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Assessing the solvency risk of insurance portfolios via a continuous time cohort model*

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Abstract

The paper describes a model that evaluates the solvency of a portfolio of assets and liabilities of an insurer subject to longevity risk and financial risks. Liabilities are evaluated at fair-value. Interest-rate risk can affect both assets and liabilities. Longevity risk is described via a continuous-time cohort model. We evaluate the impact of different investment and hedging strategies on the characteristics of the funding ratio of run-off portfolios at different time horizons. Numerical simulations, calibrated to UK historical data, show that systematic longevity risk is particularly important in the long-run and needs to be hedged. We highlight that portfolio size, investment choices and solvency requirements are deeply interconnected. Natural hedging techniques can effectively reduce the required solvency buffer when interest-rate risk is perfectly hedged.

JEL Classification: G22, G32.

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1 Introduction

The assets and liabilities owned by an insurance company or pension fund are subject to various sources of uncertainty, making the assessment of their solvency risk a challenging task. Regulators – through the Solvency II Directive – are trying to steer insurance companies towards a comprehensive accounting of the risks affecting their portfolios. This increasing attention to the soundness of risk management practices, especially in the context of the Own Risk Solvency Assessment (ORSA) process, is enhancing the level of complexity of required valuation models. Proper assessment of the solvency risk of a portfolio requires indeed the modelling of many risk sources. As a consequence, computational effort can become an issue. In this paper, we propose an ALM modelling framework which assesses the solvency risk of a run-off portfolio taking into account various sources of risk and which at the same time limits computational time, since closed form formulas for the market value of liabilities are available. We focus on the funding ratio, its variability and its sensitivity to relevant determinants, having taken into account the combined effects of investment risk, interest-rate risk, idiosyncratic as well as systematic longevity risk.

The need for the evaluation of both longevity and interest-rate risk together has become urgent, since regulators in the Solvency II framework and the recent accounting rules (IASB) stressed the importance of a market fair-valuation of liabilities in particular, which entails interest-rate risk assessment from a risk management perspective. Together with the well known randomness in the deaths of the policyholders in the portfolio (idiosyncratic risk), the recent population ageing phenomenon has clearly highlighted the exposure of annuity providers and life insurers to the uncertainty in mortality rates themselves (systematic risk). Longevity risk, in both these dimensions, can represent a relevant threat to the solvency of annuity providers and the hedging of its undiversifiable component has been recently investigated in the literature.

Given these considerations, we require liabilities to be evaluated at fair-value. Thus, both interest-rate risk and longevity risk affect them. We couple a standard description of the financial market by means of the well-known Vasicek (1977) model with a parsimonious description of mortality risk via a continuous-time cohort based stochastic model, following Luciano and Vigna (2008). This choice, in addition to being reasonably accurate in describing the evolution of mortality and interest rates, allows us to obtain the fair-value

reserves of liabilities and their sensitivities (Greeks) to relevant risk factors in closed form. This is the most important feature of our model, which permits us to reduce computational effort and to handle the simulation of large portfolios with heterogeneous products per type and cohort.
We provide a numerical evaluation of the one-year solvency probability of a portfolio of policies. Also, within the ORSA framework, we consider multi-period solvency (see Olivieri and Pitacco (2002)). We focus on annuity portfolios, thus representing the typical situation of a pension fund in the decumulation phase. Our analysis complements Delong et al. (2008), who explored – analytically – the optimal investment problem for a fund in the accumulation phase in a similar framework.
We are not the first to propose an ALM model with the aim of evaluating the solvency of insurers and annuity providers. Hari et al. (2011), in particular, focused on the characteristics of the funding ratio, as we do. Apart from selecting a different mortality modelling strategy, we extend their analysis by introducing interest-rate risk uncertainty and by accounting for the presence of life insurance policies on the liabilities side. This allows us to assess the importance of longevity risk relative to financial ones and to explore the effectiveness of natural hedging strategies. This analysis is relevant to insurers and annuity providers, as proper liability porftolio composition might be the most viable option to mitigate systematic longevity risk, given the absence of a proper market of hedging instruments.[1] Natural hedging, i.e. the offsetting between life and death benefits, has gained attention as a risk management tool (Gatzert and Wesker (2013), Stevens et al. (2011)) and its effectiveness is under scrutiny. It requires adjustments both on the laibilities side and on the asset side, as its implementation alters the risk exposures to both longevity and interest-rate risk. We assess the impact of the liability mix on solvency and bankruptcy likelihood. Gerstner et al. (2008) described a balance sheet model of the insurance company which did not include systematic mortality, but was aimed at evaluating the impact of participating policies. Stevens et al. (2011) focused on the interactions between financial and longevity risks. We complement their analysis, by providing a risk-neutral based valuation of liabilities and disentangling short-term and long-term effects.

[1] Moreover, reinsurance over-the-counter deals, which have been recently seen, might be subject to adverse selection problems (Biffis and Blake (Biffis and Blake)) or may be prove to be inefficient (Luciano and Regis (2014)).

Our numerical simulations, based on UK-calibrated mortality and interest rates, allow us to explore the impact of investment strategies, market conditions and the liability portfolio mix on the solvency of an annuity provider. First of all, they document the long-run impact of systematic longevity risk on annuity portfolios. Second, they highlight the interdependence between safety loadings, liability portfolio composition and investment choices. Third, they allow us to analyze the effects of natural hedging strategies on the funding ratio distribution and to determine the consequences for solvency.
The paper is organized as follows. Section 2 sets up our framework and describes our modelling of the risk sources. Section 3 describes our solvency measures. Section 4 presents numerical results from our simulations based on a calibrated example. Section 5 concludes.

2 Set up

In this paper, we describe the evolution of the portfolio of a life-insurer in run-off and provide an assessment of its solvency when uncertainty regarding both demographic and financial aspects is present. Liabilities are evaluated at fair-value, following a market-based approach which is consistent with recent regulatory prescriptions. We model three sources of uncertainty affecting the liability portfolio:

1. **interest-rate risk**, due to the stochastic fluctuations of the short rate;
2. **idiosyncratic longevity risk**, due to the uncertainty in the death arrival times of the individuals;
3. **systematic longevity risk**, due to the unexpected changes in the mortality intensity of the pool of policyholders.

We will assume that individuals are homogeneous, thus belong to the same generation. The premiums received by policyholders through the sale of this policies are invested in two types of assets: a stock and/or (risky or risk-less) bonds. The asset side is then affected by **financial risk**, in the form of interest-rate risk form bond investment and of **equity risk** from stock. While in principle we can have dependence between financial risks and longevity risk we will assume them to be independent.

We simulate the processes that describe these risks by discretizing them at intervals of time-length Δ such that $t_i = t_0 + i\Delta,\ i \in \mathbb{I} = \{1, 2, \ldots, N_0 \in \mathbb{N}\}$. In this section, we define more accurately our asset/liability model.

2.1 Liabilities

The liability portfolio of the insurer is composed of standard insurance policies issued - possibly - on different cohorts. The policy types are annuities (A) and temporary death contracts (D). Annuities are whole-life immediate[2], while death contracts can have different maturities. For each contract we specify the generation x to which the policyholder belongs, defined by its year of birth. Separately, for annuity contracts we specify the installment R. For death contracts, on the other hand, we specify the maturity Q and the insured amount C of the death contract.

Hence, we identify any homogeneous annuity as $A(x, R)$ and any death insurance contract by $D(x, Q, C)$. In the remainder of this section we describe the two sources of risk which affect liabilities, when they are evaluated at fair value: longevity risk and interest-rate risk.

From now on, for simplicity, let us consider a portfolio in which one type of annuity is present. To this portfolio one type of a death contract may be added. Extension to the more general case is straightforward.

Now, we denote by

$$T_A^O(x, R) = \{t_j\}, j \in J_A^O \subset \mathbb{I}$$

the set of known payment dates for the annuity $A(x, R)$ and, likewise

$$T_D^O(x, Q, C) = \{t_j\}, j \in J_D^O \subset \mathbb{I}$$

the set of known payment dates for the death contract $D(x, Q, C)$. [3]

2.1.1 Longevity risk

We model longevity risk following a well-established stream of literature, by providing a continuous-time cohort-based description of mortality. The event

[2] Extension to the case of deferred annuities is straightforward.

[3] Analogously, we define by $T_A^I(x, R) = \{t_j\}, j \in J_A^I \subset \mathbb{I}$ the set of dates at which premiums E_A are paid for the annuity $A(x, R)$ and $T_D^I(x, Q, C) = \{t_j\}, j \in J_D^I \subset \mathbb{I}$ the set of dates at which premiums E_D for the death contract $D(x, Q, C)$ are paid.

of death of an individual is modelled as the first jump time of a Poisson process with stochastic intensity. This intensity is generation-dependent. Generation x mortality intensity follows a purely-diffusive Ornstein-Uhlenbeck (OU) process

$$d\lambda_x(t) = a_x \lambda_x(t)dt + \sigma_x dW_x^{\mathbb{P}}(t), \tag{1}$$

where $a_x > 0$ and $\sigma_x \geq 0$. Indeed, the intensity can theoretically become negative, but in practical applications we make sure that the probability of that event is negligible. Our modeling choice is motivated by the many attractive features of this process

- it is parsimonious, since it requires the estimation of two parameters per generation only,
- it fits observed mortality data well and especially for older ages,
- it is a stochastic generalization of the Gompertz law, since its expected value is exponentially increasing with time,
- not only it permits to compute survival probabilities and sensitivities to mortality forecast error in closed and simple form but it also couples well with standard continuous-time models for interest rates Luciano et al. (2012a)
- it can be extended to a multiple-factor version (see Jevtić et al. (2013)) which may improve its fit and predictive capability.

In order to price insurance products following standard risk-neutral valuation, we introduce an equivalent martingale measure $\mathbb{Q}$. We assume its existence and we refer the reader to Dahl and Møller (2006) for details about existence and the properties of the involved change of measure. For simplicity, we make the assumption of absence of a risk premium for mortality, extensions to all constant risk premiums being straightforward[4] As a consequence, survival probabilities under the risk-neutral measure coincide with those under the historical one.

[4]This assumption is justified by the impossibility of calibrating the market price of risk for mortality, while a liquid market for insurance derivatives has not developed yet.

Selecting an affine process for mortality intensity allows to have a closed-form expression for survival probabilities

$$S_x(t,T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T \lambda(u)du}|\mathcal{F}_t] = e^{\alpha(T-t)+\beta(T-t)\lambda_x(t)},$$

$\mathcal{F}_t$ being the filtration generated by $W_x^{\mathbb{P}}(t)$. The functions $\alpha(t)$ and $\beta(t)$ solve a system of Riccati differential equations (see f.i. Duffie et al. (2000)) such that

$$\beta(t,T) = \frac{1}{a_x}(1-e^{a_x t}) \text{ and}$$
$$\alpha(t,T) = \sigma_x^2\left(\frac{t}{a_x^2} - \frac{e^{a_x t}}{a_x^3} + \frac{e^{2a_x t}}{4a^3} + \frac{3}{4a_x^3}\right).$$

2.1.2 Interest-rate risk

We couple our description of demographic uncertainty with a standard model for interest-rates. We select the well-known Vasicek (1977) process. Under the usual risk-neutral measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, short rate dynamics is

$$dr(t) = k(\theta - r(t))dt + \sigma_r dW_F^{\mathbb{Q}}(t), \tag{2}$$

where $r(0) = r_0 > 0$ and $k, \theta, \sigma_r > 0$. Given our (standard) independence assumption between mortality and interest-rate risk, the Brownian motions W_F and W_x are independent between them for any x. It is well known that, since the process described by equation (2) is affine, a closed-form expression for bond prices is readily available

$$B(t,T) = X(t,T)e^{-Y(t,T)r(t)} \tag{3}$$

where

$$Y(t,T) = \frac{1}{k}\left(1-e^{-k(T-t)}\right), \text{ and}$$
$$X(t,T) = \exp\left[\left(\theta - \frac{\sigma_r^2}{2k^2}\right)(Y(t,T) - T + t) - \frac{\sigma_r^2}{4k}Y(t,T)^2\right].$$

We will simulate the process under the physical measure. In order to be able to perform the change of measure, we assume a constant risk premium γ, so that the market price of risk process takes the form $\gamma(t) = \gamma r(t)$. The short rate under the historical measure takes the form

$$dr(t) = [k\theta - (k + \gamma\sigma_r)r(t)]dt + \sigma_r dW_F^{\mathbb{P}}(t),$$

where $W_F^{\mathbb{P}}$ is a Brownian motion under the physical measure $\mathbb{P}$.
Now, at each $t_i \in \mathbb{I}$ we have

$$r(t_i) = [k\theta - (k + \gamma\sigma_r)r(t_{i-1})]\,\Delta + \sigma_r[W_F^{\mathbb{P}}(t_{i-1}) - W_F^{\mathbb{P}}(t_{i-1})].$$

In order to isolate the impact of longevity risk in our analysis, we sometimes consider a flat term structure of interest rates, which allows us to disentangle the effects of longevity risk and interest-rate risk on the value of liabilities. In this case, we fix a value $r(t) = r_0$ (constant) for the interest rate and keep it fixed. The price of a bond in this case equals

$$B(t, T) = e^{-r_0(T-t)} \tag{4}$$

for each $t_i \geq t_0$ and each T.

2.1.3 Idiosyncratic risk

For simplicity we consider a portfolio made up of N_A identical annuities and N_D identical death contracts on individuals belonging to a certain cohort x, extensions to more general cases where different products are present being straightforward. Let us define with $\tau^A = \{\tau_i^A\}_{i=1}^{N_A}$ and $\tau^B = \{\tau_i^B\}_{i=1}^{N_B}$ the sets of lifetimes of the annuitants and death contracts policyholders which we assume are comprised i.i.d. random variables. Further, we define the counting processes $D_A(t)$ and $D_D(t)$ which count the number of deaths in the two portfolios i.e.

$$D_A(t) = \sum_{i=1}^{N_A} \mathbb{1}_{\{\tau_i^A < t\}} \text{ and } D_D(t) = \sum_{i=1}^{N_D} \mathbb{1}_{\{\tau_i^B < t\}}.$$

We assume that no premium is given to idiosyncratic risk, since it can be diversified away simply in large portfolios.

2.1.4 Fair-value reserving and liability portfolio value

In this section we start from our mortality and interest-rate descriptions to provide the fair-value reserves of death contracts and annuities written on a cohort x. Having at our disposal closed-form expressions for bond prices and survival probabilities, we obtain the market (fair) value of annuities and death contracts also in closed form. This leads us to a computational advantage with respect to previous works, which used different stochastic mortality models and had to rely on computational methods to assess annuity values, such as Hari et al. (2011). Fair value reserves are computed under the hypothesis that idiosyncratic risk is absent, i.e. that is diversified away.[5] Given the results in Luciano et al. (2012b), the fair-value of a whole-life annuity $A(x, d, R)$ (Z_x^A) is the present value of the future payments to the annuitant, discounted appropriately given the term structure of interest rates and survivorship

$$Z_x^A(t_i) = R \sum_{\substack{t_j \in T_A^O(x,R), \\ t_j > t_i, t_j \leq t_\omega}} B(t_i, t_j) S(t_i, t_j), \tag{5}$$

where $t_\omega \in \mathbb{I}$ is the time at which the individual reaches its terminal age, after which the survival probability is zero.
On the other hand, we consider the death contract $D(x, Q, C)$, where a single premium is paid at policy inception. We assume that payments associated with this contract are due at the end of the year in which the death event occurred. It follows that the outflow associated to a death between to successive payment dates t_{j-1} and t_j occurs at t_j.
The prospective reserve (Z_x^D) at t_i is thus:

$$Z_x^D(t_i) = C \sum_{\substack{t_j \in T_D^O(x,Q,C), \\ t_j > t_i, t_j \leq t_0 + Q}} B(t_i, t_j)[S(t_i, t_{j-1}) - S(t_i, t_j)]. \tag{6}$$

Luciano et al. (2012b) show that while the first and second order sensitivities for longevity risk of death assurances have opposite signs with respect to those of annuities, the Greeks of death assurances and annuities with respect to financial risk have the same sign. When considering natural

[5] This assumption is harmless for large portfolios. In simulations, we include this source of randomness and evaluate its impact, by computing the solvency probability for different portfolio sizes.

hedging strategies, it is important not to neglect this aspect. While annuity longevity risk can be instantaneously neutralized by issuing death contracts, additional interest-rate risk enters the portfolio in the process. Two options are available: managing interest rate risk on the bond market after having neutralized mortality risk or handling them simultaneously. In this last case, by combining insurance contracts and mortality derivatives or reinsurance appropriately, it is – at least in theory, if such products are available in the market – possible to avoid using bonds.

We have now all the ingredients to compute the value of the liabilities portfolio (L) which is simply, at each point in time, the sum of the value of all the obligations the company has outstanding. For each $t_i \in \mathbb{I}$, this value is equal to the discounted value of future benefits due to policyholders, thus

$$L(t_i) = (N_A - D_A(t_i))Z_A(t_i) + (N_D - D_D(t_i))Z_D(t_i)$$

and can be straightforwardly computed in closed-form as discussed in the previous section.

2.2 Assets

The insurance company has an initial capital available $A(0)$, comprised of its own funds prior to time t_0 and (single) premiums received. At each t_i it invests a fraction $\delta(t_i)$ of this amount in stocks, whose dynamics follow a geometric Brownian motion, while the remaining part is invested in the bond market. When interest-rate risk is present, we model it as described in the previous section. When no interest-rate risk is present, investing in the bond market results in investing in a risk-free asset whose instantaneous (constant) rate of return is r_0.

At each point in time, the company pays out the benefits associated with the liabilities portfolio to policyholders.[6] Disinvestments from the asset side cover these outflows. This fact creates a link between the asset side and the liability side. Being dependent on the net flows of the liability portfolio, also assets become subject to longevity risk. The portfolio of assets is, if possible, rebalanced at each time t_i.

Now leaving aside for a moment the outflow payment, we consider two investment choices, i.e. investment in bond portfolio and/or investment in a stock market.

[6] Analogously, it receives inflows related to premium payments made by the policyholders

2.2.1 Investment in a bond portfolio

At each point in time, the insurer holds a portfolio (P) of zero coupon bonds with different maturities T_k, $k = 1, 2, .., n \in \mathbb{N}$. The initial investment in the bond portfolio is a fraction of initial capital available, i.e. $P(0) = (1 - \delta(0))A(0)$. We denote with $\omega_k(t_i)$ the proportion of $P(t_i)$ invested, at time t_i, in the bond with maturity T_k. In addition, we denote by

$$\boldsymbol{\omega}(t) = [\omega_1(t), \ldots, \omega_n(t)], \text{ such that } \sum_{k=1}^{n} \omega_k(t) = 1,$$

the vector of shares in the bonds that enter the portfolio at time t_i. From now on, we define our investment strategy at time t_i as a couple $(\delta(t_i), \boldsymbol{\omega}(t_i))$. The number of bonds of each maturity entering the bond portfolio at time t_0 is

$$H_k(t_0) = \omega_k(t_0)P(0)/B(t_0, T_k),$$

having $k = 1, ...n$, and $\boldsymbol{\omega}(t_0)$ a priori defined. The total value of the bond portfolio, immediately prior time $t_i > t_0$, is

$$P(t_i^-) = \sum_{k=1}^{n} H_k(t_{i-1})B(t_i^-, T_k),$$

where $H_k(t_i)$ represents the number of bonds in the portfolio at time t_i having maturity T_k, i.e.

$$H_k(t_i) = \omega_k(t_i)P(t_i)/B(t_i, T_k). \tag{7}$$

Exactly at t_i, as the bond portfolio is rebalanced, $P(t_i)$ becomes

$$P(t_i) = \sum_{k=1}^{n} H_k(t_i)B(t_i, T_k). \tag{8}$$

The value of each bond at time t_i, $B(t_i, T_k)$, given $r(t_i)$, can be computed for both interest-rate models according to formulas (3) and (4).

2.2.2 Investment in a stock market

A fraction $\delta(0)$ of the inital asset value $A(0)$ is invested in the stock market, hence $M(0) = \delta(0)A(0)$. Between any two time points t_{i-1} and t_i, stock market value evolves according to a geometric Brownian motion under the physical measure $\mathbb{P}$. We discretize the diffusion process, in order to be able to simulate it at each $t_i \in \mathbb{I}$. Thus, immediately before rebalancing, we have

$$M(t_i^-) \quad = \quad M(t_{i-1}) \exp\left\{(\mu - \frac{\sigma_M^2}{2})\Delta + \sigma_M[W_M^{\mathbb{P}}(t_i^-) - W_M^{\mathbb{P}}(t_{i-1})]\right\}.$$

where $\mu > 0$ is the drift and $\sigma_M > 0$ is the diffusion coefficient of the process and $M(t_{i-1})$ is the value invested in the stock market after having rebalanced the portfolio at t_{i-1}. By continuity of the Brownian motion, $W_M^{\mathbb{P}}(t_i^-) = W_M^{\mathbb{P}}(t_i)$. In the next section we introduce payments to policyholders and we derive the value of assets.

2.2.3 Asset Value

Let us now define, at each point in time, the payments to policyholders (O) as

$$\begin{aligned} O(t_i) \quad = \quad & R\,(N_A - D_A(t_i))\,\mathbb{1}_{\{t_i \in T_A^O(x,d,R)\}} + \\ & C\,(D_D(t_i) - D_D(t_{j-1}))\,\mathbb{1}_{\{t_i = t_j \in T_D^O(x,d,R)\}}. \end{aligned}$$

$O(t_i)$ is the sum of the installments paid to annuitants which are still alive at t_i and of the insured capital paid for death contract policyholders who died between the previous payment date and t_i, if t_i is itself a payment date.[7]

[7] In a more general setting, at each point in time, we can define $I(t_i)$ as the premium payments made by policyholders to the insurer. We expand our definition of the net financial flow from the asset portfolio, to have

$$I(t_i) = E_A\,(N_A - D_A(t_i))\,\mathbb{1}_{\{t_i \in T_A^I(x,d,R)\}} + E_D D_D(t_i)\mathbb{1}_{\{t_i \in T_D^I(x,d,R)\}},$$

where E_A and E_D are the (level) premiums of the annuity and the death contract respectively. In this case the value of assets is

$$A(t_i) = M(t_i^-) + P(t_i^-) + I(t_i) - O(t_i)$$

The value of assets at time t_i is the sum of the values invested in the stock market and in the bond market net of payments to policyholders:

$$A(t_i) = M(t_i^-) + P(t_i^-) - O(t_i).$$

In order to preserve our investment strategy $(\delta(t), \boldsymbol{\omega}(t))$ as defined, we re-balance the asset portfolio at each point in time t_i:

$$M(t_i) = \delta(t_i)A(t_i),$$

$$P(t_i) = (1 - \delta(t_i))A(t_i).$$

As stated by equation (8), the portfolio of bonds is itself re-balanced according to the vector $\boldsymbol{\omega}$. This is done by adjusting at t_i the number of bonds of each maturity T_k to hold in the portfolio, according to equation (7). In practice, we value the asset portfolio at t_i^-, sell it, and buy the assets again so as to implement our portfolio strategy at t_i.[8]

3 Solvency

Given the setup we described, our aim is to analyze the behavior of the funding ratio of the portfolio (F), i.e. the ratio between asset and liability value at a certain point in time $t_i \in \mathbb{I}$ i.e.

$$F(t_i) = \frac{A(t_i)}{L(t_i)}.$$

We constrain the funding ratio to be above or equal to 1 at time t_0, assuming that the portfolio is set up at t_0 and that the selling price of the policies is at least equal to their fair value, possibly increased by a proportional safety loading η, and that the value of the funds owned by the insurance company before setting up the portolio is non-negative, so that $A(0) \geq L(0)$. The safety loading can be interpreted both as a loading on premiums and/or as an initial endowment owned by the insurance company. The entire amount $A(0)$ is invested according to strategy $(\delta, \boldsymbol{\omega})$. Following standard literature on insurance solvency[9], after having fixed an horizon T, we are interested

[8]For simplicity, we assume that no transaction costs affect our rebalancing.

[9]For example, see Pitacco et al. (2009) or Olivieri and Pitacco (2002).

in assessing the probability that the insurer is solvent at T, which we define as the probability that the funding ratio is greater than or equal to one at that time: $\mathbb{P}[F(T) \geq 1]$. To mimic – in our simplified setting – the Solvency Capital Requirement computation, we set the horizon T to 1 year and we evaluate solvency probability. We look for the safety loading that keeps this probability as high as 99.5%. We are also interested in multi-period solvency, i.e. at longer time horizons we analyze the probability that the funding ratio is above 1 at any point in time: $\mathbb{P}[\wedge_{t=1}^{T} F(t) \geq 1]$.
We look at these quantities, but also at the variability (measured by its coefficient of variation) of the Asset/Liability Ratio at a certain time. Having no new entrants in the portfolio, we also define the event of bankruptcy occurring when $A(t) < 0$ at some point in time t. This bankruptcy event is absorbing.
Before turning to our numerical exploration, let us comment on the effects of relevant parameters on the funding ratio and, consequently, on solvency probability.
Some parameters have an easily predictable effect. Higer levels of δ and σ_M (increased investment risk) increase the variability of the funding ratio, while a higher number of policyholders (reduced idiosyncratic risk) decreases it. Higher μ for fixed σ_M and δ improves future expected asset value and solvency probability as a consequence, lowering bankruptcy occurrence. The same effect is achieved by fixing a higher safety loading η.
In the next section we analyze numerically the interactions of the different risk sources, by looking different types of portfolios, investment strategies and market conditions in a calibrated example.

4 Numerical assessment of solvency - A calibrated example

This section explains our procedure to perform the numerical evaluation of the solvency probability of a portfolio and applies it to a calibrated example. We calibrate our mortality and interest rate models to UK data, while performing sensitivity analysis to portfolio size, stock market conditions and different investment strategies.

Base-case parameters			
Mortality model		**Interest-rate model**	
Parameter	Value	Parameter	Value
a_x	0.085482572	k	0.244624909
σ_x	0.000044619	θ	0.108871911
$\lambda(0)$	0.016664077	σ_r	0.096494070
		$r(0)$	0.055050247

Table 1: This Table reports the calibrated parameter for our mortality model and interes-rate model.

4.1 Set up and calibration

We calibrate our mortality intensity and interest-rate process to UK data at the beginning of year 2000. As for our OU model, we choose the parameters so as to minimize the residual sum of squares between model predicted and observed survival probabilities (source: HMD) from age 35 to the last available age observed for all the generations of interest. In particular, we consider the generations of individuals born in 1935. Similarly, we calibrate the interest rate process under measure $\mathbb{Q}$, by fitting the observed zero coupon government bond price curve at January 4th, 2000. The market price of risk is chosen so as to align the historical long-run interest rate to its 10-year average (6%). Both calibrations are performed using a Differential Evolution algorithm, whose details can be found in Jevtić et al. (2013). Resulting parameters are collected in Table 1.

We do not calibrate the stock price process, but we select a base case value $\mu = 8.5\%$, with $\sigma_M = 20\%$. We then explore sensitivity of the investment strategies to different values for μ and σ_M. We recall our independence assumption between stock market and interest-rates.

We consider two types of portfolios in run-off. The first is composed by annuities only, describing the typical situation of an annuity provider in the run-off decumulation phase. The second introduces life insurance policies along with annuities and represents the portfolio of an insurer who can exploit natural hedging opportunities. To be more precise, in the first case we consider N_A $A(1935, 1)$ policies. When life insurance policies are introduced, we consider N_D death contracts $D(1935, 10, 100)$ in the portfolio.

We question the effectiveness of naturally hedged portfolios held by life insurers and benchmark them with portfolios built neutralizing mortality Delta, following Luciano et al. (2012b). Coverage is not rebalanced in time. We analyze sensitivity of our results to relevant parameters, either considering both interest-rate and longevity risk at the same time or isolating longevity risk by assuming that reserves are computed using the flat term structure of interest rates. We explore sensitivity to:

- portfolio size;
- liability mix, considering both annuities and life insurance policies;
- safety loading;
- investment strategy: different investment strategies, under various market conditions (σ_M and q).

For each experiment, we run 10000 simulations of the whole life of the insurance portfolio, until time t_ω. Diffusions are properly discretized at monthly intervals. At each time t, we:

1. Determine the current value of the short rate $r(t)$, of mortality intensity $\lambda_x(t)$ and of the stock $M(t)$ by simulating the respective diffusions;
2. given the actual mortality intensity $\lambda_x(t)$, we simulate the number of deaths occurred in the portfolio, in order to obtain the number of annuitants still alive $(N_A - D_A(t))$ and the number of deaths occurred in the life insurance portfolio $(D_D(t_i) - D_D(t_{i-1})$;
3. obtain the term structure of interest rates and the current value of assets and liabilities.
4. Rebalance the portfolio in order to keep our chosen investment strategy constant, such that

$$\delta = \delta(0) = \delta(t_i) \text{ for each } t_i \in \mathbb{I}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}(0) = \boldsymbol{\omega}(t_i) \text{ for each } t_i \in \mathbb{I}.$$

   The value invested in the stock market and the number of bonds in the portfolio are properly re-adjusted to this end as described in Section 2.2.3.

The insurer operates on the bond market purchasing a 5-year maturity bond, rolled over at each rebalancing date. This particular maturity was chosen because our simulations indicate it to be the one that minimizes funding ratio dispersion. Figure 1 compares its performance to bonds with different maturities when the asset portfolio is invested entirely in the bond market. An analogous consideration applies when analyzing 1-year solvency probability.

[Insert Here Figure 1]

Through our calibrated example, we investigate three main issues: the relevance of systematic longevity risk in the assessment of portfolio solvency, the interactions between investment risk, safety loadings and portfolio size and the potential effectiveness of natural hedging due to liability mix composition.

4.2 Importance of systematic longevity risk and its hedging

First, our analysis highlights the importance of considering a comprehensive view of the risks to which an insurance portfolio is subject, including systematic longevity risk. Let us discuss on its relative importance with respect to the other risk sources.

Table 3 reports the coefficients of variation of the funding ratio, of assets and liabilities when idiosyncratic longevity risk only is present (first column), when systematic longevity risk is added (second column) and when interest-rate risk is present as well (third column), after 1and 20 years, for different annuity portfolio sizes. The first column reports results obtained when mortality is modelled following the traditional deterministic Gompertz law, obtained setting $\sigma_x = 0$. When interest-rate risk is introduced, it enters both the liabilities' side, as insurance contracts are evaluated at market value, and the asset side, as investment in the 5-year bond is risky. Assets are invested entirely in the bond market.

First of all, the table allows to appraise the impact of idiosyncratic longevity risk in both assets and liabilities. Increasing portfolio size to 100000 is sufficient to diversify it almost entirely, as the first column of Table **??** shows. Table 2 reports the realized standard deviations of survivors for annuity portfolios with different size. Figures are very close for the two largest portfolios, suggesting that an effective diversification level is achieved already by the

Standard deviation of realized survival rates				
Horizon	**Portfolio size**			
	1000	10000	50000	100000
1	0.0041	0.0013	0.0001	0.0001
5	0.0095	0.0033	0.0020	0.0018
10	0.0141	0.0062	0.0049	0.0048
15	0.0173	0.0093	0.0081	0.0075
20	0.0187	0.0115	0.0104	0.0105
30	0.0118	0.0077	0.0071	0.0072
40	0.0019	0.0001	0.0001	0.0001

Table 2: This table reports the standard deviation of realized survival rates for annuity portfolios of different sizes.

50000 annuitants portfolio.

Figure 2 compares the simulated distributions of the realized survival rates of a small portfolio (1000 annuitants) and of a large one (100000 annuitants). It confirms that a small portfolio is subject to a non-negligible level of additional variability due to (idiosyncratic) longevity risk.

[Insert Here Figure 2]

The importance of accounting for systematic longevity risk can be observed by comparing the three columns of Table 3. For a well-diversified 100000 annuitants portfolio the 20-year dispersion of the funding ratio is 0.117 when systematic longevity risk is accounted for, while it is as small as 0.019 when only idiosyncratic risk is considered. Longevity uncertainty is shown to have an impact on assets as well, since the occurrence of net flows is influenced by actual mortality. The coefficient of variation of assets rises sharply with time, even for well-diversified, i.e. large enough, portfolios.
Even though it is generally believed that interest-rate risk is by far the most important risk source in the valuation of liabilities (at fair-value), our simulations highlight that systematic longevity risk is relevant as well. When the time horizon is small (1 year) the relative importance of systematic longevity risk is (expectedly) quite low, reflecting the low degree of uncertainty in short-term mortality forecasts. The 1-year cv of liabilities when no interest-rate risk is present is 0.0043 only, while it is 0.2707 when both systematic

longevity and interest-rate risk are present. In the short-term, the market value of liabilities is far more influenced by interest-rate risk than longevity risk , as one can expect.

However, when the time horizon lenghtens to 20 years, the impact of systematic longevity risk turns out to be relevant. The coefficient of variation of liabilities with no interest-rate risk is 0.044 and it is 0.35 when we consider uncertainty in the bond market as well. The dispersion of the funding ratio is around 4 and 7 times smaller when no interest rate risk is present (0.0063 vs. 0.0148 and 0.117 vs. 0.7494) after 1 and 20 years respectively. Figure 3 compares the evolution of the value of liabilities of the same portfolio of 50000 annuitants when longevity risk only is present and when both longevity and interest-rate risk are accounted for. The figure confirms that even though interest-rate risk is responsible for the largest part of the variability of earlier years (upper panel), the long-term impact of systematic longevity risk is considerable. The lower panel, which compares the paths starting from the twentieth year, highlights that longevity risk largely affects variability of annuity prices for old-aged policyholders. Especially in the long-run, thus, the importance of trying to mitigate longevity risk, as well as interest-rate risk – which is usually handled by life offices – seems to be remarkable.

[Insert Here Figure 3.]

Risk sources effect									
	IL			**IL + SL**			**IL + SL + IR**		
Portfolio size	$\frac{Std[F_T]}{\mathbb{E}[F_T]}$	$\frac{Std[A_T]}{\mathbb{E}[A_T]}$	$\frac{Std[L_T]}{\mathbb{E}[L_T]}$	$\frac{Std[F_T]}{\mathbb{E}[F_T]}$	$\frac{Std[A_T]}{\mathbb{E}[A_T]}$	$\frac{Std[L_T]}{\mathbb{E}[L_T]}$	$\frac{Std[F_T]}{\mathbb{E}[F_T]}$	$\frac{Std[A_T]}{\mathbb{E}[A_T]}$	$\frac{Std[L_T]}{\mathbb{E}[L_T]}$
T=1									
1000	0.0046	0.0004	0.0042	0.0063	0.0004	0.0060	0.0148	0.2584	0.2664
10000	0.0015	0.0001	0.0013	0.0045	0.0001	0.0044	0.0146	0.2626	0.2711
50000	0.0007	0.0001	0.0006	0.0043	0.0001	0.0042	0.0142	0.2574	0.2651
100000	0.0005	0.00005	0.0004	0.0043	0.00005	0.0042	0.0145	0.2621	0.2708
T=20									
1000	0.1887	0.1623	0.0376	0.2200	0.1765	0.0582	0.7677	0.9271	0.3623
10000	0.0601	0.0517	0.0119	0.1296	0.0885	0.0462	0.7554	0.9104	0.3518
50000	0.0270	0.0231	0.0053	0.1171	0.0750	0.0443	0.7535	0.9180	0.3540
100000	0.0188	0.0161	0.0038	0.1170	0.0745	0.0450	0.7495	0.9202	0.3484

Table 3: This table reports the coefficient of variation of the funding ratio, assets and liabilities after 1 and 20 years when three sources of risk (idiosyncratic and systematic longevity risk, interest-rate risk) are taken into account.

4.3 Investment risk, idiosyncratic risk and safety loadings

Having commented on the relevance of longevity risk in the previous section, we now focus on the interactions among investment strategy, portfolio size and the safety loading. Investment risk has a sheer impact on solvency and on the funding ratio variability. As expected, the higher the investment in stock, the higher the variability of the funding ratio and the higher the probability of bankruptcy at longer horizons. Also, solvency increases with μ, everything else being fixed, and decreases with σ_M for fixed investment strategy.

Since there is usually higher volatility in the stock market than in the bond one, increasing the investment in the riskier stock asset can increase the variability of assets and, as a consequence, of the funding ratio. It is interesting to explore how portfolio size and investment strategies combine. Their interaction is shown in Figure 4. While solvency probability is higher for larger portfolios when δ is low, this is not anymore the case for riskier investment strategies. Variability due to investment risk overcomes idiosyncratic risk in riskier portfolios. This happens both in the absence and in the presence of a safety loading. The investment strategy that maximizes short-term solvency probability for the well-diversified portfolio has $\delta = 9\%$, given our basecase parameters, when no interest-rate risk is accounted for. Unreported experiments show that this figure lowers to 6% when financial risk affects the valuation of liabilities, as well as the asset side.

Adding a safety loading on premiums – invested according to the selected investment strategy – increases solvency more at the margin when the investment strategy is less risky, as Figure 4 shows.[10]

[Inser here Figure 4]

The presence of a solvency constraint – either regulatory-based or internal to the fund – leads, as expected, to a trade-off between:

1. less conservative investment strategies;

2. larger safety loadings, for constant capital buffer.

[10]Portfolio size may indeed affect the "expenses" loading, which can be affected by the presence of economies of scale, which our model cannot incorporate.

Also, when the investment strategy is less risky, the better diversified the annuity portfolio, the more effective the loading in increasing the solvency probability. Indeed, absent interest-rate and investment risk, the required level η which gives a 99.5% 1-year solvency probability for annuity portfolios decreases with its size. It is 11% in a well-diversified 50000 annuitants portfolio, 12% in a 10000 annuitants one and 16% when the number of policyholders is 1000. A similar result holds when interest-rate risk is present and the insurer invests entirely in a 5-year risky bond. Investment strategy, capital buffer and premiums charged to annuitants in order to reach a certain solvency threshold influence each other and have to be jointly determined by the insurer. Our fast assessment of solvency probabilities and variability of the funding ratio can constitute then an extremely useful tool to accompany the decision process of a life insurance company.

4.4 Portfolio composition: natural hedging effectiveness

As a market for mortality derivatives is still in the making, we consider important to question the systematic longevity hedging potential of the liability portfolio mix. We analyze the effect of natural systematic longevity risk immunization techniques, whose design and performance have been addressed in some recent papers, such as Wang et al. (2010), Luciano et al. (2012b) and (Gatzert and Wesker, 2013). Addressing the issue of evaluating the effects of natural hedging on the funding ratio and on solvency probability is not an easy task, since the liability composition affects the asset side as well through the payments made to policyholders. We compare the solvency risk and characteristics of portfolios with different liability mixes. We consider a well-diversified portfolio of 50000 annuitants and different life insurance portfolio sizes. We focus on the case in which liabilities are subject to longevity risk only, thus neglecting interest-rate risk at first. We consider no investment risk and premiums loading, in order to disentagle any other effect from the liability mix one. Figure 5 reports the coefficient of variation of assets, liabilities and of the funding ratio as a function of the size of the life insurance portfolio at different time horizons.

[Insert Here Figure 5]

It shows that at all horizons, the funding ratio coefficient of variation reaches a minimum, which fluctuates between 10000 and 15000 life insurance policies. This figure is close to the portfolio size suggested by a Delta-hedging technique in the spirit of Luciano et al. (2012b).[11] The relative effect of properly choosing the hedging portfolio size is particularly evident for short horizons. For longer horizons, asset variability increases sharply with the size of the life insurance portfolio, driving the behavior of the funding ratio too, while liabilities are less affected by portfolio composition. Obviously, this is expected, since the liability mix is not rebalanced and the life insurance policies expire after the tenth year. Summarizing, introducing life insurance policies in the portfolio of the company may then reduce its riskiness, if the size of the natural hedging component of the portfolio is chosen appropriately.

We investigate how the effectiveness of a natural hedging strategy depends on portfolio size and investment strategy. We consider a static Delta-hedged portfolio, whose optimal dimension N_D is 26% of the annuity portfolio one. Our results show that, while very effective for large portfolios, natural hedging can be problematic for small portfolios. Table 4.4 reports the safety loading level required to reach the 99.5 % 1-year Solvency level. It is lower in naturally hedged portfolios than in non-hedged ones when size is large enough to diversify the additional idiosyncratic risk provided by the life insurance portfolio. It lowers to 10% (from 12%) for a 10000 annuitants portfolio and – remarkably – to 4.5% (from 11%) for the 50000 annuitants' one, supporting the empirical finding of Cox and Lin (2007) who found a competitive advantage for natural hedgers. However, the safety level increases from 16% to 34% in the small 1000 annuitants portfolio, reflecting the increased riskiness of the naturally hedged position.

Figure 6 shows that the asset investment choice do impact on the performance of the natural hedging strategy. Natural hedging (which is performed as a static Delta-hedging) increases 1-year solvency probability sharply – and decreases the dispersion of the funding ratio – when investing a small amount in the stock, while the difference with respect to the solvency of the annuity portfolio is small (even negative) for riskier strategies. This is due to the fact that investing in stocks influences the asset side and affects the distribution of the funding ratio as well.

[Insert here Figure 6]

[11] The difference in suggested optimal size with respect to a Delta-hedged portfolio is due both to the fact that a static and unrebalanced hedging is performed and to second order effects.

Required safety loading		
Portfolio size	η	
	No hedge	Natural Hedge
1000	16%	34%
10000	12%	10%
50000	11%	4.5%

Table 4: This Table reports the inital safety loading required to reach the 99.5% level in 1-year Solvency probability in an annuity only portfolio (no hedge) and in a portfolio which includes life insurance policies too (Natural hedge) Natural hedged portfolio is built neutralizing time 0 mortality Delta and it includes $N_D = 26\% N_A$ policies.

Evaluating the impact of the liability mix when interest-rate risk is accounted for requires more care. While systematic longevity risk is offset when life insurance policies are introduced in the annuity portfolio, additional interest-rate risk enters the portfolio. This can be hedged through bonds. We compare the performance of three portfolios: an annuity portfolio (A), a (static) naturally Delta-hedged portfolio in which interest-rate is not managed (DL) ($N_D = 26\%$ and all premiums are invested in the 5-year bond), a static Delta-hedged portfolio in which first-order sensitivity to both longevity and interest-rate risk are neutralized (DLIR) ($N_D = 26\%$ and premiums invested in part in a 4-year and in part in a 5-year one, so that interest-rate Delta is neutralized).[12]
We compare the performances of these three portfolios in Table 4.4. The DL portfolio clearly produces the worst outcome after one year, showing the lowest solvency probability (53.09%) and the highest funding ratio coefficient of variation (0.0153). This happens because adding the life insurance portfolio, while offsetting longevity risk, increases interest-rate risk. Proper management of the financial risk inherent the liability portfolio, either within the liability mix, using – if necessary and possible – mortality derivatives or reinsurance, has very important consequences for solvency. The DLIR portfolio performs best in terms of variability reduction (0.0139 vs. 0.0143 of the

[12]These are indeed a sub-optimal hedging strategy, since the portfolio should be rebalanced continuously in order to get a perfect hedge.

Natural Hedge effectiveness			
Portfolio	1-year F CV	1-year Solvency	10-year bankruptcy
No hedge (A)	0.0143	57.01%	0.37%
Natural hedge, (DL)	0.0153	53.09%	4.34 %
Natural hedge, (DLIR)	0.0139	55.43%	2.78%

Table 5: This table compares the 1-year funding ratio CV, the 1-year solvency and the 10-year bankruptcy probability of three portfolios: an unhedged annuity portfolio (A), a naturally hedged portfolio where interest rate risk is not managed (DL) and a naturally hedged portfolio in which interest-rate risk is also Delta-hedged (DLIR).

A portfolio) at the long – without any rebalancing – 1-year term. Solvency reduces instead with respect to the unhedged portfolio (55.43% vs. 57.01%) because of a slightly lowered mean funding ratio (1.0009 vs. 1.0006). The additional interest-rate risk in the DL portfolio produces a long-term effect, increasing the default event probability to 4.34% after 10-years, from 0.37% of the A portfolio. The static hedging technique on which the DLIR builds upon also leads to an undesirable increase in the catastrophic event likelihood (2.78%). Rebalancing the portfolio at least at yearly basis seems then advisable. Our analysis calls for further scrutiny of the performance of dynamically hedged portfolios.

5 Conclusions and further research

Our paper presents a computationally efficient method to assess the solvency probability of a life insurance portfolio in run-off. Our model takes an asset-liability management perspective and accounts for investment risk, interest-rate risk and both systematic and idiosyncratic longevity risk. Our continuous-time modeling framework, appropriately discretized, allows us to reduce computational effort and provide an assessment which can be consistent with the ORSA approach.

We employed our model to analyze numerically non-trivial parameter effects on the funding ratio and its variability. The simulations, whose mortality and interest rate processes are calibrated to UK historical data, show some interesting facts and results. First of all, they support the idea that systematic longevity risk is responsible for a large part of the long-run funding ratio variability and that pension funds and life offices cannot neglect it. Second, we investigated the interconnectedness of portfolio size, investment strategy and the effectiveness of premium loadings. Third, we numerically explored the impact of liability portfolio mix on solvency. We found that well-diversified annuity portfolios can benefit from natural hedging and require a lower safety loading when they want to keep insolvency probability under control. We highlighted the importance of managing the additional interest-rate risk present in naturally hedged portfolios. Frequent rebalancing of the bond hedging strategy may be crucial to natural hedging effectiveness. We left some issues for future development. First, we would like to explore the performance of dynamic hedging strategies rather than static ones. Second, we would like to consider heterogenous portfolios, with products written on different generations. Finally, our parsimonious description of the risk sources, would allow us also to compute easily (and in closed form) the one-year VaR of the liabilities portfolio at each time instant. Computing, monitoring and using this measure as a risk management tool and driver of investment choices would be an interesting task.

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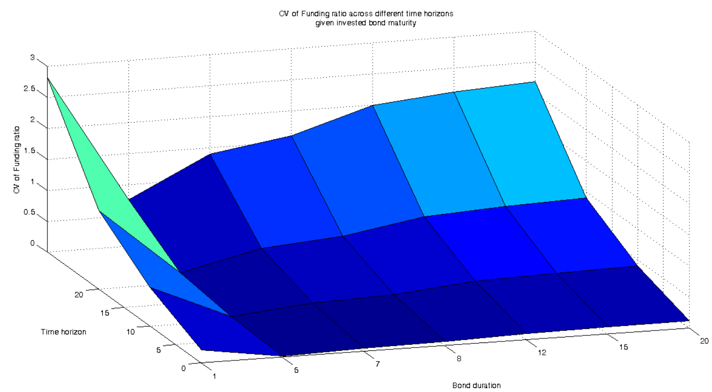


Figure 1: This figure shows the funding ratio coefficient of variation for different time horizons (1,5,10,20) years as a function of the maturity of the bond in which the whole asset portfolio is invested.

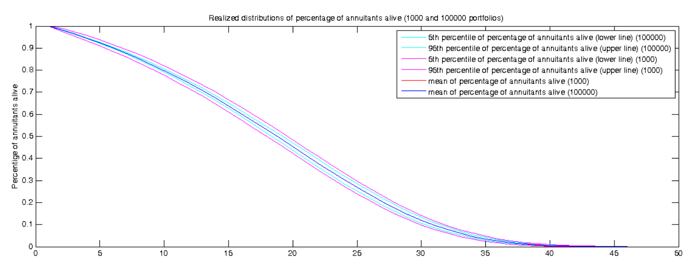
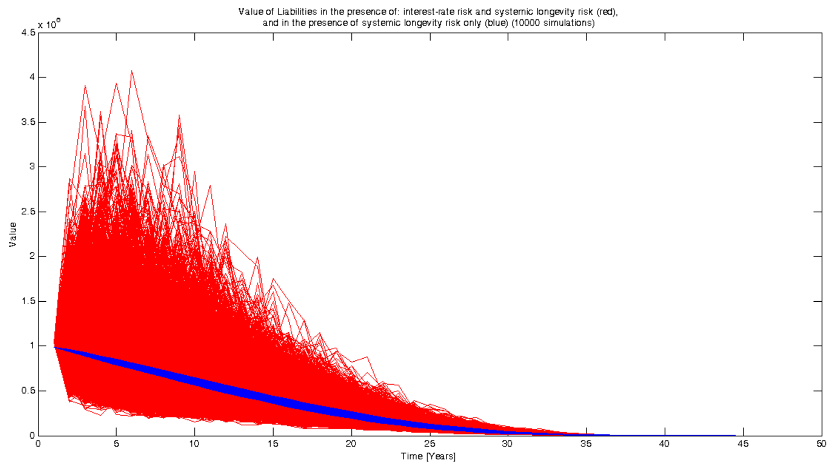
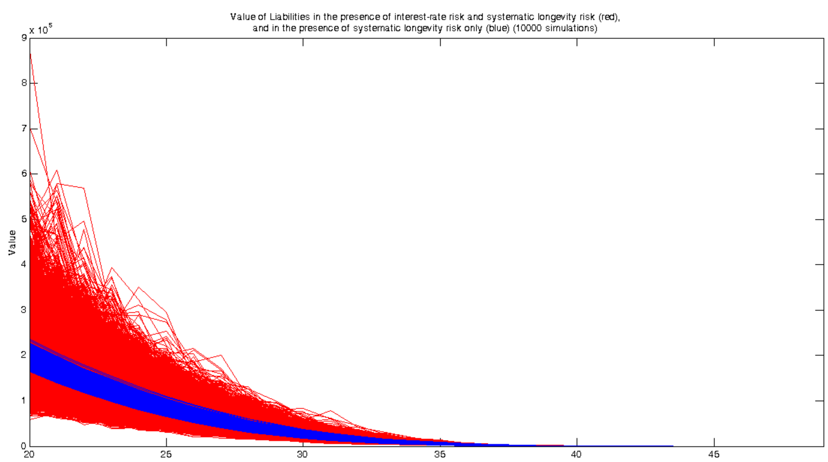


Figure 2: This figure compares the distribution of realized survival rates for two portfolios composed of 1000 annuitants and 100000 annuitants respectively. Pink (red) lines depict 95% and 5% confidence intervals (mean) for the 1000 annuitants portfolios, cyan (blue) ones for the 100000 one.



(a) toFill



(b) toFill

Figure 3: This figure shows the simulated paths of the value of liabilities in a portfolio with 50000 annuitants when longevity risk only (blue) and both longevity and interest-rate risk (red) are present. The upper panel shows the whole simulation period, while the lower panel starts after the 20th year.

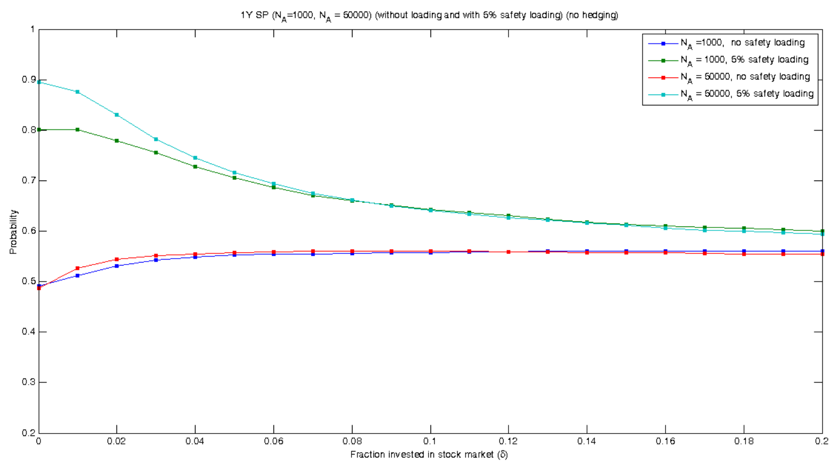
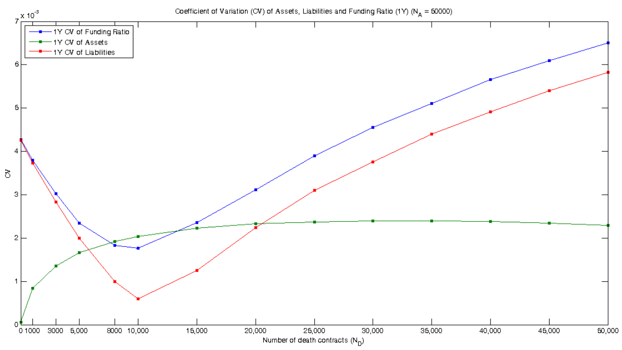
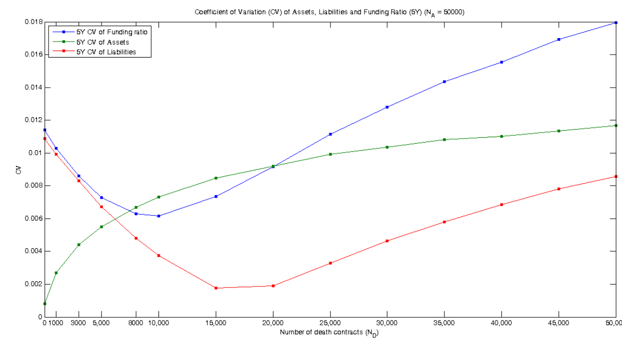


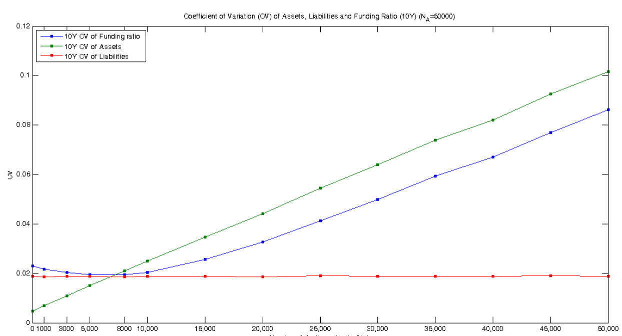
Figure 4: This figure shows the 1-year solvency probability for two annuity portfolios, $N_A = 1000$ and $N_A = 50000$, for different investment strategies δ with no loading and when a 5% loading on pure premiums is present.



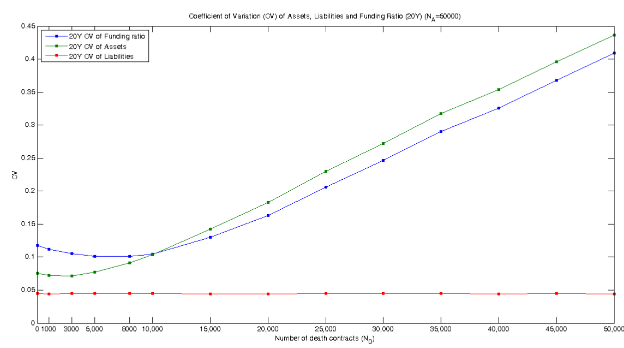
(a) 1-year time horizon



(b) 5-years time horizon



(c) 10-years time horizon



(d) 20-years time horizon

Figure 5: The figure shows the coefficient of variation of the assets, liabilities and the funding ratio at different time horizons in relation to the life insurance portfolio size.

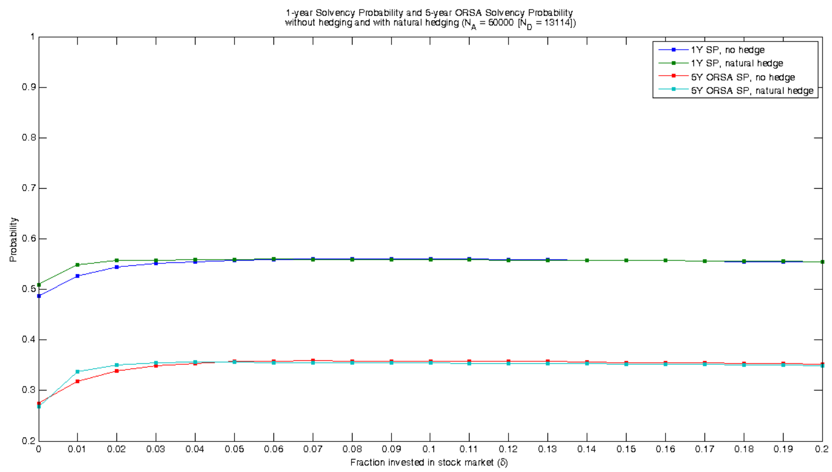


Figure 6: This figure shows the effectiveness of natural hedging under different investment strategies δ, by showing the difference between the 1-year solvency probability and the 5-year ORSA solvency probability of a portfolio of 50000 annuitants only (no hedge) and the one with 13114 life insurance policyholders also (natural hedge).