [W_t - C_t] + (1 - \tau_F)Y_{t+1}, \text{ for } t = 0, \ldots, T - 1$$

$$Y_t = Y, \text{ for } t = 0, \ldots, T - 1,$$

$$Y_T = 0,$$

$$0 \leq C_t \leq W_t, \text{ for } t = 0, \ldots, T - 1,$$

$$0 \leq f_t \leq 1, \text{ for } t = 0, \ldots, T - 1,$$

where

$$R_t(f) := \tilde{r} + f \tilde{\sigma} \tilde{\lambda} - \frac{1}{2} f^2 \tilde{\sigma}^2 + f \tilde{\sigma} Z_t, \text{ for all } f \in [0, 1],$$  \hspace{1cm} (15)$$
with $Z_t \sim_{i.i.d.} N(0,1)$, and where $\beta$ is the subjective discount factor, $g_{t+1}$ is the growth rate on income, and the terminal value $\bar{u}(W_T)$ is the utility of final wealth on retirement date if the optimal decumulation strategy is used. The optimal decumulation strategies with and without annuitization are derived in Section 3.3, and the corresponding values of $\bar{u}(W_T)$ are derived in Appendix B.

### 3.2.2 Optimal accumulation for pension account holder

We now consider the problem of a pension account holder. The optimization problem of the pension account holder is similar to that of the free wealth account holder in (8)-(14), with two differences in the constraints. First, the individual has both a free wealth account (used for immediate consumption) and a pension account (used to save for retirement benefits). Different tax rates may apply to these two accounts, which implies that the wealth dynamics in (9) and (10) may be different. Second, the pension account holder cannot withdraw wealth from the pension account for current consumption. This affects the liquidity constraints in (13).

First, we consider the effect on the wealth dynamics. In each period $t = 0, 1, \ldots, T - 1$, the pension account holder divides her wealth between an amount put in the free wealth account in order to consume in period $t$, and an amount put in the pension account to save for consumption after retirement. If she wants to consume $C_t$ in period $t$, she needs to put the amount $C_t/(1 - \tau_F)$ in the financial account. The remainder, $Y_t - C_t/(1 - \tau_F)$, is put in the pension account. Hence, the after-tax amount added to the pension account in period $t$ equals:

$$
(1 - \tau_{PC}) \left[ Y_t - \frac{C_t}{1 - \tau_F} \right] = (1 - \tau_{PC})Y_t - \left( \frac{1 - \tau_{PC}}{1 - \tau_F} \right) C_t. \tag{16}
$$

We let $W_t$ denote the wealth in the pension account just after $(1 - \tau_{PC})Y_t$ is added but before $(1 - \tau_{PC})/(1 - \tau_F)C_t$ is subtracted. Then, the initial wealth is given by:

$$
W_0 = (1 - \tau_{PC})Y_0, \tag{17}
$$
and the wealth dynamics of the pension account are:

\[ W_{t+1} = e^{R(f_t)} \cdot \left[ W_t - \left( \frac{1 - \tau_{PC}}{1 - \tau_F} \right) C_t \right] + (1 - \tau_{PC})Y_{t+1}, \]

(18)

for \( t = 0, \ldots, T - 1 \). In an EET system, pension contributions are tax exempt, and so \( \tau_{PC} = 0 \). In contrast, in an TTE account, they are taxed and so \( \tau_{PC} > 0 \). Moreover, in the special case where \( \tau_{PC} = \tau_F \), the wealth dynamics of the pension account holder are equal to those of the free wealth account holder.

Second, we impose different liquidity constraint to a pension account holder. In the case of the free wealth account, the investor can freely withdraw the accumulated wealth from her account and thus the upper bound of consumption is given by \( W_t \) as in (13). However, the accrued contributions in the pension account cannot be withdrawn until retirement. The maximum amount of consumption for a pension account holder is therefore restricted to the after-tax labor income in each period. Hence, the liquidity constraint of a pension account holder is given by:

\[ 0 \leq C_t/(1 - \tau_F) \leq Y_t. \]

(19)

We conclude that the optimal consumption and investment strategy of the pension account holder in the accumulation phase solves optimization problem (8)-(14), with constraints (9), (10), and (13) replaced by (17), (18), and (19), respectively.

3.2.3 Numerical optimization

The optimal strategies in the accumulation phase need to be determined numerically using dynamic programming. For both types of account holders, we determine the optimal consumption and investment strategies by numerically solving the Bellman equations with \( \pi(W_T) = W_T^{1-\gamma} \cdot EU_{decum}^*(1) \), where \( EU_{decum}^*(1) \) is the expected discounted utility of the optimal decumulation path, given \( W_T = 1 \). The values of \( EU_{decum}^*(1) \) for the case of no annuitization and the case of annuitization, respectively are given in Appendix B. A detailed
description of the numerical optimization method is presented in Appendix A.

3.3 Optimal Strategies in the Decumulation Phase

In this section we determine the optimal consumption and investment strategies in the decumulation phase. We first consider the case where the individual optimally plans her consumption and investment strategies without annuitizing her wealth. Next, we consider the case where she decumulates her retirement wealth by buying an optimal variable annuity.

3.3.1 Optimal decumulation without annuitization

Let \( t = D \) be the last period in which the individual can still be alive. Then, for any given available wealth \( W_T \) at retirement date \( t = T \), she will plan her consumption and investment strategy so as to maximize the expected discounted utility of her consumption over the periods \( t = T, T + 1, \ldots, D \). Specifically, the optimal investment and consumption strategy solve the following optimization problem:

\[
\sup_{C_t, f_t} \mathbb{E}_T \left[ \sum_{t=T}^{D} e^{-\beta t} \cdot i_p T \cdot U(C_t) \left| W_T \right. \right],
\]

s.t. \( W_{t+1} = e^{R_t(f_t)} \cdot [W_t - C_t], \) for \( t = T, \ldots, D, \)

\( C_t \leq W_t, \) for \( t = T, \ldots, D, \)

\( f_t \in [0, 1], \) for \( t = T, \ldots, D. \) (20)

We will determine the optimal solution without liquidity and shortsales constraints, and show that these constraints are not binding.

Due to the homothetic property of the utility function and the linearity of the constraints, the fraction of wealth that is optimally consumed in period \( t \) is independent of the level of wealth. This implies that the dynamic optimization problem in (20) can be simplified to a static problem in which the investor on retirement date \( T \) divides the total available wealth \( W_T \) in \( D - T + 1 \) buckets reserved for consumption in periods \( t = T, T + 1, \ldots, D \), and decides on the investment strategy for each bucket (Grebenchtchikova et al. (2017) [25]). We denote \( W_T(t) \) for the bucket that is reserved for consumption in period \( t \), and \( f_{[T,t]} \) for
the fraction of that bucket that is invested in the risky asset in periods \( h = T, T+1, \ldots, t-1 \). The resulting value on date \( t \) (which is \( \left( \prod_{h=T}^{t-1} e^{R_h(f_{[T,t]})} \right) \cdot W_T(t) \)) is used for consumption in period \( t \). Hence, the optimal bucket sizes and risky investment fractions solve the following optimization problem:

\[
\max_{W_T(t), f_{[T,t]}} \mathbb{E}_T \left[ \sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t^{-\gamma} \cdot p_t \cdot U(C_t) \right] \quad (21)
\]

s.t. \( C_t = \left( \prod_{h=T}^{t-1} e^{R_h(f_{[T,t]})} \right) \cdot W_T(t) \)

\[
\sum_{t=T}^{D} W_T(t) = (1 - \tau_{WB}) \cdot W_T. \quad (23)
\]

It follows immediately from (21) and (22) that the optimal investment strategy for each bucket is the Merton strategy (see, e.g., Karatzas and Shreve, 1998). After correcting for taxes, this implies that the optimal fraction of bucket \( W_T(t) \) that is invested in the risky asset is independent of time and given by (Dammon et al. (2001) [17]):

\[
f^*_t = f^* := \frac{\tilde{\lambda}}{\gamma \sigma} = \frac{\mu - r}{\gamma \sigma^2 (1 - \tau_{CG})}, \quad (24)
\]

for all \( t = T, T+1, \ldots, D \). Moreover, a straightforward extension of Balter and Werker (2020) [2] to take into account survival probabilities in the objective function in (21) yields that the optimal value of the bucket reserved for consumption in period \( t \) is given by:

\[
W^*_T(t) = (1 - \tau_{WB}) \cdot W_T \cdot \left( \frac{b^*(t) \cdot t^{-\gamma} \cdot p_t}{\exp(\beta(t-T))} \right)^{\frac{1}{\gamma}} / \left( \sum_{k=T}^{D} \left( \frac{b^*(k) \cdot k^{-\gamma} \cdot p_k}{\exp(\beta(k-T))} \right)^{\frac{1}{\gamma}} \right), \quad (25)
\]

where

\[
b^*(t) = \mathbb{E}_0 \left[ \left( \prod_{h=T}^{t-1} e^{R_h(f^*)} \right)^{1-\gamma} \right] = \exp \left\{ \left( \tilde{r} + \frac{\tilde{\lambda}^2}{2\gamma} \right) (1 - \gamma)(t - T) \right\}. \quad (26)
\]

The second equality follows immediately from the fact that returns are iid and follow a
Log-normal distribution. Details are provided in Appendix B.

The optimal consumption path now follows from (22) combined with (25)-(26). By construction, the optimal consumption level $C_t^*$ in period $t$ is weakly lower than the available wealth in that period. Combined with the fact that (7) implies that $f^* \leq 1$, this implies that the optimal investment and consumption strategies satisfy the liquidity constraint and the shortsales constraint, respectively.

3.3.2 Optimal decumulation with annuities

In this section we consider the case where at retirement date $T$, the individual converts her accumulated wealth $W_T$ in an optimal variable annuity. At the beginning of each period, the individual receives an annuity payoff, consumes an amount, and invests her remaining wealth. The stochastic payoff of a variable annuity is fully specified once one determines, for $t = T, T+1, \ldots, T+D$, the size of the bucket $W_T(t)$ reserved for annuity payments in period $t$ and the corresponding fraction $f_{[T,t]} \in [0, 1]$ of this bucket that is invested in the risky asset. The payoff of the annuity in period $t$ then equals $(\prod_{h=T}^{t-1} e^{R_h(f_{[T,h]})}) \cdot W_T(t)$ (see, e.g., Horneff et al., 2010; Maurer et al., 2013). Because the optimal consumption path without annuities (as characterized by (22) combined with (25)) equals the payoff of a variable annuity, the optimal strategy if variable annuities are available is to buy a variable annuity and to consume in each period the payoff of the annuity in that period. That is, it is not optimal to save some of the annuity payoff for later consumption. Hence, without loss of generality, we determine the optimal variable annuity assuming that consumption in period $t$ equals the payoff of the annuity in that period.

As was the case without annuitization, the investor now chooses the size of the buckets $W_T(t)$, and the corresponding investment fractions $f_{[T,t]}$ so as to optimize the objective function in (21). The constraints, however, are different. First, because annuity benefits are taxed, consumption equals after-tax annuity payoffs. Second, because the individual now uses her available wealth at retirement date to buy a variable annuity, the budget

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constraint requires that the price of the variable annuity equals the available wealth $W_T$. Hence, constraints (22) and (23) are replaced by:

$$C_t = (1 - \tau_{AB}) \cdot \left( \prod_{h=T}^{t-1} e^{R_h(f_{h,t}, t)} \right) \cdot W_T(t),$$

$$\left(1 + \psi\right) \cdot \left( \sum_{t=T}^{D} t^{-\tau_{PT}} \cdot W_T(t) \right) = W_T,$$

where $\psi$ is a proportional cost loading in the price of the variable annuity.

The investor now chooses the size of the buckets $W_T(t)$, and the corresponding investment fractions $f_{[T,t]}$ so as to optimize the objective function in (21) subject to (27) and (28). As was the case without annuities, it is optimal to invest the buckets according to the after-tax Merton strategy, and so $f^*_{[T,t]} = f^*$ as defined in (24) for all buckets $t = T, T + 1, \ldots, D$. Moreover, the optimal value of the buckets is given by:

$$W_T^*(t) = W_T \cdot \left( \frac{b^*(t)}{\exp(\beta(t - T))} \right)^{\frac{1}{2}} \left( (1 + \psi) \cdot \sum_{k=T}^{D} \frac{b^*(k)}{\exp(\beta(k - T))} \right)^{\frac{1}{2}} \cdot k^{-\tau_{PT}},$$

where the function $b^*(\cdot)$ is as defined in (26). The optimal consumption path is obtained by combining (27) with (29). For detailed derivation of the optimal solutions in (25) and (29), see Appendix B.

### 3.3.3 Optimal consumption levels with and without annuities

We conclude by comparing the optimal consumption levels in the decumulation phase with and without annuitization. The optimal consumption levels depend on: the size of the buckets in absence of taxes, the corresponding investment strategy for each bucket, the capital gains tax on the investment returns, the annuity costs and the tax rate on annuity payments in case of annuitization, and the tax rate on lump sum withdrawals in case of no annuitization.

First, ceteris paribus, the mortality credit included in the price of the annuity (i.e., the...
fact that the net price of the bucket reserved for consumption in period $t$ is $\iota_{-T}p_t \cdot W_T(t)$, see (28)) implies that optimal buckets sizes are higher with annuitization than without annuitization, which in turn leads to higher consumption levels. However, the proportional transaction cost $\psi$ included in the price of the annuity has the opposite effect. Second, both with and without annuitization all buckets are invested according to the time-constant after-tax Merton fraction $f^*$. However, differences in capital gains tax rates between different accounts (e.g., a free wealth account without annuitization and an EET pension account with annuitization) do cause differences in investment strategy. Specifically, as can be seen from (24), $f^*$ is higher when the capital gain tax $\tau_{CG}$ is higher, i.e., capital gains taxes make investment in the risky asset relatively more attractive. Ceteris paribus, this makes consumption levels higher in expectation, but more volatile. Finally, whereas consumption levels are reduced by the tax rate $\tau_{AB}$ on annuity benefits in case of annuitization, they are reduced by the tax rate $\tau_{WB}$ on lump sum withdrawals in case of no annuitization.

Combined these three factors imply that the effect of annuitization on the optimal consumption levels is ambiguous. We investigate these effects numerically in Section 4.3.

### 3.4 Quantifying Welfare Effects of Taxation

We will use the certainty equivalent as a measure to evaluate welfare effects of different accumulation and decumulation strategies. The certainty equivalent of a given stochastic consumption stream is defined as the deterministic amount of wealth for which the utility is equal to the discounted expected utility of the stochastic consumption stream.

Let $[C^*] := (C^*_0, C^*_1, \ldots, C^*_D)$ be the optimal consumption pattern for one of the four strategies displayed in Table 1. Because $U^{-1}(x) = [(1 - \gamma)x]^{1/(1-\gamma)}$, the certainty equivalent wealth of the stochastic consumption stream $[C^*]$ for the accumulation phase and for the
two phases combined are given by:

\[
CE_{\text{accum}}[C^*] := \left(1 - \gamma \right) \cdot \sum_{t=0}^{T-1} \exp(-\beta t) \cdot t^p_0 \cdot \frac{\mathbb{E}_0[(C^*_t)^{1-\gamma}]}{1 - \gamma} \right)^{\frac{1}{1-\gamma}}, \tag{30}
\]

\[
CE_{\text{total}}[C^*] = \left(1 - \gamma \right) \cdot \sum_{t=0}^{D} \exp(-\beta t) \cdot t^p_0 \cdot \frac{\mathbb{E}_0[(C^*_t)^{1-\gamma}]}{1 - \gamma} \right)^{\frac{1}{1-\gamma}}. \tag{31}
\]

Because the optimal consumption strategies in the accumulation phase are determined numerically, closed form expressions for (30) and (31) are not available. We therefore use simulation to approximate these values. Details are available in Appendix C.

For the decumulation phase, we determine both the date-0 and the date-\(T\) certainty equivalent of the optimal consumption patterns with and without annuitization. Let \(W_T^*\) be the level of wealth at retirement date if the optimal strategy is used during the accumulation phase. We show in Appendix C that the date-0 certainty equivalent equals:

\[
CE_{\text{decum,0}}[C^*] := \left( \sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t_{-TP_T} \cdot \mathbb{E}_0 \left[ (W_T^*)^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}},
\]

\[
= \tilde{\eta} \cdot \mathbb{E}_0 \left[ (W_T^*)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \cdot \left( \sum_{t=T}^{D} \beta_t \cdot \left( \frac{\alpha_t}{\beta_t} \right)^{\frac{\gamma}{1-\gamma}} \right)^{\frac{1}{1-\gamma}}, \tag{32}
\]

where

\[
\alpha_t = e^{-\beta(t-T)} \cdot t_{-TP_T} \cdot b^*(t), \tag{33}
\]

with \(b^*(t)\) as defined in (26), and where \(\beta_t\) and \(\tilde{\eta}\) depend on whether the individual annuitizes. Specifically,

\[
\beta_t = \begin{cases} 
1, & \text{in case of no annuitization}, \\
 t_{-TP_T}, & \text{in case of annuitization}, 
\end{cases} \tag{34}
\]

and

\[
\tilde{\eta} = \begin{cases} 
1 - \tau_{WB}, & \text{in case of no annuitization}, \\
(1 - \tau_{AB})/(1 + \psi), & \text{in case of annuitization}. 
\end{cases} \tag{35}
\]
The date-0 certainty equivalent of the decumulation phase $CE_{decum,0}[C^\ast]$ depends on the strategies in the accumulation phase through their effects on the value of $E_0 [(W_T^\ast)^{1-\gamma}]^{\frac{1}{1-\gamma}}$. To isolate the effect of the consumption and investment strategies in the decumulation phase on the investor’s certainty equivalent, we therefore also determine the date-$T$ certainty equivalent utility of consumption in the decumulation phase per dollar of accumulated wealth at retirement date $T$, which we denote $CE_{decum,T}[C^\ast]$. We show in Appendix C that

$$CE_{decum,T}[C^\ast] = \left( \sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t^{-T} \cdot \beta t \cdot (\frac{\alpha_t}{\beta t})^{\frac{1}{\gamma}} \right)^{\frac{1}{1-\gamma}},$$

where the parameters $\alpha_t, \beta_t, \text{ and } \tilde{\eta}$ are as defined in (33), (34), and (35). By comparing certainty equivalents of different strategies/tax systems, we can quantify welfare gains and losses, using the concept of certainty equivalent utility loss. Specifically, let $C_1^\ast$ and $C_2^\ast$ each be the optimal consumption stream for one of the six strategies mentioned in Table 1, with $C_2^\ast$ yielding higher expected utility in phase $j \in \{accum, decum_0, decum_T, total\}$ than $C_1^\ast$. Then, the certainty equivalent utility loss $L_j$ defined as:

$$L_j(C_1^\ast, C_2^\ast) = 1 - \frac{CE_j[C_1^\ast]}{CE_j[C_2^\ast]}, \quad \text{for } j \in \{accum, decum_0, decum_T, total\},$$

yields the percentage by which the wealth of an individual with stream $C_2^\ast$ can be reduced such that in phase $j$, she is indifferent between consumption stream $C_2^\ast$ with the reduced wealth and $C_1^\ast$ with unreduced wealth.
4 The Effect of Taxation on Optimal Decisions and Welfare Implications

In this section, we analyze empirically the effect of taxation on the optimal consumption and investment choices of an investor by applying our life-cycle model under different pension tax treatment of the US, Australia, Netherlands, and Korea. We introduce the specific pension tax rules of the four countries in Section 4.1. The parameters used in our analysis are outlined in Section 4.2, and Section 4.3 discusses the results.

4.1 Pension Tax Treatments in Different Countries

In our analysis, we employ pension tax treatments of four different countries: US (401(k) plan\(^5\)), Australia (superannuation\(^6\)), Netherlands (Collective DC), and Korea (individual DC). The pension tax regimes of US, Netherlands, and Korea\(^7\) are following the EET system while Australia has implemented the TTE system. Although the EET-type countries are classified as the same tax regime, the specific features of the pension tax treatments such as capital gains taxation, pension income taxation, lump-sum option also come in a variety of forms. The overall taxation rules for the ordinary taxable free wealth account and the tax-privileged pension account are presented in [Table 2].

\(^5\) Among several types of 401(k) plans, we use the traditional 401(k) plan in our analysis.
\(^6\) There are many complex details across different types of pension plans, however, we focus solely on the defined contribution superannuation fund and we abstain from other complexities in the Australian pension regulation such as voluntary deductible contributions, state contributions, super co-contribution, etc.
\(^7\) The pension tax treatment in Korea is usually known as EET, however, there are also some aspects of the TET system due to the tax credit policy implemented in 2014. Under the new rule, the contributions are subject to marginal labor income tax rates but workers are eligible to tax credits. Thus, the contribution put in the pension account is actually taxed, however, workers can claim a tax rebate afterwards. This is why the Korean regime is sometimes categorized as TET, however, we define the system as EET since the workers eventually do not have to pay the income tax on contribution if the tax credits are refundable.
Table 2: Taxation Rules for Standard Account and Pension Account

<table>
<thead>
<tr>
<th>Country</th>
<th>Standard Account</th>
<th>Pension Account</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Income</td>
<td>Capital</td>
</tr>
<tr>
<td>US</td>
<td>Marginal income tax</td>
<td>Long-term Capital gains tax</td>
</tr>
<tr>
<td>- Traditional 401(k)</td>
<td>Marginal income tax</td>
<td>Capital gains tax¹</td>
</tr>
<tr>
<td>Australia</td>
<td>Marginal income tax</td>
<td>Marginal income tax</td>
</tr>
<tr>
<td>- Superannuation</td>
<td>Marginal income tax</td>
<td>Marginal income tax</td>
</tr>
<tr>
<td>Netherlands</td>
<td>Marginal income tax</td>
<td>Marginal income tax</td>
</tr>
<tr>
<td>- Collective DC</td>
<td>Marginal income tax</td>
<td>Marginal income tax</td>
</tr>
<tr>
<td>Korea</td>
<td>Marginal income tax</td>
<td>Marginal income tax</td>
</tr>
<tr>
<td>- Individual DC</td>
<td>Marginal income tax</td>
<td>Marginal income tax</td>
</tr>
</tbody>
</table>

¹ The investment return from an asset with a holding period less than a year is treated as salary; subject to long-term capital gains tax otherwise.

² All traditional 401(k) withdrawals are subject to federal income tax.

³ For a superannuation fund, the investment returns are taxed during the accumulation period only.

⁴ In Dutch taxation system, every individual is assumed to earn “deemed return”, which is 1.89-5.69% of the taxable wealth depending on the wealth level. A flat rate of 30% is applied on this deemed return.

⁵ A recent proposal of Dutch pension reform includes a lump-sum option of maximum 10% out of total retirement wealth.

⁶ These tax rates are age-dependent.

⁷ Pension income tax rates are equally applied to any withdrawals other than annuity payments.

4.2 Parameters Used in the Analysis

There are five important sets of parameters in our model: mortality rates, financial market parameters, coefficients for individual preferences, income levels, and tax rates. First of all, two types of mortality tables can be employed: a population mortality table and an annuitant mortality table. A population life table represents the mortality rates of the whole population on average, whereas an annuitant table is based on mortality rates of a specific subpopulation group who has purchased annuities voluntarily. In our analysis, we employ the annuitant life tables of each country. The rationale is that the investor in our model faces decision
to annuitize at retirement, and an individual with population mortality tend to be highly reluctant to purchase an annuity that is priced based on a healthier mortality group.

The financial market parameters and individual preferences also play a significant role in determining optimal behavior of individuals. The parameters are set as \( r = \beta = 0.02, \mu = 0.06, \sigma = 0.2 \) and hence \( \lambda = 0.2 \). In addition, the one-off annuity cost \( \psi \) is assumed to be 0.1 except for Netherlands where the annuity payments are directly provided by pension funds. Thus, the annuity cost is only applied to the annuitization through a free wealth account in Dutch case. Regarding the risk aversion coefficient, we test \( \gamma = \{2, 5\} \) which represents a less risk-averse and a highly risk-averse individual, respectively, to test individuals with different risk tolerance levels. This is also to embrace both relatively low estimated coefficients from experimental studies and relatively high estimated coefficients from portfolio selections of individuals (Bovenberg and Mehlkopf (2014) [5]).

The remaining parameters to be determined are income levels and tax rates for each country case. These two parameters are closely related to each other as every income bracket has its corresponding marginal tax rate. For the income level, we set a constant income level throughout the working period and do not consider any life-cycle income profile although the change in income level may lead to different marginal income tax rates. Specifically, we use a median income based on most recent available data in each country. Note that we set a hypothetical rate on a lump-sum withdrawal from the pension account although it is not allowed to make non-annuitized withdrawals in Netherlands. The corresponding tax rate parameters of each country are presented in [Table 3]. The corresponding tax rates are applied based on the taxation rules of each country as of 2022, presented in [Table 2].
Table 3: Parameters Used in the Analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>US</th>
<th>Australia</th>
<th>Netherlands</th>
<th>Korea</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Annuittant Table</td>
<td>Pri-2012(^1)</td>
<td>UK-AUS Adjusted(^2)</td>
<td>AG2020</td>
<td>KIDI 9th</td>
</tr>
<tr>
<td>(r) Risk-free rate</td>
<td>0.02</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\mu) Return on risky asset</td>
<td>0.06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\sigma) Standard deviation of risky asset</td>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta) Subjective discount factor</td>
<td>0.02</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\gamma) Risk aversion coefficient</td>
<td>{2.5}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\psi) Annuity cost</td>
<td>0.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\bar{Y}) Constant labor income</td>
<td>$35,805(^3) (≈ $44,892)</td>
<td>AU$62,400(^4) (≈ $37,342)</td>
<td>€34,728(^5) (≈ $23,105)</td>
<td>W29 million(^6)</td>
</tr>
<tr>
<td>(\tau_F) Labor income tax</td>
<td>11.44%</td>
<td>17.22%</td>
<td>37.07%</td>
<td>11.28%</td>
</tr>
<tr>
<td>(\tau_{PC}) Pension contribution tax</td>
<td>N/A</td>
<td>15.00%</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>(\tau_{CG}) Capital gains tax</td>
<td>0.00%</td>
<td>10.00%</td>
<td>0.57%</td>
<td>15.40%</td>
</tr>
<tr>
<td>(\tau_{AB}) Pension income tax</td>
<td>10.00%</td>
<td>0.00%</td>
<td>37.07%</td>
<td>4.40%</td>
</tr>
<tr>
<td>(\tau_{WB}) Tax on withdrawals</td>
<td>37.00%</td>
<td>0.00%</td>
<td>49.50%</td>
<td>8.80%</td>
</tr>
</tbody>
</table>

\(^1\) Pri-2012 annuitant rates are published as a period mortality table. On 28/04/2022, the US Treasury Department and the Internal Revenue Service have proposed a cohort mortality table to be used for DB pension plans based on the Pri-2012 rates. The updated rates also include mortality improvement scales for valuation dates in 2023.

\(^2\) Actuaries Institute of Australia (2018) have suggested an indicative annuitant mortality table for Australian population based on relevant UK mortality rates. As recommended in the report, we apply the mortality improvement factors from ALT-2010-12 table to the indicative table to obtain cohort annuitant mortality rates of Australia.

\(^3\) Real median personal income in 2020, US Census Bureau.

\(^4\) Median salary before tax for all workers in 2021, Bureau of Statistics.

\(^5\) Median gross income in 2022, Centraal Planbureau (CPB).

\(^6\) Median gross income in 2020, Statistics Korea.

4.3 Results

4.3.1 Optimal Consumption and Investment Decisions

We begin our discussion by investigating how the optimal consumption and investment decisions are affected by tax rules. [Figure 1] illustrates the optimal median paths for \(C_t^*\) and \(f_t^*\) in the case of annuitization, and the corresponding wealth accumulation and decumulation...
paths are presented in [Figure 2]. Each figure contains three paths for optimal consumption (wealth) and investment paths, indicating the case of no taxation, the free wealth account holder, and the pension account holder. In addition to the median consumption paths, the 10th and 90th percentile values of our simulation are plotted for the two types of savings accounts. When taxes are not considered, the optimal consumption patterns are similar across the countries except for the Netherlands where there is no small drop at retirement since $\psi = 0$. Also, the optimal investment paths without taxation commonly suggest that individuals should fully invest in stocks until around the age of 40 and gradually reduce the risky position until retirement, converging to the tax-adjusted Merton fraction.

In the presence of taxes, however, the optimal patterns depend on the tax regime of each country. First, in the US case, the optimal consumption path of the pension account holder is slightly above that of the free wealth account holder. This is due to the fact that the optimal investment strategies are almost identical because of the zero long-term capital gains tax, and that the tax benefit from deferred pension income tax (10%) has marginal advantage over the labor income tax (11.44%). As a result, a US pension account holder will consume $124 more on average during the accumulation phase and $262 more on average during the decumulation phase, compared to the free wealth case. Second, in the Netherlands where the capital gains taxes are not heavily imposed, there is almost no difference in the optimal risky investment fractions with and without tax consideration similar to the US case. The consumption pattern, however, does show a difference due to the annuity cost that is only charged to a free wealth account holder who buys an annuity from the private market. This leads to an increased income of €338 and €862 during the accumulation and decumulation phase respectively for an individual with a pension account. The optimal median wealth paths of US and NL in [Figure 2] shows that the wealth path of the savings in a pension account is as much steep as that of no taxation case, while a free wealth account holder will accumulate and decumulate her wealth more slowly. This is because of the taxation structure of the EET system where pension contributions are accumulated without taxes
which allows a relatively higher accumulation of wealth before retirement, but the pension withdrawals are taxed during the decumulation phase and thus the wealth reduction is also large compared to the free wealth case. Despite the similar wealth accumulation paths, the consumption opportunities under no taxation and the pension tax rule have a significant gap as can be seen in [Figure 1].

Next, the Australian case also provides interesting findings. The most notable difference between the Australian regime compared to the others is that the pension tax rules follow the TTE system. That is, the timing of taxation is exactly the same for both types of accounts and therefore the way the taxation affects the optimal decisions is also identical. The only difference is the tax rate on labor income (17.2%) and pension contributions (15%), while a 10% capital gains tax is equally imposed on the investment returns from both accounts. Hence, we observe only a small difference in the optimal consumption and investment patterns between the two savings accounts. The figure on the top-right panel shows that the optimal risky investment paths of both types of accounts almost coincide with each other and the asset allocation should be about 1-7%p more aggressive compared to the case without taxes. The resulting wealth accumulation paths are also different from other countries in that the optimal wealth levels are similarly reduced from the path without taxation. Nevertheless, a pension account holder in Australia can additionally consume AU$241 and AU$538 on average in each year before and after retirement. Lastly, in the Korean case, the differences in the optimal paths are more pronounced. In Korea, capital gains tax rate is the highest (15.4%) among the four countries and this leads to roughly a 3-year delay of the decrease in the stock investment over time. For instance, a Korean worker who accumulates retirement wealth through a pension account would optimally retain more than half of her financial wealth in risky assets until the age of 47, but a worker with free wealth account will keep $f_t^* > 50\%$ until the age of 50. During the accumulation phase, the largest difference in $f_t^*$ between the free wealth account and the pension account is 13%p around the age of 45. Moreover, the preferential tax rate on pension income (4.4%) over the
labor income (11.28%) generates an increasing gap between the optimal consumption paths of the two accounts. This allows on average 4% higher consumption for a pension account holder over the life-cycle.

In all country cases, it is commonly observed that the optimal consumption path in the optimistic scenario (the 90th percentile) becomes flat for a pension account holder around the age of 45. This is because of the liquidity constraint, i.e. the accumulated wealth in the pension account cannot be withdrawn until retirement once the contributions are made. Hence, the consumption opportunity is binding to the after-tax labor income $Y(1 - \tau_F)$ and we see a jump at retirement. The magnitude of the jump depends on the number of years the consumption amount is capped, and the flat consumption flows are compensated by higher annuity benefits in decumulation phase compared to the free wealth case. This implies that using a pension account as a wealth accumulation vehicle has tax advantages, but at the same time the individual would face a significant liquidity problem in a good market condition which may result in about 20% reduction in consumption compared to the consumption level that would have been optimal without the liquidity constraint.
Figure 1: Optimal Paths for $C^*_t$ and $f^*_t$: Annuitization

Note: Each figure shows optimal paths for $C^*_t$ and $f^*_t$ for individual with $\gamma = 5$ in the case of annuitization. “NoTax” refers to the case without tax consideration, “PW” is the pension wealth under the pension tax treatment, and “FW” is the free wealth under the ordinary tax treatment. Also, “$m$”, “10th”, and “90th” denote $C^*_t$ at median, the 10th percentile, and the 90th percentile, respectively. The median paths for $C^*_t$ ($f^*_t$) are presented with solid (dashed) lines following the y-axis on the left (right), and the 10th and 90th percentile paths for $C^*_t$ are presented with dotted lines.
Figure 2: Optimal Median Paths for $F^*_t$ and $f^*_t$: Annuitization

*Note:* Each figure shows optimal median paths for $F^*_t$ and $f^*_t$ for individual with $\gamma = 5$ in the case of annuitization. “NoTax” refers to the case without tax consideration, “PW” is the pension wealth under the pension tax treatment, and “FW” is the free wealth under the ordinary tax treatment. $F^*_t$ $(f^*_t)$ is presented with solid (dashed) lines following the y-axis on the left (right).

We now turn to the case where the individual decides not to annuitize, presented in [Figure 3](#). In non-annuitization case, the most important factor affecting the consumption level of an individual is the tax penalty given to the lump-sum withdrawal from the pension account. As $\tau_{WB}$ is relatively higher in the US (37%) and Netherlands (49.5%), accumulating through a tax-privileged pension account leads to a lower optimal consumption path.
compared to a free wealth account and the consumption level even drops at retirement. This wealth reduction effect by the tax penalty is clearly visible in [Figure 4], suggesting that the individual should save significant amount before retirement in order to be prepared for the heavy taxation on the lump-sum withdrawal. Despite the tax penalty is set to be the highest in the Netherlands, the consumption gap between two accounts is larger in the US case due to the higher difference in the tax rate on labor income and on the lump-sum withdrawals. The optimal investment decisions are also affected by the tax penalty to the pension account. It can be seen that $f_t^*$ of the pension account holder is overall below that of no taxation or free wealth case once it starts to decrease from full risky investment. Since the marginal (dis)utility of consumption is higher in lower consumption levels, an individual should optimally choose to manage her portfolio safely so as to reduce the volatility of the future consumption path. All effects combined, a US citizen is expected to consume 6.9% lower in the accumulation phase and 13.1% lower after a consumption drop in the decumulation phase, if she does not annuitize in the pension account. Also, if a Dutch person would have been able to non-annuitize the retirement wealth, she would have had to optimally reduce her consumption by 4.4% and 8.5% before and after retirement respectively, compared to the consumption path of a free wealth account holder.

In Korea, however, using a pension account is still a dominant strategy over a free wealth account even for a non-annuitizing individual because of the low penalty on the lump-sum withdrawal (8.8%) compared to other countries. Accordingly, a consumption drop is not found at retirement and the pension account holder have more consumption opportunities over the life-cycle. The optimal investment fraction of the pension account is also not less than no taxation case during the accumulation phase, and after retirement the individual should increase the risk exposure to the tax-adjusted Merton fraction because the capital gains tax is not exempt once the wealth is withdrawn out of the pension account. Lastly, the Australian case shows the least difference compared to the optimal strategies with annuitization. As stated previously, the taxation structure is identical to both types
of savings account under the TTE system. The consumption patterns are therefore similar with the annuitization case except for the fact that there is no annuity costs and that the consumption starts to decrease around the age of 85. As in the Korean case, using a pension account provides better consumption opportunities since the tax rate on pension contribution is lower and no tax penalty is given to the lump-sum withdrawal of the retirement wealth in the pension account.
Figure 3: Optimal Paths for $C_t^*$ and $f_t^*$: No Annuitization

*Note:* Each figure shows optimal paths for $C_t^*$ and $f_t^*$ for individual with $\gamma = 5$ in the case of non-annuitization. “NoTax” refers to the case without tax consideration, “PW” is the pension wealth under the pension tax treatment, and “FW” is the free wealth under the ordinary tax treatment. Also, “m”, “10th”, and “90th” denote $C_t^*$ at median, the 10th percentile, and the 90th percentile, respectively. The median paths for $C_t^*$ ($f_t^*$) are presented with solid (dashed) lines following the y-axis on the left (right), and the 10th and 90th percentile paths for $C_t^*$ are presented with dotted lines.
Figure 4: Optimal Median Paths for $F^*_t$ and $f^*_t$: No Annuitization

Note: Each figure shows optimal median paths for $F^*_t$ and $f^*_t$ for individual with $\gamma = 5$ in the case of non-annuitization. “NoTax” refers to the case without tax consideration, “PW” is the pension wealth under the pension tax treatment, and “FW” is the free wealth under the ordinary tax treatment. $F^*_t$ ($f^*_t$) is presented with solid (dashed) lines following the y-axis on the left (right).

4.3.2 Welfare Analysis

We have shown that how the different types of taxation rules can affect financial decisions of individuals over the life-cycle. To evaluate the impact of taxes more accurately, we quantify the welfare effect by comparing the certainty equivalent welfare losses. First of all, the welfare
loss from not having access to annuities is reported in [Table 4]. Specifically, we calculate the CE of consumption paths for annuitization and non-annuitization strategies and compare the CE per category: no taxation, a free wealth account, and a pension account. Note that in Netherlands any withdrawal options other than annuitization are not allowed in the pension account, hence corresponding welfare effects must be interpreted as hypothetical ones. As we have described in the previous section, we compute four kinds of certainty equivalent utility losses: in the accumulation phase, decumulation phase, “per unit” of decumulation phase, and in the full horizon.

Table 4: Welfare Losses from Not Having Access to Annuities (Unit: %)

<table>
<thead>
<tr>
<th></th>
<th>γ = 2</th>
<th></th>
<th>γ = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No Taxation</td>
<td>Free Wealth</td>
<td>Pension</td>
</tr>
<tr>
<td>US</td>
<td>Acc.</td>
<td>2.15</td>
<td>2.15</td>
</tr>
<tr>
<td></td>
<td>Dec. per unit</td>
<td>21.10</td>
<td>21.10</td>
</tr>
<tr>
<td></td>
<td>Full horizon</td>
<td>4.39</td>
<td>4.39</td>
</tr>
<tr>
<td>AUS</td>
<td>Acc.</td>
<td>0.98</td>
<td>1.31</td>
</tr>
<tr>
<td></td>
<td>Dec.</td>
<td>5.74</td>
<td>7.44</td>
</tr>
<tr>
<td></td>
<td>Dec. per unit</td>
<td>8.74</td>
<td>11.25</td>
</tr>
<tr>
<td></td>
<td>Full horizon</td>
<td>2.02</td>
<td>2.69</td>
</tr>
<tr>
<td>NL</td>
<td>Acc.</td>
<td>2.42</td>
<td>1.55</td>
</tr>
<tr>
<td></td>
<td>Dec. per unit</td>
<td>22.19</td>
<td>14.53</td>
</tr>
<tr>
<td></td>
<td>Full horizon</td>
<td>4.93</td>
<td>3.18</td>
</tr>
<tr>
<td>KR</td>
<td>Acc.</td>
<td>2.03</td>
<td>2.58</td>
</tr>
<tr>
<td></td>
<td>Dec. per unit</td>
<td>18.37</td>
<td>21.90</td>
</tr>
<tr>
<td></td>
<td>Full horizon</td>
<td>4.15</td>
<td>5.23</td>
</tr>
</tbody>
</table>

We first look at the welfare-enhancing effect of annuitization without taxation. To make it comparable with the standard literature, we first focus on the per unit welfare effect in the decumulation phase which indicates the annuity value conditional on the retirement date with a normalized wealth level. For an individual with a modest risk aversion coefficient $(\gamma = 5)$, the welfare losses from non-annuitization are about 24-30%. This is comparable

\*\*In Australia, the welfare loss is given by 14.59%. This relatively low effect is due to the limiting age
to Mitchell et al. (1999) [48], although estimated with a fixed annuity, who have reported
the welfare gains of annuitization of around 35%. Also, a recent analysis from Peijnenburg
et al. (2015) [53] have estimated the welfare gains of 31%, consistent with our results.

When taxes are taken into account, we find that the welfare losses are more pronounced in
most cases. First of all, non-annuitization within a free wealth account entails larger welfare
losses than the case without taxation in Australia and Korea, while it generates equal or less
welfare losses in the US and Netherlands. In the US, \( \tau_{CG} = 0\% \) and therefore the income tax
equally reduces the level of consumption only, resulting in the same welfare losses from not
having access to annuities. In the Netherlands, annuitization in the private market requires
annuity costs and this makes it less attractive to annuitize in the free wealth account than in
the pension fund. In Australia and Korea, however, investors can avoid capital gains taxes
by annuitization and thus it is more appealing to purchase an annuity.

It is not surprising that a decision not to annuitize the retirement wealth in the pension
account leads to a significant welfare loss if an individual faces a high tax rate on the
lump-sum withdrawals. For instance, the welfare losses are estimated to be almost doubled
in the US and Netherlands. This suggests that a strict tax penalty given to the lump-sum
withdrawal of the accumulated pension contributions at retirement might be an effective tool
to incentivize investors to annuitize, particularly for those who seek to make tax-optimized
decisions. In contrast, the welfare losses remain similar if the tax penalty is small as shown in
the Korean cases. Furthermore, the difference in the welfare effects of annuitization remains
negligible in the Australian cases since the consumption opportunities do not vary much
under the TTE regime, as discussed previously. For a less risk-averse individual with \( \gamma = 2\),
annuities still generates substantial value over the life-cycle. The present results in [Table 4]
provide a strong support for the claim that ignoring taxation leads to a huge distortion in
welfare implications, which constrasts to the findings of Fischer et al. (2013) [21].

We also report the welfare losses from not using pension account in the case of both
in the Australian mortality table which is about 10 years less than other tables, making annuitization less
attractive.
annuitization and non-annuitization in [Table 5]. To evaluate the welfare effects, we first compute the CEs derived from the optimal consumption paths of annuitization and non-annuitization for both types of savings accounts. We then obtain the certainty equivalent welfare losses by comparing the CEs of the same strategy but using different types of accounts. Thus, this analysis allows us to judge whether the pension tax rule in each country is appropriately designed to achieve its policy goal by inducing people to annuitize through the pension account and keeping people from using the pension account only as a savings vehicle. Here, we do not report the per unit welfare effects in the decumulation phase since the income tax is only given to the wealth in the pension account and hence it does not give a fair comparison.

Table 5: Welfare Losses From Not Using Pension Account (Unit: %)

<table>
<thead>
<tr>
<th></th>
<th>γ = 2</th>
<th></th>
<th>γ = 5</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Annuitization</td>
<td>No Annuitization</td>
<td>Annuitization</td>
<td>No Annuitization</td>
</tr>
<tr>
<td>US</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Acc.</td>
<td>-2.10</td>
<td>-5.42</td>
<td>0.06</td>
<td>-7.81</td>
</tr>
<tr>
<td>Dec.</td>
<td>3.60</td>
<td>-22.58</td>
<td>0.78</td>
<td>-16.77</td>
</tr>
<tr>
<td>Full horizon</td>
<td>-0.96</td>
<td>-8.55</td>
<td>0.20</td>
<td>-9.91</td>
</tr>
<tr>
<td>AUS</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Acc.</td>
<td>-1.21</td>
<td>-1.13</td>
<td>0.43</td>
<td>0.53</td>
</tr>
<tr>
<td>Dec.</td>
<td>3.24</td>
<td>3.11</td>
<td>1.20</td>
<td>1.23</td>
</tr>
<tr>
<td>Full horizon</td>
<td>-0.15</td>
<td>-0.07</td>
<td>0.60</td>
<td>0.70</td>
</tr>
<tr>
<td>NL</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Acc.</td>
<td>-1.02</td>
<td>-4.05</td>
<td>1.48</td>
<td>-4.89</td>
</tr>
<tr>
<td>Dec.</td>
<td>7.84</td>
<td>-13.54</td>
<td>4.10</td>
<td>-10.37</td>
</tr>
<tr>
<td>Full horizon</td>
<td>0.81</td>
<td>-5.85</td>
<td>1.98</td>
<td>-6.17</td>
</tr>
<tr>
<td>KR</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Acc.</td>
<td>0.61</td>
<td>0.58</td>
<td>2.69</td>
<td>2.55</td>
</tr>
<tr>
<td>Dec.</td>
<td>11.45</td>
<td>8.89</td>
<td>5.74</td>
<td>4.53</td>
</tr>
<tr>
<td>Full horizon</td>
<td>2.97</td>
<td>2.60</td>
<td>3.39</td>
<td>3.16</td>
</tr>
</tbody>
</table>

As we have seen from the figures in the previous section, using a pension account has a disadvantage from the liquidity constraint in the accumulation phase and this is partially compensated by higher annuity benefits after retirement. Combined effects seem to be not very significant in the US and Australia as the welfare losses from annuitization through a free wealth account instead of a pension account are less than 1% in the full horizon for individuals
with $\gamma = 5$. In the Netherlands, annuitization through a pension account has a merit of about 2% since no annuity costs are charged, compared to annuitization via a free wealth account. The Korean tax regime turns out to be the most favorable to annuitization through a pension account as it provides a tax exempt for capital gains in the accumulation phase and a low pension income tax (4.4%) in the decumulation phase. Ironically, Korean investors can earn comparable amount of welfare gains by using the pension account without annuitization, due to the low tax penalty on the lump-sum withdrawal at retirement. This explains why the voluntary annuity market is thin and people often use the tax-privileged pension account as a savings vehicle (Lee, 2013 [40]). On the other hand, our analysis confirms that the tax penalty effectively discourage people from withdraw their money from the pension account without annuitization. If a person decides not to annuitize, using a free wealth account gives a welfare gain of 9.91% and 6.17% in the US and Netherlands respectively, compared to the non-annuitization with a pension account. Lastly, the welfare effects of the Australian case consistently shows that the structure of the TTE system, without any other consideration, is not very effective in inciting people to a certain direction.

5 Conclusion

In this paper, we study how the pension tax treatments of different regimes affect optimal decisions and welfare implication of individuals under the life-cycle framework. We find that optimal consumption and investment choice significantly alters when individuals take into account the taxation rules. Depending on tax rate parameters and taxation structure, we show that the merit of using a pension account as a vehicle for annuitization can be higher or lower than using a free wealth account even under the same pension tax treatment. Our results suggest that financial advisors should be very careful about the tax rules applied to their clients which would potentially lead to severe welfare losses.

There are several limitations of our study. First of all, we do not consider time-varying
income profile of workers. Second, several factors that would affect decision to annuitize such as bequest motive, medical costs, family or marriage status, etc. are not incorporated in our model. Third, we do not test different mortality groups. Fourth, we do not consider other pension regulations such as limits on risky investment, maximum withdrawal amount per year, etc. All these features are important factors that would affect optimal decisions significantly. Further research is needed to better understand the complex association between taxation and optimal decisions over the life-cycle.
References


[50] OECD. (2021), “Annual Survey of Investment Regulation of Pension Funds and Other Pension Providers.” OECD.


Tables
Figures
Appendix A. Numerical Technique Used in the Accumulation Phase

In this appendix we describe the numerical method that we use to determine the optimal consumption and investment strategy in the accumulation phase, for a free wealth account holder and a pension account holder, respectively.

For any (future) time \( t = 0, 1, \ldots, T - 1 \), the indirect utility \( J(W,Y,t) \) is defined as:

\[
J(W,Y,t) = \sup_{C_s, f_s} \mathbb{E}_{W,t} \left[ \sum_{s=t}^{T-1} e^{-\beta(s-t)} \cdot s_{-t} p_t \cdot u(C_s) + e^{-\beta(T-t)} \cdot t_{-t} p_t \cdot \bar{u}(W_T) \right],
\]

subject to the constraints in (9)-(14) in case of a free wealth account, and subject to (11)-(12) and (17)-(19) in case of a pension wealth account. The end value is given by:

\[
J(W,Y,T) = \bar{u}(W) = W^{1-\gamma} \cdot EU^*_{decum}(1),
\]

where \( EU^*_{decum}(1) \) is the discounted expected utility of consumption in the decumulation phase, given \( W_T = 1 \). The expression for \( EU^*_{decum}(\cdot) \) in case of annuitization and in case of no annuitization is derived in Appendix B.

We define the following normalized variables:

\[
c_t \equiv C_t / W_t, \quad (A.3)
\]
\[
y_t \equiv Y_t / W_t. \quad (A.4)
\]

Then, the homothetic property of the utility function and the linearity of the constraints implies that:

\[
J(W_t,Y_t,t) = W_t^{1-\gamma} \cdot J(1, y_t, t), \quad (A.5)
\]

for all \( t = 0, 1, \ldots, T \). Now let the random variable \( h(f_t, c_t, y_t) \) be defined as the wealth \( W_{t+1} \) as a function of the decision variables \( f_t \) and \( c_t \), and the state variable \( y_t \), conditional on \( W_t = 1 \) i.e.,

\[
h(f_t, c_t, y_t) = \begin{cases} 
    e^{R_c(f_t)} \cdot (1 - c_t) + (1 - \tau_F) y_t, & \text{for a free wealth account,} \\
    e^{R_c(f_t)} \cdot \left( 1 - \frac{1 - \tau_{PC}}{1 - \tau_F} \right) c_t + (1 - \tau_{PC}) y_t, & \text{for a pension account.}
\end{cases}
\]

(A.6)

Then, the normalized indirect utility, which is defined as \( j(y_t, t) := J(1, y_t, t) \) for \( t =
0, 1, . . . , T, satisfies:

\[ j(y_t, t) = \sup_{c_t, f_t} \left\{ U(c_t) + pt \cdot e^{-\beta} \cdot \mathbb{E}_t \left[ h(f_t, c_t, y_t)^{1-\gamma} \cdot j(y_{t+1}, t+1) \right] \right\}, \tag{A.7} \]

subject to

\begin{align*}
  y_0 &= 1/(1 - \tau), \tag{A.8} \\
  y_{t+1} &= y_t / h(f_t, c_t, y_t), \quad \text{for } t = 0, \ldots, T - 2 \tag{A.9} \\
  y_T &= 0, \tag{A.10} \\
  0 &\leq c_t \leq \bar{c}_t, \tag{A.11}
\end{align*}

where \( \tau = \tau_F \) and \( \bar{c}_t = 1 \) for a free wealth account pension wealth account, and \( \tau = \tau_{PC} \) and \( \bar{c}_t = (1 - \tau_F)y_t \) for a pension wealth account. For both types of accounts, the end value is given by:

\[ j(y_T, T) := \bar{u}(1) = EU_{decum}^*(1), \tag{A.12} \]

where \( EU_{decum}^*(\cdot) \) is given in (B.11).

We determine the optimal consumption and investment strategies by numerically solving the Bellman equations in (A.7)-(A.12). It follows from (A.6) and (A.9) that the state variable \( y_t \) is restricted to the interval \([0, 1]\) in case of a free wealth account holder and to \([0, 1/(1 - \tau_F)]\) in case of a pension account holder. We discretize these intervals using a grid with step size 0.001. To compute the date-\( t \) expectation of \( h(f_t, c_t, y_t)^{1-\gamma} j(y_{t+1}, t+1) \) for given values of \( f_t, c_t, y_t \), we use Gauss-Hermite quadrature with a 5-point quadrature, combined with cubic spline interpolation to next period’s value function \( j(y_{t+1}, t+1) \) for values of \( y_{t+1} \) that are not on the grid. The optimal consumption and investment pattern is given by \( (C_t^*, f_t^*) = (c_t^* \cdot W_t, f_t^*) \).

**Appendix B. Optimal Consumption in the Decumulation Phase**

In this appendix, we first derive the optimal consumption pattern in the decumulation phase with and without annuitization, for any given value of wealth \( W_T \) at retirement date. Next, we determine closed form expressions for the expected present value of the optimal consumption pattern in the decumulation phase in both cases.

Throughout this appendix, we let the parameters \( \alpha_t \) and \( \beta_t \) for \( t = T, T + 1, \ldots, D \) be as defined in (33) and (34), and let \( \tilde{\eta} \) be as defined in (35). Moreover, we let the parameters \( \xi \),
and $\eta$ be given by:

$$\xi = \begin{cases} 1, & \text{in case of no annuitization,} \\ 1 - \tau_{AB}, & \text{in case of annuitization,} \end{cases} \quad (B.1)$$

and

$$\eta = \begin{cases} 1 - \tau_{WB}, & \text{in case of no annuitization,} \\ 1/(1 - \Psi), & \text{in case of annuitization.} \end{cases} \quad (B.2)$$

Note that this implies that $\tilde{\eta}$ defined in (35) equals $\xi \eta$.

**Proposition 1.** The optimal consumption levels in the decumulation phase are given by:

$$C^*_t = \xi \left( \prod_{h=T}^{t-1} e^{R_h(f^*)} \right) \cdot W_T^*(t), \quad (B.3)$$

for $t = T, T+1, \ldots, D$, where $W_T^*(t)$ are given by:

$$W_T^*(t) = \eta W_T \cdot \left( \frac{\alpha_t}{\beta_t} \right)^{\frac{1}{\gamma}} / \left( \sum_{k=T}^{D} \left( \frac{\alpha_k}{\beta_k} \right)^{\frac{1}{\gamma}} \cdot \beta_k \right). \quad (B.4)$$

**Proof.** For any given value of the bucket $W_T(t)$ for $t = T, T+1, \ldots, D$, the consumption in period $t$ if a fraction $f^*$ is invested in the risky asset is given equals $C_t = \xi \left( \prod_{h=T}^{t-1} e^{R_h(f^*)} \right) \cdot W_T(t)$. Hence, the date-$T$ discounted expected utility of this consumption pattern equals:

$$EU_{decum} := \sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t^{-\beta_T} \cdot P_T \cdot \mathbb{E}_T \left[ \left( \frac{(C_t)^{1-\gamma}}{1-\gamma} \right) \right],$$

$$= \frac{1}{1-\gamma} \cdot \sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t^{-\beta_T} \cdot P_T \cdot \mathbb{E}_T \left[ \left( \xi \left( \prod_{h=T}^{t-1} e^{R_h(f^*)} \right) W_T(t) \right)^{1-\gamma} \right],$$

$$= \frac{1}{1-\gamma} \cdot \xi^{1-\gamma} \cdot \sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t^{-\beta_T} \cdot P_T \cdot b^*(t) \cdot W_T(t)^{1-\gamma},$$

$$= \frac{1}{1-\gamma} \cdot \xi^{1-\gamma} \cdot \left( \sum_{t=T}^{D} \alpha_t \cdot W_T(t)^{1-\gamma} \right). \quad (B.5)$$

Therefore, it follows from constraint (23) in case of no annuitization and constraint (28) in case of annuitization that the optimization problem for the optimal values of the buckets
\( W_T(t) \) in the decumulation phase is given by:

\[
\max_{W_T(t)} \left\{ \sum_{t=T}^{D} \alpha_t \cdot W_T(t)^{1-\gamma} \right\} \tag{B.6}
\]

s.t. \( \sum_{t=T}^{D} \beta_t \cdot W_T(t) = \eta \cdot W_T. \tag{B.7} \]

The first order conditions for (B.6)-(B.7) are:

\[
(1 - \gamma) \cdot \alpha_t \cdot W_T(t)^{-\gamma} - \zeta \cdot \beta_t = 0, \quad \text{for all } t = T, T + 1, \ldots, D, \tag{B.8}
\]

\[
\zeta \cdot \left( \sum_{t=T}^{D} \beta_t \cdot W_T(t) - \eta \cdot W_T \right) = 0, \tag{B.9}
\]

where \( \zeta \) is the Lagrange multiplier. Solving (B.8)-(B.9) yields (B.4).

The next proposition presents expressions for the discounted expected utility of the optimal consumption path in decumulation with and without annuitization, as a function of the available wealth at retirement date, which is defined as:

\[
EU_{\text{decum}}^*(W_T) := \sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t^{-TP_T} \cdot \mathbb{E}_T \left[ \left( \frac{(C_t^*)^{1-\gamma}}{1-\gamma} \right) \right], \tag{B.10}
\]

where \( C_t^* \) for \( t = T, T + 1, \ldots, D \) is given by Proposition 1.

**Proposition 2** For any given level of wealth \( W_T \) on retirement date, the date-\( T \) discounted expected utility of the optimal consumption pattern in the decumulation phase with or without annuitization equals:

\[
EU_{\text{decum}}^*(W_T) = \frac{1}{1 - \gamma} \cdot (\bar{\eta} W_T)^{1-\gamma} \cdot \left( \sum_{k=T}^{D} \left( \frac{\alpha_k}{\beta_k} \right)^{\frac{1}{\gamma}} \cdot \beta_k \right)^{\gamma}. \tag{B.11}
\]

**Proof.** It follows from (B.5) that for any given level of wealth \( W_T \) on retirement date, the date-\( T \) discounted expected utility of the optimal consumption pattern equals:

\[
EU_{\text{decum}}^*(W_T) = \frac{1}{1 - \gamma} \cdot \xi^{1-\gamma} \left( \sum_{t=T}^{D} \alpha_t \cdot W_T^*(t)^{1-\gamma} \right), \tag{B.12}
\]

where the buckets \( W_T^*(t) \) for \( t = T, T + 1, \ldots, D \) are given by (B.4). Substituting (B.4) in
(B.12) yields

\[
EU^\ast_{\text{decum}}(W_T) = \frac{1}{1-\gamma} \cdot (\xi \eta W_T)^{1-\gamma} \cdot \sum_{t=T}^D \alpha_t \cdot \left( \frac{\alpha_t}{\beta_t} \right)^{1-\gamma} \div \left( \sum_{k=T}^D \left( \frac{\alpha_k}{\beta_k} \right)^{1-\gamma} \cdot \beta_k \right)^{1-\gamma},
\]

\[
= \frac{1}{1-\gamma} \cdot (\tilde{\eta} W_T)^{1-\gamma} \cdot \left( \sum_{k=T}^D \left( \frac{\alpha_k}{\beta_k} \right)^{1-\gamma} \cdot \beta_k \right)^{\gamma}. \tag{B.13}
\]

We conclude with a derivation of the value of \(b^\ast(t)\).

**Proposition 3** It holds that

\[
b^\ast(t) = \exp \left\{ \left( \tilde{\tau} + \frac{\tilde{\lambda}^2}{2\gamma} \right) (1-\gamma)(t-T) \right\}, \quad \text{for all } t \in \{T, T+1, \ldots, D\}. \tag{B.14}
\]

**Proof.** Let \( t \in \{T, T+1, \ldots, D\} \) be given. Substituting the value of \( f^\ast \) from (24) in (15) yields

\[
R_h(f^\ast) = \tilde{\tau} + f^\ast \tilde{\sigma} \tilde{\lambda} - \frac{1}{2} (f^\ast)^2 \tilde{\sigma}^2 + f \tilde{\sigma} Z_h = \tilde{\tau} + \frac{\tilde{\lambda}^2}{2\gamma^2} (2\gamma - 1) + \frac{\tilde{\lambda}}{\gamma} Z_h, \tag{B.15}
\]

with \( Z_h \sim \text{i.i.d. } N(0,1) \) for \( h = T, T+1, \ldots, t-1 \). Now let \( Z = (1-\gamma) \sum_{h=T}^{t-1} Z_h \). Then, it holds that:

\[
b^\ast(t) = \mathbb{E}_T \left[ \left( \prod_{h=T}^{t-1} e^{R_h(f^\ast)} \right)^{1-\gamma} \right]
\]

\[
= \exp \left\{ \left( \tilde{\tau} + \frac{\tilde{\lambda}^2}{2\gamma^2} (2\gamma - 1) \right) (1-\gamma)(t-T) \right\} \mathbb{E}_T [\exp (Z)]
\]

\[
= \exp \left\{ \left( \tilde{\tau} + \frac{\tilde{\lambda}^2}{2\gamma^2} (2\gamma - 1) \right) (1-\gamma)(t-T) \right\} \exp \left\{ \left( \frac{\tilde{\lambda}^2}{2\gamma^2} \right) (1-\gamma)^2(t-T) \right\}
\]

\[
= \exp \left\{ \left( \tilde{\tau} + \frac{\tilde{\lambda}^2}{2\gamma} \right) (1-\gamma)(t-T) \right\}.
\]

The third equality follows from the fact that \( Z \sim N(0,\frac{\tilde{\lambda}}{\gamma} (1-\gamma) \sqrt{t-T}) \).  ■
Appendix C. Certainty Equivalents

Recall that for the accumulation phase, the dynamic optimization procedure yields $f^*_t$, and $C^*_t$ as a function of the state variable $y_t$, which takes values on the grid $[0,0.001,1]$ (see Appendix A). In contrast, the optimal investment fraction as well as the optimal consumption patterns in the decumulation phase as a function of the available wealth at retirement age (i.e., date $T$) are available in closed form expression (see Appendix B). To determine the certainty equivalents in the accumulation phase and in the two phases combined, we use the following algorithm, with number of scenarios equal to $S = 100,000$ and $Y = 1$.

1. Set $W_0 = (1 - \tau_F)Y$ in case of a free wealth account holder and $W_0 = (1 - \tau_{PC})Y$ in case of a pension account holder, and let $y_0 = Y/W_0$.

2. Generate $S$ scenarios for i.i.d. returns in periods $[t, t + 1)$, for $t = 0, 1, 2, \ldots, D - 1$.

3. (Consumption in Accumulation phase):
   For each scenario $s = 1, 2, \ldots, S$, let $f^*_0(s) = f^*_0$ and $C^*_0(s) = c^*_0 \cdot W_0$.

   For $t = 0, 1, 2, \ldots, T - 2$, repeat the following steps:

   (a) Determine in each scenario $s = 1, 2, \ldots, S$ the corresponding wealth $W^*_t(s)$ on date $t + 1$, using the wealth dynamics in (10) in case of a free wealth account and in (18) in case of a pension account.

   (b) Determine in each scenario $s = 1, 2, \ldots, S$ the corresponding value of the state variable $y^*_t(s) = Y/W^*_t(s)$, and the corresponding values of $f^*_t(s)$ and $c^*_t(s)$. In case $y^*_t(s)$ is not an element of the grid, we use cubic spline interpolation to determine the corresponding values of $f^*_t(s)$ and $c^*_t(s)$. We set $C^*_t(s) = c^*_t(s) \cdot W^*_t(s)$.

   For $t = T - 1$: determine $W^*_T(s)$ in each scenario $s = 1, 2, \ldots, S$, using the wealth dynamics in (10) with $Y_T = 0$ in case of a free wealth account and using the wealth dynamics in (18) with $Y_T = 0$ in case of a pension account.

4. (Consumption in decumulation phase): for each scenario $s = 1, \ldots, S$, determine $C^*_t(s)$ using (22) with $W^*_T = W^*_T(s)$, combined with (25) in case of a free wealth account holder, and (27) with $W^*_T = W^*_T(s)$ combined with (29) in case of a pension account holder.

5. (Certainty equivalents): Determine discounted expected utility as the average value of $\sum_{t=0}^{T-1} \left\{ e^{-\beta t} \cdot t_{p0} \cdot U(C^*_t(s)) \right\}$ and of $\sum_{t=0}^{D-1} \left\{ e^{-\beta t} \cdot t_{p0} \cdot U(C^*_t(s)) \right\}$ over the $S$ scenarios, and the take the inverse utility of these values.
For the certainty equivalent utility of the optimal strategies in the decumulation phase, there are closed form expressions. Specifically, for any given level of wealth $W_T$ at retirement date, let $[\mathbf{C}^*] := (C^*_T, C^*_{T+1}, \ldots, C^*_T)$ be the optimal consumption pattern in the decumulation phase, as characterized by (B.3) and (B.4). The certainty equivalent utility of the decumulation phase on retirement date equals:

$$X = U^{-1}(EU^*_{decum}(W_T)), \quad \text{on retirement date},$$

$$= \left(1 - \gamma\right)(EU^*_{decum}(W_T))^{\frac{1}{1-\gamma}},$$

$$= \xi \eta W_T \left(\sum_{t=T}^{D} \beta_t \cdot \left(\frac{\alpha_t}{\beta_t}\right)^{\frac{1}{\gamma}}\right)^{\frac{\gamma}{1-\gamma}}.$$ 

The last equality follows from (B.11). Hence, the certainty equivalent on retirement date per unit of wealth available on date $T$ equals:

$$CE_{decum,T}[\mathbf{C}^*] = \xi \eta \left(\sum_{t=T}^{D} \beta_t \cdot \left(\frac{\alpha_t}{\beta_t}\right)^{\frac{1}{\gamma}}\right)^{\frac{\gamma}{1-\gamma}}. \quad \text{(C.1)}$$

The certainty equivalent of the decumulation phase on date zero equals:

$$CE_{decum,0}[\mathbf{C}^*] = U^{-1} \left(\sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t^{-TPT} \cdot \mathbb{E}_0 \left[\left(C^*_t\right)^{1-\gamma}\right]\right),$$

$$= U^{-1} \left(\frac{1}{1-\gamma} \cdot \sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t^{-TPT} \cdot \mathbb{E}_0 \left[\xi^{1-\gamma} \cdot \left(\prod_{h=T}^{t-1} e^{R_h(f^*)}\right)^{1-\gamma} \cdot W^*_T(t)^{1-\gamma}\right]\right),$$

$$= U^{-1} \left(\frac{1}{1-\gamma} \cdot \xi^{1-\gamma} \cdot \sum_{t=T}^{D} e^{-\beta(t-T)} \cdot t^{-TPT} \cdot \mathbb{E}_0 \left[\left(\prod_{h=T}^{t-1} e^{R_h(f^*)}\right)^{1-\gamma} \cdot \mathbb{E}_0 \left[W^*_T(t)^{1-\gamma}\right]\right]\right),$$

$$= U^{-1} \left(\frac{1}{1-\gamma} \cdot \xi^{1-\gamma} \cdot \sum_{t=T}^{D} \alpha_t \cdot \mathbb{E}_0 \left[W^*_T(t)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}. \quad \text{(C.2)}$$

Recall that returns are independent of each other. In the third equality we use that this implies that $\prod_{h=T}^{t-1} e^{R_h(f^*)}$ and $W^*_T(t)$ are independent of each other. In the fourth equality we use that it implies that $\mathbb{E}_0\left[\left(\prod_{h=T}^{t-1} e^{R_h(f^*)}\right)^{1-\gamma}\right] = \mathbb{E}_T\left[\left(\prod_{h=T}^{t-1} e^{R_h(f^*)}\right)^{1-\gamma}\right] = b^*(t).$
Substituting (B.4) in (C.2) yields

\[ CE_{\text{decum},0}[C^*] = \xi \eta \mathbb{E}_0 \left[ (W_T^*)^{1-\gamma} \right] \frac{1}{\frac{1}{\gamma}} \cdot \left[ \sum_{t=T}^{D} \alpha_t \cdot \left( \frac{\alpha_t}{\beta_t} \right)^{\frac{1-\gamma}{\gamma}} \right] \frac{1}{\frac{1}{\gamma}} / \left( \sum_{k=T}^{D} \left( \frac{\alpha_k}{\beta_k} \right)^{\frac{1}{\gamma}} \cdot \beta_k \right), \]

\[ = \xi \eta \mathbb{E}_0 \left[ (W_T^*)^{1-\gamma} \right] \frac{1}{\frac{1}{\gamma}} \cdot \left[ \sum_{t=T}^{D} \beta_t \cdot \left( \frac{\alpha_t}{\beta_t} \right)^{\frac{1}{\gamma}} \right] \frac{1}{\frac{1}{\gamma}} / \left( \sum_{k=T}^{D} \left( \frac{\alpha_k}{\beta_k} \right)^{\frac{1}{\gamma}} \cdot \beta_k \right), \]

\[ = \xi \eta \cdot \mathbb{E}_0 \left[ (W_T^*)^{1-\gamma} \right] \frac{1}{\frac{1}{\gamma}} \cdot \left( \sum_{t=T}^{D} \beta_t \cdot \left( \frac{\alpha_t}{\beta_t} \right)^{\frac{1}{\gamma}} \right) \frac{1}{\frac{1}{\gamma}}. \quad (C.3) \]