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Abstract

We provide a robust optimal hedging strategy in an incomplete market. This policy can protect the investor from parameter uncertainty. The investor aims to minimize a function of hedging error under the worst case scenario by means of solving a min-max robust optimization problem. We apply this methodology to the asset and liability management and employ an expected shortfall hedging criterion as our value function. The robust policy is more conservative than the naive one when the fund is facing solvency risk. The investor can benefit from the robust policy when the expected return is overestimated.

Keywords: Model misspecification, robust optimization, uncertainty set, incomplete market, dynamic hedging, explicit finite difference, expected shortfall.

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1 Introduction

Pricing and hedging pension or insurance liabilities confront with two barriers. On one hand, the market is incomplete. Liability risks are typically not (actively) traded in the financial market. For instance, a pension liability is exposed to long-term interest rate risk and longevity risk. Due to the missing market, these risks cannot be hedged by constructing a replicating portfolio. To eliminate such kind of risks involves a tradeoff between risk and expected return.

On the other hand, expected asset return is very difficult to estimate based on historical data. [1] Suppose we have a series of stock price S_t with $t \in [0, T]$. The expected return can be estimated by $\frac{\ln(S_T)-\ln(S_0)}{T} + \frac{1}{2}\sigma^2$. If we have 100 years of historical data and the market volatility is 16%, then the standard error of the equity premium is 1.6% $\left(\frac{16\%}{\sqrt{100}}\right)$ leading to an approximate 95% confidence interval span of 6.3%($\pm 1.96 \times 1.6\%$).[2] Although the interval shrinks with the square root of time period for estimation, it is hard to keep the same data generating process during the entire period. Therefore, the investor is exposed to estimation error in both expected asset returns and also in the expected evolution of the liabilities. A poorly estimated expected return will lead to a suboptimal portfolio choice.

Despite many studies on pricing and hedging in incomplete market, for instance De Jong (2008) surveys more than five existing methods to value pension liabilities in incomplete markets. Very few studies consider the effect of parameter uncertainty. However, literatures on portfolio choice under estimation risk can date back to 1970s. One of the pioneer work by Bawa et al. (1979) has already noticed that the optimal hedge differs from the classic one if estimation risk is taken into consideration. It seems a waste to ignore either side since they both are very likely to occur in practise at the same time.

We provide a robust hedging strategy in incomplete markets for fixed income securities. The main goal is to design a hedging strategy that not only works well when the underlying model is correct but also performs reasonably well under model misspecification. To be more specific, we assume that the agent makes an investment decision to hedge the downside liability risk of which the payoff cannot be fully replicated due to market incompleteness. Meanwhile, the agent is pessimistic towards the underlying model. In order to neutralize the effect of model ambiguity, the agent follows a robust policy that ensures him against a worst case scenario. We benefit from the robust policy in the sense that the investment decision is less sensitive to the estimation error. The additional guarantee makes the robust policy more expensive but also makes it robust against the uncertainty from Mother Nature.

[1] Kandel and Stambaugh (1996) and Barberis (2000) show that regressing stock returns on a set of "predictive" variables, such as the lagged dividend yield has almost zero explanatory power ($R^2 < 4\%$).

[2] Peijnenburg (2011) and Pelsser (2011) use a similar example to motivate their work on optimal portfolio under parameter uncertainty.

There are two ways to understand the preference for robustness. First, market incompleteness creates an unknown market price of risk. This unobservable market price of risk leads to the rationale of Cochrane and Saa-Requejo (1996)'s Good-Deal-Bound that constrains the maximum Sharpe ratio of the market. The true market price of risk could be anywhere within this Good-Deal-Bound. Second, the investor who fears for parameter misspecification is ambiguity averse. He believes that the true model parameters differ from the estimated ones. To formulate model misspecification, Hansen and Sargent (2001) and Hansen and Sargent (2007) employ a relative entropy factor. This relative entropy captures the perturbation between the estimated model and the unobservable true model. Although the economic interpretations of the two are different, technically the two motives are identical. Both interpretations can be understood as requiring an additional premium to represent the estimation error. Pelsser (2011) states that mathematically, the two interpretations are identical.

We design a stylized robust hedging problem for asset and liability management that integrates the Good-Deal-Bound rationale into Hansen-Sargent approach. To model our robust optimization problem involves four elements. First is an incomplete market. We introduce two uncorrelated risk drivers in our model, one is hedgeable and the other is not. Both risks are univariate standard Brownian motions. The unhedgeable risk captures the incompleteness of the market. The asset market is exposed to hedgeable risk only, but the liability side is exposed to both types of risk. Second element is the parameter estimation error. We introduce two perturbation time series processes, one on the asset side but both on the liability side. Each of the two is defined as an additional drift term on the Brownian motion. Economically, an additional drift on the Brownian motion can be understood as the unobservable market price of risk which is in line with Cochrane and Saa-Requejo (1996)'s explanation. Technically, the two parameters measure the discrepancies between alternative probability distributions. This explanation goes along with Hansen and Sargent (2007)'s relative entropy approach. Third element is the uncertainty set of the perturbations. We apply statistical distribution theory to construct an ellipsoid uncertainty set under the assumption that the estimation error is normally distributed. This approach differs from Cochrane and Saa-Requejo (1996) which requires the investor to specify a subjective value of the bound. Last element is the objective function. We take the expected shortfall of the liability at given terminal period as our value function. Hence the robust optimization problem is to solve a min-max expected shortfall function.

Investors benefit from our optimal hedging model in two aspects. On one hand, we use a coherent risk measure instead of a well known utility function. Hence we avoid the risk aversion puzzle and the debate of proper utility functions. On the other hand, we consider downside risks only instead of both, for it is not realistic to set a punishment on the upside for most financial institutions. It is not a bad news at all to be over funded.

We solve the robust hedging problem in both static and dynamic environment and find in both cases that the robust policy is more conservative than the naive policy when the financial institution is facing solvency risk. This result is inline with Brennan (1998)'s conclusion but is limited to the risk aversion investors. When funding ratio is low, the agent will increase the risk exposure to the stock market so as to gamble out of trouble. However, the more risky assets are held, the more estimation error is exposed. The robust agent is particularly afraid of a downside shock on the risky assets hence he will put less wealth in the stock market compared with the agent who disregards the estimation error. We also find that for both robust and non-robust policy, the risky portfolio increases over time if the instantaneous funding ratio is low and is another way around if over funded.

Further, we also evaluate the robust policy by means of comparing its expected loss from estimation error with the non-robust policy. The loss function is defined as the difference between the cost of hedging conditional on the estimated expected return and the true minimum cost. We find that the agent can benefit from a robust policy in two aspects. First, the expected loss from estimation error is less sensitive to the estimated parameters. Second, if the expected return is over estimated, the robust policy has a lower hedging cost.

2 Literature Review

The past decade has witnessed a growing attention on the studies of pricing and hedging in incomplete markets. We categorize these studies into three approaches. The first approach is related to the arbitrage pricing theory. Cochrane and Saa-Requejo (1996) weaken the arbitrage pricing theory by using a "good deal bound" (GDB) as an acceptability criterion to role out high Sharpe ratio portfolios. This method calculates price for non-trade securities by making sure that the risk-return trade-off (Sharpe ratio) of any asset does not excess a given "good deal" Sharpe ratio. Carr et al. (2001) introduce an alternative criterion which links the arbitrage pricing with the utility - based valuation. They find a security can be hedged and financed if and only if the expected gain under each probability measure is no less than its non-negative associated floor. He and Pearson (1991) introduce a minimax local martingale measure to solve the optimal consumption problem in incomplete markets. They suggest to complete the market by adding an additional parameter on the pricing kernel so that the demand for the new claim is zero. Similar technique is also employed by Schachermayer (2000) and Schachermayer (2004).

The second approach is based on utility indifference pricing principle. This approach is built on investor's preference towards risks that cannot be replicated. An investor is willing to purchase a claim if her maximum happiness with or without this investment stays the same. Early work by Hodges and Neuberger (1989) has considered using indifference pricing principe to value European options in the presence of transaction cost. More recently, Young and Zariphopoulou (2002) extend the study by pricing the dynamic insurance risk. Young (2004) calculates the price of an insurance claim which is exposed to catastrophe risk and stochastic interest rate risk. Mostly these studies are surveyed by Henderson and Hobson (2004).

The final approach aims to minimize the hedging error using a suitable risk measurement. Hedging error is defined as the mismatched payoff between asset and liability and it can be transformed in several ways. Föllmer and Leukert (1999) introduce a quantile hedging strategy that maximize the probability of a successful hedge. Their investment policy is to minimize the required cost of hedging conditional on a fixed shortfall probability. Their another paper Föllmer and Leukert (2000), look at the duel problem in which an agent aims to minimize the shortfall probability with given hedging cost. As an extension, Rudloff (2009) considers a dynamic hedging problem that minimize the shortfall using a coherence risk measure. Basak and Chabakauri (2011) provide an analytical solution of dynamic hedging in incomplete markets. Their hedging criterion is based on the minimum variance of hedging error.

There is an increasing amount of work on hedging under model uncertainty. Early works are dominated by Bayesian paradigm. To be a Bayesian, one always has to specify an objective probability distribution regardless of subjective beliefs. To incorporate Bayesian's approach with estimation risk, one has to summarize a conditional distribution of parameters given the data. [3] Klein and Bawa (1976) is one of the first studies consider the effect of model uncertainty on portfolio choice. They look at a two-period model and find out that in the presence of estimation risk, the optimal hedging portfolio differs from the traditional analysis. Barberis (2000) extend the study in a multi-period economy setting while keeping the hedging decision static and find that ignoring estimation risk may result in an over aggressive portfolio. Brennan (1998) incorporates learning with parameter uncertainty and finds out that risk lovers put more wealth on risky assets after learning while the risk averse investors are more conservative with their portfolio.

A more recent approach is the max-min expected utility paradigm developed by Gilboa and Schmeidler (1989). Gilboa and Marinacci (2011) claim that Gilboa-Schmeidler's axiom is a neo-Bayesian paradigm because it allows decision makers to have a set of subjective priors. The agent aims to maximize her utility under the least preferred prior so as to display an aversion to uncertainty. As an extension, Hansen and Sargent (2001) managed to transform Gilboa-Schmeidler's static theory to a dynamic version through the techniques of robust control theory which already been broadly used in engineering

[3] Technically, our uncertainty set design is related to Bayesian paradigm, but the philosophy behind is completely opposite. Bayesian assumes historical data to be objective. Hence estimation of parameters has to be conditional on the given data. However, we follow GMM, and maximum likelihood estimation approach which assumes an existing true value of parameter.

and applied mathematics since 1980s. Maenhout (2004) is one of the first studies employ Hansen-Sargent framework to restudy Merton (1969) optimal portfolio problem and find out that due to the uncertainty aversion, investors reduce their risk exposure on stock market dramatically.[4] Uppal and Wang (2003) extend Maenhout's model in a multi-dimensional setting.

The rest of the paper is planed as follows. Section 3, introduces the general economic setup. In this section, we formulate the market incompleteness, model misspecification and construct the uncertainty set. We also define the robust optimization problem. In Section 4, we derive the value function analytically under a static environment and numerically calculate the static robust portfolio. We also evaluate the robust policy via the loss function. Section 5, provides dynamic analysis of the robust policy. We numerically solve the partial differential equation using explicit finite difference approach. Section 6 summaries and concludes.

3 Model

We construct a continuous-time incomplete market with a finite trading horizon $[0, T]$. The uncertainty is modeled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which are defined two uncorrelated risk factors, a hedgeable risk W_1 and a unhedgeable risk W_2. Both W_1 and W_2 are univariate standard Brownian motions and we consider $\{\mathcal{F}_t : t \in [0, T]\}$ as the completion of filtration generated by W_1 and W_2. A hedgeable risk means we can replicate the payoff of such kind of risk perfectly. The payoff for a unhedgeable risk is not replicable because it is not traded.

3.1 Asset and Liability Model

On the asset side, we have a risk-free money-market account B_t, which earns a deterministic risk-free rate of interest r, so $dB_t = rB_t dt$. We also have a stock market. The stock price follows a geometric Brownian Motion process $dS_t = \mu S_t\, dt + \sigma S_t\, dW_1$. The agent can only invest in the money-market account and the stock market. Denote the value of the assets at time t as A_t. The investor puts wA_t amount in the stock market at time t. The remaining part of the assets $(1 - w)A_t$ is put into the money-market account. The asset diffusion process follows as

$$dA_t = \big(r + w(\mu - r)\big)\, A_t\, dt + w\sigma\, A_t\, dW_1, \tag{1}$$

[4] In order to obtain a closed form solution, Maenhout (2004) modifies Hansen-Sargent's framework by transforming the constant Lagrange multiplier into a function of state variables. However, Pathak et al. (2002) argues that this modification breaks the link to the world of Gilboa-Schmeidler.

where w is the possibly time varying hedging strategy. We do not set constraint on w, therefore, short position in our economy is allowed.

The liability is exposed to both hedgeable risk W_1 and unhedgeable risk W_2. We assume that the diffusion process of the liability L_t follows an exogenously given geometric Brownian motion with constant drift term and constant volatility,

$$dL_t = a\, L_t\, dt + b\, L_t \left(\rho\, dW_1 + \sqrt{1-\rho^2}\, dW_2\right), \tag{2}$$

where a is the drift of the liability and b is its volatility. The non-traded risk driver, dW_2, represents the incomplete part of the market. We introduce a correlation parameter $\rho \in [-1, 1]$ between asset risk and liability risk. It controls the risk exposure to W_2 of the liability. If $\rho = \pm 1$, then the non-traded risk W_2 disappears from the liability side. The liability in this case can be perfectly hedged by a replicating portfolio. We are interested in the case when ρ is strictly between -1 and 1.

3.2 Robust Asset and Liability Model

We use the Hansen and Sargent (2007) framework to integrate the preference for robustness to the asset-liability model (1) and (2). With the preference for robustness, the agent treats (1) and (2) as an approximate model towards the unknown true state evolution of A_t and L_t. We design a particular form of model misspecification by limiting the parameter uncertainty to the drift terms μ and a only. We also assume that the volatility term σ is known. The approximate model only provides an estimated value of the drift terms, but since the expected return is relatively volatile, it is hard to obtain an accurate estimate of the expected return. Therefore, the expected return is subject to estimation error. For example, suppose we have a series of stock price S_t with $t \in [0, T]$. The expected return can be estimated by $\frac{\ln(S_T)-\ln(S_0)}{T} + \frac{1}{2}\sigma^2$. If we have 100 years of historical data and the market volatility is 16%, then the standard error of the equity premium is 1.6% $\left(\frac{16\%}{\sqrt{100}}\right)$ leading to an approximate 95% confidence interval span of 6.3%($\pm 1.96 \times 1.6\%$). Although the interval shrinks with the square root of time period for estimation, it is hard to keep the same data generating process during the entire period. However, the constant variance terms σ and b can be estimated via high frequent observations and therefore are not subject to parameter misspecification. Hence, our model misspecification problem is reduced to the uncertainty about the drift terms of the state variables.

In Hansen and Sargent framework, the robust model contains an unknown drift term on the Brownian motion, so in our case the robust model dW_1 and dW_2 in (1) and (2) are replaced by $dW_1 + \lambda_1 dt$ and $dW_2 + \lambda_2 dt$. The two drift terms λ_1 and λ_2 are defined as two perturbation time series process that quantify the misspecification of the underlying model. The values of λ_1 and λ_2 are constrained by an uncertainty set. We provide two interpretations of these two additional terms. First, they shift the mean distribution of the asset and the liability diffusion process by a unit of $w\sigma\lambda_1$ and $b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2$ respectively. Hence they specify a set of alternative measures referring to different specification of the stochastic process. We also call this set of alternative measures a Girsanov kernel. Second explanation states that if the expected return is poorly estimated, then the market price of risk is also misspecified. The two additional drifts correct the estimation error of the market price of risk. The perturbed evolution of the state variable is given by:

$$dA_t = \big(r + w(\mu - r)\big)\, A_t\, dt + w\sigma\, A_t\, (dW_1 + \lambda_1 dt)\,, \tag{3a}$$

$$dL_t = a\, L_t\, dt + b\, L_t\, \Big(\rho\, (dW_1 + \lambda_1 dt) + \sqrt{1-\rho^2}\, (dW_2 + \lambda_2 dt)\Big)\,, \tag{3b}$$

The perturbation of the model is bounded by an uncertainty set $\mathbb{S}$. The larger the uncertainty set $\mathbb{S}$, the more pessimistic the agent is towards the accuracy of the underlying model. To describe the uncertainty set, we introduce some additional notation. Let δ be the vector of the estimated drift terms from the approximate model, and let δ_1 be the estimation error which is a vector of the additional drift terms created from the perturbation processes λ_1 and λ_2. Hence the true expected return is $\delta_0 = \delta + \delta_1$. Let Σ be the covariance matrix with $\Gamma\Gamma' = \Sigma$.

$$\delta = \begin{pmatrix} \mu \\ a \end{pmatrix} \quad \delta_1 = \begin{pmatrix} \sigma\lambda_1 \\ b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2 \end{pmatrix} \quad \Gamma = \begin{pmatrix} \sigma & 0 \\ b\rho & b\sqrt{1-\rho^2} \end{pmatrix}$$

Hence we have $\delta_1 = \Gamma\lambda$ where $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ is the perturbation vector. Suppose we have N observations. The difference between the true expected return and the estimated expected return $\delta_0 - \delta$ is normally distributed with mean zero and variance $\frac{\Sigma}{N}$ where $\Sigma = \begin{pmatrix} \sigma^2 & b\rho\sigma \\ b\rho\sigma & b^2 \end{pmatrix}$. We also know that $\delta_1' \left(\frac{\Sigma}{N}\right)^{-1} \delta_1$ is a Chi-square distribution with two degrees of freedom, $\chi^2(2)$. Denote the critical value at α significance level as CV_α, then we have a probability of $1-\alpha$ that

$$\delta_1' \Sigma^{-1} \delta_1 \leq \kappa^2 \tag{4}$$

where $\kappa^2 = \frac{\mathrm{CV}_\alpha}{N}$. Equation (4) provides a natural boundary of the perturbation parameters. Simplify (4) further, we get

$$(\Gamma\lambda)' \left(\Gamma\Gamma'\right)^{-1} (\Gamma\lambda) \leq \kappa^2$$

and it becomes $\lambda'\lambda \leq \kappa^2$. Hence our uncertainty set is as follows,

$$\mathbb{S} = \left\{\lambda_1, \lambda_2 | \lambda_1^2 + \lambda_2^2 \leq \kappa^2\right\} \tag{5}$$

Our uncertainty set has a circular shape in λ space centered by zero. Hence, we can write the confidence interval of δ_0 as,

$$\delta_0 \in \{\delta + \Gamma\lambda \mid \mathbb{S}\} \tag{6}$$

The true drift term δ_0 is constrained by an ellipsoid uncertainty set centered by δ and it can be at any point within this set. The size of the uncertainty set depends on two factors, one is the significance level α and the other is the sample size N. If the agent has infinite observations, then the uncertainty set shrinks to the point estimate δ. However, the agent only obtains limited observations which means δ moves away from δ_0. The fewer observations, the higher κ^2 will be and the more parameter uncertainty the agent is exposed to.

Our stylized uncertainty set is related to the Good Deal Bounds rational proposed by Cochrane and Saa-Requejo (1996). Its idea is to add a confidence interval surrounding the observable market price of risk, and constrain the total market of risk to a reasonable value. Equation (4) could also be understood as the Good Deal Bounds constraint in which we only put a boundary on the unobservable part of the market price of risks, λ_1 and λ_2. If we completely follows the Good Deal Bounds method, our uncertainty set should be as follows $(\delta_0 - \iota r)' \left(\frac{\Sigma}{T}\right)^{-1} (\delta_0 - \iota r) \leq \mathrm{G}$ where $\iota = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and G is the value of the Good Deal Bounds. Denote $\bar{\delta} = \delta - \iota r$ the demeaned point estimate. Then the Good Deal Bounds condition becomes $\left(\bar{\delta} + \delta_1\right)' \Sigma^{-1} \left(\bar{\delta} + \delta_1\right) \leq \frac{G}{T}$. The function can be further simplified to $\bar{\delta}'\Sigma^{-1}\bar{\delta} + 2\bar{\delta}'\Sigma^{-1}\delta_1 + \lambda_1^2 + \lambda_2^2 \leq \frac{\mathrm{G}}{T}$.

The uncertainty set we propose differs from the Good Deal Bounds in the way that our uncertainty set is derived from the distribution theorem. Therefore, the uncertainty set parameter κ is endogenous depending only on the statistic factors, namely α and T. However, the Good Deal Bounds method is inspired by an economic believe that the total market price of risk in an incomplete market has to be bounded.

3.3 Robust Optimization Problem

Define utility at time t as a function of A_t and L_t. We employ an optimal hedging strategy that maximizes the utility function $U(A_t, L_t, t)$. The optimization problem is given by

$$\max_{w} U(A_t, L_t, t) \tag{7}$$

We call the hedging strategy which does not consider model misspecification a naive policy denoting w_{na}. The agent is completely confident with the estimated model and his hedging criterion is to maximize the utility function by making an instantaneous investment decision w_{na}.

If the agent is afraid of the model misspecification, he seeks a robust policy defined as:

$$\max_{w} \min_{\lambda_1, \lambda_2} U(A_t, L_t, t) \tag{8}$$

subject to state variable evolutions (3). The control variables λ_1 and λ_2 are subject to the uncertainty set $\mathbb{S}$. This is a robust control problem. The minimized utility is the worst case scenario. The max-min optimization problem is according to Anderson et al. (2003) a Nash Markov equilibrium of a two-player game. Player one is the robust agent. He makes an instantaneous investment decision w_{rob} to maximize the utility function at time t. Player two is the (imaginary) Mother Nature. Given the decision from player one, Mother nature attempts to minimize the utility by making an instantaneous choice of λ_1 and λ_2. This imaginary Mother Nature represents agent's fear for model misspecification. We call the optional portfolio choice w_{rob} from (8) a robust decision.

If the utility function $U(A_t, L_t, t)$ is convex and satisfies the assumptions of **Theorem 2.6** of Rudloff (2006), then function (8) and its dual problem [5] are equivalent.

4 Static Robust Optimization

Given the information at time t, our hedging strategy is defined over the hedging error $L_T - A_T$ at a predetermined time T. Suppose our utility function is in forms of the shortfall risk $-[L_T - A_T]^+$, which specifies the downside risk on the liability shortfall. The lower the shortfall risk is, the higher the agent's utility will be. Hence, we employ an optimal hedging strategy that minimizes the expected shortfall at time T. Hence, the

[5] The dual problem is: $\max_{\lambda_1, \lambda_2} \min_{w} U(A_t, L_t, t)$

naive optimization problem is given by

$$\min_{w} \mathbb{E}[(L_T - A_T)^+ \mid \mathcal{F}_t] \tag{9}$$

and the robust optimization is

$$\min_{w} \max_{\lambda_1,\lambda_2} \mathbb{E}[(L_T - A_T)^+ \mid \mathcal{F}_t] \tag{10}$$

The expected shortfall function is convex, hence (10) and its dual problem $\max_{\lambda_1,\lambda_2} \min_{w} \mathbb{E}[(L_T - A_T)^+ \mid \mathcal{F}_t]$ has the same optimal solution.

In the following two sections, we will show how to solve the robust optimization problem and how the robust solution differs from the naive one, and also, how can we benefit from the robust decision. We start with the relatively simple case when both agent and Mother nature only make decisions now without rebalancing until the expiration period T. The static case is technically easy to solve and also provides us some intuition about the robust policy. However, the static solution is not optimal, if the agent is able to rebalance before T. We therefore, also provide a dynamic solution in the next section in which λ_1 and λ_2 are time series processes.

4.1 Pricing in the Incomplete Market

If the control variables w, λ_1 and λ_2 are static, our criterion function $\mathbb{E}\left[(L_T - A_T)^+\right]$ is very similar to the value of an "exchange option" (see e.g.Hull (2009)) which changes one asset for another at time T. We can also consider this payoff function as a European option with a floating strike price. Margrabe (1978) provides a formula for valuing this type of options. The problem in our case is more complicated, because we are in an incomplete market, which means the equivalent martingale is not unique, or in other words, the so called risk-neutral $\mathbb{Q}$ measure is not unique, but depending on λ_1 and λ_2.

To facilitate calculation, let

$$\mu_S = \mu + \sigma\lambda_1, \tag{11a}$$

$$\mu_A = r + w(\mu_s - r), \tag{11b}$$

$$\mu_L = a + b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2, \tag{11c}$$

representing the drift terms of the stock market, the asset and the liability respectively.

There are many ways to solve this static criterion function. We use the change of probability measure technique. By multiplying and dividing $\mathbb{E}\left[L_T\right]$ inside the valuation function, we can create a Radon-Nikodym process $\frac{L_T}{\mathbb{E}[L_T]}$ that changes the probability measure from $\mathbb{P}$ to a new measure called $\mathbb{L}$. That is to say $\frac{L_T}{\mathbb{E}[L_T]}$ is a positive $\mathbb{P}-$ martingale with $\mathbb{E}\left(\frac{L_T}{\mathbb{E}[L_T]}\right) = 1$, hence the expected shortfall function can be rewritten as,

$$\mathbb{E}\left[(L_T - A_T)^+\right] = \mathbb{E}\left[L_T\right]\mathbb{E}\left[\frac{L_T}{\mathbb{E}\left[L_T\right]}\left(1 - \frac{A_T}{L_T}\right)^+\right] = \mathbb{E}\left[L_T\right]\mathbb{E}^{\mathbb{L}}\left[(1 - C_T)^+\right], \quad (12)$$

We denote the coverage ratio at time t as $C_t = \frac{A_t}{L_t}$. It is a common criterion used to describe the performance of a financial institution. If the coverage ratio is smaller than one, the the fund is facing a solvency risk. Hence, we reconstruct our "exchange option" to a product of the expected value of L_T under $\mathbb{P}$ and the value of a European put option under measure $\mathbb{L}$. The first term $\mathbb{E}\left[L_T\right]$ is equal to $L_0 \exp\left(\mu_L T\right)$, and the second term is known from Margrabe (1978).

Applying Ito's lemma on dC_t, we can derive the diffusion process of C_t. We show the calculation in Appendix 7.1. We also show how to get the process $W_1^{\mathbb{L}}$ and $W_2^{\mathbb{L}}$ under the new probability measure. The diffusion process of the coverage ratio C_t under the new measure $\mathbb{L}$ is given by

$$dC_t = C_t\left[\left(-\mu_L + \mu_A\right)dt + \left(w\sigma - b\rho\right)dW_1^{\mathbb{L}} - b\sqrt{1-\rho^2}dW_2^{\mathbb{L}}\right] \quad (13)$$

Notice that dC_t process under $\mathbb{L}$ has a drift term, containing μ_A and μ_L only. The variance of coverage ratio is $\sigma_C^2 = \left(w\sigma - b\rho\right)^2 + b^2\left(1-\rho^2\right)$. Therefore, the analytical solution of our objective function under the static case is given by $\bar{L}\left[\Phi(-d_2) - \bar{C}\Phi(-d_1)\right]$, or

$$\mathbb{E}\left[(L_T - A_T)^+\right] = \bar{L}\Phi\left(-d_2\right) - \bar{A}\Phi\left(-d_1\right) = \bar{L}\left(\Phi\left(-d_2\right) - \bar{C}\Phi\left(-d_1\right)\right) \quad (14)$$

where

$$\begin{aligned} \bar{L} &= L_0 \exp\left(\mu_L T\right) \\ \bar{A} &= A_0 \exp(\mu_A T) \\ \bar{C} &= C_0 \exp\left[\left(\mu_A - \mu_L\right)T\right] \end{aligned}$$

and

$$\begin{aligned} d_1 &= \frac{\ln \bar{C} + \frac{\sigma_C^2}{2}T}{\sigma_C\sqrt{T}} \\ d_2 &= d_1 - \sigma_C\sqrt{T} \end{aligned}$$

The function Φ is the cumulative probability distribution function for a standard normal distribution. Therefore, for given λ_1 and λ_2, the optional hedge disregarding the preference for robustness is the solution of the first order condition for maximizing $\mathbb{E}^{\mathbb{L}}\left[(1-C_T)^+\right]$ with respect to w (see Appendix (7.2)),

$$\frac{\partial\left[\Phi(-d_2)-\bar{C}\Phi(-d_1)\right]}{\partial w}=-\Phi\left(-d_1\right)\bar{C}\left(\mu-r\right)T+\bar{C}\phi\left(d_1\right)\sqrt{T}\frac{w\sigma^2-b\rho\sigma}{\sigma_C}=0 \tag{15}$$

where function ϕ denotes the standard normal probability density function. Note that $-\Phi\left(-d_1\right)$ is the delta of the BS put-option which is always less than zero, and $\bar{C}\phi\left(d_1\right)\sqrt{T}$ denotes the vega of the BS option which is always positive. Therefore, we see from (15) that the optimal w strikes a balance between the "delta effect" that reduces the value of the option and the "vega effect" that increases the value of the option. There is a special case when $\mu=r$ when the "delta-effect" disappears and the optimal w is then given by the minimum variance solution $w=\frac{b\rho}{\sigma}$.

4.2 Static Robust Portfolio Choice

In this subsection, we provide numerical solution for the static robust optimization problem. We can rewrite the objective function (9) as

$$\min_{w}\max_{\lambda_1,\lambda_2} L_0\exp\left(\mu_L T\right)\Phi(-d_2)-A_0\exp\left(\mu_A T\right)\Phi(-d_1)=0 \tag{16}$$

subject to $\lambda_1^1+\lambda_2^2\leq\kappa^2$. Notice that μ_A and μ_L are functions of λ.

This problem can only be solved numerically. As a benchmark scenario, we assume $\mu=0.04$, $\sigma=0.16$, $r=0$, $a=0$, $b=0.1$, $\rho=0.5$, $\kappa=0.25$. We assume that the stock return μ is higher than the liability return a because the stock volatility σ is normally higher than the volatility of the liability b.

As we have discussed in Section 3.2 that uncertainty set parameter κ depending on the significance level α and the sample size N, hence is fixed and state variable independent. We choose the significance level $\alpha=0.05$ at which the corresponding χ^2 value with 2 degrees of freedom is 5.99. The choice of κ is also based on an implicit assumptions that the risk premium $\frac{\mu_S-r}{\sigma}$ is always positive, which means $\kappa\leq\frac{\mu-r}{\sigma}=0.25$. Hence, the sample size N has to be larger than 96 so as to satisfy this implicit assumption.

The robust hedge provides the optimal solution under the worst case scenario. Under the benchmark scenario, the worst case scenario occurs when the true expected asset return is lower than the estimated value, $\mu_A<r+w\left(\mu-r\right)$, and the true liability return is higher than the estimated result $\mu_L>a$. In this case, the true hedging error will be much higher than expected. A negative λ_1 and a positive λ_2 lead to the worst case scenario. To see this, we first look at the asset side. If λ_1 is negative, then the expected asset return is reduced. But a negative λ_1 also reduces the liability return since λ_1 also plays a role on the liability side except when $w\sigma - b\rho = 0$, the effect of λ_1 on the liability side cancels out the effect of λ_1 on the asset side completely. In this particular case, the decision of λ_2 is independent of the choice of λ_1. Besides, the effect of λ_2 is partially canceled and it has to be sufficiently big so as to beat negative effect of λ_1 so as to push up the liability return. In short, the agent is afraid of a negative λ_1 and a positive λ_2.

In Figure 1, we show the static optimal portfolio choice at time $t = 0$ as the function

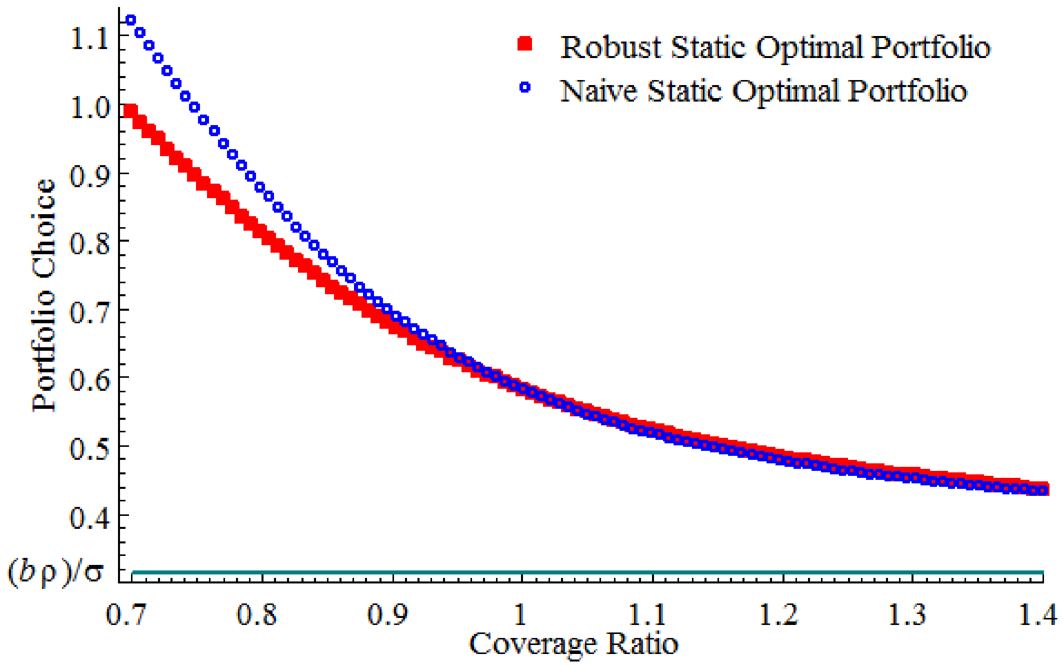


Figure 1: Static optimal portfolio choice. This figure compares the robust and naive static optimal hedging policies. The investor makes an investment decision at time $t = 0$ with given current funding ratio C_0 so as to minimize the expected shortfall at time period T. The naive policy replies completely on the estimation parameters. The robust policy takes the parameter uncertainty into consideration and insures against the worst case scenario. The horizontal axis depicts the present funding ratio. The results are based on the benchmark estimation parameters $\mu = 0.04$, $\sigma = 0.16$, $r = 0$, $a = 0$, $b = 0.1$, $\rho = 0.5$, $\kappa = 0.25$, and $T = 5$.

of the current funding ratio, C_0. The solid-dot curve depicts the robust portfolio decision involving the awareness of model misspecification and the open-dot curve shows the naive optimal portfolio choice with $\lambda_1 = \lambda_2 = 0$. When there is underfunding, which means $C_0 < 1$, the robust and naive policy differs. Both take substantial risk, but the robust portfolio is more conservative than the naive one. For example, if the current funding ratio equals to 80%, then the robust policy will reduce the risky asset exposure by approximately 6% based on the naive policy. A naive investor is completely relying on the underlying model and he believes that there is a positive expected asset return of 4%.

However, the robust investor is not so sure about the asset return and is afraid that the true expected asset return is not as high as the model estimated. Therefore the robust investor is more conservative with his portfolio choice.

However, robust hedges are not always more conservative than naive hedges. If the fund is already balanced, or even over funded with $C_0 \geq 1$, the two policies are almost identical, which means the robustness effect diminishes if C_0 goes up. The two curves converges to a hedging ratio, $\frac{b\rho}{\sigma} = 0.3125$ if C_0 is sufficiently big. The ratio $\frac{b\rho}{\sigma}$ is the result of a minimum-variance hedge that neutralizes the traded part of the liability risk. [6] The resulting volatility becomes $b\,(1-\rho^2)$ which is the unhedgeable part of the liability risk. Also, this position neutralizes the λ_1 effect such that the misspecification of asset return does not influence the performance of hedges.

The decision of Mother Nature is displayed in Figure 2. We show the movement of λ_1

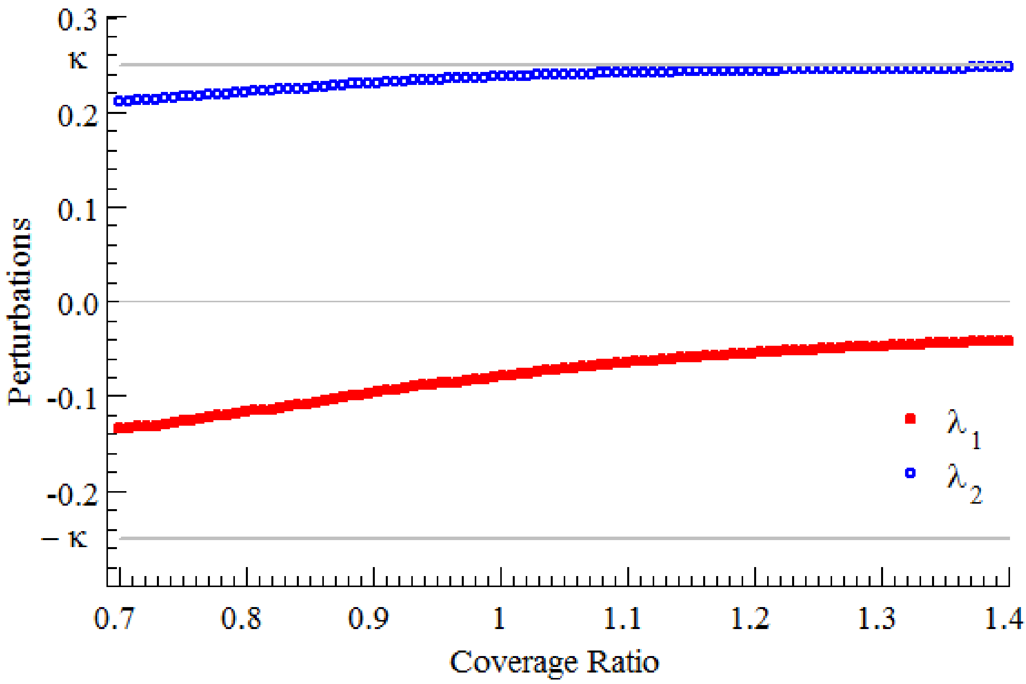


Figure 2: Static optimal perturbations λ_1 and λ_2. This figure depicts the optimal λ_1 and λ_2 as functions of the present coverage ratio C_0 under the benchmark scenario with $\mu = 0.04$, $\sigma = 0.16$, $r = 0$, $a = 0$, $b = 0.1$, $\rho = 0.5$, $\kappa = 0.25$, and $T = 5$. Mother nature makes decisions of λ_1 and λ_2 at time 0 under the constraint $\lambda_1^2 + \lambda_2^2 \leq \kappa^2$ so as to maximize the expected shortfall at period T.

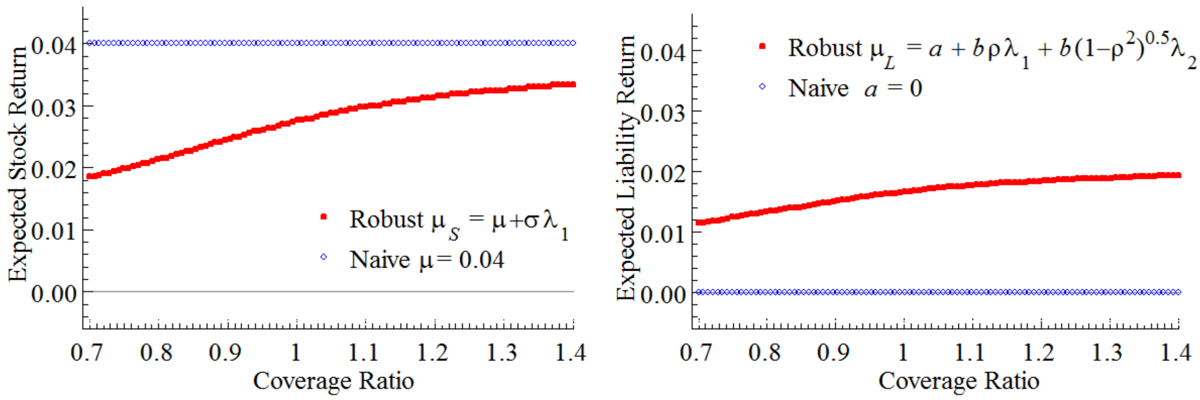
an λ_2 as a function of present coverage ratio C_0. To facilitate the comparison, we put the two perturbations in one graph. With given C_0, each combination of λ_1 and λ_2 leads to the biggest perturbations between models. We obtain that λ_1 is always beneath the zero

[6] To minimize the volatility of the hedging error, $\sigma_C^2 = (w\sigma - b\rho)^2 + b^2\left(1-\rho^2\right)$, with respect to w, the optimal hedging ratio w equals to $\frac{b\rho}{\sigma}$.

line given any coverage ratio level and is close to zero when C_0 is high, but λ_2 is always positive and converges to κ. We also find that the optimal choice of λ_1 and λ_2 are always on the circle $\lambda_1^2 + \lambda_2^2 = \kappa^2$, which means the worst case scenario is always at the boundary of the uncertainty set.

The resulting negative λ_1 represents the fear for an over estimated asset return. Hence, the absolute value of λ_1 is increasing with the exposure to the stock market, w. We have know from Figure 1 that risk exposure and the coverage ratio are negatively related. The lower the coverage ratio is, the higher the risk exposure will be and therefore, the more negative value of λ_1 will be. However, if the coverage ratio is sufficiently high, the investor will put less wealth in the risky asset, as penalty from λ_1 is smaller.

However, the penalty term λ_1 also plays a role in the liability return. A negative λ_1 can benefit the agent by bringing down the expected liability return . However, the agent is afraid of an under estimated liability return. To represent the fear for an increase of the liability return, the nature has to choose a positive λ_2 so as to compensate the negative effect from λ_1 an to increase the liability return. In other words, the perturbed drift term of the liability $\mu_L = a + b\left(\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2\right)$ must be higher than the estimated value a. In contrast to a declining fears of a misspecified asset return, the penalty term λ_2 is climbing over C_0.



(a) Robust vs. Naive μ_S

(b) Robust vs. Naive μ_L

Figure 3: Mean rate of stock and liability return with and without the preference for robustness. This figure displays the expected stock and liability returns before and after considering parameter uncertainty as functions of the present coverage ratio. Panel 3a: comparing the robust stock drift $\mu_S = \mu + \sigma\lambda_1$ with the naive drift term $\mu_S = \mu$. Panel 3b: comparing the robust liability drift term $\mu_L = a + b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2$ with the naive one $\mu_L = a$ under the benchmark scenario.

Continuously, we now investigate how do the perturbation terms impact the expected returns. Figure 3 displays both naive and robust mean rate of the stock return and the liability return as functions of C_0. Without the preference for robustness, both drift terms are constant at the benchmark level disregard the movement of C_0. However, if the investor is aware of the model misspecification and aims to insure against the worst case scenario. The perturbed expected stock return is dragged down by $|\sigma\lambda_1|$ amount and the worst case liability drift is pushed up by $|b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2|$ amount.

The robust policy differs from the naive one in the way that the agent adds an additional guarantee onwards the naive contract so as to neutralize himself from the estimation error. This additional insurance makes the robust policy more expensive. In Appendix 7.3, we show the cost of hedge following the two difference policies.

4.3 Policy Evaluation

The agent is motivated to follow the robust policy because the robust policy is less sensitive to the estimation error. In this section, we will show how and when the agent can benefit from the robust policy. Let $Q(w,\delta)$ be the cost of hedging following a certain policy w, where δ is introduced in Section 3.2. In our case, the cost of hedge is defined by

$$Q(w,\delta) = \mathbb{E}_t\left[(L_T - A_T)^+ | w, \delta\right] \tag{17}$$

The optimal hedging policy has a cost $q(\delta) = \min_w Q(w,\delta)$ for given δ. Let δ_0 be the true value of δ, with $\delta_0 = \begin{pmatrix} \mu_0 \\ a_0 \end{pmatrix}$ and denote q_0 as the minimum hedging cost when the investor implements the associated optimal hedging policy w_0 under the true value δ_0. Therefore, it is trivial that any other alternative hedging policies $w_a\,(w_a \neq w_0)$ are more expensive than q_0.

Define the loss function $K(w_a|\delta_0)$ as the difference between the cost of hedging following a suboptimal policy $Q(w_a,\delta_0)$ and the true minimum cost:

$$K(w_a|\delta_0) = Q(w_a,\delta_0) - q_0 \tag{18}$$

In case δ is misspecified with $\delta \neq \delta_0$, the agent is facing estimation error, therefore $w_a \neq w_0$ and $K(w_a|\delta_0) > 0$.

Suppose the agent does not know the true value of the drift terms δ_0, given the estimated drift terms δ, he can choose between two alternative hedging policies, a robust policy w_{rob} and a naive policy w_{na}. At the benchmark scenario when the present coverage ratio $C_0 = 80\%$, we have solved that $w_{\text{rob}} = 0.81$ and $w_{\text{na}} = 0.87$. When $C_0 = 90\%$, we have that $w_{\text{rob}} = 0.67$ and $w_{\text{na}} = 0.69$.

The robust policy out performs the naive one if its loss is smaller, or

$$K(w_{\mathrm{rob}}|\delta_0) < K(w_{\mathrm{na}}|\delta_0) \tag{19}$$

Next, we will investigate in what circumstance the robust policy can beat the naive policy. We display the loss indifference curves in Figure 4 when the present coverage ratio is

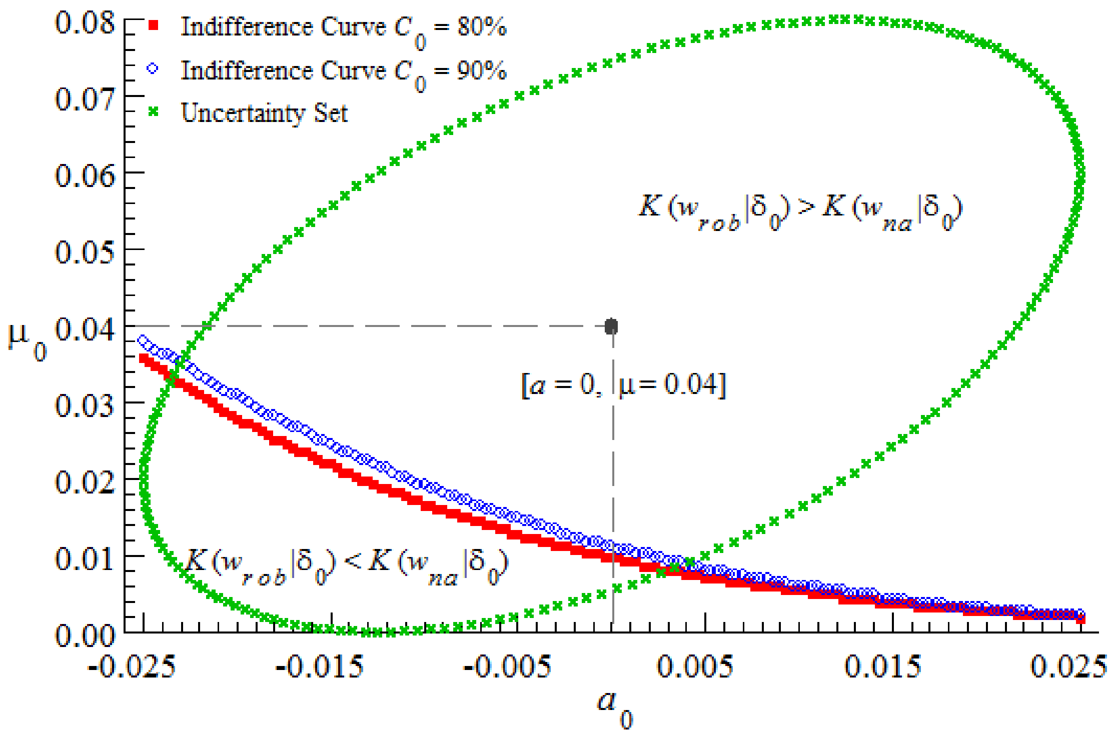


Figure 4: Loss function equivalent curves. The figure plots the indifference curve of the loss when $K(w_{\mathrm{rob}}|\delta_0) = K(w_{\mathrm{na}}|\delta_0)$. $y-$axis is the true value of expected stock return μ_0 and $x-$axis is the true value of the liability drift a_0. The estimated value is $\mu = 0.04$ and $a = 0$. The solid-dot indifference curve represents the case when $C_0 = 80\%$ and the open-dot curve is the when $C_0 = 90\%$. In the region beneath the curve, the robust policy outperforms the naive one and in the region above is another way around.

80% and 90%. The x-axis and y-axis represent the true value of liability return a_0 and asset return μ_0 respectively. The $[a = 0, \mu = 0.04]$ spot represents the estimated expected return δ. We also display the ellipsoid uncertainty set of the true drift term δ_0 in the figure. The area outside the ellipse is assumed infeasible.

On the policy indifference curve, when $K(w_{\mathrm{rob}}|\delta_0) = K(w_{\mathrm{na}}|\delta_0)$, the two policies are equally expensive. In the region beneath the indifference curve for both scenarios (when $C_0 = 80\%$ and 90%), the robust policy is cheaper then the naive policy. The value of δ_0 in this region is lower than the estimated value δ. Hence we can conclude that when the true drift term δ_0 is over estimated, the robust policy is better off.

Also, we find that this beneficial region is positively related to the present coverage ratio C_0. In Figure12, we have shown that the additional cost of hedging following the robust policy is increasing with a decreasing C_0 is low. Therefore a lower C_0 leads to a smaller beneficial region.

4.4 Sensitivity Analysis

The correlation parameter ρ, standing for the completeness of the market, plays an important role in our economy. If $\rho = \pm 1$, and $\lambda_1 = \lambda_2 = 0$, then our underlying market becomes complete and the unhedgeable risk driver W_2 does not play a role. In this section, we investigate how sensitive the optimal hedges are with respect to a change of ρ.

In Figure 5, we show an extreme case when $\rho = 1$. The non-traded risk driver W_2

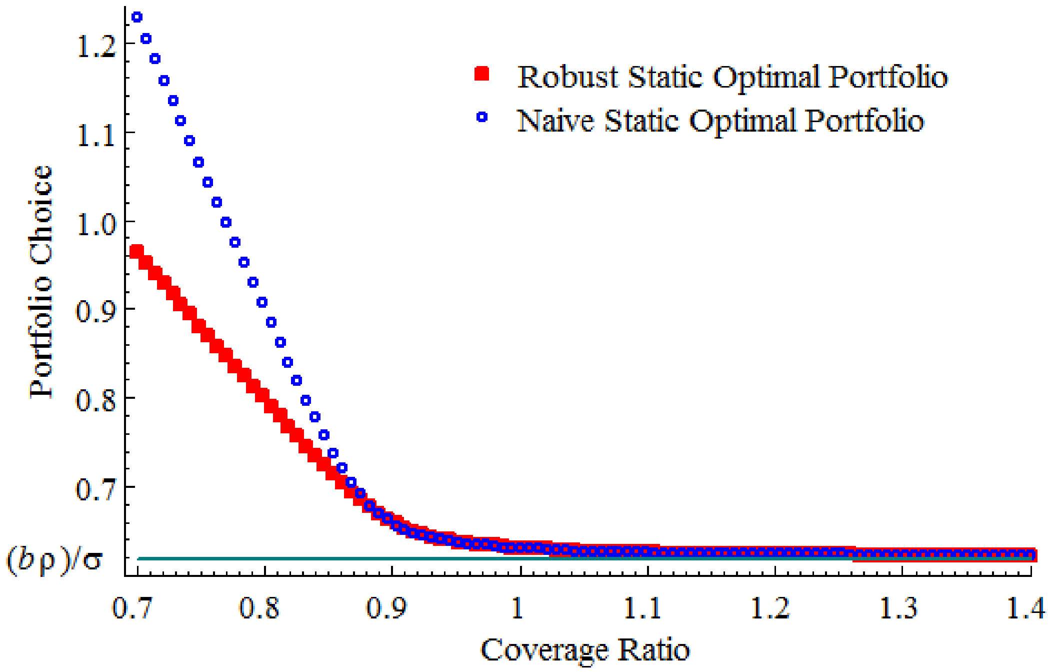


Figure 5: Sensitivity analysis with $\rho = 1$. The figure depicts the optimal portfolio choice when $\rho = 1$. The rest parameters stay at the benchmark level. The solid-dot line represents the robust policy and the empty dotted curve is the naive policy. The naive agent considers such an economy a complete market if $\lambda_1 = \lambda_2 = 0$, since the non-tradable risk driver W_2 is gone. However, the robust agent still stays in the incomplete market, because the model misspecification ($\lambda_1 \neq 0\, \lambda_2 \neq 0$) is also considered as another source of market incompleteness.

disappears from the liability diffusion process and the perturbation parameter λ_2 does not play a role neither. Mother nature can only control over λ_1 to maximize the expected shortfall at period T. The naive agent consider this as a complete market. However, the robust agent still faces another source of incompleteness which is caused by model misspecification.

If there is insufficient wealth in the fund, the robust policy deviates from the naive one much more severe compared with the benchmark case. When the asset risk and the liability risk are perfectly correlated, Mother nature will choose a more negative λ_1 so as to maximize the expected shortfall. Although a negative λ_1 reduces the expected liability return as well, the liability drift term is less sensitive to the change of λ_1 than the expected asset return does, since $\sigma > b$. As the result the robust investor's fear for an over estimated asset return is stronger than the benchmark level.

In the case of overfunding, the two policies are identical. The hedging error volatility becomes $\sigma_C^2 = (w\sigma - b\rho)^2$. The investor can fully replicate the liability by following a Delta-neutral strategy $w = \frac{b\rho}{\sigma} = 62\%$ if he has sufficient asset. In that case robustness does not play a role because the Delta hedge neutralizes the λ_1 effect.
In Figure 6 we show the two hedging policies as a function of correlation parameter ρ.

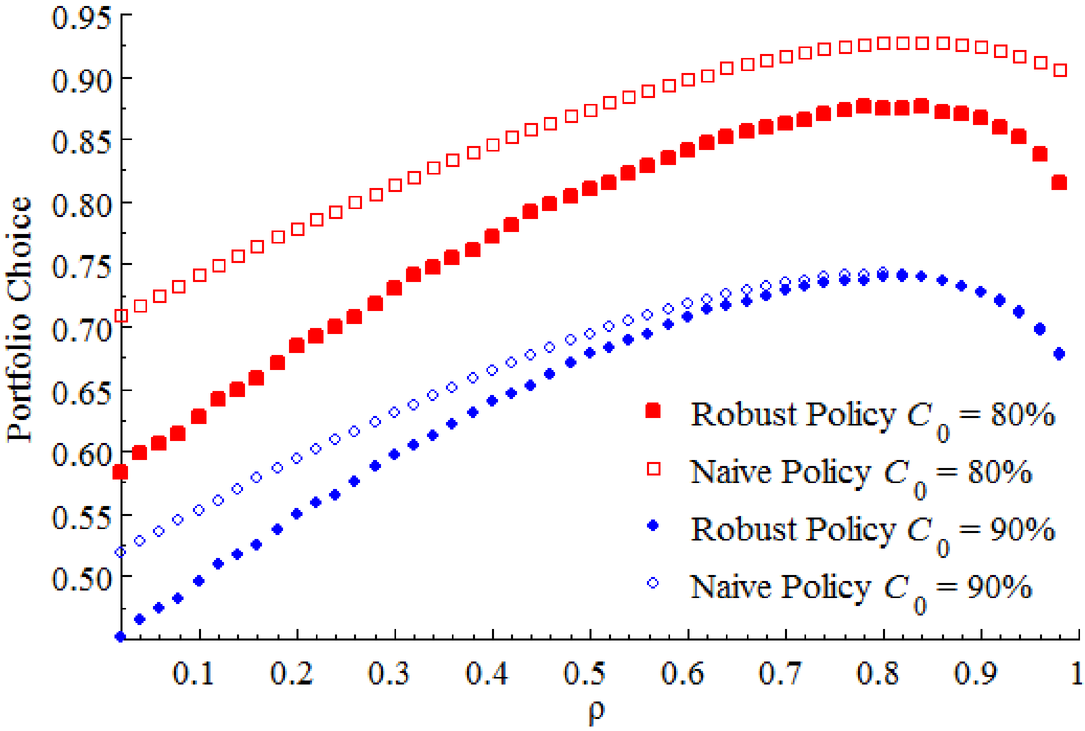


Figure 6: Sensitivity analysis with respect to ρ . The figure plots the optimal naive and robust hedging policy as a function of correlation parameter ρ. We show two pairs of comparison. The red pair with cube-dot is when the present coverage ratio C_0 is 80%, and the blue pair with circle-dot is when $C_0 = 90\%$. The solid-dot curves represent the robust policy and the empty-dot curves are the naive policy.

We display two scenarios, one is when $C_0 = 80\%$ and the other is when $C_0 = 90\%$. The relation between the optimal portfolios and ρ is not monotonous, but is hump shaped.

This is because of the volatility of the value function σ_C is a quadratic function of ρ.

The optimal portfolio is firstly increasing with ρ for either policies because the liability market is more exposed to the tradable risk driver W_1. Therefore, the risky portfolio has to increase as well in order to hedge the traded liability risk. Next, the optimal portfolio reaches the peak where ρ maximizes the total volatility σ_C. After the peak, the risky portfolio goes down with ρ, because after the peak, any higher level of correlation will reduce σ_C. From Figure 6, we can also see that the difference between the two policies under the lower coverage ratio is wilder than under the higher C_0.

5 Dynamic Robust Optimization

In this section, we will extend the problem to a dynamic world. The robust investor still aims to minimize the final-period expected shortfall under the worse case scenario, but instead of making a static portfolio choice, he is now considering a dynamic optimal portfolio. The nature also needs to rebalance her choice of λ_1, λ_2 instantaneously given the intertemporal decision of w. We employ dynamic programming technique to solve this robust optimization problem. The structure of this section is as follows: we first formulate the dynamic robust optimization problem and discuss the analytical solution at the extreme case. Then, we employ numerical methods to solve the partial differential equation. Last, we will investigate the dynamic effect on the policy indifference curve analyzed in Section 4.3.

5.1 Dynamic Programming

Let the value function at time t be $U(A, L, t)$, then using Feynman-Kaç we can derive the Hamilton-Jacobi-Bellman equation (henceforth HJB) or partial differential equation (pde) for the investor's min-max problem:

$$0 = \min_{w} \max_{\lambda_1,\lambda_2} U_t + U_A A\left(r + w(\mu - r) + w\sigma\lambda_1\right) + U_L L\left(a + b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2\right) + \tfrac{1}{2}U_{AA}w^2\sigma^2A^2 + \tfrac{1}{2}U_{LL}b^2L^2 + U_{AL}b\rho w\sigma AL \quad (20)$$

with the boundary condition: $\lambda_1^2 + \lambda_2^2 \leq \kappa^2$. We employ the method of Lagrange by introducing the multiplier ν and forming the Lagrangian function:

$$0 = \min_{w} \max_{\lambda_1,\lambda_2} U_t + U_A A\left(r + w(\mu - r) + w\sigma\lambda_1\right) + U_L L\left(a + b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2\right) + \tfrac{1}{2}U_{AA}w^2\sigma^2A^2 + \tfrac{1}{2}U_{LL}b^2L^2 + U_{AL}b\rho w\sigma AL - \frac{1}{2}\nu\left(\lambda_1^2 + \lambda_2^2 - \kappa^2\right) \quad (21)$$

By solving a linear system of equations based on the first order condition of (21) with respect to w, λ_1 and λ_2, we have

$$w^* = -\frac{(\mu - r)\, U_A A \nu}{\sigma^2 \left(U_{AA} A^2 \nu + U_A^2 A^2\right)} - \frac{U_{AL} A L b \rho \sigma \nu + U_L U_A A L b \rho \sigma}{\sigma^2 \left(U_{AA} A^2 \nu + U_A^2 A^2\right)} \tag{22a}$$

$$\lambda_1^* = -\frac{(\mu - r)\, U_A^2 A^2 \sigma}{\sigma^2 \left(U_{AA} A^2 \nu + U_A^2 A^2\right)} - \frac{b\rho \left(U_{AL} A L U_A A - U_L L U_{AA} A^2\right)}{\left(U_{AA} A^2 \nu + U_A^2 A^2\right)} \tag{22b}$$

$$\lambda_2^* = \frac{U_L L b \sqrt{1-\rho^2}}{\nu} \tag{22c}$$

The sign of optimal λ_2 must be positive since it increases the expected liability return but does not influence the pension asset. The sign of λ_1 is however not defined. A positive λ_1 does not only increase the liability but also the asset, but the net effect depends on the value of other input variables.

The structure of the solution (22) is very interesting. The investor's dynamic portfolio choice w and lifetime discrepancy between the reference model and perturbations, $\lambda's$, are dueling against each other. The dynamic optimal investment strategy w^* is a tradeoff between hedging and speculation. We can see this by considering the extreme case when $\nu \to 0$ and $\nu \to \infty$.

When ν is 0 For $\nu \to 0$, the discrepancy parameters $\lambda's$ have more freedom to choose an arbitrarily large aversion pair of drift for the Brownian Motions, or in other words, the agent is extremely pessimistic towards the approximation model. When $\nu \to 0$, we have

$$w^*_{\nu \to 0} = -\frac{U_L L}{U_A A} \frac{b\rho}{\sigma}, \tag{23}$$

This is a pure hedging portfolio, where the agent invests an amount into risky asset such that the change in value $U(A, L, t)$ due to L is (as much as possible) offset by a change in value due to A. It is not possible to completely eliminate the volatility of L. This is because the liabilities are exposed to both hedgeable risk W_1 and unhedgeable risk W_2, but only the hedgeable part W_1 can be eliminated.

The optimal value for λ_1^* when $\nu \to 0$ is given by

$$\lambda^*_{1,\nu \to 0} = -\frac{\mu - r}{\sigma} - \frac{b\rho \left(U_{AL} U_A - U_L U_{AA}\right) L}{U_A^2} \tag{24}$$

which contains two terms. The first term is the observable market-price of risk which we can see from the Black-Scholes setup. The second term in more interesting. Notice that $-\frac{b\rho(U_{AL}U_A - U_L U_{AA})L}{U_A^2} == \sigma \frac{\partial (w^*_{\nu \to 0} A)}{\partial A} = w^*_{\nu \to 0}\sigma + \sigma A \frac{\partial w^*_{\nu \to 0}}{\partial A}$, it reflects to what extend the agent's best possible hedging strategy is influenced by the instantaneous wealth level A_t.

When ν is infinity On the other extreme, when we consider the case $\nu \to \infty$, then both λ_1 and λ_2 shrink to zero, so $\kappa = 0$. This corresponds to the case when the agent faces no model misspecification. Hence we recover the "classical" Merton's solution for the optimal portfolio choice:

$$w^*_{\nu\to\infty} = -\frac{\mu - r}{\sigma^2}\frac{U_A}{U_{AA}A} - \frac{U_{AL}L}{U_{AA}A}\frac{b\rho}{\sigma} \tag{25}$$

The first term is a speculative portfolio where the agent invests in the stock market to obtain the optimal trade-off between the observable market price of risk $\frac{\mu-r}{\sigma^2}$ and the local risk aversion $-\frac{U_A}{U_{AA}A}$. The second term is the intertemporal hedging component, but the optimal amount to hedge is now measured in terms of the "CAPM-beta". That is, the optimal hedge is the local covariance term $b\rho\sigma$ divided local variance term σ^2, i.e. the stock market investment that minimizes locally the (unhedgeable) variance in the portfolio.

5.2 Numerical Solution

Due to the complexity of our problem, we cannot solve the PDE analytically. We employ an explicit finite difference method to solve the PDE. In Appendix 7.4, we elaborate the numerical procedure of solving our dynamic programming problem. In this section, we will show the numerical result of the dynamic optimization problem.

In Figure 7, we have shown the dynamic robust investment policy as a function of the instantaneous coverage ratio C_t and time. We can conclude from the figure that on one hand, if the coverage ratio is continuously low, the investor will increase the risk exposure over time. The reasoning is that given the unpleasant performance of the fund, the investor is afraid of a even poorer funding ratio in the next period. We have learnt from the static case that the optimal risk exposure is supposed to be high when the coverage ratio is low. On the other hand, if the liability payoff is already fully covered with $C_t > 1$, the investor will decrease his risk exposure over time. and the optimal portfolio converges faster to the hedging ratio Delta ($\frac{b\rho}{\sigma}$) when t is approaching to expiration.

Next, we would like to investigate the difference between the robust and the naive policy in dynamic version. In Figure 8, we show the two investment policies as a function of instantaneous coverage ratio at two selected time periods, $t = 0$ and $t = 3$. We highlight two findings from the figure. First, the robust policy is always less risky than the naive one as long as the instantaneous coverage ratio is lower than 1. Second, we find that the difference between the two policies is increasing over time if the instantaneous

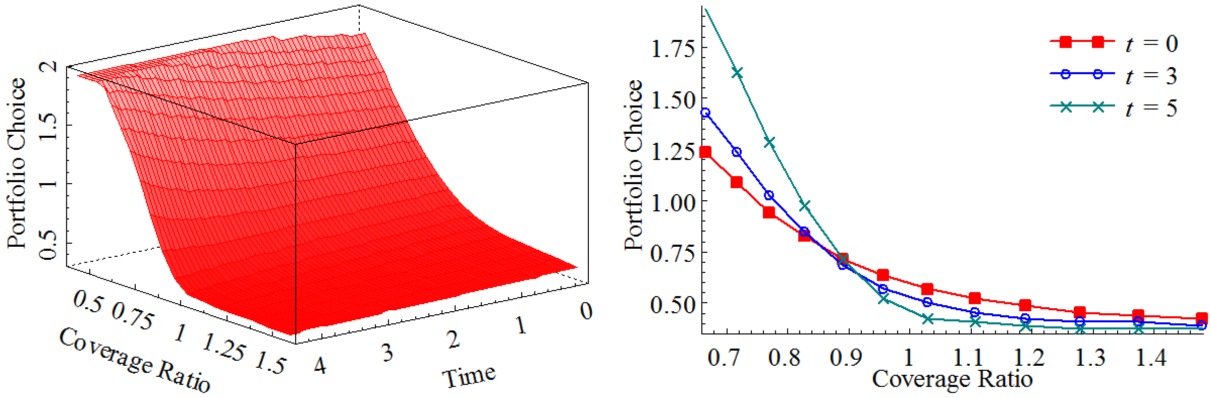


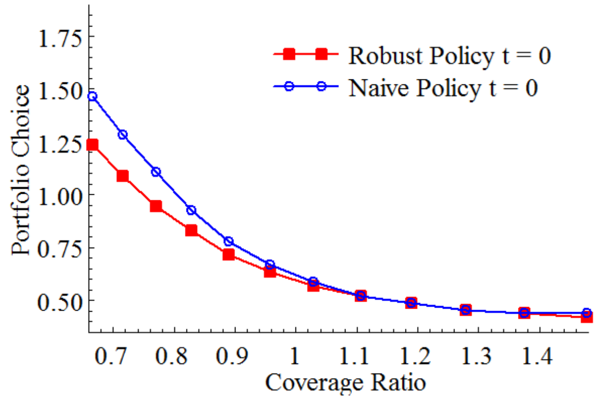
Figure 7: Dynamic robust optimal hedging strategy. This figure displays the robust optimal investment policy as a function of time and the instantaneous coverage ratio C_t with benchmark input parameters. Panel 7a plots the robust portfolio choice as a function of the instantaneous coverage ratio and the time movement. Panel 7b reduces one dimension from Panel 7a and plots the robust solution as a function of the instantaneous coverage ratio C_t through period 0 to period T, therefore it is in three dimensions. Panel 7b only depict the solutions at period $t = 0, 3, 5$. For technical limitations, our grid searching interval of the risky portfolio w has to be smaller than 1.95, otherwise we will confront a negative probability problem in some trinomial trees.

coverage ratio is low. The growing difference between the two policies is caused by the accumulated λ_1 effect.

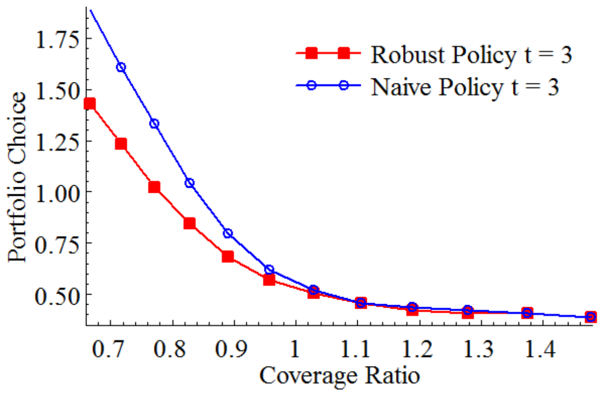
Next, we investigate the dynamic optimal perturbation process. Figure 9 shows the dynamic optimal λ_1, λ_2 as functions of the coverage ratio at three different time period. It is no longer fresh that λ_1 is always negative and λ_2 is always positive. We now focus on the dynamic effect of the processes.

We first look at the low coverage ratio region ($C_t < 1$). If C_t is low, λ_1 is decreasing over time. We provide two intuitive explanations of this finding. First, if a fund is continuously under performing, the agent would be more and more pessimistic towards the underlying model and is more likely to believe that the true expected asset return can be lower, hence is negative λ_1 effect is growing. Second, the agent is expecting a decreasing instantaneous coverage ratio, hence his risky portfolio increases over time,as a result,the exposure to the estimation error is increasing over time as well. An increasing λ_1 effect is always accompanying with a decreasing λ_2 effect due to our specific uncertainty set design.

If the instantaneous coverage ratio is high, the agent is approaching a Delta hedge so as to neutralize the λ_1 effect. Therefore, λ_1's negative effect is diminishing over time as $t \to T$. In contrast, λ_2 effect is growing over time and converging quickly to κ, such that Mother nature can maximize the shortfall risk.



(a) $t = 0$



(b) $t = 3$

Figure 8: Dynamic robust and naive optimal hedging strategy as a function of instantaneous coverage ratio at selected time periods . In this figure we display both robust and naive investment policies as functions of the instantaneous coverage ratio at different time period. Panel 8a plots the movement at period 0. Panel 8b show the results at time $t = 3$.

Now let us look at the movement of dynamic perturbed drift terms displayed in Figure 10. Panel 10a plots the perturbed expected stock return process μ_S as a function of C_t and at time $t = 0, 3, 5$. Since $\mu_S = \mu + \sigma\lambda_1$ is a linear function of λ_1, it shares common characteristics of λ_1. In short, μ_s decreases over time if underfunding and the other way around if $C_t > 1$.

Panel 10b shows the movement of μ_L. When C_t is low, the perturbed expected liability return μ_L decreases over time because λ_1 and λ_2 is reducing over time. For large C_t, the negative effect of λ_1 diminishes over time and λ_2 converges to κ, therefore, μ_L goes up overt time and converges to $a + b\sqrt{1-\rho^2}\kappa$.

5.3 Dynamic Policy Evaluation

We have known from the static case that the robust policy is better off when the drift term is over estimated. In this section, we will investigate the policy indifference curve as a function of time.

Figure 11 displays the policy indifference curve in different time period. Panel 11a displays the scenario when $C_t = 80\%$ and Panel 11b shows the plots at $C_t = 90\%$. Each panel plots the inference curve at time $t = 0, 3, 5$. In the area beneath the indifference curve, the robust policy is better in the sense that the cost of hedging with given amount of estimation error is lower. In line with the finding from Figure 4, we also observe from Figure 11 that the robust policy's beneficial region has a positive relation with coverage

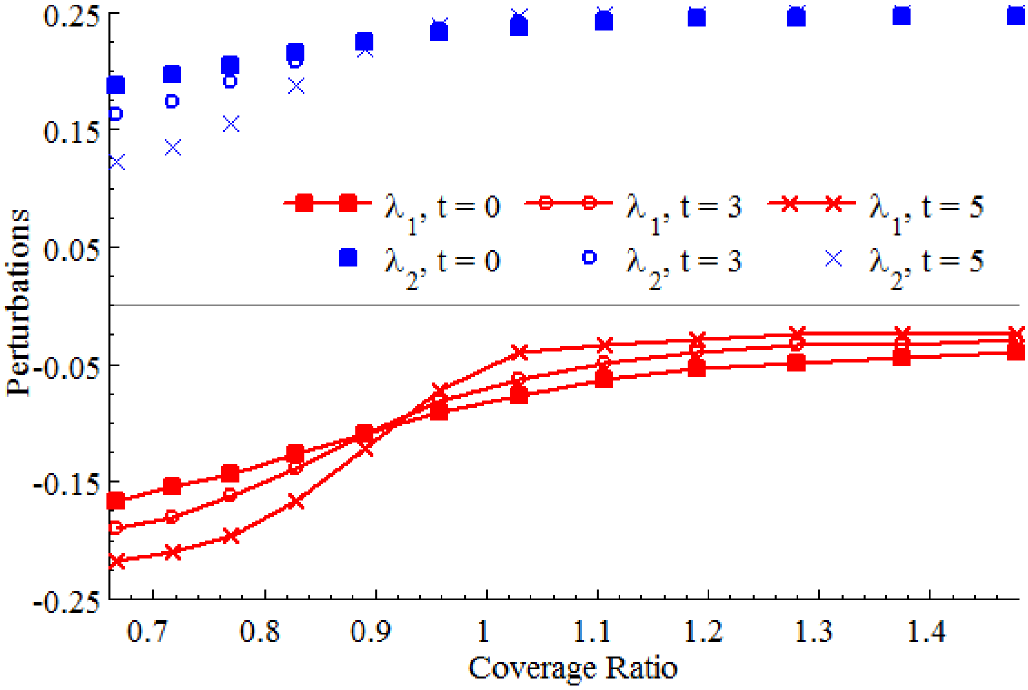


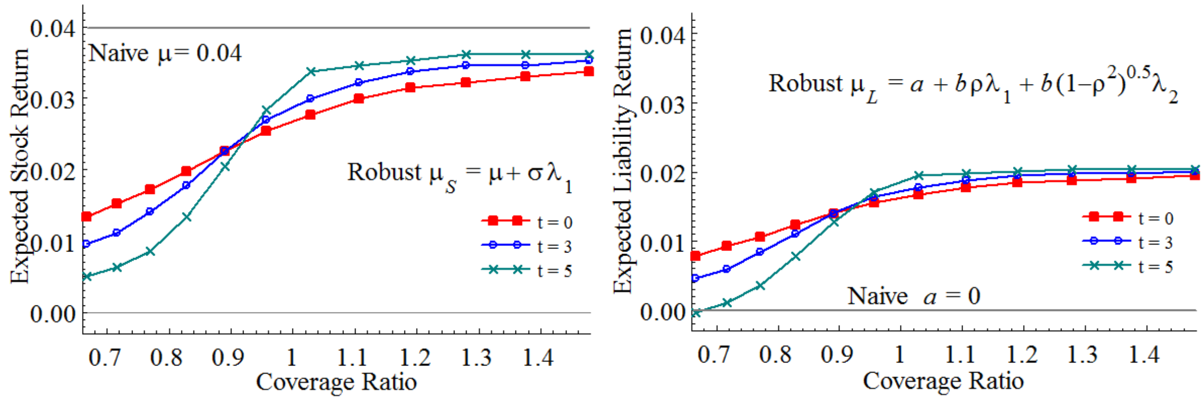
Figure 9: Dynamic optimal perturbation processes.In this figure we show the optimal perturbation processes λ_1 and λ_2 as functions of the instantaneous coverage ratio at time $t = 0, 3, 5$. The solid lines are the movement of λ_1 and λ_2 at time $t = 0$, the dashed curves are at time $t = 3$ and the dotted curves are at time $t = 5$. The upper panel with positive perturbations are the optimal results of λ_2. The negative part of the figure are the optimal solutions of λ_1.

ratio C_t. Besides, we also obtain that the beneficial region decrease over time. However this the dynamic effect is relatively weaker in Panel 11b when the instantaneous coverage ratio is relatively high.

6 Conclusion

In this paper we provide a robust hedging strategy under the condition that the market is incomplete and the underlying model is misspecified. From our analysis, we summarize two major characteristics of the robust policy. We first find that the robustness effect strongly depends on the instantaneous coverage ratio. The preference for robustness only influence the hedging policy when the coverage ratio is low, if the fund asset is big enough to cover the liability payoff, then the robust policy and the naive one are identical. Second, we find that agent can benefit from the robust policy when the expected return is over estimated.

In future research, we should extend the source of uncertainty. Instead of considering additional drift terms in Geometric Brownian Motion only, we shall also study the uncertainty of the drift term with mean reversion. We shall also consider a time varying



(a) μ_S (b) μ_L

Figure 10: Dynamic perturbation effect on drift terms. In this figure, we plot the dynamic movement of the perturbed drift terms as functions of the instantaneous coverage ratio at time $t = 0, 3, 5$. Panel 10a depicts the movement of $\mu_S = \mu + \sigma\lambda_1$ and Panel 10b shows $\mu_L = a + b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2$.

correlation parameter.

7 Appendix

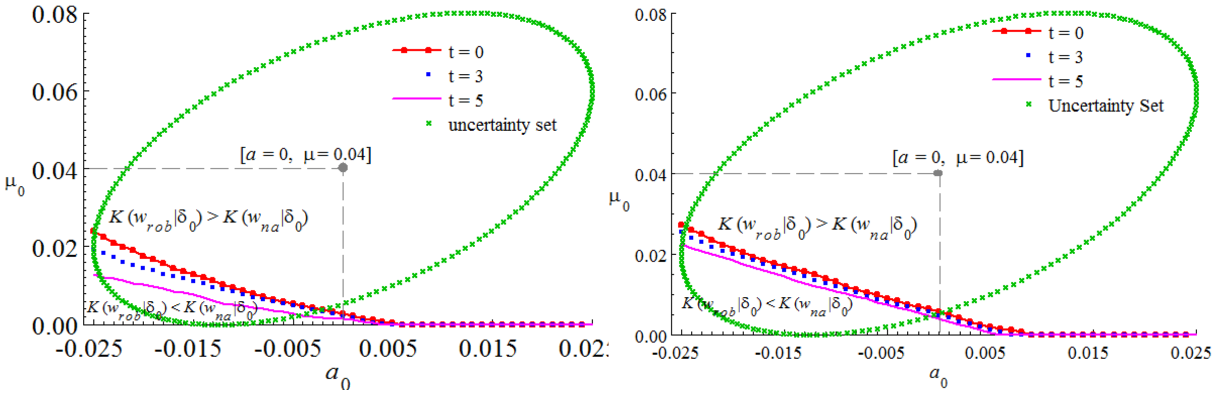
7.1 Change of Measure

We use a change of probability measure approach to solve $\mathbb{E}\left[(L_T - A_T)^+\right]$. The model is given by

$$
\begin{aligned}
dA_t &= \mu_A A_t dt + \sigma_A A_t dW_1, \\
dL_t &= \mu_L L_t dt + b\rho L_t dW_1 + b\sqrt{1-\rho^2} L_t dW_2
\end{aligned}
$$

First we need to construct a Radon-Nikodym process θ that changes measure from $\mathbb{P}$ to a new measure $\mathbb{L}$. According to Pelsser (2000), with given measure $\mathbb{P}$ and $\mathbb{L}$, if there is a random variable $\theta = \frac{d\mathbb{L}}{d\mathbb{P}}$ such that $\mathbb{E}^{\mathbb{L}}[X] = \mathbb{E}^{\mathbb{P}}[\theta X]$ for all random variables X, then we say θ is the Radon-Nikodym process of $\mathbb{L}$ with respect to $\mathbb{P}$. The expectation under $\mathbb{P}$ of θ must be equal to 1 because $\mathbb{E}^{\mathbb{P}}[\theta] = \mathbb{E}^{\mathbb{L}}[1] = 1$. By following the derivation below

$$
\begin{aligned}
\mathbb{E}\left[(L_T - A_T)^+\right] &= \mathbb{E}\left[L_T\left(1 - \frac{A_t}{L_t}\right)^+\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[L_T\right]\frac{L_T}{\mathbb{E}\left[L_T\right]}\left(1 - C_T\right)^+\right] \\
&= \mathbb{E}\left[L_T\right]\mathbb{E}\left[\frac{L_T}{\mathbb{E}\left[L_T\right]}\left(1 - C_T\right)^+\right] \\
&= \mathbb{E}\left[L_T\right]\mathbb{E}^{\mathbb{L}}\left[\left(1 - C_T\right)^+\right]
\end{aligned}
$$



(a) $C_t = 80\%$ (b) $C_t = 90\%$

Figure 11: Dynamic loss function equivalent curve.The figure shows the policy indifference curve at a function of the true drift terms (μ_0, a_0) at three difference time periods $t = 0, 3, 5$. Panel The upper panels plots the scenario when the instantaneous coverage ratio equals to 80%, and the bottom panel is the case when $C_t = 90\%$. The robust policy are better off in the area beneath the indifference curves. The dynamic policies w_{rob} and w_{na} are determined based on the estimated drift terms with value $\mu = 0.04$ and $a = 0$.

we construct a Radon-Nikodym process $\{\theta_t\} = \frac{L_t}{\mathbb{E}[L_t]}$ that changes the probability measure from $\mathbb{P}$ to $\mathbb{L}$. Notice that $\{\theta_t\}$ is strictly positive and its expectation under the $\mathbb{P}$ measure is $\mathbb{E}(\theta_t) = 1$.

According to Ito's Lemma [7]

$$dC_t = d[A_t \left(L_t^{-1}\right)] = A_t d(L_t^{-1}) + L_t^{-1} dA_t + d[A_t, L_t^{-1}] \tag{27}$$

where

$$\begin{aligned}
A_t d(L_t^{-1}) &= A_t \left[-\frac{dL}{L^2} + \frac{d[L, L]}{L^3} \right] \\
&= C_t \left[\left(-\mu_L + b^2\right) dt - b\rho dW_1 - b\sqrt{1-\rho^2} dW_2 \right] \\
L_t^{-1} dA_t &= C_t \left[\mu_A dt + w\sigma dW_1\right] \\
d[A_t, L_t^{-1}] &= -C_t b\rho w\sigma dt
\end{aligned}$$

Hence the diffusion process of the coverage ratio C_t under $\mathbb{P}$ measure is given by

$$dC_t = C_t \left[\left(-\mu_L + \mu_A + b^2 - b\rho w\sigma\right) dt + (w\sigma - b\rho)\, dW_1 - b\sqrt{1-\rho^2} dW_2 \right] \tag{28}$$

Next, Girsanov Theorem is applied to define the new measure. We show here how to

[7] We apply multivariate Ito's rule for computing quadratic covariation. Suppose x and y are functions with finite quadratic variation. Define $\phi(X, Y) = XY$ a product function of the two variables. Then

$$d(XY) = Y dX + X dY + d[X, Y] \tag{26}$$

where $d[X, Y]$ is the quadratic covariations between X and Y.

find the scalar process λ_t in our model. Given that $\theta_t = \frac{L_t}{\mathbb{E}[L_t]}$ with the expectation under measure $\mathbb{P}$, we can derive that

$$\begin{aligned} d\theta_t &= d\frac{L_t}{\mathbb{E}[L_t]} = \frac{dL_t}{\mathbb{E}[L_t]} - d\mathbb{E}[L_t]\frac{L_t}{\mathbb{E}[L_t]^2} \\ &= \theta_t\left[b\rho dW_1 + b\sqrt{1-\rho^2}dW_2\right] \end{aligned}$$

Hence, according to Girsanov Theorem , the Radon-Nikodym process is given by

$$d\theta_t = -\theta_t\left(-b\rho, -b\sqrt{1-\rho^2}\right)\begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix}$$

so $\lambda_t - \begin{bmatrix} -b\rho \\ -b\sqrt{1-\rho^2} \end{bmatrix}$, and the new Brownian motion under the $\mathbb{Q}^L$ measure is given by

$$\begin{aligned} dW_1^{\mathbb{L}} &= -b\rho dt + dW_1 \\ dW_2^{\mathbb{L}} &= -b\sqrt{1-\rho^2}dt + dW_2 \end{aligned}$$

Then we can rewrite equation (28) under measure $\mathbb{L}$ as,

$$dC_t = (\mu_A - \mu_L)\,C_t dt + (w\sigma - b\rho)\,C_t dW_1^{\mathbb{L}} - b\sqrt{1-\rho^2}C_t dW_2^{\mathbb{L}}$$

Next we show how to calculate $\mathbb{E}^{\mathbb{L}}\left[(1-C_T)^+\right]$,

$$\mathbb{E}^{\mathbb{L}}\left[(1-C_T)^+\right] = \mathbb{E}^{\mathbb{L}}\left[\mathbb{1}_{1\geq C_T}\right] - \mathbb{E}^{\mathbb{L}}\left[C_T\mathbb{1}_{1\geq C_T}\right]$$

where $\mathbb{1}$ is an indicator function.

$$\begin{aligned} & 1 \geq C_T \\ \Rightarrow\ & \ln\left(\frac{1}{C_0}\right) \geq \mu_A - \mu_L - \frac{1}{2}\left((w\sigma - b\rho)^2 + \left(1-\rho^2\right)b^2\right)T + (w\sigma - b\rho)\sqrt{T}W_1^{\mathbb{L}} - b\sqrt{1-\rho^2}\sqrt{t}W_2^{\mathbb{L}} \\ \Rightarrow\ & \ln\left(\frac{1}{C_0}\right) \geq \mu_A - \mu_L - \frac{1}{2}\sigma_C^2 T + \sigma_C\sqrt{T}Z \\ \Rightarrow\ & Z \leq \frac{-\ln\bar{C} + \frac{1}{2}\sigma_C^2 T}{\sigma_C\sqrt{T}} = -d_2 \end{aligned}$$

where $\sigma_C^2 = (w\sigma - b\rho)^2 + (1-\rho^2)\,b^2$, $Z \sim N(0,1)$ and $\bar{C} = C_0\exp\left((\mu_A - \mu_L)\,t\right)$. Hence

$\mathbb{E}^{\mathbb{L}}\left[\mathbb{1}_{1\geq C_T}\right] = \Phi\left(-d_2\right)$.

$$\begin{aligned}
\mathbb{E}^{\mathbb{L}}\left[C_T\mathbb{1}_{1\geq C_T}\right] &= \bar{C}\exp\left(-\frac{1}{2}\sigma_C^2 T + \sigma_C\sqrt{T}Z\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_2}\exp\left(-\frac{1}{2}Z^2\right)dZ \\
&= \bar{C}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_2}\exp\left(-\frac{1}{2}\left(Z-\sigma_C\sqrt{T}\right)^2\right)dZ \\
&= \bar{C}\Phi\left(-d_2-\sigma_C\sqrt{T}\right)
\end{aligned}$$

Let $-d_2 - \sigma_C\sqrt{T} = -d_1$, or $d_1 = d_2 + \sigma_C\sqrt{T}$, hence $\mathbb{E}^{\mathbb{L}}\left[C_T\mathbb{1}_{1\geq C_T}\right] = \bar{C}\Phi\left(-d_1\right)$.

Last, we have

$$\begin{aligned}
\mathbb{E}\left[L_T\right] &= \mathbb{E}\left[L_0\exp\left(\left(\mu_L - \frac{1}{2}b^2\right)T + b\rho\sqrt{T}W_1 + b\sqrt{1-\rho^2}\sqrt{T}W_2\right)\right] \\
&= L_0\exp\left[T\left(\mu_L - \frac{1}{2}b^2 + \frac{1}{2}\left(b\rho\right)^2 + \frac{1}{2}\left(b\sqrt{1-\rho^2}\right)^2\right)\right] \\
&= L_0\exp\left(\mu_L T\right) = \bar{L}
\end{aligned}$$

To summarize

$$\mathbb{E}\left[L_T\right]\mathbb{E}^{\mathbb{L}}\left[\left(1-C_T\right)^+\right] = \bar{L}\left(\Phi(-d_2) - \bar{C}\Phi(-d_1)\right)$$

7.2 Static Optimal Solution

We show in this section the first order condition of the static value function with respect to each control variable. We first introduce some properties that will be applied in the calculation.

Property 1: $d_2 = d_1 - \sigma_C\sqrt{T}$.

Property 2: $d_2^2 = d_1^2 - 2\ln\bar{C}$.

Property 3: $\Phi'\left(-d_1\right) = \Phi'\left(d_1\right)$

Property 4: $\Phi'\left(-d_2\right) = \Phi'\left(-d_1\right)\bar{C}$

Property 5: $\frac{\partial(-d_2)}{\partial w} - \frac{\partial(-d_1)}{\partial w} = \frac{\partial\left(\sigma\sqrt{T}\right)}{w} = \sqrt{T}\frac{w\sigma^2 - b\rho\sigma}{\sigma_C}$

Property 6: $\frac{\partial(-d_2)}{\partial\lambda_{1,2}} - \frac{\partial(-d_1)}{\partial\lambda_{1,2}} = \frac{\partial\left(\sigma\sqrt{T}\right)}{\partial\lambda_{1,2}} = 0$

where $\Phi'\left(d_1\right) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}d_1^2\right)$

- FOC w.r.t w

$$
\begin{aligned}
\frac{\partial\left[\bar{L}\Phi\left(-d_2\right)-\bar{A}\Phi\left(-d_1\right)\right]}{\partial w} &= \bar{L}\frac{\partial\left[\Phi\left(-d_2\right)-\bar{C}\Phi\left(d_1\right)\right]}{\partial w} \\
&= \bar{L}\left[\frac{\partial\Phi\left(-d_2\right)}{\partial w}-\frac{\partial\bar{C}}{\partial w}\Phi\left(-d_1\right)-\bar{C}\frac{\partial\Phi\left(-d_1\right)}{\partial w}\right] \\
&= \bar{L}\left[\Phi'\left(-d_1\right)\bar{C}\left(\frac{\partial\left(-d_2\right)}{\partial w}-\frac{\partial\left(-d_1\right)}{\partial w}\right)-\frac{\partial\bar{C}}{\partial w}\Phi\left(-d_1\right)\right] \\
&= \bar{L}\left[\Phi'\left(d_1\right)\bar{C}\sqrt{T}\frac{w\sigma^2-b\rho\sigma}{\sigma_C}-\bar{C}\Phi\left(-d_1\right)\left(\mu-r+\sigma\lambda_1\right)T\right]
\end{aligned}
$$

- FOC w.r.t. λ_1

$$
\begin{aligned}
\frac{\partial\left[\bar{L}\Phi\left(-d_2\right)-\bar{A}\Phi\left(-d_1\right)\right]}{\partial\lambda_1} &= \frac{\partial\bar{L}}{\partial\lambda_1}\Phi\left(-d_2\right)+\bar{L}\frac{\partial\Phi\left(-d_2\right)}{\partial\lambda_1}-\frac{\partial\bar{A}}{\partial\lambda_1}\Phi\left(-d_1\right)-\bar{A}\frac{\partial\Phi\left(-d_1\right)}{\partial\lambda_1} \\
&= \frac{\partial\bar{L}}{\partial\lambda_1}\Phi\left(-d_2\right)-\frac{\partial\bar{A}}{\partial\lambda_1}\Phi\left(-d_1\right)+\bar{A}\Phi'\left(-d_1\right)\left(\frac{\partial\left(-d_2\right)}{\partial\lambda_1}-\frac{\partial\left(-d_1\right)}{\partial\lambda_1}\right) \\
&= \frac{\partial\bar{L}}{\partial\lambda_1}\Phi\left(-d_2\right)-\frac{\partial\bar{A}}{\partial\lambda_1}\Phi\left(-d_1\right) \\
&= \bar{L}b\rho T\Phi\left(-d_2\right)-\bar{A}w\sigma T\Phi\left(-d_1\right) \qquad (30a)
\end{aligned}
$$

- FOC w.r.t. λ_2

$$
\begin{aligned}
\frac{\partial\left[\bar{L}\Phi\left(-d_2\right)-\bar{A}\Phi\left(-d_1\right)\right]}{\partial\lambda_2} &= \frac{\partial\bar{L}}{\partial\lambda_2}\Phi\left(-d_2\right)+\bar{A}\left[\Phi'\left(-d_1\right)\left(\frac{\partial\left(-d_2\right)}{\lambda_2}-\frac{\partial\left(-d_1\right)}{\partial\lambda_2}\right)\right] \\
&= \frac{\partial\bar{L}}{\partial\lambda_2}\Phi\left(-d_2\right)=\bar{L}b\sqrt{1-\rho^2}\Phi\left(-d_2\right) \qquad (31a)
\end{aligned}
$$

7.3 Cost of Preference for Robustness

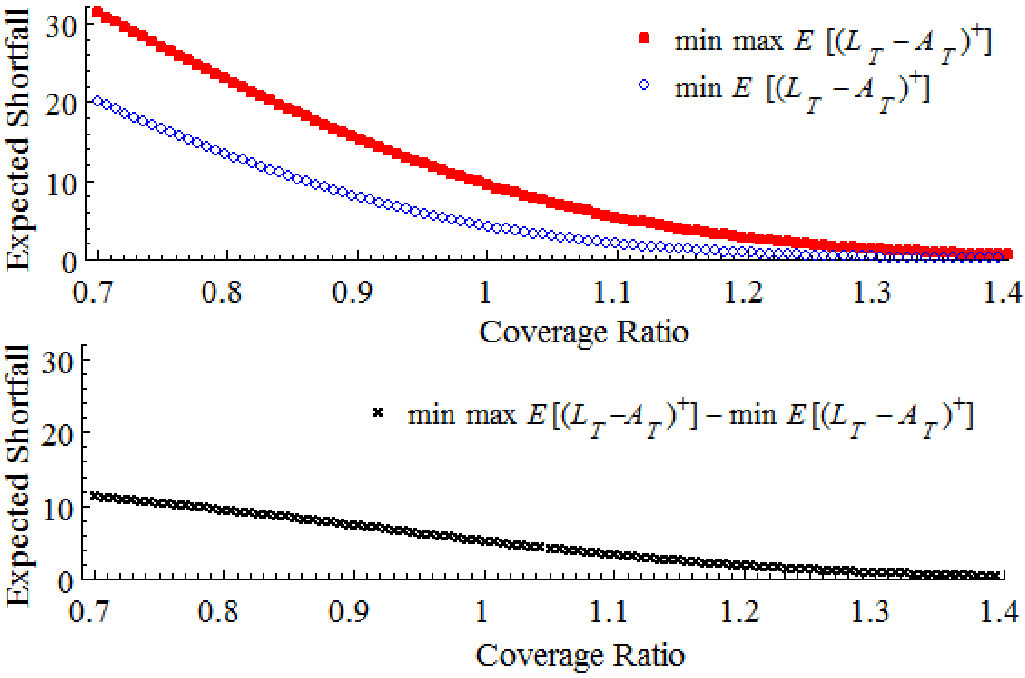


Figure 12: Cost of Preference for Robustness$T$. The upper panel plots the expected shortfall at period T as a function of the present coverage ratio by following different investment policies under the benchmark scenario. The solid-dotted line is the worst case scenario with robust optimal w, λ_1 and λ_2. The empty dotted curve is the expected shortfall following naive investment policy without perturbations. The bottom panel plots the gap of the two curves. It measures the cost of the preference for robustness.

In Figure 12, we show the additional expense required from the robust policy. The upper panel shows the expected shortfall of the two policies at period T as a function of the present coverage ratio C_0. The solid-dot curve is the robust result and the dotted curve is the naive result. Remark, the two curves are not comparable. It is trivial that the robust policy is more expensive because the agent buys an additional insurance to neutralize himself from the model misspecification. The bottom panel measures the cost of adding the preference for robustness on the hedging contract which is the gap between the two curve.

We assume the present liability value is $L_0 = 100$, if coverage ratio is 80% which means the current value of asset is $A_0 = 80$ and the current mismatch of the fund is $L_0 - A_0 = 20$, then after T periods the naive expected shortfall is 13 but the worst case scenario ends up with 23, which means the agent has to pay additional 10 to guarantee himself against the estimation error. The gap between the two curves captures the robust agent's pessimism in two directions: the penalty cost from an over estimated asset return and the penalty cost from an under estimated liability drift.

7.4 Numerical Methodology

In this appendix, we will elaborate the numerical procedure of dynamic programming problem. We simplify our optimization problem in two steps then we employ explicit finite difference method to show the problem.

7.4.1 Simplification Stage 1

We first simplify the optimization problem (10) to

$$\min_{w} \max_{\lambda_1,\lambda_2} L_t \mathbb{E}[(1 - C_T)^+ \mid \mathcal{F}_t] \tag{32}$$

where $C_t = \frac{A_t}{L_t}$ and the control variables are functions of time. The simplification does not influence the final solution of the control variables since L_t is not affected by the agent's decision.

We now show the proof this property. Denoting the value function by $U(A_t, L_t, t)$ with boundary condition $U(A_T, L_T, T) = \mathbb{E}_T[(L_T - A_T)^+]$. We also define value function $V(C_t, t)$ with boundary condition $V(C_T, T) = \mathbb{E}_T\left[(1 - C_T)^+\right]$. We want to prove that $U(A_t, L_t, t) = L_t V(C_t, t)$. If this property holds, then the simplification of the optimization problem states in (32)is valid.

The partial differential equation for $U(A_t, L_t, t)$ can be written as

$$U_t + U_A A\left(r + w(\mu - r) + w\sigma\lambda_1\right) + U_L L\left(a + b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2\right) + \tfrac{1}{2}U_{AA}w^2\sigma^2A^2 + \tfrac{1}{2}U_{LL}b^2L^2 + U_{AL}b\rho w\sigma AL = 0 \tag{33}$$

Assume $U(A_t, L_t, t) = L_t V(C_t, t)$ is valid, then L_t on the right side only has a scale effect and we would expect a univariate PDE for $V(C_t, t)$ that does not depend on L_t. The partial derivative of $U(A_t, L_t, t)$ can be expressed in terms of $V(C_t, t)$

$$U_A = V_C \tag{34a}$$

$$U_L = V - V_C C \tag{34b}$$

$$U_{AA} = V_{CC}\frac{1}{L} \tag{34c}$$

$$U_{LL} = V_{CC}\frac{C^2}{L} \tag{34d}$$

$$U_{AL} = -V_{CC}\frac{C}{L} \tag{34e}$$

$$U_t = LV_t \tag{34f}$$

Replace (34) into (33) we get

$$V_t + V_C C\left(r + w(\mu - r) + w\sigma\lambda_1\right) + (V - V_C C)\left(a + b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2\right)$$
$$+ \tfrac{1}{2}V_{CC}C^2w^2\sigma^2 + \tfrac{1}{2}V_{CC}C^2b^2 - V_{CC}C^2b\rho w\sigma = 0 \quad (35)$$

PDE (35) is what we would have expected by using the new lognormal process dC see (28). Since (35) is derived from (33) and is L neutral, we should be able to get the same optimal w, $\lambda's$ from the two PDE's, but the latter one is easier to solve numerically since it is a univariate PDE.

7.4.2 Simplification Stage 2

The second simplification procedure is related to the numerical aspect. It is more efficient to use finite difference methods with $\ln C$ instead of C. [8] Define $Z = \ln C$, then we have

$$dZ = \left(\mu_A - \mu_L + \frac{1}{2}b^2 - \frac{1}{2}w^2\sigma^2\right)dt + (w\sigma - b\rho)\, dW_1 - b\sqrt{1-\rho^2}dW_2 \quad (36)$$

and the corresponding simplified HJB equation (35) in terms of $U(Z,t)$ is given by

$$-U_t = \min_{w} \max_{\lambda_1,\lambda_2,\nu} U_Z\left(\mu_A - \mu_L - \frac{1}{2}\sigma_1^2\right) + U(\mu_L) + U_{ZZ}\left(\frac{1}{2}\sigma_1^2\right) \quad (37)$$

using the following relations,

$$U_Z = V_C C \quad (38a)$$
$$U_{ZZ} = V_{CC}C^2 + V_C C \quad (38b)$$

If we replace the explicit optimal solution for λ_1 and λ_2 (see (22)) into the uncertainty set constraint $\lambda_1^2 + \lambda_2^2 = \kappa^2$, we will get a fourth order polynomial equation of ν

$$(\varsigma_0\iota_1 + \varsigma_1\iota_0)^2\nu^2 + \varsigma_2^2(\nu\,\iota_0 + \iota_1^2)^2 = \kappa^2\nu^2(\nu\,\iota_0 + \iota_1^2)^2, \quad (39)$$

where $\varsigma_0 = (U_{ZZ} - U_Z)\, b\rho\sigma - U_Z(\mu - r)$, $\varsigma_1 = -(U - U_Z)\, b\rho$, $\varsigma_2 = -(U - U_Z)\, b\sqrt{1-\rho^2}$, $\iota_0 = (U_{ZZ} - U_Z)\,\sigma$ and $\iota_1 = U_Z\sigma$.

Equation (39) contains four roots, but only one root gives us the min-max solution.[9] Also, this specific ν has to be positive, this is because, on one hand the optimal λ_2^* (see

[8]The advantage of using Z instead of C (under the condition that C_t is a Geometric Brownian motion process) as the state variable of the value function is that we can turn the partial differential equation state-variable neutral, so that we can get a relatively simple version of PDE equation.

[9]The remaining three solutions of equation (39) result in the min-min, max-max and max-min value of the Bellman's equation.

equation (22)) is a function of ν, and we have discussed that λ_2 has to be positive; and on the other hand the numerator of the fraction λ_2^* is positive. Hence that results in a positive ν.

The partial differential equation for $U(Z,t)$ is given by

$$U_t + U_Z\left(r + w(\mu - r) + w\sigma\lambda_1\right) + (U - U_Z)\left(a + b\rho\lambda_1 + b\sqrt{1-\rho^2}\lambda_2\right) + (U_{ZZ} - U_Z)\left(\frac{1}{2}w^2\sigma^2 + \frac{1}{2}b^2 - b\rho w\sigma\right) = 0 \quad (40)$$

as we can see, PDE (40) is state variable free, so compared with the PDE (35) from $U(C,t)$, it is relatively more efficient to use log coverage ratio.

7.4.3 Explicit Finite Difference

We generate a stylized finite difference gird. We divide time T into a certain N (unknown) equally spaced interval with length $\Delta t = \frac{T}{N}$. Let $C_{\min}$ and $C_{\max}$ be the two extremes of the coverage ratio with corresponding log extremes $Z_{\min}$ and $Z_{\max}$. We divide the interval $[Z_{\min}, Z_{\max}]$ into M particular spaced intervals. with length ΔZ. According to Hull (2009), it is more efficient to set $\Delta Z = \sigma\sqrt{3\Delta t}$. Therefore, we set the length $\Delta Z = \frac{3T\sigma^2}{Z_{\max} - Z_{\min}}$.[10]

[10] Given that $M = N$ and $M = \frac{Z_{\max} - Z_{\min}}{\Delta Z}$ and $\Delta Z = \sigma\sqrt{3\Delta t} = \sigma\sqrt{3\frac{T}{N}}$, we have $\Delta Z^2 = \sigma^2 \frac{3T}{N} = \sigma^2 \frac{3T\Delta Z}{Z_{\max} - Z_{\min}}$.

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