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# Funding Ratio Options

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# FUNDING RATIO OPTIONS

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**Abstract**

This paper defines an approximation to the value of funding ratio put options for pension funds. This option is, by construction, the ideal option to hedge the risk of a funding ratio falling below some required minimum level. It's value can be used for several applications, for example as a risk measure for internal risk management or regulation, as a benchmark for (other) derivative solutions to hedge insolvency risks, or to value guarantees made by sponsors to eliminate a funding shortfall. A numerical example shows that the impact of the presence of mortality volatility risk on the value of funding ratio put options is significant.

*JEL classification:* G23, G24, G28

*Keywords: pension fund, funding ratio, insolvency risk, regulation*

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# 1. Introduction

The funding ratio is one of the first characteristics mentioned in a discussion on the financial situation of a pension fund, and in many European countries also regulatory requirements are based on the funding ratio. One of the main goals of pension fund management is therefore to maintain at least some minimum level of the funding ratio.

Since funding ratios are recognized to be very sensitive to interest rate and stock price movements, the investment policy and overlays are often chosen such that interest rate and equity market risks are hedged to some extent. The hedging instruments that are most often applied are plain vanilla swaps and swaptions to hedge interest rate risk, and equity put options to hedge equity market risk.

But, hedging interest rate risk and equity market risk separately is sub-optimal for a pension fund, since this hedging combination is overprotecting in some situations. One example is the case where interest rates are increasing, thereby raising the funding ratio, such that the pension fund can handle some decrease in stock prices at the same time before falling below the required minimum funding ratio level.

Noticing this sub-optimality investment banks nowadays offer more exotic solutions where the derivatives are designed to pay off when interest rates and stock prices fall simultaneously. For example: ‘Traffic Light Options’ (see Jørgensen, 2007) are designed for and bought by Danish pension funds, and ‘Equity Linked Swaptions’ (Cardano, 2008) are designed for and bought by Dutch pension funds.

In this paper we analyze the ‘ideal’ option to protect the funding ratio from falling below some minimum level: a funding ratio put option. Theoretically, the value of a funding ratio put option can be considered as the minimum price one has to pay in order to hedge the risk of a funding ratio falling below the required minimum level. The value of a funding ratio put option can be used as a benchmark when studying other derivative solutions or insurance to minimize the risk of a low funding ratio. Or to value the guarantee a sponsor has made to eliminate a funding shortfall.

Another interesting application of the value of a funding ratio put option would be as a risk measure for regulation or internal risk management. The value of a funding ratio put option where the minimum level is 100%, will give the (theoretical) minimum price to hedge this risk, and could therefore be seen as a minimum required level for the surplus of the pension fund under consideration.

In existing literature concerning the valuation of options in the context of pension schemes the focus is often on the (comparison of) values of options written by stakeholders in different pension schemes, see for example Bagehot (1972), Sharpe (1976), Blake (1998) and Kocken (2006). In this paper we are not concerned about solidarity issues between stakeholders or the actual setup of the pension scheme. Instead, given the pension scheme we are concerned with the value of an option to protect the current financial position of the pension fund.

Most of the subsequent results are based on derivatives pricing literature where both stocks and interest rates are assumed to be stochastic (see in the context of pension funds for example Briys and de Varenne, 1997, Jørgensen, 2007) and on literature on combining financial models with stochastic mortality models, see e.g. Møller (1998), Dahl (2004) and Schrager (2006). Our contribution consists of the application of these combined financial and mortality models to the specific risk management needs of pension fund managers. We derive the partial differential equation of the inverse of the funding ratio in the presence of both financial risks (stochastic asset prices and stochastic interest rates) and mortality risks under a fixed but arbitrary equivalent martingale measure, such that options on the funding ratio can be priced. We propose a closed form formula for the value of a funding ratio put option and show the impact of the presence of mortality risk.

In contrast to Briys and de Varenne (1997), Schrager (2006), and Jørgensen (2007) we consider option values given asset price risk, interest rate risk and mortality risk simultaneously. As opposed to e.g. Møller (1998) and Dahl (2004) our focus is not on insurance companies but on pension funds and their specific need to 1) protect their solvency position as measured by the funding ratio, and 2) manage financial risks and mortality risks simultaneously.

The remainder of this paper is organized as follows: first we introduce a stylized balance sheet and the dynamics of the assets and liabilities. Then we look at the value of funding ratio put options and show a numerical example where we hedge only financial risk, and an example taking also into account mortality risk. We end with conclusions.

## 2. Funding ratio options

We are interested in the valuation of options designed for pension funds. We first introduce the following stylized balance sheet of the pension fund at time $t$:

| Assets | Liabilities |
|---|---|
| $A_t$ | $L_t$ |
| | $E_t$ |

where $A_t$ are the assets, $L_t$ the liabilities and $E_t$ is the equity of the pension fund at time $t$. Given this balance sheet we define the funding ratio at time $t$ as:

$$FR_t = \frac{A_t}{L_t} \tag{2.1}$$

The funding ratio is one of the first characteristics mentioned when one discusses the financial situation of a pension fund, and in many European countries also regulatory requirements are based on the funding ratio. One of the main goals of pension fund management is therefore to maintain at least some minimum level of the funding ratio.

Since funding ratios are recognized to be very sensitive to interest rate and stock price movements, the investment policy and overlays are often chosen such that interest rate and equity market risks are hedged to some extent. The hedging instruments that are most often applied are plain vanilla swaps and swaptions to hedge interest rate risks, and equity put options to hedge equity market risk. But investment banks also offer more exotic solutions where the derivatives are designed to pay off when interest rates and stock prices fall simultaneously. For example ‘Equity Linked Swaptions’ (Cardano, 2008) and ‘Traffic Light Options’ (Jørgensen, 2007).

An Equity Linked Swaption (ELS) is an interest rate receiver swaption where the strike $K_{ELS}$ depends on the stock price at time of maturity, $S_{T_O}$, relative to the stock price at start, when *t=0*, $S_0$, as follows:

$$K_{ELS} = K - a\left(\frac{S_{T_O}}{S_0} - 1\right) \tag{2.2}$$

where $K$ and $a$ are constants. When the stocks perform well, the strike of the interest rate receiver swaption is lower: the pension fund gets (and needs) less interest rate protection.

For Traffic Light Options (TLO) the payoff at time of maturity $T_O$ is

$$(K_r - r_{T_O})_+ \cdot (K_S - S_{T_O})_+ \tag{2.3}$$

where $K_r$ and $K_S$ are constants, and $(x)_+ := \max(x,0)$. This payoff is positive when both interest rate and stock price are below their respective (constant) strike levels $K_r$ and $K_S$ at time of maturity.

The idea behind products like the ELS and TLO is that offsetting effects in interest rates and equity movements do not need to be hedged. Products that take into account these offsetting effects should result in a smaller option premium then buying interest rate and equity market risk hedges separately. ELS and TLO are bought to protect the funding ratio against falling below some minimum level (for example 100% or a minimum level indicated by regulation), where equity risk and interest rate risks are recognized as the main risk factors. We suggest that ideally a pension fund would like to buy a Funding Ratio put Option (FRO) to hedge the risk of a funding ratio falling below some minimum level.

The payoff of a European style funding ratio put option that pays at time $T_O$ in case the funding ratio falls below some minimum level $FR^{\min}$ is equal to:

$$L_{T_O} \cdot (FR^{\min} - \frac{A_{T_O}}{L_{T_O}})_+ \tag{2.4}$$

where $A_{T_O}$ and $L_{T_O}$ denote the assets and liabilities at time of maturity respectively.

Figure 1 and figure 2 illustrate the hedging effect of the three options (ELS, TLO, FRO) on the funding ratio. Figure 1 on the left depicts the funding ratio given an unhedged balance sheet of a stylized pension fund under instantaneous interest rate and stock price movements. The minimum required funding ratio is this example is 100%. Ideally, the hedged balance sheet results in figure 1 on the right, which actually depicts the funding ratio when a FRO with strike level 100% is bought.

Figure 2 shows the funding ratio given instantaneous stock price and interest rate movements, using an ELS (left) respectively a TLO (right) as hedging instrument. As one can see these options are overprotecting when stock prices and interest rates fall considerably simultaneously. This extra upside potential will result in a higher price for these options compared with the FRO. In practice a collar construction will be used, but still the collar construction will not be as ideal as the FRO when hedging insolvency risk.

Theoretically, the FRO gives the minimum price for hedging against a funding ratio falling below some minimum level. Furthermore, figures 1 and 2 only show stock price and interest rate movements, which are the only risks the ELS and TLO do hedge. But when defining a FRO, we can also take into account other risks, for example mortality risk.

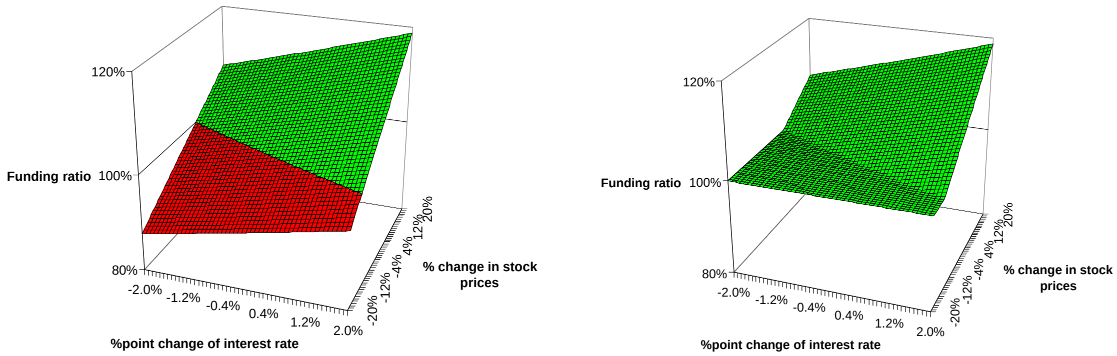


Figure 1: Left - unhedged funding ratio as a function of stock price movements and interest rate movements. Right - the funding ratio when a Funding Ratio put Option is bought.

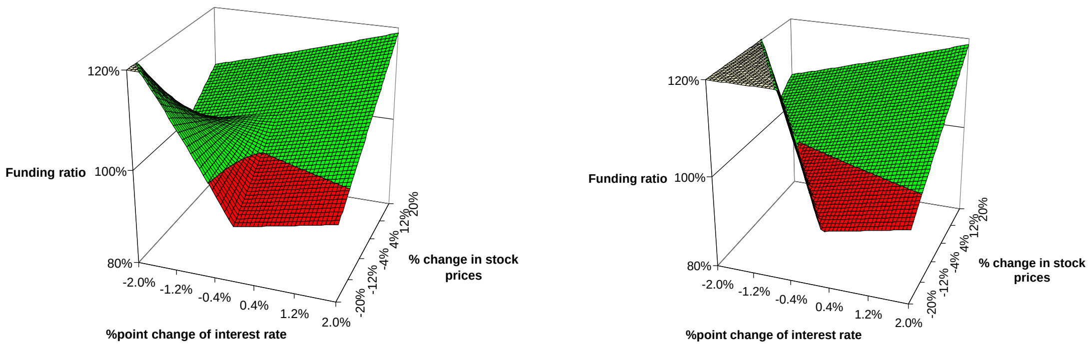


Figure 2: Left - the funding ratio when a Traffic Light Option is bought. Right - the funding ratio when an Equity Linked Swaption is bought.

The price of a FRO can be used as a benchmark for other derivatives solutions, to value a guarantee given by a sponsor, or for example as a risk measure in the context of an asset liability management model. Even regulation could be based on the price of FROs, since the price could be seen as a minimal required level of free equity, see Joseph and de Jong (2010). In the next sections we will approximate the value of FROs.

## 3. *Model Development*

To price funding ratio put options we need to take into account the dynamics of the assets and liabilities. The main drivers of the assets and liabilities are often recognized as financial market risks, but we would also like to take into account actuarial risks, such as e.g. mortality risks. Our model setup is therefore inspired by literature on frameworks which can capture actuarial risks, but are still compatible with financial option pricing models, as given in e.g. Møller (1998), Dahl (2004) and Schrager (2006).

We consider a continuous time model on a (finite) time interval $[0,T]$ that contains both financial and actuarial information. The model is described by a probability space $(\Omega, H, P)$ with filtration $(H_t)_{0\le t\le T}$ satisfying the usual conditions, and where $P$ denotes the real world probability measure.

### *3.1. Assets of the pension fund*

First we look at the dynamics of the assets of the pension fund. Let the financial market consist of $l$ tradable assets $A_t^i$ $i = 1,...,l$. The price processes of these assets are defined on the above introduced probability space. Their real-world dynamics is given by

$$dA_t^i = A_t^i \cdot \left(m^{Ai} dt + \sigma^{Ai} ' dW_t^P\right),\ i = 1,...,l \tag{3.1}$$

Where the $m^{Ai}$ and $\sigma^{Ai}$ are progressively measurable uniformly bounded processes and $(W_t^P)_{0 \le t \le T}$ is a vector Brownian Motion on the interval $[0,T]$ under $P$. We denote the $P$-augmentation of the natural filtration generated by the tradable assets by $(F_t\ )_{0 \le t \le T}$. In the following we use vector notation $m^A = (m^{A1},...,m^{Al})'$ and matrix notation $\Sigma^A = (\sigma^{A1},...,\sigma^{Al})'$.

The total asset value of the pension fund is determined by the trading strategy, which is assumed to be a predictable process vector $w_t = \left(w_t^1,..., w_t^l\right)',\ \forall t \ge 0$. At any time $t \in [0,T]$, $w_t^i$ represents the number of asset $A_t^i$ held in the portfolio, where $i = 1,...,l$. Given this trading strategy, the investment portfolio of the pension fund, denoted by $A_t$, follows the process:

$$dA_t = A_t \left(w_t ' m^A\ dt + w_t ' \Sigma^A dW_t^P\right) \tag{3.2}$$

In the examples to come, the investment portfolio will consist of risky assets and discount bonds, and the trading strategy is assumed to be self-financing. The suitable dynamics of the assets in the investment portfolio depend on the options under consideration. For example, if one would like to consider options with long maturity, which is well possible in the context of pension contracts, several authors suggest that a stochastic volatility model for risky assets would be more suitable than constant volatility models. See for example Ballotta and Haberman (2003).

### *3.2. Liabilities of the pension fund*

The process of the liabilities is defined on the probability space $(\Omega, H, P)$ as:

$$dL_t = L_t \cdot \left(m^L dt + \sigma^L dW_t^P + \sigma^{L\varepsilon} dW_{\varepsilon t}^P\right) \tag{3.3}$$

Where $m^L$, $\sigma^L$, and $\sigma^{L\varepsilon}$ are progressively measurable uniformly bounded processes. Further $W_{\varepsilon t}^P$ is a vector Brownian motion on interval $[0,T]$ under measure $P$ which represents other sources of uncertainty than financial market uncertainty. One can think of mortality risk, growth of the overall workforce and uncertainty in the invalidity rates.

The latter risks are very specific to the particular pension fund under consideration, and have to be modeled differently in any individual case, while mortality affects all pension funds. Furthermore, in general we expect the effect of mortality risk to be extensive compared to the effect of other uncertainties in the pension fund, therefore we will only consider mortality risk from here on. We will write $W_{\mu t}^{P}$ instead of $W_{\varepsilon t}^{P}$ where the $\mu$ refers to mortality and the mortality intensity process is driven by $W_{\mu t}^{P}$. Let $(M_t)_{0\le t\le T}$ be the $P$-augmentation of the natural filtration generated by mortality intensity. The liabilities $L_t$ are then equal to the sum of till pension date deferred annuities over all participants still alive at time $t$.

The development of the pension contract is assumed to be a right continuous Markov process with a finite number of jumps on a finite state space, which if it was only for one individual and now that we consider only mortality as actuarial risk, consists of two states: alive or death. The $P$-augmentation of the natural filtration generated by the pension contract is denoted by $(I_t)_{0\le t\le T}$. This filtration contains information on which individuals in the pension contract are still alive. We assume that the earlier defined filtration $H_t$ equals $F_t \vee M_t \vee I_t$. $H_t$ thus contains all other information concerning mortality, $M_t$, financial markets, $F_t$, and the pension contract $I_t$. The economy is assumed to be stochastically independent of mortality and the development of the pension contract.

The model setup is fairly general. The diffusion setting is appealing because it enjoys the desirable property of simplicity, allowing one to fully understand various mechanisms affecting the financial position of a pension fund. But, because of the simplicity, the model does not allow for a full account of uncertainty facing a pension fund. For example within our specification there are no jumps in the assets and liabilities of the pension fund to capture e.g. the effect of default events or mortality catastrophe. Some jump components could be added to the model, but this is left for further research.

In current model setup, the suitable dynamics of the liabilities depend mainly on the funding ratio under consideration. Regulatory requirements in for example The Netherlands and Denmark are based on the nominal funding ratio. In that case the market value of the liabilities does not take into account future indexation, and a liability model with as financial risk only the risks due to nominal interest rates might be suitable.

From a participants perspective real funding ratios might be more interesting. In this case the funding ratio takes into account purchasing power of the obligations. When one would like to study options on the real funding ratio it is necessary to model not only nominal interest rates but also real interest rates for the liabilities. See for example the Jarrow-Yildirim (2003) model.

### *3.3. Change of measure*

Now that we defined the dynamic processes of assets and liabilities, we can concentrate on the value of funding ratio put options. Note that we modeled the real world behavior of assets and liabilities, which is often used in asset liability management models. But for option pricing we need the behavior of the processes under the pricing measure. Since the combined financial and pension market is incomplete there are infinitely many pricing measures or equivalent martingale measures. And therefore pension contracts cannot be priced uniquely by a no arbitrage argument.

However we can choose an arbitrary but fixed equivalent martingale measure $Q^{Assets}$ to derive possible prices for funding ratio put options, which are consistent with absence of arbitrage.

We follow Dahl (2004) and Schrager (2006), we take into account mortality as non-financial risk in the pension contract and we assume that individual mortality risk can be diversified away, so that only systematic mortality risk remains. To construct a new measure $Q^{Assets}$ we define a likelihood process by:

$$d\ln\Delta_t = h_t^A dW_t^P + h_t^\mu dW_{\mu t}^P,\ \Delta_0 = 1 \tag{3.4}$$

where $h$ 's are adapted processes. $h^A$ changes the drift term of the assets, $h^\mu$ changes the drift term of mortality. We can now define a measure $Q^{Assets}$ by $dQ^{Assets}/dP = \Delta_T$.
Girsanovs theorem shows that under measure $Q^{Assets}$ $W_t^{Q^{Assets}} = W_t^P - \int_0^t h_u^A du$ and $W_{\mu t}^{Q^{Assets}} = W_{\mu t}^P - \int_0^t h_u^\mu du$ are independent $Q^{Assets}$-Brownian motions. Considering the financial assets only, then there is only one $h^A$ possible such that the tradable assets are $Q^{Assets}$ martingales. Contracts contingent on mortality intensity however, are scarcely traded on the financial market, and prices of traded contingent claims are not publically available, thus there are no further conditions for $h^\mu$ available from a market perspective.

The liabilities $L_t$ of the pension fund are equal to the expected value of discounted future benefits under some arbitrary but fixed market measure $Q^{Assets}$. One could choose $h^\mu$ equal to zero, so that the market is risk neutral with respect to systematic mortality risk. The resulting martingale measure is known as the minimal martingale measure, cf. Schweizer (1991), and has been applied in pricing literature on insurance contracts, see e.g. Møller (1998).

Other choices are for example a market price of risk such that mortality intensity still follows a similar process as under the real-world measure, but the parameters are such that survival probabilities are more prudent in the eyes of the counterparty with respect to the real world measure survival probabilities; see for example Dahl (2004). In this paper we choose the market price of mortality risk such that the process under consideration is a martingale under the pricing measure $Q^{Assets}$. In the analysis below we could equally well apply any of the martingale measures resulting from other choices of the market price of mortality risk. However other choices would complicate the calculations in the next sections.

## *4. Funding Ratio Put Options*

Now we turn to the valuation of a funding ratio put option. The formulation of the model of assets and liabilities in (3.2) and (3.3) is very general; to add the required interpretation to the valuation formulas we will be more specific on the dynamics of the assets and liabilities. The asset portfolio (3.2) is a combination of risky assets, denoted $S_t$, and zero coupon bonds $D_T$ with different maturities $T$. The risky assets are assumed to follow a geometric Brownian motion. With respect to the interest rates we assume an affine term structure model (see i.e. Duffie and Kan, 1996). Under the risk neutral measure the model is given by

$$\begin{aligned} r_t &= g_0^r + \vec{g}_Y^{\,r}{}'\vec{Y}_t^r \\ d\vec{Y}_t^r &= \underline{\underline{A}}^r(\vec{\theta}_Y^r - \vec{Y}_t^r)dt + \underline{\underline{\Sigma}}_Y^r\sqrt{\underline{\underline{X}}_t^r}\,d\vec{W}_{Y^r t}^Q \end{aligned} \tag{4.1}$$

where $\vec{W}_{Y^r t}^Q$ is a vector Brownian motion and $\underline{\underline{X}}_t^r$ is diagonal matrix with elements $\left\langle \underline{\underline{X}}_t^r \right\rangle_{ii} = \alpha_i^r + \vec{\beta}_i^r{}'\vec{Y}_t^r$. Here and in the following, we use notation $\vec{x}$ for a vector, and $\underline{\underline{X}}$ for matrices. It is well known that under affine term structure models a discount bond with maturity $T$ can be written as $D(t,T) = e^{A(t,T)-\vec{B}(t,T)\vec{Y}_t^r}$, where $\vec{B}$ is a row vector. In vector notation the assets of the pension fund are given by:

$$A_t = \vec{w}_t^{S}{}'\vec{S}_t + \vec{w}_t^{D}{}'\vec{D}_t^A \tag{4.2}$$

The liabilities depend on the nominal interest rate and mortality, and payments are at fixed preset times. Mortality intensity of a person aged $x$ at time $t$ also follows an affine process[2] under the real-world measure, as introduced by Schrager (2006):

$$\begin{aligned} \mu_{x+t} &= g_0^\mu(x) + \vec{g}_Y^{\,\mu}(x)'\vec{Y}_t^\mu \\ d\vec{Y}_t^\mu &= \underline{\underline{A}}^\mu(\vec{\theta}_Y^\mu - \vec{Y}_t^\mu)dt + \underline{\underline{\Sigma}}_Y^\mu\sqrt{\underline{\underline{X}}_t^\mu}\,d\vec{W}_{Y^\mu t}^P \end{aligned} \tag{4.3}$$

where $g_0, \langle \vec{g}_Y \rangle_i : x \to R_+$ and $\vec{W}_{Y^\mu t}^P$ is a vector Brownian motion under the real world measure, and $\underline{\underline{X}}_t^\mu$ is diagonal matrix with elements $\left\langle \underline{\underline{X}}_t^\mu \right\rangle_{ii} = \alpha_i^\mu + \vec{\beta}_i^\mu{}'\vec{Y}_t^\mu$. The survival probability at time $t$ up till time $T$ of an $x$ year old is given by ${}_{T-t}p_x(t) = E^P[\exp(-\int_t^T \mu_{x+s}ds) \mid M_t]$, and can similar to the discount bond be written as ${}_{T-t}p_x(t) = e^{A^\mu(x,t,T)-\vec{B}^\mu(x,t,T)\vec{Y}_t^\mu}$.

Liabilities at time $t$ are then given by:

[2] One of the drawbacks of this model is that mortality intensity can get zero or even negative. On the other hand parameters can be chosen such that the probability of negative values is very small.

$$L_t = \sum_{x=0}^{\varpi} w_t^x \sum_{j=0}^{\varpi} I_{x+j\geq xp} \; {}_{j+\lceil t \rceil - t}p_x(t) \cdot D(t, \lceil t \rceil + j) = \vec{w}_t^{L}{}' \underline{\underline{\mathrm{P}_t}} \vec{D}_t^L \tag{4.4}$$

where $w_t^x = \sum_i I_{\lceil x_i(t) \rceil = x} w_t^i$ gives the sum of promised benefits for all individuals $i$ with (almost) [3] age $x$ ., and payments are due at the end of a yearly (monthly) period if the individual is still alive. Here $I$ is an indicator function and we assume $w_t^i \; \forall i$ are constants. The elements of the matrix respectively vectors are given by $\left\langle \underline{\underline{\mathrm{P}_t}} \right\rangle_{x,j} = I_{x+j\geq xp} \; {}_{j+\lceil t \rceil - t}p_x(t)$, $\left\langle \vec{D}_t \right\rangle_j = D(t, \lceil t \rceil + j)$, and $\left\langle \vec{w}_t^L \right\rangle_x = w_t^x$ where $x = 0,...,\varpi \;\; j = 0,...,\varpi$ , and $xp$ represents the retirement age.

Given the dynamics of the assets and liabilities (4.2) and (4.4), the funding ratio under consideration is:

$$FR_t = \frac{A_t}{L_t} = \frac{\vec{w}_t^{S}{}' \vec{S}_t + \vec{w}_t^{D}{}' \vec{D}_t^A}{\vec{w}_t^{L}{}' \underline{\underline{\mathrm{P}_t}} \vec{D}_t^L} \tag{4.5}$$

The funding ratio put option is bought at time zero, which can be done without loss of generality. As a consequence of the setup of the model we can reduce the valuation of a claim conditional upon survival such as the liabilities to that of a nondefaultable claim, so we do not actually need to take into account which of the participants survive in our formulas, see Schrager(2006). The liabilities can then be seen as promised benefits discounted by the sum of the mortality rate and the nominal interest rates. Under the risk neutral measure $Q$, the price of this option at time 0 can be expressed as

$$V(A_0, L_0) = E^Q \left\{ e^{-\int_0^{T_O} r_t dt} \cdot L_{T_O} \left( FR^{\min} - FR_{T_O} \right)_+ \mid H_0 \right\} \tag{4.6}$$

But to value the funding ratio put option, it is more convenient to rewrite (4.6) by changing the numeraire. We choose the assets as the numeraire. We rewrite (2.4) as

$$L_{T_O} \cdot (FR^{\min} - FR_{T_O})_+ = A_{T_O} \cdot FR^{\min} \cdot \left( \frac{1}{FR_{T_O}} - \frac{1}{FR^{\min}} \right)_+ \tag{4.7}$$

Under the new measure $Q^{Assets}$ the value of the option at time 0 is

$$V(A_0, L_0) = A_0 \cdot FR^{\min} \cdot E^{Q^{Assets}} \left\{ \left( \frac{1}{FR_{T_O}} - \frac{1}{FR^{\min}} \right)_+ \mid H_0 \right\} \tag{4.8}$$

With this change of numeraire the discounting term is now outside the expectation operator. We will value the funding ratio put option using equation (4.8). So we need the dynamics of the inverse funding ratio under the new measure $Q^{Assets}$. As mentioned in section 3 we assume the market price of risk with respect to mortality uncertainty is such that the inverse of the funding ratio is a martingale under the pricing measure. Using Ito's lemma we get the stochastic differential equation of the inverse funding ratio:

[3] To be correct we should add survival probabilities which represent the probability of individuals to become x years old, here this chance of survival is set equal to one.

$$\frac{d(1/FR)_t}{(1/FR)_t} = -\frac{\vec{w}_t^{S}{}'\underline{\underline{\Sigma}}^S\left(\vec{S}_t\right)}{A_t} d\vec{W}_{S^A t}^{Q^{Assets}} + \left( \frac{\vec{w}_t^{D}{}'\underline{\underline{B}}_t^A \vec{D}_t^A}{A_t} - \frac{\vec{w}_t^{L}{}'\underline{\underline{\mathrm{P}_t B_t^L}} \vec{D}_t^L}{L_t} \right) \underline{\underline{\Sigma}}_Y^r \sqrt{\underline{\underline{X}}_t^r} d\vec{W}_{Y^r t}^{Q^{Assets}} - \frac{\vec{w}_t^{L}{}'\underline{\underline{\mathrm{P}_t B_t^\mu}} \vec{D}_t^L}{L_t} \underline{\underline{\Sigma}}_Y^\mu \sqrt{\underline{\underline{X}}_t^\mu} d\vec{W}_{Y^\mu t}^{Q^{Assets}} \tag{4.9}$$

Where the diagonal matrices $\underline{\underline{B_t}}$ consist of the elements equal to $\left\langle \underline{\underline{B_t}} \right\rangle_{j,j} = \vec{B}_t(t,j)$.

There are three sources of uncertainty in the funding ratio: due to risky asset prices, interest rates and mortality. Notice from the second term on the right hand side of equation (4.9) that a cash flow matching investment mix will reduce the volatility of the inverse funding ratio.

Given the stochastic differential equation of the inverse of the funding ratio under $Q^{Assets}$ we can use (4.8) to calculate the value of the funding ratio put options explicitly. Note however that the volatility terms in the process of the inverse funding ratio are stochastic processes themselves, which makes it complicated, if not impossible to write down exact closed formed solutions for the option value. Inspired by approximations in the context of interest rate derivatives, see Schrager and Pelsser (2006), we propose to freeze some stochastic elements in the volatility of the inverse funding ratio at their time zero value. So that formula (4.9) is approximated by:

$$\frac{d(1/FR)_t^*}{(1/FR)_t^*} = -\frac{\vec{w}_0^{S}{}'\underline{\underline{\Sigma}}^S\left(\vec{S}_0\right)}{A_0} d\vec{W}_{S^A t}^{Q^{Assets}} + \left( \frac{\vec{w}_0^{D}{}'\underline{\underline{B}}_t^A \vec{D}_0^A}{A_0} - \frac{\vec{w}_0^{L}{}'\underline{\underline{\mathrm{P}_0 B_t^L}} \vec{D}_0^L}{L_0} \right) \underline{\underline{\Sigma}}_Y^r \sqrt{\underline{\underline{X}}_0^r} d\vec{W}_{Y^r t}^{Q^{Assets}} - \frac{\vec{w}_0^{L}{}'\underline{\underline{\mathrm{P}_0 B_t^\mu}} \vec{D}_0^L}{L_0} \underline{\underline{\Sigma}}_Y^\mu \sqrt{\underline{\underline{X}}_0^\mu} d\vec{W}_{Y^\mu t}^{Q^{Assets}} \tag{4.10}$$

This formula shows that regarding the investment portfolio, we actually assume a rebalancing portfolio in the approximation[4]: we freeze the percentage invested in the individual risky assets ($w_{i,t}^S{}' S_{i,t} / A_t$ is approximated by $w_{i,0}^S{}' S_{i,0} / A_0$) and the percentage invested in zero coupon bonds of different maturities as at time 0 ($w_{j,t}^D{}' D_{j,t}^A / A_t$ is approximated by $w_{j,0}^D{}' D_{j,0}^A / A_0$). This may not be a very restricting assumption since in practice the portfolio will be rebalanced on a regular basis. With respect to the liabilities, in the approximation we freeze the individual 'with mortality discounted' cash flows with different maturities as a percentage of total liabilities. So the implicit assumption is that the liabilities are stable over time.

Now $\left(1/FR\right)_t^*$ is lognormally distributed. In matrix notation:

[4] We could also use $w_{i,t}^S{}' S_{i,0} / A_0$ respectively $w_{j,t}^D{}' D_{j,0}^A / A_0$ as an approximation if some other investment strategy would be at hand, since *w* is predictable.

$$d(1/FR)_t^* = \begin{pmatrix} \partial_S (1/FR)_t^* \\ \partial_r (1/FR)_t^* \\ \partial_\mu (1/FR)_t^* \end{pmatrix}' \begin{pmatrix} \underline{\underline{\Sigma}}^S\left(\vec{S}_0\right) & 0 & 0 \\ 0 & \underline{\underline{\Sigma}}_Y^r \sqrt{\underline{\underline{X}}_0^r} & 0 \\ 0 & 0 & \underline{\underline{\Sigma}}_Y^\mu \sqrt{\underline{\underline{X}}_0^\mu} \end{pmatrix} \underline{\underline{C}} \begin{pmatrix} d\tilde{\vec{W}}_{S^A t}^{Q^{Assets}} \\ d\tilde{\vec{W}}_{Y^r t}^{Q^{Assets}} \\ d\tilde{\vec{W}}_{Y^\mu t}^{Q^{Assets}} \end{pmatrix} = \partial_{All} (1/FR)_t^* {}' \underline{\underline{\Sigma}}^{All} \underline{\underline{C}} d\tilde{\vec{W}}_t^{Q^{Assets}} \tag{4.11}$$

Where the $\tilde{W}$ denote uncorrelated Brownian motions, and $\underline{\underline{C}}$ results from the Choleski decomposition. So that we get:

$$\begin{aligned} 1/FR_{T_O}^* &\sim \log N(m_{1/FR^*}, \sigma_{1/FR^*}^2) \\ m_{1/FR^*} &= \ln \tfrac{1}{FR_0} - \tfrac{1}{2}\sigma_{1/FR^*}^2 \\ \sigma_{1/FR^*}^2 &= \int_0^{T_O} \left( \partial_{All} (1/FR)_u^* {}' \underline{\underline{\Sigma}}^{All} \underline{\underline{C}}\,\underline{\underline{C}}' \underline{\underline{\Sigma}}^{All} \partial_{All} (1/FR)_u^* \right) du \end{aligned} \tag{4.12}$$

Where $\sigma_{1/FR^*}^2$ is the volatility of the inverse funding ratio. The option price at time 0 is given by:

$$\begin{aligned} V(FR_0,0) &= A_0 FR^{\min} E^{Q_{Assets}} \left\{ \left( \tfrac{1}{FR}{}_{T_O}^{*} - \tfrac{1}{FR^{\min}} \right)_+ \mid H_0 \right\} \\ V(FR_0,0) &= A_0 FR^{\min} \left[ \tfrac{1}{FR_0} \cdot \Phi(d_1) - \tfrac{1}{FR^{\min}} \cdot \Phi(d_2) \right] \\ d_1 &= \frac{\ln(FR^{\min} / FR_0) + \tfrac{1}{2}\sigma_{1/FR^*}^2}{\sigma_{1/FR^*}} \\ d_2 &= d_1 - \sigma_{1/FR^*} \end{aligned} \tag{4.13}$$

$\varphi$ and $\Phi$ are the probability density and cumulative probability density function respectively of the Gaussian distribution with $m = 0, \sigma^2 = 1$.

If one assumes mortality to be deterministic, formulas (4.9)-(4.13) are adapted by changing the weights of the liabilities to $\vec{w}_t^{L*}{}' = \vec{w}_t^{L}{}' \underline{\underline{\mathrm{P}}}_t$, and let the term due to uncertainty in mortality vanish.

## 5. Example

In this section we illustrate the approximation to the value of funding ratio put options with a numerical example.

Within the stylized pension fund used in the example the liabilities are represented by a single payment at $T_L$ for the average cohort of all participants aged $x$ at time $t = 0$. This is an assumption that is often made when obtaining first insights into the mechanisms of pension fund dynamics.

On the asset side of the balance sheet of the stylized pension fund we assume the fund invests in a well-diversified reference portfolio of risky assets and in zero coupon bonds with maturity $T_A$. With respect to the reference portfolio of risky assets, no assumptions are made regarding the decomposition of this portfolio (equities, real estate, credits, alternatives) or regarding the dynamics of the individual components. We simply work on an aggregate level. The portfolio is assumed to be self-financing: there are no contributions and no withdrawals up until time $T_L$ when the fixed single payment takes place.

The dynamics of assets and liabilities are determined by a Black Scholes Vasicek model for risky assets and interest rates respectively, where the latter is a special case of (4.1) where $g_0^r = 0, g_Y^r = 1, \alpha_1^r = 1, \beta_1^r = 0$, and a Gaussian Makeham model for mortality intensity, which is a special case of (4.3) where $g_0^\mu = 0, g_{Y_1^\mu}^\mu(x) = 1, g_{Y_2^\mu}^\mu(x) = c^x$.

Under the risk neutral measure $Q$, the Black-Scholes Vasicek model is given by:

$$\begin{aligned} dr_t &= a(\theta - r_t)dt + \sigma_r dW_{rt}^Q \\ dS_t &= r_t S_t dt + \sigma_S S_t dW_{St}^Q \\ dW_{rt}^Q dW_{St}^Q &= \rho dt \end{aligned} \tag{5.1}$$

where $W_{rt}^Q$ and $W_{St}^Q$ are correlated Brownian Motions. The time $t$ price of a zero coupon bond with maturity $T$ can be written as $D(r_t, t, T) = e^{A(t,T) - B(t,T) r_t}$, where $B(t,T) = [1 - e^{-a(T-t)}]/a$.

Under the real-world measure $P$, the Gaussian Stochastic Makeham model is given by:

$$\begin{aligned} \mu_{x+t} &= Y_{1t} + Y_{2t} \cdot c^x \\ dY_{it}^\mu &= a_{Y_i}(\theta_{Y_i} - Y_{it})dt + \sigma_{Y_i} dW_{Y_i t}^P, i = 1,2 \end{aligned} \tag{5.2}$$

where for simplicity we assume $dW_{Y_1 t}^P dW_{Y_1 t}^P = 0$ The survival probability at time $t$ up till time $T$ of an $x$ year old is given by ${}_{T-t}p_x(t) = e^{A^\mu(x,t,T) - B_1^\mu(x,t,T)Y_{1t} - B_2^\mu(x,t,T)Y_{2t}}$.

The funding ratio can then be written as

$$FR_t = \frac{w_t^S S_t + w_t^D D(r_t, t, T_A)}{w^L{}_{T_L - t}\, p_x(t) D(r_t, t, T_L)} \tag{5.3}$$

Where $w^L > 0$ is a constant. The dynamics of the inverse funding ratio (4.9) is then given by

$$\frac{d(1/FR)_t}{(1/FR)_t} = \left( \frac{w_t^D D(r_t, t, T_A)}{A_t} B(t, T_A) - B(t, T_L) \right) \sigma_r dW_{rt}^{Q^{Assets}} - \frac{w_t^S S_t}{A_t} \sigma_S dW_{St}^{Q^{Assets}} - \vec{B}^{\mu}(t, T_L) \Sigma_{\mu} d\vec{W}_{\mu t}^{Q^{Assets}} \tag{5.4}$$

Where the drift term is zero since the inverse funding ratio is a martingale under $Q^{Assets}$. Notice that $w_t^S S_t / A_t$ is the percentage of risky assets in the portfolio at time $t$ and equivalently $w_t^D D(r_t, t, T_A) / A_t$ is the percentage of bonds at time $t$. The approximation of the dynamics of the inverse funding ratio (4.10) is given by

$$\begin{aligned} \frac{d(1/FR)_t^*}{(1/FR)_t^*} &= \left( \frac{w_0^D D(r_0, 0, T_A)}{A_0} B(t, T_A) - B(t, T_L) \right) \sigma_r dW_{rt}^{Q^{Assets}} - \frac{w_0^S S_0}{A_0} \sigma_S dW_{St}^{Q^{Assets}} - B^{\mu}(t, T_L) \Sigma_{\mu} d\vec{W}_{\mu t}^{Q^{Assets}} \\ &= \left( (1 - \tilde{w}^S) B(t, T_A) - B(t, T_L) \right) \sigma_r dW_{rt}^{Q^{Assets}} - \tilde{w}^S \sigma_S dW_{St}^{Q^{Assets}} - B^{\mu}(t, T_L) \Sigma_{\mu} d\vec{W}_{\mu t}^{Q^{Assets}} \end{aligned} \tag{5.5}$$

Where $\tilde{w}^S$ denotes the percentage invested in the well-diversified reference portfolio of risky assets and $(1 - \tilde{w}^S)$ is invested in the zero coupon bonds with maturity $T_A$. Actually if the pension fund has a continuously rebalancing portfolio the approximation formula (5.5) equals (5.4).

The standard deviation of the inverse of the funding ratio (4.12) equals

$$\begin{aligned} \sigma^2_{1/FR^{/\mu^*}} &= \int_0^{T_O} \left( (1 - \tilde{w}^S) B(t, T_A) \sigma_r - B(t, T_L) \sigma_r - \tilde{w}^S \sigma_S \rho \right)^2 dt + \int_0^{T_O} \left( \tilde{w}^S \sigma_S \sqrt{1 - \rho^2} \right)^2 dt \\ &+ \int_0^{T_O} \left( B_1^{\mu}(t, T_L) \sigma_{Y_1} \right)^2 dt + \int_0^{T_O} \left( B_2^{\mu}(t, T_L) \sigma_{Y_2} \right)^2 dt \end{aligned} \tag{5.6}$$

The value of the funding ratio put option is given by (4.13)

The values of the funding ratio put options are given in table 1, dependent on the moneyness at time 0, the percentage of risky assets in the portfolio, the correlation between risky assets and interest rates, and the time to maturity. Also the difference between a financial hedge including and excluding mortality are given. The latter represents a contingent claim which is comparable with other over the counter products such as the Equity Linked Swaptions and Traffic Light Options mentioned earlier.

The parameter values that are fixed in table 1 are as follows. The maturity of the liabilities $T_L$ is 20 years, the maturity of the discount bonds in the investment portfolio $T_A$ is 5 years. The average cohort of all participants is aged 50.

For the short term interest rate we choose an interest rate at start, $r_0$ equal to 4%, a volatility equal to 2%, a speed of mean reversion $a$ of 0.25, and $\theta$ equal to 0.048, which implies a zero coupon interest rate curve sloping upwards from 4% and converging towards 4.48% as time goes to infinity. The volatility of the equity portfolio is chosen equal to 20%.

The parameters for mortality intensity volatility are borrowed from Schrager (2006), who estimated the parameters for the Gaussian Makeham model using mortality coefficients of the Dutch male population from 1950 to 2002, which are available at the Dutch Central Bureau for Statistics. The parameter values are as follows $c = 1.11$, $a_{Y_1} = 0.028$, $a_{Y_2} = 0.0046$, $\sigma_{Y_1} \cdot 10^5 = 2$, $\sigma_{Y_2} \cdot 10^7 = 4$.

The strike level of the funding ratio is 100%. We will consider options with short maturities (1 and 3 yrs), because in practice options with long maturity are not often issued, regulation requirements are often short term requirements, and the simple model we have here is expected to be less accurate on the long term (for example, on the long term we may better choose a model with stochastic volatility for risky assets). The results using (4.13) are given in table 1.

| | | Funding ratio put option values as % of liabilities<br>values only financial risk / values including mortality<br>Dependence on %risky assets, correlation, funding ratio at start, time to maturity | | |
|---|---|---|---|---|
| | | $T_O$=1 | | |
| | **correlation** | **$FR_0$=95 ITM** | **$FR_0$=100 ATM** | **$FR_0$=105 OTM** |
| **$\%S_0$=0%** | **-0.5** | 5.02% / 6.21% | 1.02% / 3.17% | 0.03% / 1.34% |
| | **0** | 5.02% / 6.21% | 1.02% / 3.17% | 0.03% / 1.34% |
| | **0.5** | 5.02% / 6.21% | 1.02% / 3.17% | 0.03% / 1.34% |
| **$\%S_0$=25%** | **-0.5** | 5.29% / 6.48% | 1.82% / 3.51% | 0.34% / 1.63% |
| | **0** | 5.73% / 6.83% | 2.53% / 3.92% | 0.82% / 2.01% |
| | **0.5** | 6.14% / 7.16% | 3.08% / 4.30% | 1.27% / 2.35% |
| **$\%S_0$=50%** | **-0.5** | 6.44% / 7.40% | 3.46% / 4.58% | 1.59% / 2.61% |
| | **0** | 7.33% / 8.15% | 4.50% / 5.41% | 2.54% / 3.40% |
| | **0.5** | 8.09% / 8.81% | 5.35% / 6.13% | 3.34% / 4.10% |
| **$\%S_0$=75%** | **-0.5** | 7.95% / 8.68% | 5.19% / 5.99% | 3.19% / 3.96% |
| | **0** | 9.18% / 9.78% | 6.53% / 7.18% | 4.49% / 5.13% |
| | **0.5** | 10.21% / 10.74% | 7.63% / 8.20% | 5.58% / 6.14% |
| **$\%S_0$=100%** | **-0.5** | 9.57% / 10.14% | 6.95% / 7.57% | 4.90% / 5.51% |
| | **0** | 11.09% / 11.57% | 8.57% / 9.08% | 6.51% / 7.01% |
| | **0.5** | 12.38% / 12.80% | 9.92% / 10.36% | 7.86% / 8.31% |
| | | $T_O$=3 | | |
| | **correlation** | **$FR_0$=95 ITM** | **$FR_0$=100 ATM** | **$FR_0$=105 OTM** |
| **$\%S_0$=0%** | **-0.5** | 5.61% / 9.93% | 2.36% / 7.34% | 0.69% / 5.28% |
| | **0** | 5.61% / 9.93% | 2.36% / 7.34% | 0.69% / 5.28% |
| | **0.5** | 5.61% / 9.93% | 2.36% / 7.34% | 0.70% / 5.28% |
| **$\%S_0$=25%** | **-0.5** | 6.32% / 10.26% | 3.31% / 7.69% | 1.46% / 5.64% |
| | **0** | 7.47% / 10.89% | 4.65% / 8.36% | 2.68% / 6.29% |
| | **0.5** | 8.40% / 11.47% | 5.68% / 8.97% | 3.66% / 6.91% |
| **$\%S_0$=50%** | **-0.5** | 8.69% / 11.66% | 6.00% / 9.17% | 3.97% / 7.11% |
| | **0** | 10.48% / 12.95% | 7.92% / 10.52% | 5.86% / 8.47% |
| | **0.5** | 11.93% / 14.09% | 9.45% / 11.71% | 7.39% / 9.67% |
| **$\%S_0$=75%** | **-0.5** | 11.47% / 13.73% | 8.97% / 11.33% | 6.91% / 9.28% |
| | **0** | 13.73% / 15.59% | 11.34% / 13.27% | 9.29% / 11.25% |
| | **0.5** | 15.59% / 17.21% | 13.28% / 14.95% | 11.25% / 12.96% |
| **$\%S_0$=100%** | **-0.5** | 14.38% / 16.15% | 12.01% / 13.85% | 9.97% / 11.83% |
| | **0** | 17.04% / 18.51% | 14.78% / 16.30% | 12.78% / 14.32% |
| | **0.5** | 19.28% / 20.56% | 17.09% / 18.41% | 15.13% / 16.47% |

Table 1: Funding ratio put option values as percentage of the liabilities

In table 1 we see, as expected, that the option price is higher when the percentage of risky assets is higher, and also when the time to maturity is longer, 3 years instead of 1 year. Further, the option price is higher when correlation between risky assets and bonds is positive. When the percentage of risky assets is zero the option price is independent of the correlation parameter: the funding ratio depends on the interest rate only.

Taking mortality risk into account, table 1 shows that option prices are much higher, especially when the percentage of risky assets is low, the option is ATM or OTM and/or maturity of the option is longer. It is clear that mortality risk is extensive, despite the fact that we didn't take into account any specific future trend mortality risk/ longevity risk. In this model the trend uncertainty effect can be estimated by adjusting the strike level of the funding ratio put options.

The mortality risk effects in table 1 are of a more subtle form, they just result from volatility risk or the discrepancy between the present trend and observed mortality. Actually the mortality volatility risk works through on the option price as if the interest rate had a higher volatility. For practical applications one could translate the volatility of mortality into an implied volatility impact on the interest rate.

Notice that if the investment portfolio is continuously rebalanced then formula's (5.5) and (5.6) are exact. In table 2 we show the Monte Carlo results if the investment portfolio is static in the sense that the number of risky assets and the number of zero coupon bonds is fixed. In this case the approximation (5.5, 5.6) is not exact any more, which results in the differences between the approximated values in table 1 and the Monte Carlo results in table 2.

If the investment portfolio exists of risky assets (zero coupon bonds) only, then the approximation is exact, because the rebalancing assumption holds again. Thus, the approximation works better when the percentage of risky assets is low or high, because then the rebalancing assumption has less impact.

For 'in the money' options we see that the approximation slightly underestimates the value, while for 'out of the money' options the approximation slightly overestimate the option value. Whether the approximation is good enough to use depends on the specific application. The approximation method is of course not as exact as Monte Carlo simulation, but the calculation is easy and one can save a considerable amount of time.

**Monte Carlo Funding ratio put option values as % of liabilities**
**values only financial risk / values including mortality**
**(standard errors between brackets, rounded at two decimals)**
**Dependence on %stocks, correlation, funding ratio at start, time to maturity**

**$T_O=1$**

| | corr. | $FR_0=95$ ITM | $FR_0=100$ ATM | $FR_0=105$ OTM |
|---|---|---|---|---|
| $\%S_0=0\%$ | -0.5 | 5.02% (0%) / 6.21% (0%) | 1.02% (0%) / 3.17% (0%) | 0.03% (0%) / 1.34% (0%) |
| | 0 | 5.02% (0%) / 6.21% (0%) | 1.02% (0%) / 3.17% (0%) | 0.03% (0%) / 1.34% (0%) |
| | 0.5 | 5.02% (0%) / 6.21% (0%) | 1.02% (0%) / 3.17% (0%) | 0.03% (0%) / 1.34% (0%) |
| $\%S_0=25\%$ | -0.5 | 5.34% (0%) / 6.53% (0%) | 1.81% (0%) / 3.51% (0%) | 0.29% (0%) / 1.58% (0%) |
| | 0 | 5.8% (0%) / 6.92% (0%) | 2.52% (0%) / 3.92% (0%) | 0.74% (0%) / 1.92% (0%) |
| | 0.5 | 6.23% (0.01%) / 7.25% (0.01%) | 3.07% (0%) / 4.29% (0%) | 1.16% (0%) / 2.24% (0%) |
| $\%S_0=50\%$ | -0.5 | 6.52% (0.01%) / 7.47% (0.01%) | 3.46% (0%) / 4.58% (0%) | 1.51% (0%) / 2.53% (0%) |
| | 0 | 7.41% (0.01%) / 8.22% (0.01%) | 4.5% (0.01%) / 5.4% (0.01%) | 2.44% (0%) / 3.3% (0%) |
| | 0.5 | 8.18% (0.01%) / 8.89% (0.01%) | 5.33% (0.01%) / 6.11% (0.01%) | 3.23% (0.01%) / 3.99% (0.01%) |
| $\%S_0=75\%$ | -0.5 | 7.98% (0.01%) / 8.71% (0.01%) | 5.18% (0.01%) / 5.99% (0.01%) | 3.15% (0.01%) / 3.93% (0.01%) |
| | 0 | 9.2% (0.01%) / 9.81% (0.01%) | 6.52% (0.01%) / 7.18% (0.01%) | 4.44% (0.01%) / 5.09% (0.01%) |
| | 0.5 | 10.23% (0.01%) / 10.77% (0.01%) | 7.63% (0.01%) / 8.19% (0.01%) | 5.52% (0.01%) / 6.08% (0.01%) |
| $\%S_0=100\%$ | -0.5 | 9.56% (0.01%) / 10.15% (0.01%) | 6.96% (0.01%) / 7.56% (0.01%) | 4.89% (0.01%) / 5.52% (0.01%) |
| | 0 | 11.1% (0.01%) / 11.58% (0.01%) | 8.57% (0.01%) / 9.09% (0.01%) | 6.49% (0.01%) / 7% (0.01%) |
| | 0.5 | 12.39% (0.01%) / 12.8% (0.01%) | 9.92% (0.01%) / 10.36% (0.01%) | 7.84% (0.01%) / 8.31% (0.01%) |

**$T_O=3$**

| | corr. | $FR_0=95$ ITM | $FR_0=100$ ATM | $FR_0=105$ OTM |
|---|---|---|---|---|
| $\%S_0=0\%$ | -0.5 | 5.61% (0%) / 5.61% (0%) | 2.36% (0%) / 7.34% (0%) | 0.69% (0%) / 5.28% (0%) |
| | 0 | 5.61% (0%) / 5.61% (0%) | 2.36% (0%) / 7.34% (0%) | 0.69% (0%) / 5.28% (0%) |
| | 0.5 | 5.61% (0%) / 5.61% (0%) | 2.35% (0%) / 7.33% (0%) | 0.69% (0%) / 5.28% (0%) |
| $\%S_0=25\%$ | -0.5 | 6.42% (0.01%) / 6.42% (0.01%) | 3.31% (0%) / 7.7% (0%) | 1.37% (0%) / 5.54% (0%) |
| | 0 | 7.58% (0.01%) / 7.58% (0.01%) | 4.64% (0.01%) / 8.33% (0.01%) | 2.52% (0%) / 6.13% (0%) |
| | 0.5 | 8.52% (0.01%) / 8.52% (0.01%) | 5.64% (0.01%) / 8.92% (0.01%) | 3.45% (0.01%) / 6.69% (0.01%) |
| $\%S_0=50\%$ | -0.5 | 8.81% (0.01%) / 8.81% (0.01%) | 5.99% (0.01%) / 9.17% (0.01%) | 3.81% (0.01%) / 6.95% (0.01%) |
| | 0 | 10.59% (0.01%) / 10.59% (0.01%) | 7.88% (0.01%) / 10.48% (0.01%) | 5.68% (0.01%) / 8.29% (0.01%) |
| | 0.5 | 12.03% (0.01%) / 12.03% (0.01%) | 9.39% (0.01%) / 11.62% (0.01%) | 7.18% (0.01%) / 9.44% (0.01%) |
| $\%S_0=75\%$ | -0.5 | 11.51% (0.01%) / 11.51% (0.01%) | 8.96% (0.01%) / 11.32% (0.01%) | 6.82% (0.01%) / 9.18% (0.01%) |
| | 0 | 13.76% (0.01%) / 13.76% (0.01%) | 11.31% (0.01%) / 13.23% (0.01%) | 9.2% (0.01%) / 11.15% (0.01%) |
| | 0.5 | 15.63% (0.02%) / 15.63% (0.02%) | 13.23% (0.02%) / 14.92% (0.02%) | 11.12% (0.01%) / 12.85% (0.01%) |
| $\%S_0=100\%$ | -0.5 | 14.37% (0.02%) / 14.37% (0.02%) | 12.03% (0.01%) / 13.85% (0.01%) | 9.95% (0.01%) / 11.86% (0.01%) |
| | 0 | 17.06% (0.02%) / 17.06% (0.02%) | 14.77% (0.02%) / 16.32% (0.02%) | 12.76% (0.02%) / 14.29% (0.02%) |
| | 0.5 | 19.3% (0.02%) / 19.3% (0.02%) | 17.09% (0.02%) / 18.42% (0.02%) | 15.12% (0.02%) / 16.45% (0.02%) |

Table 2: Funding ratio put option values as percentage of the liabilities

## 6. Conclusions

This paper analyzes funding ratio put options in an asset liability management framework, taking into account financial risks and actuarial risks. The value of a funding ratio put option which is based on financial risks only can be used as a benchmark price for financial derivative solutions, such as interest rate swaps/swaptions, equity put options and more exotic products such as Equity Linked Swaptions and Traffic Light Options.

Using the value of funding ratio put options with broader risk management applications in mind however it is advisable to also take into account non-financial risks. Numerical examples indicate that ignoring the impact of mortality volatility risk can be significant. A pragmatic way to take into account mortality volatility risk is just by adjusting the interest rate volatility, while a higher strike of the option can be used to approximate the effect longevity risks.

The price of a funding ratio put option starting from a funding ratio of 100% can be seen as a risk measure that indicates the minimum required level of free equity to protect the rights of the participants. Our results indicate the extent to which mortality volatility risks effect the price to 'insure' the pension fund against insolvency.

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