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Intra-group Risk Sharing under Financial Fairness

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Abstract

We examine intra-group risk sharing under the assumption that prices are given exogenously. Two situations are considered: one in which the entities that constitute the group choose to trade only among each other, and one in which the entities have limited access to the external market. In the first situation we prove that, if agents are expected utility maximizers, there is a unique risk-sharing rule that preserves value according to the exogenous rule and that is Pareto optimal among feasible allocations. This provides an extension of a result by Balasko (1979). In the second situation we give necessary and sufficient conditions for the internal market to be completable in the sense that all agents can reach all positions, subject only to the budget constraint, by combining their own partial access to the external market with internal trades under the given pricing rule.

Keywords: fixed-price equilibrium, internal trading, risk sharing, Pareto efficient allocations.

JEL codes: D51, D52, D53, C62.

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1 Introduction

In the theory of general equilibrium (Arrow and Debreu, 1954), agents are typically thought of as being neutral to one another, with no ex-ante preferred trading partnerships. Yet there are many examples of groups of economic agents who explicitly or implicitly form sub-markets within the broader context of the economy as a whole. One may think for instance of partners in a joint venture, participants in a pension fund, business units within a corporation, or heirs to an estate. Reasons for the existence of such groups can be legal, geographical, socio-cultural, or purely economic (related to transaction costs). The formation process of economic collectives is studied in institutional economics (Coase, 1937; Ostrom, 1990). Given the existence of such collectives, we focus in this paper on intra-group trading, and in particular on sharing of financial risks. Compared to the existing literature on risk sharing, the main new element that we consider is the assumption that an exogenous pricing rule is given under which all trades take place. Such a rule may be derived from external market prices, or, in some instances, may be imposed by an external authority; it may also be the result of a prior negotiation process between the participants in the collective. Under the supposition that the external prices are accepted by the participants as fair, we shall refer to the condition that trades must take place at exogenous prices as "financial fairness".

Risk sharing has been studied by Borch (1962) for collectives of agents whose preferences are described by von Neumann-Morgenstern utility functions, and who hold common probabilistic beliefs. Borch shows that Pareto efficient risk sharing for such collectives must be described by a reallocation rule that takes the aggregate endowment as input, and he provides a parametrization of all efficient reallocation rules. In later work, the nonuniqueness of efficient risk sharing has generally been viewed as giving rise to a "bargaining set", from which an element is chosen by an additional rule; exogenously given prices are not usually part of the framework. The notion of individual rationality reduces the bargaining set, but in general does not lead to uniqueness (cf. for instance Azrieli and Lehrer (2007)).

The problem of finding Pareto efficient reallocation rules may be approached from the perspective of general equilibrium theory. Indeed, in the classical Arrow-Debreu-McKenzie theory the existence is proven of pricing rule which is such that the allocation rule that is obtained by allowing all of the agents to maximize their utilities under this pricing rule provides a risk-sharing rule that is both feasible (in the sense that it defines a redistribution of the aggregate endowment) and Pareto efficient. The pricing rule is *endogenous* in this setting; it depends on the agents' utility functions. Several authors have studied equilibrium concepts in which prices are subject to *exogenous* constraints. In particular, Drèze (1975) has formulated an equilibrium concept in which prices must lie between certain bounds, and Benassy (1975) studied equilibria with fixed prices. The place of the pricing mechanism as an instrument of reconciling feasibility with utility maximization is taken in these studies by rationing schemes. It was shown by Herings and Konovalov (2009) that Pareto efficiency is in general not achieved by reallocation rules generated by fixed-price Drèze equilibria.

Given that utility maximization does not appear to be helpful in finding efficient risk-sharing rules in a fixed-price environment, one may ask whether such rules may still be found if individual optimality is disregarded (so that we should speak of *allocations* rather than of *equilibria*). A positive answer to this question has been provided by Balasko (1979). Balasko does not require that the preferences of agents can be described by von Neumann-Morgenstern utilities. Using methods of differential topology, he proves that fixed-price efficient allocations do exist. Moreover, he shows that such allocations are unique when the imbalance between aggregate demand and aggregate supply for given asset prices and agents' initial endowments is sufficiently small. For general price-income vectors, he proves that the number of allocations is piecewise constant as a function of the price-income vector, and that the number is odd outside a set of measure zero.

In the first part of this paper, we specialize the model of Balasko to the situation that was considered by Borch: collectives consisting of von Neumann-Morgenstern agents with common probability beliefs. In this setting, we sharpen Balasko's result by showing that, for any given initial endowments and exogenous pricing functional, there exists a *unique* allocation that is both financially fair and Pareto efficient among feasible allocations. To arrive at this result, we employ nonlinear Perron-Frobenius theory (Lemmens and Nussbaum, 2012) rather than differential topology. We also provide an iterative algorithm by which the unique Pareto efficient and financially fair allocation can be computed, and give a proof of its convergence. In the second model that we consider in this paper, we assume that agents each have access to part of the external market, and the purpose of internal trading is to allow agents to achieve individual optimality rather than Pareto optimality. A graph-theoretic criterion is given under which this can be achieved.

The fixed-price model was motivated by Balasko with reference to economies with centrally regulated prices. He also proposed the model as a theoretical tool to investigate whether Pareto efficiency is compatible with given prices and given income distributions. Additional motivation for fixed-price models comes from "internal markets" that can be formed by collectives. Below we briefly discuss a few examples of such situations.

An internal market may be formed within the bounds of a large corporation that consists of a number of divisions, each of which may have several branches that may be further divided into business units, and so on. Such entities are all part of one corporation, but nevertheless they may act to some extent under their own discretion, and in particular they may have their own budgets. Indeed the model of large enterprises as syndicates of "corporate entrepreneurs" has gained some popularity (Birkinshaw, 2000; Ellig, 2001). Entities within a corporation may share risks among each other on the internal market. Accounting rules may impose that the contracts concluded between different entities should be of zero market value as measured by external prices, so as to prevent perceived cross-subsidies. Even when entities would be allowed to effect similar trades outside the corporation, they may prefer internal trades in order to avoid transaction costs, such as the costs of intermediation and costs of credit insurance.

For another example of intra-group risk sharing, one can think of a collective pension fund in which rules for benefits and premium adjustment essentially imply contracts between younger and older generations. Again it would be reasonable to impose financial fairness on an ex-ante basis; in fact, it has been argued that collective systems are not sustainable if financial fairness is not imposed (Kocken, 2012). The gain from internal trading may again lie in the avoidance of transaction costs, and, to the extent that participation is mandatory, in inclusion of future generations into the risk-sharing collective.

Yet another situation that exhibits a form of intra-group risk sharing occurs when a given risk is divided among a number of agents in such a way that each agent receives a predefined market value. When the agents have different attitudes towards risk, it may be advisable to use a nonproportional sharing rule. An example of such risk sharing is given in Section 4.3 below (cf. Fig. 1).

The results of our paper may also be applied in the context of stochastic bankruptcy games as defined by Habis and Herings (in press). If the size of the inheritance is uncertain, then even when proportional division is imposed in terms of expected value, the actual division given a particular outcome may use different proportions depending on the amount that is to be divided. The allocation rule that would be obtained from this paper depends on the probability distribution of the size of the estate and the utility functions of the claimants; it may well happen that small claimants receive a relatively larger portion when the size of the inheritance is small, as it is the case in the Talmud rule (Aumann and Maschler, 1985).

A consequence of the use of exogenous prices is that, in contrast to the situation in the Arrow-Debreu-McKenzie setting, the notion of initial endowment is not essential in the framework that we use; only the *value* of the initial endowment (as given by the exogenous pricing rule) is of importance. In fact it is sufficient that the "fair share" of each agent in the market value of the collective endowment is well-defined. The size of the share may in some cases be prescribed in other ways than by the market value of an individual endowment. For instance, one may think of a project that has an uncertain financial outcome and to which agents contribute in kind, for example by spending a certain number of hours of work on the project. The fair share of a given agent may then be determined by the number of hours worked by this agent, relative to the total number of hours of work spent on the project by all participants. The framework of this paper can still be applied. In view of this interpretation, we sometimes use the term "contribution" below instead of "market value of initial endowment".

Limited access to external markets is essential in the context of this paper. As noted above, there can be various reasons why economic agents would share risks within a collective rather than through external markets, and below we take this situation as given. Transaction costs may play a role, and one may think of modeling these explicitly in terms of bid-ask spreads, so that it becomes possible to indicate at which level of transaction costs agents will choose to trade internally; we shall however not pursue this here. Furthermore, in the model that we consider in this paper, agents do not have decision power with regard to the risks that are given to them. These risks are given, and the agents can change their risk exposures only by trading at exogenous prices. Consequently, we do not address here the theory of syndicates as developed by Wilson (1968), and leave to future work the extension of the mathematical machinery developed in this paper towards group decision problems.

The paper is organized as follows. Notation and assumptions are established in the section following this introduction. The theorem concerning the existence and uniqueness of Pareto efficient and financially fair allocations is in Section 3. Some examples are discussed in Section 4. In Section 5, we turn to a model in which agents have limited access to the external market. A graph-theoretic criterion for effective completeness is obtained. Finally, conclusions follow in Section 6. Some auxiliary results and most of the proofs are in the appendix.

2 Notation and assumptions

We start from a probability space $(\Omega, \mathcal{F}, P)$. We assume that random variables $X_i \in L^\infty(\Omega, \mathcal{F}, P)$ $(i = 1, \ldots, n)$ are given, which represent the initial risks carried by n agents. The total risk of all agents is denoted by $X := \sum_{i=1}^n X_i$. Our sign convention is that positive values of X indicate gains and negative values indicate losses, so that the term "risk" is to be understood as "uncertain outcome" without necessarily a negative connotation.

It will be assumed that agents are expected utility maximizers. The utility function of agent i is $u_i(\cdot)$, which is a function defined on an interval of the form (b_i, ∞) with

$b_i \in [-\infty, \infty)$. It will be assumed throughout that all utility functions are continuously differentiable, strictly increasing, and strictly concave, and that the Inada conditions for marginal utility are satisfied:

$$\lim_{x \downarrow b_i} u_i'(x) = \infty, \qquad \lim_{x \to \infty} u_i'(x) = 0. \tag{2.1}$$

As a result of the above assumptions, the marginal utility is a continuous and strictly decreasing function that takes all values on the positive real axis. The inverse marginal utility of agent i will be denoted by $I_i(\cdot)$. That is to say, I_i is the function from $(0, \infty)$ to (b_i, ∞) that is defined implicitly by

$$u_i'(I_i(z)) = z \qquad (z > 0).$$

The inverse marginal utility is a continuous and strictly decreasing function that has the interval (b_i, ∞) as its image. We write

$$b := \sum_{i=1}^{n} b_i \tag{2.2}$$

and

$$D := (b, \infty) \tag{2.3}$$

with the convention that $b = -\infty$ as soon as there is an index i such that $b_i = -\infty$. The space of continuous functions from D to $\mathbb{R}$ will be denoted by $C(D, \mathbb{R})$.

We will discuss the problem of redistributing the total risk among the agents. If all b_is are finite and if the total risk X is such that $P(X \leq \sum_{i=1}^{n} b_i) > 0$, then there is no way to redistribute the risk in such a way that the expected utility of each agent is finite. Therefore it is reasonable to require that $P(X \in D) = 1$. For convenience, we shall work under the stronger assumption that there exists a compact set A such that

$$A \subset D, \qquad P(X \in A) = 1. \tag{2.4}$$

The condition above implies that the total risk X is essentially bounded away from the critical limit b.

For vectors c, $d \in \mathbb{R}^n$, the notation $c < d$ will be used to indicate that $c_i < d_i$ for all i, whereas $c \lneq d$ means that $c \leq d$ and $c_i < d_i$ for at least one index i. Similar notation will be used for real-valued functions: in particular, for functions f, $g \in C(D, \mathbb{R})$ we write $f < g$ when $f(x) < g(x)$ for all $x \in D$. The nonnegative cone $\{c \in \mathbb{R}^n \mid c \geq 0\}$ is denoted by $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$ indicates the positive cone $\{c \in \mathbb{R}^n \mid c > 0\}$. When c is a given vector in $\mathbb{R}^n$ and $S = \{i_1, \ldots, i_k\}$ is a nonempty subset of the index set $\{1, \ldots, n\}$, we write $c_S := (c_{i_1}, \ldots, c_{i_k})$. If $(c^k)_{k=1,2,\ldots}$ is a sequence of vectors in $\mathbb{R}^n$, the notation $c^k \to \infty$ means that $c^k_i \to \infty$ for all $i = 1, \ldots, n$.

All equalities and inequalities involving stochastic variables are taken to hold P-almost surely. In addition to the measure P which is interpreted as an objective measure which is used in computing expected utilities, we assume that another measure Q is given which is equivalent to P and which represents a pricing measure; that is to say, the market price of any risk $Y \in L^\infty(\Omega, \mathcal{F}, P)$ is given by $E^Q Y$. To make the redistribution problem that we will consider meaningful, it will be assumed that the agents' initial endowments X_i satisfy

$$E^Q X_i > b_i \tag{2.5}$$

for $i = 1, \ldots, n$.

3 Financial fairness and Pareto efficiency

In the work of Borch (1962) on Pareto optimal reallocations of risk for collectives of von Neumann-Morgenstern agents, it is shown that efficient reallocations depend only on the outcome of total risk, rather than on the outcomes of individual risks, so that attention may be limited to reallocation functions that depend on a single variable. We use the term *risk-sharing rule* to refer to collections of functions $(y_1, \ldots, y_n)$ in $C(D, \mathbb{R})$ which satisfy the redistribution property:

$$\sum_{i=1}^{n} y_i(x) = x \qquad (x \in D). \tag{3.1}$$

The risk of agent i after reallocation is

$$Y_i := y_i(X)$$

and the corresponding utility for agent i is $E^P u_i(Y_i)$. A risk-sharing rule $(y_1, \ldots, y_n)$ is said to be *Pareto efficient* if there does not exist another risk-sharing rule $(\tilde{y}_1, \ldots, \tilde{y}_n)$ with associated post-redistribution risks $(\tilde{Y}_1, \ldots, \tilde{Y}_n)$ such that $(E^P u_1(\tilde{Y}_1), \ldots, E^P u_n(\tilde{Y}_n)) \gneq (E^P u_1(Y_1), \ldots, E^P u_n(Y_n))$. Borch (1962) obtained a parametrization of Pareto efficient risk-sharing rules in the following form.

Theorem 3.1 (Borch) *A risk-sharing rule $(y_1(x), \ldots, y_n(x))$ is Pareto efficient for any given total risk taking values in the domain D if and only if there exist a continuous function $J : D \to \mathbb{R}_+$ and positive constants $c_1, \ldots, c_n$ such that*

$$c_i u_i'(y_i(x)) = J(x) \qquad (x \in D) \tag{3.2}$$

for all $i = 1, \ldots, n$.

The theorem is proved by using the fact that, under the assumptions on utility functions as stated in Section 2, all Pareto efficient solutions can be found by optimizing weighted sums, with positive weights, of the individual utility functions. Moreover, the optimization can be done conditionally on the outcome x. The function $J(x)$ that appears in (3.2) is the Lagrange multiplier corresponding to the redistribution constraint (3.1). For detailed arguments, see for instance DuMouchel (1968).

Remark 3.2 The Borch condition (3.2) can be rewritten as follows:

$$y_i(x) = I_i\big(J(x)/c_i\big). \tag{3.3}$$

Since the functions $y_i(x)$ must satisfy the redistribution property, the following condition has to be satisfied for all $x \in D$:

$$\sum_{i=1}^{n} I_i\big(J(x)/c_i\big) = x. \tag{3.4}$$

For given $(c_1, \ldots, c_n)$ and given x, the above equation determines $J(x)$ uniquely since the function $z \mapsto \sum_{i=1}^n I_i(z/c_i)$ is strictly decreasing. Conversely, if the equation above is satisfied, then the functions $y_1(x), \ldots, y_n(x)$ determine a Pareto efficient risk-sharing rule. In this way, Borch's theorem provides a parametrization of Pareto efficient risk-sharing rules in terms of the positive constants $c_1, \ldots, c_n$. The effective number of parameters is in fact $n-1$ rather than n, since the reallocation rule that is generated by a positive vector $(c_1, \ldots, c_n)$ does not change if all numbers c_i are multiplied by the same positive constant, because the corresponding function $J(x)$ is then multiplied by the same constant so that the ratios $J(x)/c_i$ remain the same.

Remark 3.3 Given a positive vector $(c_1, \ldots, c_n)$, let $(y_1(x), \ldots, y_n(x))$ denote the Pareto efficient risk-sharing rule defined through (3.3) and (3.4). We can introduce a "group utility" $u(x)$ corresponding to the weights $(c_1, \ldots, c_n)$ by

$$u(x) = \sum_{i=1}^n c_i u_i(y_i(x)). \tag{3.5}$$

Under the assumption that the utility functions u_i are twice continuously differentiable, the inverse marginal utilities are continuously differentiable; it follows that the function J, being the inverse of the mapping $z \mapsto \sum_{i=1}^n I_i(z/c_i)$, is differentiable as well. Consequently, the redistribution functions y_i defined by (3.3) are likewise differentiable. We can then write (following Xia (2004))

$$u'(x) = \sum_{i=1}^n c_i u_i'(y_i(x)) y_i'(x) = J(x) \sum_{i=1}^n y_i'(x) = J(x) \tag{3.6}$$

where the second equality follows from Borch's condition (3.2) and the third uses the redistribution property (3.1). The function $J(x)$ therefore can be given the interpretation of a marginal group utility corresponding to a given set of weights $(c_1, \ldots, c_n)$.

Remark 3.4 For given weights $(c_1, \ldots, c_n)$, the relation between the group utility $u(x)$ as introduced above and the individual utility functions $u_i(x)$ can be described as follows: first define the function J in terms of the inverse marginal utilities by (3.4), then define the redistribution functions by (3.3), and finally define the group utility by (3.5). A more direct relationship can be written down in terms of the conjugate functions $u^*(z)$ and $u_i^*(z)$, which are defined by

$$u_i^*(z) = \sup_{x>b_i}(u_i(x) - zx), \qquad u^*(z) = \sup_{x>b}(u(x) - zx).$$

Indeed, computation shows that the following equality holds:

$$u^*(z) = \sum_{i=1}^{n} c_i u_i^*(z/c_i). \tag{3.7}$$

The group utility $u(x)$ can be retrieved from its conjugate by the standard formula $u(x) = \inf_{z>0}(u^*(z) + zx)$.

We have adopted the standard assumption of mathematical finance that market values of risks are given by a pricing measure Q, which is equivalent to the real-world measure P. The concept of a financially fair redistribution is introduced as follows.

Definition 3.5 A risk-sharing rule $(y_1(x), \ldots, y_n(x))$ for the given total risk X is *financially fair* if the market value of the post-redistribution risk of each agent is equal to the market value of the agent's pre-redistribution risk, that is to say, if

$$E^Q y_i(X) = E^Q X_i \qquad (i = 1, \ldots, n). \tag{3.8}$$

There are n equations in (3.8); however, given the market clearing requirement (3.1), the number of effective constraints is only $n - 1$. This is equal to the number of effective degrees of freedom in Borch's parametrization of Pareto efficient risk-sharing rules. It was already shown by Balasko (1979) that a Pareto efficient and financially fair (PEFF) allocation exists. Our main result states that, when the preferences of agents can be given by expected utility, this allocation is in fact unique.

Theorem 3.6 *In the situation described in Section 2, there is a unique risk-sharing rule that is both Pareto efficient and financially fair.*

The proof, which is provided in the appendix, follows a line of reasoning that is quite different from the arguments of Balasko (1979), which are based on differential topology. The proof also shows existence independently of Balasko's theorem. An outline of the argument can be given as follows. On the basis of the representation (3.3), the problem of finding a Pareto efficient and financially fair risk-sharing rule comes down to finding a pair (c, J) consisting of a positive vector $c \in \mathbb{R}^n_+$ and a function $J : D \to \mathbb{R}_+$ such that the following two equations are satisfied:

$$\sum_{i=1}^{n} I_i\big(J(x)/c_i\big) = x \qquad (x \in D) \tag{3.9}$$

$$E^Q I_i\big(J(X)/c_i\big) = E^Q X_i \qquad (i = 1, \ldots, n). \tag{3.10}$$

As noted above, the equation (3.9) determines the function $J(\cdot)$ uniquely when the positive vector $c = (c_1, \ldots, c_n)$ is given. This equation therefore defines a mapping from the positive cone $\mathbb{R}^n_{++}$ to the set of strictly decreasing functions from D to $\mathbb{R}_+$. On the other hand, given a strictly decreasing function J, the relation (3.10) can be used, for each $i \in \{1, \ldots, n\}$ separately, to determine $c_i > 0$. A mapping φ from the positive cone into itself is defined by combining the two operations. This mapping is homogeneous of degree 1 and monotonic. Solutions to the problem of finding a Pareto efficient and financially fair risk-sharing rule correspond to fixed points of the mapping φ in $\mathbb{R}^n_{++}$. It can be shown that if $c \in \mathbb{R}^n_{++}$ and $\lambda > 0$ are such that $\varphi(c) = \lambda c$, then necessarily λ equals 1. Therefore, to establish existence and uniqueness of the Pareto efficient and financially fair risk-sharing rule, it is sufficient to prove that the mapping φ has a uniquely defined (up to scalar multiplication) positive eigenvector with a positive eigenvalue. Conditions for existence and uniqueness of such eigenvectors are provided by nonlinear Perron-Frobenius theory. In particular we can use a theorem of Oshime (1983) which guarantees the existence of a unique positive eigenvector. To apply the theorem, we have to show that the mapping φ can be extended continuously to the nonnegative cone, and that it satisfies a number of additional properties. This program is worked out in the appendix.

Remark 3.7 It should be noted that Pareto efficiency, as defined above, holds with respect to all possible risk-sharing rules, not just the ones that are financially fair.

Remark 3.8 The applicability of the nonlinear Perron-Frobenius theory in the fixed-price framework rests on the monotonicity of the mapping φ. It is possible to phrase Walrasian equilibrium similarly in terms of a mapping φ, but in the case of competitive prices this mapping is not in general guaranteed to be monotonic.

Remark 3.9 Even when agents do not have access to the external market, the PEFF rule may not be attractive to them since it may well happen for some agents that the utility derived from their post-redistribution risk is less than the utility from their pre-redistribution risk. Agents might however be forced to take part in the redistribution by an institutional setting. As noted in the introduction, the notion of initial endowment or pre-redistribution risk is in fact not essential to the setting of this paper, and consequently the comparison of pre-redistribution and post-redistribution utility may take a different form. For instance, in the case of a group project to which each of the group members contributes by means of labor, participation in the group is attractive for a particular agent if the expected utility of post-redistribution income of this agent exceeds the expected utility of income that the agent could otherwise obtain from the same labor as would be contributed to the joint venture. Such a condition is not unlikely to be satisfied in a well-conceived project.

Given that a Pareto efficient and financially fair allocation exists and is unique, it is of interest to have an algorithm to compute the allocation in specific cases. Our method of proof in fact naturally suggests such an algorithm. As discussed above, the Pareto efficient and financially fair risk-sharing rule of Thm. 3.6 can be found by determining the unique positive eigenvector of the mapping φ that is defined as the composition of the mappings defined by (3.9) (where c is given) and (3.10) (where J is given). An associated normalized mapping ψ from the open unit simplex $\{(c_1, \ldots, c_n) \in \mathbb{R}^n_{++} \mid \sum_{i=1}^n c_i = 1\}$ into itself is defined by

$$\psi(c) = \frac{\varphi(c)}{\|\varphi(c)\|_1} \tag{3.11}$$

where $\|w\|_1 = \sum_{i=1}^n |w_i|$ is the 1-norm of a given vector $w \in \mathbb{R}^n$. Positive eigenvectors of the mapping φ correspond to fixed points of the mapping ψ. It is a natural idea to use an iterative method in order to find the fixed point of ψ, and the following theorem shows that this strategy is in fact always successful. The proof of the theorem is again given in the appendix.

Theorem 3.10 *Take $c^0 \in \mathbb{R}^n_{++}$, and define $c^1, c^2, \ldots$ successively by $c^{k+1} = \psi(c^k)$ where ψ is the normalized mapping given by (3.11). The sequence $(c^k)_{k=1,2,\ldots}$ converges to a limit, which is given by the unique positive eigenvector of φ.*

The theorem provides an effective way to compute the Pareto efficient and financially fair risk-sharing rule. The normalization at every step serves to simplify the proof of convergence. Since the eigenvalue associated to the positive eigenvector is equal to 1, as noted above, problems with divergence of iterates based on the original mapping φ are not to be expected in general. Numerical experience suggests that the iteration algorithm runs as well, or perhaps even better, without normalization.

In a few cases, the PEFF allocation can be found analytically. Some examples will be discussed in the next section. The simplest case is the one in which agents are identical. The proposition below shows that the PEFF risk-sharing rule is then the one that should be expected: everybody gets an equal share.

Proposition 3.11 *In the situation of Section 2, suppose that all agents have equal utility functions and equal wealths. In this case, the unique Pareto efficient and financially fair sharing rule is given by*

$$y_i(x) = \frac{x}{n} \qquad (i = 1, \ldots, n).$$

Proof. The equations (3.9) and (3.10) are satisfied by $J(x) = u_0'(x/n)$, where $u_0(x)$ is the common utility function, and $c = (1, 1, \ldots, 1)$. □

4 Examples

In this section we present three examples of Pareto efficient and financially fair risk-sharing rules. The first two examples show cases in which analytic formulas can be given, namely the situation in which all agents have power utility with the same coefficient of relative risk aversion, and the situation in which all agents have an exponential utility function. In the third example we consider agents who have power utility with different coefficients of relative risk aversion, and the risk-sharing rule is computed numerically.

4.1 Power utility with equally risk averse agents

Assume n agents who all have the same utility function given by

$$u_i(x) = \frac{x^{1-\gamma}}{1-\gamma} \qquad (x > 0,\ i = 1, \dots, n) \tag{4.1}$$

where $\gamma > 0$ and $\gamma \neq 1$, or by $u_i(x) = \log x$ for all i. In this case the Borch parametrization of Pareto efficient redistribution functions is given by

$$y_i(x) = \beta_i x \quad \text{with } \beta_i = \frac{c_i^{1/\gamma}}{\sum_{j=1}^n c_j^{1/\gamma}}$$

where γ should be taken equal to 1 in the case of the logarithmic utility function. Under the financial fairness constraint, the weights β_i are uniquely determined and we find that the redistribution functions

$$y_i(x) = \frac{E^Q X_i}{E^Q X}\, x \qquad (i = 1, \dots, n) \tag{4.2}$$

give the unique financially fair and Pareto efficient risk-sharing rule. The result actually does not depend on the degree of risk aversion. Each agent receives a fixed percentage of the total outcome, and the size of the percentage is determined by the ratio of the market value of the agent's individual risk with respect to the market value of total risk. As an allocation rule between agents who are equally risk averse, this seems quite reasonable. However, in the example below it will be seen that the result does depend on the chosen class of utility functions; there are cases in which agents have identical utility functions but the PEFF rule does not allocate proportionally.

4.2 Exponential utility

Assume n agents with exponential utility functions $u_i(x) = -\alpha_i e^{-x/\alpha_i}$, $i = 1, \ldots, n$. The coefficients $\alpha_i > 0$ are risk *tolerance* parameters rather than risk aversion parameters, i.e. larger values of α_i correspond to lower aversion against risk. Computation shows that the Borch parametrization of Pareto efficient redistribution functions is in this case given by

$$y_i(x) = \frac{\alpha_i}{\alpha} x + w_i$$

where $\alpha := \sum_{i=1}^n \alpha_i$ and

$$w_i = \alpha_i \big(\log c_i - \sum_{j=1}^n \frac{\alpha_j}{\alpha} \log c_j \big).$$

Under the financial fairness constraint, we must have $\frac{\alpha_i}{\alpha} E^Q X + w_i = E^Q X_i$, so that the unique Pareto efficient and financially fair risk-sharing rule in the exponential case is given by

$$y_i(x) = \frac{\alpha_i}{\alpha} x + \Big(\frac{E^Q X_i}{E^Q X} - \frac{\alpha_i}{\alpha} \Big) E^Q X. \tag{4.3}$$

In words, the post-redistribution risk of each agent consists of a fixed part and a part that is proportional to the total group result. The proportionality factor for each agent is not determined by the relative contribution of that agent as in the case of power utility with equally risk averse agents, but by agent's risk tolerance relative to the total risk tolerance of the group. The fixed part adds a compensation so that the scheme becomes financially fair. So even if the agents are equally risk averse, the sharing rule differs from the rule that was found in (4.2), except when the contributions of the agents are also the same. More generally, a proportional risk sharing rule is obtained when the relative risk tolerances of all agents are equal to their relative contributions. In the situation in which the relative contributions of the agents are comparable but one of them is far more risk tolerant than the others, then the rule (4.3) states that this agent should essentially absorb all of the risk and should compensate ("buy out") all other agents.

The iteration algorithm of Thm. 3.10 behaves in a particular way in the case of exponential utilities. Let a positive start vector $(c_1^0, \ldots, c_n^0)$ be given. The inverse marginal utilities are given by $I_i(z) = -\alpha_i \log z$, so that the equation (3.9) which determines $J(x)$ in terms of the c_i^0's is given by

$$-\sum_{i=1}^{n} \alpha_i \log \big(J(x)/c_i^0\big) = x.$$

Solving for $J(x)$ leads to

$$J(x) = \exp\Big(\sum_{i=1}^{n} \frac{\alpha_i}{\alpha} \log c_i^0\Big) \exp(-x/\alpha).$$

This shows that the function $J(x)$ depends on the start vector c^0 only through a multiplicative constant. Since such constants are inessential, this means that the vector c^1 that we find after one step of the iteration is already the fixed point that we are looking for. In other words, the algorithm converges after one step. Also in the case of equally risk averse CRRA agents that was discussed above, it can be verified that the iteration algorithm converges in one step.

4.3 Tranching between CRRA agents

When we consider CRRA agents with different coefficients of relative risk aversion, an analytical solution is in general no longer possible and numerical methods have to be used. Consider a situation in which a lognormal[1] risk is divided between three agents whom we will refer to as "senior", "mezzanine", and "equity", and who have coefficients of relative risk aversion that are equal to 2, 1, and 0.5 respectively. Specifically the total risk is of the form $X = \exp Z$ where Z is, under the pricing measure Q, a standard normal variable. The three agents take equal shares of this risk in terms of market value.

For the purposes of computation, the normal variable Z was replaced by a discrete variable on a equally spaced grid with step size 0.01, ranging from -4 to 4, so that the number of grid points N is equal to 801. Iteration was carried out according to the algorithm of Thm. 3.10, switching back and forth between the iterates c and the iterates

[1] A suitable truncation may be applied to satisfy the boundedness assumptions under which we have been working.

J. The latter were represented as vectors of length N. The iteration was started at the point $c^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$; successive iterates c^i were not renormalized. After five iterations, a solution was found which satisfies the market clearing constraint (3.9) up to 0.01% (in the sense that the L^∞ norm of the random variable $\sum_{i=1}^{3} y_i(X)/X - 1$, computed on the basis of the discretization as indicated, is less than 10^{-4}), whereas the fairness constraints (3.10) are satisfied up to machine precision by the design of the algorithm. After another three iterations, the L^∞ error in the market clearing constraint is less than 0.0001%. The resulting risk-sharing rule is shown in Fig. 1. Outcomes of total risk X are represented on the horizontal scale in terms of their Q-quantile; on the vertical scale it is shown how a particular outcome is divided among the three agents. For instance, the median outcome is distributed between the senior, mezzanine, and equity tranches approximately according to a 50:40:10 rule.

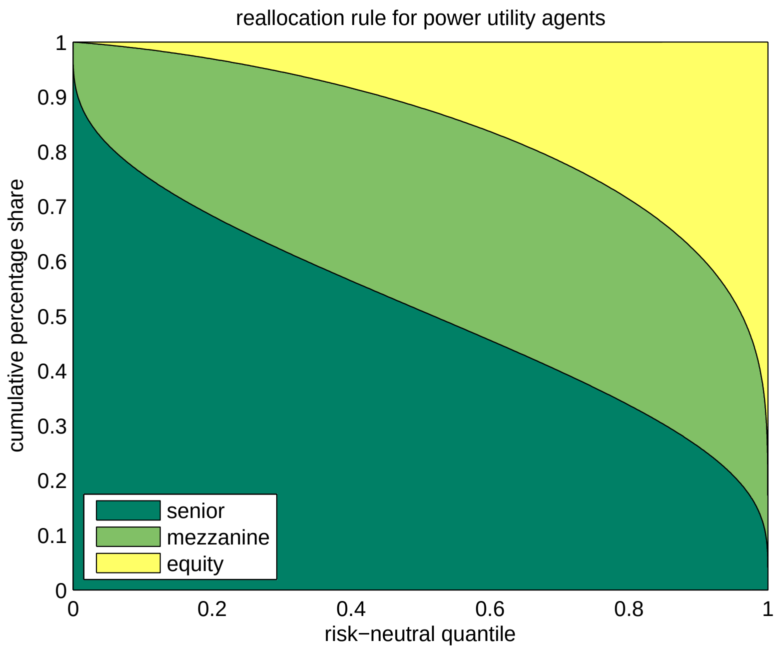


Figure 1: Pareto efficient and financially fair distribution of a lognormal risk among three agents with equal market shares and different coefficients of constant relative risk aversion.

5 Limited access and individual optimality

In this section we consider a situation in which agents have limited access to the market. Again we assume that agents can conclude contracts among each other at market-consistent values, and we ask under which conditions the internal trading leads to effective market completeness, in the sense that all agents are capable of taking any desired position. In such a situation, agents are able to achieve individual (first-best) optimality, subject only to their budget constraints. For an example of the situation that we have in mind, consider different trading desks within a firm which each specialize in a certain area of investment. The trading desks may be able to improve their risk profiles by concluding contracts among each other. As already argued above, the contracts between the desks could be required to be of zero market value, in order to prevent cross-subsidies. Whereas before we assumed no access to the external market and we were looking for Pareto optimality, now we are assuming partial access and look for conditions that will enable agents to achieve individual optimality.

The structure that we will discuss may be described in abstract terms as follows. For simplicity, we suppose that the outcome space Ω is finite; to indicate this assumption, we speak of a "discrete market". The assets are vectors $X_1, \ldots, X_m$ in $\mathcal{X} := \mathbb{R}^{|\Omega|}$. The limited access of agents to assets is described by specifying, for each agent index $i = 1, \ldots, n$, a subset S_i of the index set $\{1, \ldots, m\}$. The access structure can alternatively be represented in the form of a bipartite graph, in which there are n "agent nodes" and m "asset nodes", and there is an edge between agent node i and asset node j if and only if asset j is accessible to agent i, that is, when $j \in S_i$. We also assume that a pricing functional is given as a linear functional defined on the space $\mathcal{X}$. The pricing functional will be denoted by E^Q as before. Before returning to this setting, we first state a more abstract definition.

Definition 5.1 A *discrete market with limited access* is specified by a finite-dimensional vector space $\mathcal{X}$, a subspace $\mathcal{Z}$ of codimension 1 which represents the space of portfolios of price zero, and a collection of subspaces $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ satisfying $\mathcal{X}_i \not\subset \mathcal{Z}$ $(1 \leq i \leq n)$ which represent the portfolios that may be held by individual agents, given sufficient budget.

The condition $\mathcal{X}_i \not\subset \mathcal{Z}$ is imposed to make sure that each agent can hold nonzero wealth. Since the subspace $\mathcal{Z}$ of zero-price portfolios has codimension 1, the condition is equivalent to the requirement

$$\mathcal{X}_i + \mathcal{Z} = \mathcal{X} \qquad (i = 1, \ldots, n). \tag{5.1}$$

Example 5.2 Suppose there are two agents and two coins that will be tossed. Agent 1 can bet on the first coin, and agent 2 can bet on the second coin. The two coin tosses together give rise to four possible outcomes, so that payoffs can be described in a four-dimensional space $\mathcal{X}$. The payoff space $\mathcal{X}_1$ is generated by the vectors [1 1 0 0] and [0 0 1 1] that correspond to betting on "heads" and on "tails" respectively, and the payoff space $\mathcal{X}_2$ is generated by [1 0 1 0] and [0 1 0 1]. The condition (5.1) is satisfied for any pricing functional $[q_1\ q_2\ q_3\ q_4]$ with $q_i > 0$ for all i.

It is of interest to find conditions under which the agents, by trading among each other at exogenously given prices, are effectively facing a complete market. A necessary condition for this to be possible is that the agents together can reach the entire market. This notion is formalized in the following definition.

Definition 5.3 A discrete market with limited access $(\mathcal{X}, \mathcal{Z}, (\mathcal{X}_1, \ldots, \mathcal{X}_n))$ is *semicomplete* if

$$\sum_{i=1}^{n} \mathcal{X}_i = \mathcal{X}.$$

Example 5.2 (continued) The coin tossing market is not semicomplete; since $\mathcal{X}_1$ and $\mathcal{X}_2$ are two-dimensional spaces that do not coincide and that both contain the riskfree asset with payoff [1 1 1 1], their sum has dimension three. If the payoff space $\mathcal{X}_2$ would be extended by giving agent 2 access for instance to the payoff [1 0 0 0] (i.e. a bet on both coins showing heads), then semicompleteness would be obtained.

The conditions for agents to achieve arbitrary positions by concluding financially fair contracts among each other are formalized as follows.

Definition 5.4 Let a discrete market with limited access $(\mathcal{X}, \mathcal{Z}, (\mathcal{X}_1, \ldots, \mathcal{X}_n))$ be given. A collection of payoffs $(Y_1, \ldots, Y_n)$ is *accessible by trading* if there exist payoffs $X_1, \ldots, X_n \in \mathcal{X}$ and $Z_1, \ldots, Z_n \in \mathcal{Z}$, with $X_i \in \mathcal{X}_i$ for all $i = 1, \ldots, n$, such that

$$X_i + Z_i = Y_i \qquad (i = 1, \ldots, n) \tag{5.2}$$

$$\sum_{i=1}^{n} Z_i = 0. \tag{5.3}$$

Definition 5.5 A discrete market with limited access $(\mathcal{X}, \mathcal{Z}, (\mathcal{X}_1, \ldots, \mathcal{X}_n))$ is *completable* if all collections of payoffs $(Y_1, \ldots, Y_n)$ $(Y_i \in \mathcal{X},\ i = 1, \ldots, n)$ are accessible by trading.

We first prove a lemma showing that for completability it is sufficient to verify accessibility of all payoffs that have zero price.

Lemma 5.6 *A semicomplete market* $(\mathcal{X}, \mathcal{Z}, (\mathcal{X}_1, \ldots, \mathcal{X}_n))$ *is completable if and only if all collections of payoffs* $(Y_1, \ldots, Y_n)$ *with* $Y_i \in \mathcal{Z}$ $(i = 1, \ldots, n)$ *are accessible by trading.*

Proof. The condition is obviously necessary. To prove the sufficiency, take a collection of payoffs $(Y_1, \ldots, Y_n)$. For all $i = 1, \ldots, n$ we can, by (5.1), find $\hat{X}_i \in \mathcal{X}_i$ such that $Y_i - \hat{X}_i \in \mathcal{Z}$. By the assumption in the statement of the lemma, there are vectors $Z_1, \ldots, Z_n \in \mathcal{Z}$ and $X'_1, \ldots, X'_n \in \mathcal{X}$, with $X'_i \in \mathcal{X}_i$ for all i, such that $\sum_{i=1}^{n} Z_i = 0$ and $Y_i - \hat{X}_i = X'_i + Z_i$ for all i. The conditions (5.2) and (5.3) are then satisfied by defining $X_i = \hat{X}_i + X'_i$. $\square$

We can now state a necessary and sufficient condition for completability. Condition (5.4) below expresses that all zero-price payoffs can be decomposed into zero-price payoffs that are accessible by the individual agents.

Theorem 5.7 *A semicomplete market* $(\mathcal{X}, \mathcal{Z}, (\mathcal{X}_1, \ldots, \mathcal{X}_n))$ *is completable if and only if*

$$(\mathcal{Z} \cap \mathcal{X}_1) + \cdots + (\mathcal{Z} \cap \mathcal{X}_n) = \mathcal{Z}. \tag{5.4}$$

Proof. First assume that (5.4) holds. Take $Y_1, \ldots, Y_n \in \mathcal{Z}$, and define $Y = \sum_{i=1}^{n} Y_i \in \mathcal{Z}$. By (5.4), there exist $X_i \in \mathcal{X}_i \cap \mathcal{Z}$ for $i = 1, \ldots, n$ such that $Y = \sum_{i=1}^{n} X_i$. Define

$Z_i = Y_i - X_i$. We then have $\sum_{i=1}^n Z_i = 0$, and $Y_i = Z_i + X_i$ with $X_i \in \mathcal{X}_i$ for all $i = 1, \ldots, n$. Due to Lemma 5.6, this is enough to prove completability.

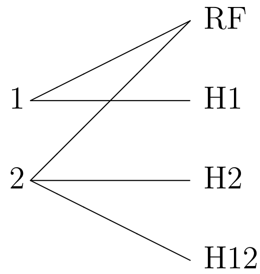
Now assume that the market is completable. The right-hand side of (5.4) clearly includes the left-hand side; it remains to prove the reverse inclusion. Take $Z \in \mathcal{Z}$, and let $(\alpha_1, \ldots, \alpha_n)$ be a collection of constants such that $\sum_{i=1}^n \alpha_i = 1$. Define the collection $(Y_1, \ldots, Y_n)$ by $Y_i = \alpha_i Z$. By the assumption of completability, there exist $Z_1, \ldots, Z_n \in \mathcal{Z}$ and $X_1, \ldots, X_n$ with $X_i \in \mathcal{X}_i$ for all i such that $\sum_{i=1}^n Z_i = 0$ and $Y_i = X_i + Z_i$ for all i. We then have $Y_i - Z_i \in \mathcal{Z} \cap \mathcal{X}_i$, and $\sum_{i=1}^n (Y_i - Z_i) = \big(\sum_{i=1}^n \alpha_i\big) Z - \sum_{i=1}^n Z_i = Z$. $\square$

We now return to a more concrete setting in which the available assets are given by $X_1, \ldots, X_m \in \mathcal{X}$, agent i can invest in assets X_j with $j \in S_i \subset \{1, \ldots, n\}$, and the subspaces $\mathcal{X}_i$ are given by $\mathcal{X}_i = \operatorname{span}\{X_j \mid j \in S_i\}$ $(i = 1, \ldots, n)$. The collection of index sets $(S_1, \ldots, S_n)$ defines a bipartite graph that will be referred to as the *access graph*. In this way we can represent a discrete market with limited access by means of a four-tuple $(\mathcal{X}, (X_1, \ldots, X_m), G, E^Q)$ where $\mathcal{X}$ is a finite-dimensional vector space, $(X_1, \ldots, X_m)$ is a collection of elements of $\mathcal{X}$, G is a bipartite graph with a vertex set consisting of m asset vertices and n agent vertices, and E^Q is a pricing functional. It will be assumed throughout that the assets X_i that are available for investment have nonzero prices, i.e. $E^Q X_j \neq 0$ for all $j = 1, \ldots, m$. The space $\mathcal{Z}$ of portfolios of price 0 is obtained as $\{X \in \mathcal{X} \mid E^Q X = 0\}$. In this context we can state the following.

Theorem 5.8 *A semicomplete market $(\mathcal{X}, (X_1, \ldots, X_m), G, E^Q)$ with linearly independent assets $X_1, \ldots, X_m$ is completable if and only if its access graph is connected.*

The proof of the theorem is given in Appendix B. We conclude with a simple example.

Example 5.2 (continued) A spanning set of assets may be chosen consisting of RF (the riskfree asset) with payoff vector [1 1 1 1], H1 (bet on heads of coin 1) with payoff vector [1 1 0 0], H2 with payoff [1 0 1 0], and H12 with payoff [1 0 0 0]. Under the assumption that agent 2 has access to asset H12, the access graph is as follows:



The graph is connected, and consequently the market is completable. None of the links can be eliminated without making the access graph disconnected.

6 Conclusions

In this paper we have analyzed intra-group trading under financial fairness in two different situations. First, we have assumed that the entities which constitute the group trade only among themselves. We have shown that in this case there is a unique Pareto efficient and financially fair (PEFF) risk-sharing rule, and we have given an algorithm by which this rule can be computed. Secondly, we have considered a situation in which the agents have limited access to the external market. Here we have given conditions under which internal trading under financial fairness allows the entities to achieve the same level of optimality as they would enjoy when they would have full access to the market.

Clearly our setting is limited. We have only looked at single-period markets whereas many interesting applications would call for a dynamic analysis. We have also assumed that the agents' risks are given, whereas frequently the risk taken by entities that form a group depends to a smaller or larger degree on decisions made by these entities. The framework of Borch (1962) has been extended to include agents' decision power by Wilson (1968) in his theory of syndicates; incorporation of the notion of financial fairness in this framework would be of interest.

It remains to be seen to what extent the uniqueness result of this paper can be extended to collectives of agents with heterogeneous beliefs and/or non-EU preferences. In recent years, the problem of efficient risk sharing for agents whose preferences are given by convex risk measures has drawn considerable interest (cf. for instance Jouini et al. (2008); Filipovic and Svindland (2008)). Due to the translation invariance of risk measures, uniqueness of efficient risk sharing under financial fairness is the same as uniqueness up to "rebalancing the cash". This property has been shown to hold under a strict convexity assumption by Filipovic and Svindland (2008). This assumption is restrictive however, and the uniqueness question remains open for several interesting classes of preferences.

In studies of the relation of group utility to individual utilities, it is customary to work with a fixed set of weights $(c_1, \ldots, c_n)$ (Hara et al., 2007; Jouini et al., 2013). When efficient risk sharing rules are formed under financial fairness, the weights become endogenous; in particular, they depend on the agents' utilities. As a result, there may be an impact on the way in which group utility changes when individual utilities change, which is the subject of Jouini et al. (2013).

On the technical side, it has been quite convenient for us to assume that the total risk confronted by the group of agents is bounded, but it should be possible to relax this assumption. In the case in which the utility functions of agents are defined on domains that are bounded from below, we have also assumed that the total risk is bounded below by a number that strictly exceeds the lower bound of total risk that could be tolerated by the collective. Again it should be possible to relax this assumption. We have presented an iterative algorithm and proved its convergence. For this algorithm to be used as a numerical method, truncations may be necessary and it would be of interest to investigate the continuity properties of the PEFF rule.

Appendix A

This appendix provides proofs in relation to Section 3. An outline of the proof of the main result Thm. 3.6 was already given above following the statement of the theorem. The proof strategy is essentially to make use of a fixed-point theorem relating to the mapping that is constructed as the composition of the mapping from vectors c to functions J defined by (3.9), and the mapping from functions J to vectors c defined by (3.10). The first mapping will be denoted by φ_1, and the second mapping by φ_2.

Recall that the domain D of group utility is defined as (b, ∞), where $b = \sum_{i=1}^{n} b_i$ and the b_is are the left limits of the domains of the utility functions of the individual agents. Within the space $C(D, \mathbb{R}_+)$ of continuous functions from D to $[0, \infty)$, equipped with the topology of pointwise convergence, we define the cone of strictly decreasing functions

$$\mathcal{L} = \{f \in C(A, \mathbb{R}) \mid f(y) < f(x) \text{ for all } x, y \in A \text{ s.t. } y > x\} \cup \{0\}.$$

Let the index set U be defined by $U = \{i \mid b_i = -\infty\}$; this set refers to the agents whose utility functions are defined on all of the real line, or in other words, agents who can tolerate unlimited losses. For $c \in \mathbb{R}_+^n$ such that $c_U > 0$, define

$$F(z, c) = \sum_{i:c_i>0} I_i(z/c_i) + \sum_{i:c_i=0} b_i. \tag{A.1}$$

This function is a continuous mapping from the product space $\mathbb{R}_{++} \times \{c \in \mathbb{R}_+^n \mid c_U > 0\}$ to $\mathbb{R}$.

Lemma A.1 *Let a vector $c \in \mathbb{R}_+^n$ be given, and suppose that $c_U > 0$. For any given $x \in D$, the equation $F(z, c) = x$ has a unique solution $z =: J_c(x) \in (0, \infty)$. The function $J_c : D \to (0, \infty)$ that is defined in this way is continuous and strictly decreasing.*

Proof. The function $F(\cdot, c) : (0, \infty) \to (b, \infty)$ is continuous and strictly decreasing, and it satisfies

$$\lim_{z \to \infty} F(z, c) = b, \qquad \lim_{z \downarrow 0} F(z, c) = \infty. \tag{A.2}$$

The function $J_c(\cdot)$ in the statement of the lemma is the inverse function of $F(\cdot, c)$. Given that $F(\cdot, c)$ is strictly decreasing and continuous, the same properties hold for J_c. □

For $c \in \mathbb{R}_+^n$, now define the function $\varphi_1(c) \in \mathcal{L}$ by

$$(\varphi_1(c))(x) = \begin{cases} J_c(x) \text{ as defined in Lemma A.1} & \text{if } c_U > 0 \\ 0 & \text{else} \end{cases} \tag{A.3}$$

for $x \in D$. It follows from the lemma above that $\varphi_1(c)$ as defined above does indeed belong to $\mathcal{L}$ for all $c \in \mathbb{R}_+^n$. The next step is to establish various properties of this mapping such as continuity and monotonicity. The term "homogeneous" is always used below in the sense of positive homogeneity.

Lemma A.2 *The mapping φ_1 is homogeneous of degree 1 and is monotonic. If $c^1 \in \mathbb{R}^n_+$ and $c^2 \in \mathbb{R}^n_+$ are such that $c^1_U > 0$ and $c^1 \gneq c^2$, then we have in fact $\varphi_1(c^1) > \varphi_1(c^2)$.*

Proof. The homogeneity is immediate from the definitions. Concerning the monotonicity, take c^1 and c^2 in $\mathbb{R}^n_+$ such that $c^1 \geq c^2$. First assume that $c^2_U > 0$; then also $c^1_U > 0$. Take $x \in D$, and let z_1 and z_2 be defined by $F(z_1, c^1) = x$ and $F(z_2, c^2) = x$. We then have $z_i = (\varphi_1(c^i))(x)$ for $i = 1, 2$. Because the function $F(\cdot, \cdot)$ is strictly increasing in each of the components of its second argument and strictly decreasing in its first argument, the vector inequality $c^1 \geq c^2$ and the equality $F(z_1, c^1) = F(z_2, c^2)$ together imply that $z_1 \geq z_2$, with strict inequality as soon as c^1 and c^2 are not equal. If $c^2_i = 0$ for some $i \in U$ while $c^1_U > 0$, then the strict inequality $\varphi_1(c^1) > \varphi_1(c^2)$ trivially holds since $\varphi_1(c^1)$ takes positive values while $\varphi_1(c^2) = 0$ by definition. Finally, if there is $i \in U$ such that $c^1_i = 0$, then $\varphi_1(c^1) = \varphi_1(c^2) = 0$. □

To show the continuity of φ_1, we make use of the following lemma.

Lemma A.3 *Let $\mathcal{Y}$ be a topological space, and let $f(\cdot, \cdot)$ be a continuous mapping from $\mathbb{R}_{++} \times \mathcal{Y}$ to $\mathbb{R}$. Suppose that for every $y \in \mathcal{Y}$ there is exactly one $x \in \mathbb{R}_{++}$ such that $f(x, y) = 0$. Let $(y_k)_{k=1,2,\ldots}$ be a sequence in $\mathcal{Y}$ that converges to $\bar{y} \in \mathcal{Y}$. Define x_k ($k = 1, 2, \ldots$) by the equations $f(x_k, y_k) = 0$, and let $\bar{x}$ be defined by $f(\bar{x}, \bar{y}) = 0$. If the collection $\{x_k \mid k = 1, 2, \ldots\}$ is bounded, then $\lim_{k\to\infty} x_k = \bar{x}$.*

Proof. By the assumed boundedness of the collection $\{x_k \mid k \in \mathbb{N}\}$, it suffices to show that any accumulation point of this collection must coincide with $\bar{x}$. Let $\tilde{x}$ be an accumulation point, and let $(k_j)_{j=1,2,\ldots}$ be such that $\lim_{j\to\infty} x_{k_j} = \tilde{x}$. From the continuity of the mapping f, we have $f(\tilde{x}, \bar{y}) = \lim_{j\to\infty} f(x_{k_j}, y_{k_j}) = 0$. The assumed uniqueness of the solution of the equation $f(x, y) = 0$ for given y then implies that $\tilde{x} = \bar{x}$. □

Lemma A.4 *The mapping $\varphi_1 : \mathbb{R}^n_+ \to \mathcal{L}$ is continuous.*

Proof. Let $(c^k)_{k=1,2,\dots}$ be a sequence of vectors in $\mathbb{R}^n_+$ converging to a vector $c \in \mathbb{R}^n_+$. Take $x \in D$; write $z_k := (\varphi_1(c^k))(x)$ and $z := (\varphi_1(c))(x)$. We need to show that $\lim_{k\to\infty} z_k = z$.

First consider the case in which $c_U > 0$. In this case we also have $c^k_U > 0$ for all sufficiently large k. By definition, the numbers z_k and z are positive and satisfy $F(z_k, c^k) = x$ and $F(z, c) = x$. Suppose there would be a subsequence $(z_{k_j})_{j=1,2,\dots}$ that tends to infinity. Since the sequences $(c_i^{k_j})_{j=1,2,\dots}$ for i with $c_i > 0$ tend to finite limits (namely c_i), the quotients $z_{k_j}/c_i^{k_j}$ tend to infinity, so that

$$x = \lim_{j\to\infty} F(z_{k_j}, c^{k_j}) = b.$$

However, we have $x \in (b, \infty)$ so that $x > b$. From this contradiction it follows that the set $\{z_k \mid k = 1, 2, \dots\}$ is bounded, and it follows from Lemma A.3 that $\lim_{k\to\infty} z_k = z$.

Now suppose that there is an index $\ell \in U$ such that $c_\ell = 0$. By definition, we then have $z = 0$. To avoid trivialities, we may assume that $c^k_U > 0$ for all k. The numbers $z^k > 0$ are then given as the solutions of $F(z_k, c^k) = x$. Take $\varepsilon > 0$, and suppose there would be a subsequence $(z_{k_j})_{j=1,2,\dots}$ such that $z_{k_j} > \varepsilon$ for all j. The quotient $z_{k_j}/c_\ell^{k_j}$ then tends to infinity because of the assumption $c_\ell = 0$, and the corresponding inverse marginal utility function $I_\ell(z)$ tends to $-\infty$ when its argument tends to infinity, because of the assumption $\ell \in U$. Because $|z_{k_j}| > \varepsilon$ for all j and the sequences $(c_i^{k_j})_{j=1,2,\dots}$ tend to finite limits, the inverse marginal utilities $I_i(z_{k_j}/c_i^{k_j})$ $(i = 1, \dots, n)$ are bounded from above. Therefore, we obtain

$$x = \lim_{j\to\infty} F(z_{k_j}, c^{k_j}) = -\infty.$$

Again we have a contradiction. It follows that $\lim_{k\to\infty} z_k = 0$, as was to be proved. □

The following lemma states a particular property of the mapping φ_1 which relates to a "nonsectionality" requirement in the fixed point theorem that will be used later on.

Lemma A.5 *Let $(c^k)_{k=1,2,\dots}$ be a sequence in $\mathbb{R}^n_+$ that has the following property: there exist complementary nonempty index sets R and S in $\{1, \dots, n\}$ and a vector $c_S \in \mathbb{R}^{|S|}_{++}$ such that $c^k_R \to \infty$ as $k \to \infty$ and $c^k_S = c_S$ for all k. Then $(\varphi_1(c^k))(x) \to \infty$ as $k \to \infty$ for all $x \in D$.*

Proof. Take $x \in D$. We can assume that all entries of c^k are positive, so that the numbers $z_k := \varphi_1(c^k)(x)$ are defined implicitly by

$$\sum_{i \in R} I_i(z_k/c_i^k) + \sum_{i \in S} I_i(z_k/c_i) = x. \tag{A.4}$$

Suppose that $(z_k)_{k=1,2,\ldots}$ has a bounded subsequence $(z_{k_j})_{j=1,2,\ldots}$. The quotients $z_{k_j}/c_i^{k_j}$ tend to zero for $i \in R$ so that the first term on the left hand side in (A.4) tends to infinity. The quotients z_{k_j}/c_i for $i \in S$ remain bounded, so that the second term at the left hand side is bounded from below. Therefore the left hand side tends to infinity as $j \to \infty$, which leads to a contradiction. The statement in the lemma follows. $\square$

We now turn to the mapping φ_2. To define this mapping, the following lemma is required.

Lemma A.6 *Take $i \in \{1, \ldots, n\}$. For any given nonzero function $J \in \mathcal{L}$, the equation*

$$E^Q I_i(J(X)/c_i) = E^Q X_i \tag{A.5}$$

has a unique solution $c_i > 0$.

Proof. Since the random variable $J(X)$ is bounded, the mapping $c_i \mapsto E^Q I_i(J(X)/c_i)$ defines a strictly increasing function with

$$\lim_{c_i \to \infty} E^Q I_i(J(X)/c_i) = \infty, \qquad \lim_{c_i \downarrow 0} E^Q I_i(J(X)/c_i) = b_i.$$

The statement in the lemma follows from the assumption that $E^Q X_i > b_i$. $\square$

The mapping φ_2 is now defined as follows:

$$(\varphi_2(J))_i = \begin{cases} c_i \text{ as defined in Lemma A.6} & \text{if } J \neq 0 \\ 0 & \text{if } J = 0 \end{cases} \tag{A.6}$$

for $i = 1, \ldots, n$.

Lemma A.7 *The mapping φ_2 is homogeneous of degree 1 and strictly monotonic in the sense that $\varphi_2(J_1) \geq \varphi_2(J_2)$ when $J_1 \geq J_2$ and $\varphi_2(J_1) > \varphi_2(J_2)$ when $J_1 > J_2$.*

Proof. The homogeneity is immediate from the definition. The strict monotonicity follows from the fact that all inverse marginal utilities $I_i(\cdot)$ are strictly decreasing; in case $J_2 = 0$, the strict monotonicity is immediate from Lemma A.6. □

Lemma A.8 *The mapping φ_2 is sequentially continuous.*

Proof. Let $(J_k)_{k=1,2,\dots}$ be a sequence in $\mathcal{L}$, converging pointwise to $J \in \mathcal{L}$. Take $i \in \{1,\dots,n\}$; we want to show that $((\varphi_2(J_k))_i)_{k=1,2,\dots}$ converges to $(\varphi_2(J_k))_i$. Write $c_i^k := (\varphi_2(J_k))_i$ and $c_i := (\varphi_2(J))_i$.

First assume that the limit function J is nonzero. Consider the function $F : \mathbb{R}_{++} \times \mathcal{L} \to \mathbb{R}$ defined by

$$F(c_i, J) = E^Q I_i(J(X)/c_i).$$

It follows from the bounded convergence theorem that this function is continuous. By definition, we have

$$E^Q I_i(J_k(X)/c_i^k) = E^Q X_i$$

for all $k \in \mathbb{N}$. The collection of random variables $J_k(X)$ is uniformly bounded by $\sup_k J_k(\min A)$. Therefore, if there would exist a subsequence $(c_i^{k_j})_{j=1,2,\dots}$ converging to infinity, it would follow that

$$E^Q X_i = \lim_{j\to\infty} E^Q I_i(J_{k_j}(X)/c_i^{k_j}) = \infty \tag{A.7}$$

which contradicts the assumptions. It follows that the collection $\{c_i^k \mid k = 1,2,\dots\}$ is bounded. An application of Lemma A.3 shows that $\lim c_i^k = c_i$.

Consider now the case in which $J = 0$. In this case, we have by definition $c_i = 0$. Take $\varepsilon > 0$ and suppose that there would exist a subsequence $(c_i^{k_j})_{j=1,2,\dots}$ such that $c_i^{k_j} > \varepsilon$ for all $j = 1,2,\dots$. The convergence of $(J_k)_{k=1,2,\dots}$ to $J = 0$ would then imply the same conclusion as in (A.7). Therefore it follows that $\lim_{k\to\infty} c_i^k = 0$ as was to be shown. □

The following lemma establishes a property that will be used in a nonsectionality argument.

Lemma A.9 *Let $(J_k)_{k=1,2,\dots}$ be a sequence in $\mathcal{L}$ such that $J_k(x) \to \infty$ for all $x \in D$ as $k \to \infty$. Then $(\varphi_2(J_k))_i \to \infty$ for all $i = 1,\dots,n$.*

Proof. Choose $i \in \{1, \ldots, n\}$. Assume that the i-th entry of $c^k := \varphi_2(J_k)$ does not tend to infinity. Then there exist a finite number M and a subsequence $(c_i^{k_j})_{j=1,2,\ldots}$ such that $c_i^{k_j} < M$ for all j. We would then have

$$E^Q X_i = \lim_{j \to \infty} E^Q I_i(J^{k_j}(X)/c_i^{k_j}) = b_i \tag{A.8}$$

which is a contradiction since it has been assumed that $E^Q X_i > b_i$. Therefore the statement of the lemma follows. □

Definition A.10 A mapping φ of R_+^n into itself is *nonsectional* if, for every decomposition of the index set $\{1, \ldots, n\}$ into two complementary nonempty subsets R and S, there exists $s \in S$ such that

(i) $(\varphi(x))_s > (\varphi(y))_s$ for all x, $y \in \mathbb{R}_+^n$ such that $x_R > y_R$ and $x_S = y_S > 0$

(ii) $(\varphi(x^k))_s \to \infty$ for all sequences $(x^k)_{k=1,2,\ldots}$ in $\mathbb{R}_+^n$ such that $x_R^k \to \infty$ while x_S^k is fixed and positive.

Lemma A.11 *The mapping $\varphi : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined by $\varphi(c) = \varphi_2(\varphi_1(c))$ is continuous, monotonic, homogeneous of degree 1, and nonsectional.*

Proof. The continuity follows from Lemmas A.4 and A.8; monotonicity and homogeneity follow from Lemmas A.2 and A.7. Consider now two nonempty complementary subsets R and S of the index set $\{1, \ldots, n\}$ as in Def. A.10. If c^1 and c^2 are such that $c_S^2 > 0$, $c_S^1 = c_S^2$, and $c_R^1 > c_R^2$, then it follows from Lemma A.2 that $\varphi_1(c^1) > \varphi_1(c^2)$. The strict inequality is preserved by the mapping φ_2 according to Lemma A.7, so that item (i) in Def. A.10 is surely satisfied. The condition in item (ii) is fulfilled due to Lemma A.5 and Lemma A.9. □

The equations (3.9) and (3.10) which define the mappings φ_1 and φ_2 are not independent; indeed, taking expectation with respect to Q in (3.9) leads to the same condition as summing over i in (3.10) does. As a consequence, we have the following additional property of the mapping φ.

Lemma A.12 *The mapping φ defined in Lemma A.11 can only have 1 as an eigenvalue corresponding to a positive eigenvector. In other words, if $c \in \mathbb{R}^n_{++}$ is such that $\varphi(c) = \lambda c$, then $\lambda = 1$.*

Proof. Let $c > 0$ be such that $\varphi(c) = \lambda c$. Since φ maps the positive cone into itself, the eigenvalue λ must be positive. Define $J = \varphi_1(c)$; then $\varphi_2(J) = \lambda c$. Note that $J(x) > 0$ for all $x \in D$. By definition, we have

$$\sum_{i=1}^{n} I_i(J(x)/c_i) = x \qquad (x \in D) \tag{A.9}$$

$$E^Q I_i(J(X)/(\lambda c_i)) = E^Q X_i \qquad (i = 1, \dots, n). \tag{A.10}$$

It follows that

$$\sum_{i=1}^{n} E^Q I_i(J(X)/(\lambda c_i)) = \sum_{i=1}^{n} E^Q X_i = E^Q X = \sum_{i=1}^{n} E^Q I_i(J(X)/c_i).$$

The claim follows by noting that the function $\lambda \mapsto I_i(J(x)/(\lambda c_i))$, for fixed x and fixed i, is strictly increasing in λ. □

The relation between the partial results proved so far and the conclusions that we aim for is established by the following result which is due to Oshime (1983).

Theorem A.13 *If a mapping φ from $\mathbb{R}^n_+$ into itself is continuous, monotonic, homogeneous of degree 1, and nonsectional, then the mapping φ has a positive eigenvector, which is unique up to scalar multiplication. In other words, there exist a constant $\alpha^* > 0$ and a vector $x^* \in \mathbb{R}^n_{++}$ such that $\varphi(x^*) = \alpha^* x^*$, and if $\alpha > 0$ and $x \in \mathbb{R}^n_{++}$ are such that $\varphi(x) = \alpha x$, then x is a scalar multiple of x^*.*

The theorem as formulated above is a compilation of facts stated in Oshime (1983). Specifically, the existence of nonnegative eigenvalues and eigenvectors for continuous, monotonic, and homogeneous mappings of $\mathbb{R}^n_+$ into itself follows from Brouwer's fixed point theorem. The number of eigenvalues is finite, in fact at most equal to $2^n - 1$ (Morishima, 1964, Appendix); therefore there is a maximal eigenvalue. For mappings that are nonsectional, in addition to the properties already mentioned, the eigenvector associated to the maximal eigenvalue is positive (Oshime, 1983, Thm. 8). Moreover, positive eigenvectors are determined uniquely up to scalar multiplication (Oshime, 1983, Thm. 8, Remark 2).

The proof of the main result Thm. 3.6 now follows from the observation that there is a one-to-one relationship between Pareto efficient and financially fair redistributions on the one hand, and pairs (c, J) that satisfy (3.9–3.10) on the other hand. The existence and uniqueness of such a pair follows by combining Lemma A.11 with Thm. A.13.

We now continue with the proof of the iteration result Thm. 3.10. The proof is based on the fact that the mapping φ is a contractive mapping with respect to a suitably defined metric. We use the Hilbert metric, which is a standard tool in nonlinear Perron-Frobenius theory. It is defined as follows.

Definition A.14 The *Hilbert metric* assigns to a pair (x, y) with $x,\ y \in \mathbb{R}^n_{++}$ the distance $d(x, y)$ given by

$$d(x, y) = \log \frac{\max_i(x_i/y_i)}{\min_i(x_i/y_i)}\,.$$

The Hilbert metric is actually a pseudometric, since the condition $d(x, y) = 0$ is equivalent to the statement that y is a positive scalar multiple of x, rather than to the statement $x = y$. Alternatively, the Hilbert metric can be viewed as a true metric on the space of positive rays, or on the open unit simplex. Indeed we have

$$d(\alpha x, \beta y) = d(x, y) \quad \text{for all } \alpha,\ \beta > 0 \tag{A.11}$$

so that points on the same ray are equivalent with respect to the metric. The following lemma is a standard fact (cf. Brooks and Schmitt (2009, Ch. 6), Lemmens and Nussbaum (2012, Ch. 2)); we provide a proof for the reader's convenience. Recall that a mapping φ from a metric space into itself is said to be *contractive* if $d(\varphi(x), \varphi(y)) < d(x, y)$ for all x, y such that $d(x, y) > 0$.

Lemma A.15 *Let φ be a mapping from $\mathbb{R}^n_+$ into itself that is homogeneous of degree 1 and strongly monotonic. Then φ is a contractive mapping with respect to the Hilbert metric.*

Proof. Take x, $y \in \mathbb{R}^n_{++}$ with $d(x,y) > 0$. Define $M := \max_i(x_i/y_i)$, $m := \min_i(x_i/y_i)$. We then have $my \lneq x \lneq My$, and by homogeneity and strong monotonicity of φ it follows that $m\varphi(y) < \varphi(x) < M\varphi(y)$. Therefore,

$$\min_i \frac{\varphi(x)_i}{\varphi(y)_i} > m, \qquad \max_i \frac{\varphi(x)_i}{\varphi(y)_i} < M$$

and hence $d(\varphi(x), \varphi(y)) < \log(M/m) = d(x,y)$. □

As a consequence of the property (A.11), the mapping ψ that is obtained from φ by normalization to the unit simplex is contractive if φ is. Using this fact, we can now prove the convergence of the iteration process proposed in Thm. 3.10 by making use of a result of Nadler (1972) who proved that, if a contractive mapping from a locally compact and connected metric space into itself has a fixed point, then every sequence of iterates of that mapping converges to the fixed point.

Proof. (of Thm. 3.10.) The mapping ψ is a contractive mapping from the open unit simplex into itself, and the simplex is a locally compact and connected metric space. Moreover we know from Lemma A.11 and Thm. A.13 that φ has a fixed point. It then follows from Nadler (1972, Thm. 1) that, for any start vector c^0 in the open unit simplex, the sequence of iterates $c^1, c^2, \ldots$ converges to the fixed point. □

Appendix B

This appendix provides a proof of Thm. 5.8. The following lemma in linear algebra will be needed.

Lemma B.1 *Let $\mathcal{X}$ be a vector space, and let $\mathcal{X}_a$, $\mathcal{X}_b$, and $\mathcal{Z}$ be subspaces of $\mathcal{X}$ such that $\mathcal{X}_a + \mathcal{Z} = \mathcal{X}$ and $\mathcal{X}_b + \mathcal{Z} = \mathcal{X}$. Then the following two statements are equivalent:*

$$(\mathcal{X}_a \cap \mathcal{X}_b) + \mathcal{Z} = \mathcal{X} \tag{B.1}$$

$$(\mathcal{X}_a + \mathcal{X}_b) \cap \mathcal{Z} = (\mathcal{X}_a \cap \mathcal{Z}) + (\mathcal{X}_b \cap \mathcal{Z}). \tag{B.2}$$

Proof. First assume that (B.1) holds. The left hand side in (B.2) clearly includes the right hand side, so it remains to prove the reverse inclusion. Take $X \in (\mathcal{X}_a + \mathcal{X}_b) \cap \mathcal{Z}$. We can then find $X_a \in \mathcal{X}_a$ and $X_b \in \mathcal{X}_b$ such that $X = X_a + X_b$. By the assumption (B.1), there exist $X_a^b \in \mathcal{X}_a \cap \mathcal{X}_b$ and $Z \in \mathcal{Z}$ such that $X_a = X_a^b + Z$. Then we have $X_a - X_a^b = Z \in \mathcal{X}_a \cap Z$, and $X_b + X_a^b = X - Z \in \mathcal{X}_b \cap \mathcal{Z}$. It follows that $X = (X_a - X_a^b) + (X_b + X_a^b) \in (\mathcal{X}_a \cap Z) + (\mathcal{X}_b \cap Z)$.

Now, assume that (B.2) holds. Take $X \in \mathcal{X}$, and write $X = X_a + Z_a = X_b + Z_b$ with $X_a \in \mathcal{X}_a$, $X_b \in \mathcal{X}_b$, and Z_a, $Z_b \in \mathcal{Z}$. We then have $X_a - X_b = Z_b - Z_a \in (\mathcal{X}_a + \mathcal{X}_b) \cap \mathcal{Z}$ so that on the basis of (B.1) there exist $\hat{Z}_a \in \mathcal{X}_a \cap \mathcal{Z}$ and $\hat{Z}_b \in \mathcal{X}_b \cap \mathcal{Z}$ such that $X_a - X_b = \hat{Z}_a - \hat{Z}_b$. Define $\hat{X} = X_a - \hat{Z}_a = X_b - \hat{Z}_b \in \mathcal{X}_a \cap \mathcal{X}_b$. From the equality $X = \hat{X} + \hat{Z}_a + Z_a$ it then follows that $X \in (\mathcal{X}_a \cap \mathcal{X}_b) + \mathcal{Z}$. $\square$

Proof. (of Thm. 5.8.) If the access graph is not connected, then there exists a nonempty index set $T \subsetneq \{1, \ldots, n\}$ such that the asset index sets $S_T := \bigcup_{i \in T} S_i$ and $S_{T'} := \bigcup_{i \notin T} S_i$ are disjoint. This implies that the corresponding subspaces $\sum_{i \in S_T} \mathcal{X}_i$ and $\sum_{i \in S_{T'}} \mathcal{X}_i$ intersect only in 0. By Lemma B.1, it follows that

$$\Big(\mathcal{Z} \cap \sum_{i \in S_T} \mathcal{X}_i\Big) + \Big(\mathcal{Z} \cap \sum_{i \in S_{T'}} \mathcal{X}_i\Big) \subsetneq \Big(\mathcal{Z} \cap \sum_{i=1}^{n} \mathcal{X}_i\Big).$$

In particular this implies that

$$\sum_{i=1}^{n} (\mathcal{X}_i \cap \mathcal{Z}) \subsetneq \mathcal{Z}$$

and therefore completability does not hold.

Now, suppose that the access graph is connected. In this case, we prove by induction the following statement: for every k with $1 \leq k \leq n$, there exists an index set $T \subset \{1, \ldots, n\}$ such that $|T| = k$ and

$$\sum_{i \in T} \big(\mathcal{X}_i \cap \mathcal{Z}\big) = \mathcal{Z} \cap \sum_{i \in T} \mathcal{X}_i. \tag{B.3}$$

The claim then follows by taking $k = n$ and applying the criterion of Thm. 5.7. For $k = 1$, the statement is trivially true. Assume now that (B.3) holds for an index set T with $|T| \leq n - 1$. Because of the assumption that the access graph is connected, there must be an asset that is accessible to one or more of the agents indexed by T and also to an agent not indexed by T. In other words, there exists $j \notin T$ and $\ell \in \bigcup_{i \in T} S_i$ such that $X_\ell \in \mathcal{X}_j$. Since it has been assumed that the assets have nonzero values, this implies

$$\Big(\mathcal{X}_j \cap \sum_{i \in T} \mathcal{X}_i\Big) + \mathcal{Z} = \mathcal{X}.$$

By Lemma B.1 and the induction assumption, it follows that

$$\Big(\mathcal{X}_j + \sum_{i \in T} \mathcal{X}_i\Big) \cap \mathcal{Z} = (\mathcal{X}_j \cap \mathcal{Z}) + \sum_{i \in T} (\mathcal{X}_i \cap \mathcal{Z}).$$

Now define $\hat{T} = T \cup \{j\}$; then (B.3) is fulfilled for $\hat{T}$. $\square$

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