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Robust Long-Term Interest Rate Risk Hedging in Incomplete Bond Markets*

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Abstract

We introduce a robust investment strategy to hedge long dated liabilities under model misspecification and incomplete bond markets. A robust agent who worries about misspecified bond premia follows a min-max expected shortfall criterion to protect against model uncertainty. We employ a backward least squares Monte Carlo method to solve this dynamic robust optimization problem. We find that both naive and robust optimal portfolios depend on the hedging horizon and current funding ratio. The robust policy suggests to take more risk when the current funding ratio is low. The robust yield curve derived through the minimum assets required to eliminate shortfall risk is lower than the naive one.

JEL Codes: G11, C61, E43

Keywords: Model misspecification, robust optimization, least squares Monte Carlo, uncertainty set, incomplete market, term structure model.

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Doubt is not a pleasant condition, but certainty is absurd. Voltaire (1767).

1 Introduction

It is important for insurance companies and pension funds to make a long-term strategic investment plan. Most people start working and participating pension schemes from age 25. The average life expectancy in most developed countries is above 80 years. Hence insurance companies and pension funds are obligated to face commitments with maturities longer than 50 years.

Pricing and hedging long dated liabilities faces two challenges. On the one hand, the market for long maturity financial instruments is incomplete. However, it is regulated by many countries (especially in Europe under Solvency II program) that pension and insurance companies should follow a "market-consistent" principle to value their liabilities, which means insurance companies have to invest very long-maturity financial instruments in order to replicate the long-dated cash flows.

The standard replication theory does not work because long-maturity financial contracts barely exist. Consols are one of the rare examples of perpetual bond issued by UK in 1917 in order to help paying for the World War I. However consols market is not very illiquid and only remains a very small part of UK government bond portfolio. The longest government bonds even in developed financial markets (such as US and Germany) have maturities no longer than 30 years. In developing countries (such as Asia, Eastern Europe and South America), the longest government bond has maturities no longer than 10 years.

On the other hand, it is difficult to avoid parameter misspecification. Due to the incompleteness of bond market, one has to extrapolating term structure of interest rates beyond the maturity of longest dated market-available instrument. Therefore, any valuation of a long dated liability with missing bond market must be model based. Conditional on a model, we could derive a term structure for all maturities. However, there are a large number of models that perfectly fit bond prices up to maturity of 20 years but yet imply different prices for the longer maturity bonds.

Despite much work on term structure models, very few studies consider the impact of model uncertainty on long dated liability valuation. In this paper, we propose a robust optimal hedging policy that minimize the shortfall of long dated liabilities in the presence of parameter uncertainty and missing bond market. Our replicating portfolio is robust to model misspecification in the sense that the investment policy is less sensitive to the choice of model. A second aim is to propose an alternative term structure that is robust against long-term interest rate risk.

The challenge of pricing in incomplete markets and model misspecification can be closely related. First, market incompleteness creates an unknown market price of risk.

This unobservable market price of risk leads to the rationale of Cochrane and Saa-Requejo (1996)'s Good-Deal-Bound that constrains the maximum Sharpe ratio of the market. The true market price of risk could be anywhere within this Good-Deal-Bound.

Second, the investor who fears for parameter misspecification is ambiguity averse. She believes that the true model parameters differ from the estimated ones. To formulate model misspecification, Hansen and Sargent (2001) and Hansen and Sargent (2007) employ a relative entropy factor. This relative entropy captures the perturbation between the estimated model and the unobservable true model. Although the economic interpretations of the two are different, technically the two motives are identical. Both interpretations can be understood as requiring an additional premium to represent the estimation error. Pelsser (2011) states that mathematically, the two interpretations are identical.

We solve a dynamic robust optimization problem by employing Hansen and Sargent (2007)'s framework that min-maxing agent's utility. In the model, agent allocates her instantaneous wealth between a short-term and a median-term bond market so as to minimize the expected shortfall of a long maturity commitment. On the other hand, Mother nature perturbs the parameters in order to maximize the expected shortfall given the decision of the agent. The robust optimal portfolio is therefore, robust against model ambiguity.

We propose a feasible region of uncertainty set that gives Mother nature a reasonably bounded freedom of decision marking. We use generalized method of moments (GMM) approach to estimate the one-factor affine term structure model. By assuming the estimation errors to be asymptotically normal, one can use the property that standardized error term square becomes a Chi-squared distribution to shape a joint confidence interval for estimation parameters.

To solve our robust optimization problem, we introduce a new regression-based method. The new method belongs to the family of backward least squares Monte Carlo method . It combines, in essence the method of Brandt et al. (2005), Koijen et al. (2007) proposed to approximate the conditional expectation that we encounter in solving the dynamic program by polynomial expansions in the state variables. While in addition, I add the preference for robustness into the algorithm.

We find that the preference for robustness induces a large demand for long-term bond when solvency ratio is low. A robust investor worries about model uncertainty in the sense that she is afraid that mother nature would choose a lower bond premia than she expected, hence she needs a risker portfolio so as to gamble out of trouble. Both naive and robust optimal portfolios depend on the hedging horizon. The longer the horizon, the more risk exposure on long-term bond markets. We also find that robust policy requires more initial wealth than the naive one in order to meet the shortfall fall target. In other words, the robust yield curve is always lower than the naive yield curve. However, when spot rate is low, both policy-based yield curves are higher than the Vasicek yield curve.

Concerning the area of robust asset allocation in incomplete markets for long-term asset and liability investors, this paper is related to three themes. The first is about optimal portfolio choice under model uncertainty. Early works are dominated by Bayesian paradigm. Klein and Bawa (1976) is one of the first studies consider the effect of model uncertainty on portfolio choice. They look at a two-period model and find out that in the presence of estimation risk, the optimal hedging portfolio differs from the traditional analysis. Barberis (2000) extend the study in a multi-period economy setting while keeping the hedging decision static and find that ignoring estimation risk may result in an over aggressive portfolio. Brennan (1998) incorporates learning with parameter uncertainty and finds out that risk lovers put more wealth on risky assets after learning while the risk averse investors are more conservative with their portfolio.

A more recent approach is the max-min expected utility paradigm developed by Gilboa and Schmeidler (1989). Gilboa and Marinacci (2011) claim that Gilboa-Schmeidler's axiom is a neo-Bayesian paradigm because it allows decision makers to have a set of subjective priors. The agent aims to maximize her utility under the least preferred prior so as to display an aversion to uncertainty. As an extension, Hansen and Sargent (2001) managed to transform Gilboa-Schmeidler's static theory to a dynamic version through the techniques of robust control theory which already been broadly used in engineering and applied mathematics since 1980s. Our robust optimization problem is closely related to Maenhout (2004). This is one of the first studies employ Hansen-Sargent framework to restudy Merton (1969) optimal portfolio problem and find out that due to the uncertainty aversion, investors reduce their risk exposure on stock market dramatically.[1] Uppal and Wang (2003) extend Maenhout's model in a multi-dimensional setting.

Our article also relates to the long-term investment literature. Early work by Brennan and Xia (2002) and Campbell and Viceira (2002) study the optimal hedge demand for long-term bond assuming constant and specified bond premia. Koijen et al. (2010) and Sangvinatsos and Wachter (2005) allows for time-varying bond risk premia and both find that time-varying risk premia induce a large hedging demand for long-term bonds. None of the papers above consider bond premia misspecification and incompleteness of the very long-term bond markets.

Studies on dynamic hedging in incomplete market are also related to our work. Basak and Chabakauri (2012), Föllmer and Leukert (1999) and Föllmer and Leukert (2000) all provide a closed form solution of dynamic hedge in incomplete market while assuming the underlying models are correctly specified. Basak and Chabakauri (2012) introduce an unhedgeable risk driver to represent market incompleteness and employ minimum-variance as hedging criterion and conclude that a larger hedging demand is required due

[1] In order to obtain a closed form solution, Maenhout (2004) modifies Hansen-Sargent's framework by transforming the constant Lagrange multiplier into a function of state variables. However, Pathak et al. (2002) argues that this modification breaks the link to the world of Gilboa-Schmeidler.

to speculative uncertainty. Föllmer and Leukert (1999) maximize the success of hedge and Föllmer and Leukert (2000) minimize the probability of shortfall risk. They define incomplete market as a system with multiple martingales. Both studies find that the optimal solution depends on the shortfall target, which is also discovered in our study.

The rest of the paper proceeds as follows. Section 2 describes the one-factor affine term structure model employed in our economy. Section 3 uses GMM method to estimate the structural parameters. The dynamic robust optimization problem is explained in Section 4. Section 5 elaborates the regression-based techniques used in solving our dynamic programming problem. Section 6 discusses the robust optimal solution and we provide policy evaluation of the robust policy. Section 7 concludes.

2 Term Structure Model

Our term structure model follows the framework of Duffee (2002) and Duffie and Kan (1996). We assume that the spot rate r follows the one-factor Vasicek model

$$dr = \kappa (\theta - r) dt + \sigma dW, \tag{1}$$

where θ isthe unconditional mean, κ is the mean reversion, σ is a volatility parameter and dW a univariate Brownian motion. Let $P(t, T)$ be the time t price of a discount bond maturing at time T. The Vasicek model implies that bond prices follow the diffusion

$$dP = rPdt + B\sigma P (\lambda dt + dW) \tag{2}$$

where B is the volatility of long-term bond returns relative to the spot rate volatility and where λ is the price of risk. In the original Vasicek (1977) model, the price of risk is constant. In the essential affine extension of Duffee (2002), the price of risk also depends on the spot rate,

$$\lambda = \Lambda_0 + \Lambda_1 r \tag{3}$$

In our model we will assume that the volatilities σ and B can be estimated very precisely, whereas the expected excess return parameters Λ_0 and Λ_1 are subject to model uncertainty.

3 Model Calibration

The model parameters are estimated using standard GMM method. We use Euro area nominal government bonds with issuing rate of triple A, obtained from the ECB statistical data warehouse. We use daily annualized data on 3 constant maturity zero rates with

maturities of 3 months, and 5 and 10 years for the period September 6, 2004 to November 15, 2013.[2]

Table 1: **Summary Statistics**
Means, standard deviation and autocorrelations of daily yield curve spot rate with three different maturities and their difference. The variable r_t denotes three-month spot rate. ADF denotes the Augmented Dickey-Fuller unit root statistics with a 5% critical value of -3.43.

Variable	Mean	Std	Std Δ	ADF
r_t	1.62%	1.45%	3.28bp	-0.61
$Y_t(5)$	2.61%	1.10%	4.27bp	-0.40
$Y_t(10)$	3.36%	0.80%	3.97bp	-0.94

Summary statistics are displayed in Table 1. The average three-month rate is 1.62% with a standard deviation of 1.45%. 5-year bond has mean rate of 2.61% with standard deviation of 1.1%. 10-year bond has highest average rate 3.36% but lowest volatility 0.8% among the three. The ADF test shows that we cannot reject the unit root hypothesis for any of the three series.

Table 2: **GMM Estimates** .
The term structure model is estimated by GMM method using daily European data. The bond maturities used in estimated are three-month, five-year and ten-year. The parameters are expressed in annual terms. We choose 20 as the number of lags applied in Newey West estimator.

Parameter	Estimate	Std error	Correlation Matrix				
κ	17.16%	17.43%	1.0000	-0.5893	0.8674	0.0380	-0.9987
θ	1.62%	1.24%	-0.5893	1.0000	-0.6587	0.7808	0.5937
σ	0.52%	0.14bp	0.8674	-0.6587	1.0000	-0.1627	-0.8629
$\sigma\Lambda_0$	-0.16%	0.36%	0.0380	0.7808	-0.1627	1.0000	-0.0363
$\sigma\Lambda_1$	-14.19%	17.49%	-0.9987	0.5937	-0.8629	-0.0363	1.0000

In Table 2, we report the parameter estimation result for full sample. The parameters are expressed in annual terms. The first two columns of Table 2 reports the estimated value and standard deviation of each structural parameter. The unconditional mean of spot rate, θ is about 162 basis points. The mean reversion coefficient κ implies half-life innovation of 4 years.

The rest of the table reports the correlation matrix of parameters. First, the low volatility of σ implies an accurate estimation performance on spot rate volatility. Hence we ignore the parameter uncertainty on σ. Second, we find an extremely high correlation

[2]The yield data is available at http://sdw.ecb.europa.eu/browse.do?node=3570581. Sample size is 2360, and there are 9.2 (approximately) years in our sample, hence the average yearly number of trading days is $\frac{2360}{9.2} \approx 255$.

between κ and Λ_1, and between θ and Λ_0. We also check the eigenvalues of the variance covariance matrix of the estimating parameters, We find two positive eigenvalues 0.0609, 0.0001 and the rest three are very close to zero.

In this paper, we consider the misspecification of the two risk-price factors. As a result, we fix the dynamics of the spot rate and only consider the effect of uncertainty in the risk premium of long-term bonds. In other words, only the drift of $\frac{dP}{P} - rdt$ is effected by the parameter uncertainty and the volatility is fixed.

We assume price of risk factors are asymptotically joint normal with mean $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$ and variance covariance matrix denotes Ω.

$$\begin{pmatrix} \Lambda_0 \\ \Lambda_1 \end{pmatrix} \sim \mathbb{N}\left(\begin{pmatrix} \hat{\Lambda}_0 \\ \hat{\Lambda}_1 \end{pmatrix}, \Omega \right) \tag{4}$$

Denote c_0 and c_1 the estimation errors of the two misspecified parameters. We know that standardized error terms square is a Chi-squared distribution with two degrees of freedom under the condition of normality. This leads to a confidence interval of Λ_0 and Λ_1, denoting $\mathbf{S}$

$$\mathbf{S} := \left\{ \begin{pmatrix} \hat{\Lambda}_0 \\ \hat{\Lambda}_1 \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \mid \begin{pmatrix} c_0 \, c_1 \end{pmatrix} \Omega^{-1} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \leq \gamma^2 \right\} \tag{5}$$

where γ^2 represents the level of preference for robustness. When $\gamma^2 = 0$, the uncertainty set $\mathbf{S}$ shrinks at the point estimates. The bigger γ^2 is, the more a robust investor worries about model uncertainty. Statistically, γ^2 is the critical value of a Chi-squared distribution with two degrees of freedom, which implies γ^2 should be equal to 5.99 at 5% significance level.

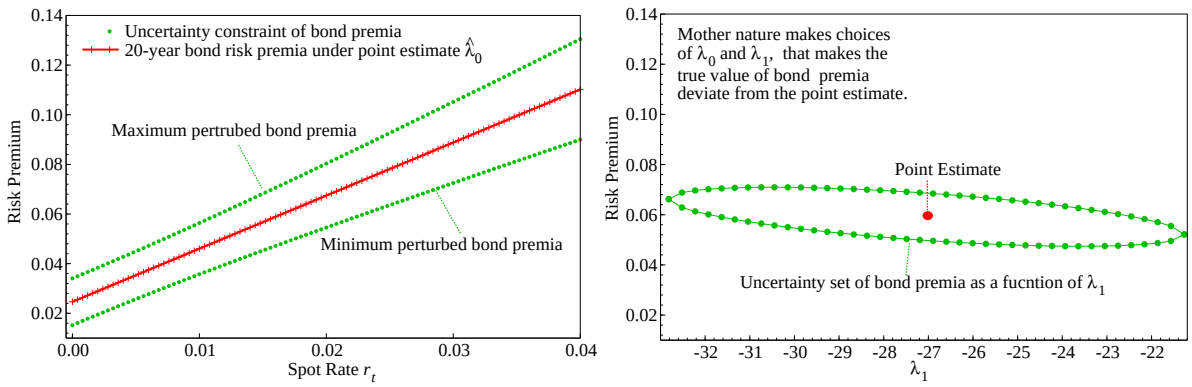
However, the statistical value of γ^2 is too big in reality, for it allows for a huge negative bond premia. Therefore, we choose a feasible value of γ^2 such that even at the worse case scenario, we can still guarantee a none negative bond premia $-B\lambda\sigma$. A feasible boundary of γ^2 is given by

$$0 \leq \gamma^2 \leq \frac{\left(\hat{\Lambda}'\alpha\right)^2}{\alpha'\Omega^{-1}\alpha} \tag{6}$$

where $\alpha = (1, r)'$ and $\hat{\Lambda} = \left(\hat{\Lambda}_0, \hat{\Lambda}_1\right)'$. Eq.(6) shows that the feasible value of γ^2 depends on the spot rate. The lower the spot rate, the lower feasible γ^2 will be. To scan out negative bond premia scenarios means under any plausible value of spot rate, Eq.(6) holds. The lowest upper bound occurs at the lowest spot rate level. Our sample data shows the lowest three-month spot is -0.017% and the corresponding value of γ^2 is 0.22. Therefore, a feasible region is $[0, 0.22]$. As a benchmark, we choose $\gamma^2 = 0.03$.

Figure 1: **Bond Premia**
The figure presents the 20-year bond risk premia for different spot rate under different choices of λ_0 and λ_1. The ambiguity of market price of risk constrained by set $\mathbf{S}$ is shown in Eq.(5) with $\gamma^2 = 0.03$. Panel 1a presents the minimum and maximum values of bond premia among the feasible region for different r. The point estimate premia locates in between the two extremes. Panel 1*b* plots the feasible region of bond premia for different choices of λ_1 under the unconditional expectation of spot rate.

(a) (b)



The impact of term structure factor r on bond premia is governed by Λ_1, if Λ_1 is estimated with an error, then the impact of spot rate is also ambiguous. Figure 1 presents 20-year nominal bond risk premia for a realistic range of spot rates. First, we find risk premia are increasing with spot rate. Second, misspecification of market price of risk results in an perturbation of bond premium around 200 basis points either downside or upside.

4 Robust Optimal Portfolio Choice

4.1 Robust Hedging

The agent with wealth X_t at time $t = 0$, aims to hedge a long dated liability with payoff equal to one at maturity time T by investing in two financial instruments: a zero coupon bond with maturity $\tau_2 < T$; and a short term bond. Suppose the agent is only interested in eliminating the downside risk. Hence our hedging criterion is defined over the expected shortfall $[1 - X_T]^+$ at time T. If the agent is not aware of the parameter uncertainty, then her hedging decision relies fully on the point estimator $\hat{\Lambda}$. The dynamic optimization problem is defined as

$$\min_{w_{t:0\leq t<T}} \mathbb{E}\left[(1 - X_T)^+ \mid \mathcal{F}_t\right] \tag{7}$$

The fraction of wealth allocated to long-term bond with maturity τ_2 at time t is indicated by w_t. The dynamics of wealth is given by

$$dX = (r - wB(\tau_2)\sigma\lambda)\, Xdt - wB(\tau_2)\sigma X dW \tag{8}$$

where we ignore the subscripts t. However, if the agent is uncertainty adverse, then she fears for a misspecified bond premia. We find from Figure 1 that a different decision on Λ would implies a different bond premia and hence will imply a different hedging position. In order to protect against parameter uncertainty, a robust investor seeks a robust policy that minimize the hedging error under worse decision of Mother nature. The robust optimization problem is given by

$$\min_{w_{t:0\leq t<T}} \max_{(\Lambda_0,\Lambda_1)\in\mathbf{S}} \mathbb{E}\left[(1 - X_T)^+ \mid \mathcal{F}_t\right] \tag{9}$$

We assume mother nature controls price of risk parameters Λ_0 and Λ_1 and her action is centered around the point estimators $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$ bounded the uncertainty set $\mathbf{S}$. The dynamic robust optimization problem can be interpreted as a two-players game. At each rebalancing time t, player one, the robust invest who is aware of bond premia uncertainty, makes a hedging decision that minimizes the expected shortfall at the end of hedging horizon conditional on mother nature's action. Player two is mother nature. Conditional one player one's choice, she makes a decision of bond premia so as to maximize the expected shortfall at time T. The equilibrium of the game gives us an instantaneous robust hedge.

5 Numerical Technique

Hedging with expected shortfall objective function as well as robustness does not allow for analytical solution, so we use numerical approach instead. Our method combines, in essence the method of Brandt et al. (2005), Koijen et al. (2007) proposed to approximate the conditional expectation that we encounter in solving the dynamic program by polynomial expansions in the state variables. We also follow Diris (2011)'s approach to parameterize the coefficients of the approximation function by a quadratic function of portfolio weights, such that we can calculate the optimal portfolio under each path analytically and further, this method allows us to reach an accurate result using a very small grid of testing portfolio. As an extension, we integrate robustness with the standard simulated based algorithm.

We start with simulating a large number N sample paths with length T years of bond returns using Euler discretisation. We also choose a $M-$ dimensional grid for financial wealth before investment decision. The wealth grid points are indicated by X_j,

$j = 1, \cdots, M$. In total, we have $(M \times N)$ grid points at each point in time.

The algorithm for robust policy consists of two parts. We first solve mother nature's decision analytically. Mother nature's decision only influence the long-term bond premia. Therefore, under the assumption that $w_t \geq 0$, maximizing expected shortfall boils down to minimize bond premia. Therefore, we can analytically solve for mother nature's decision.

The optimal c_0, c_1 are the solution of following quadratic programming problem.

$$\begin{aligned} \min_{c_t} \quad & -B(\tau_2)\sigma\left(\hat{\Lambda} + c_t\right)' \alpha, \\ s.t \quad & c_t' A c_t = 1 \end{aligned}$$

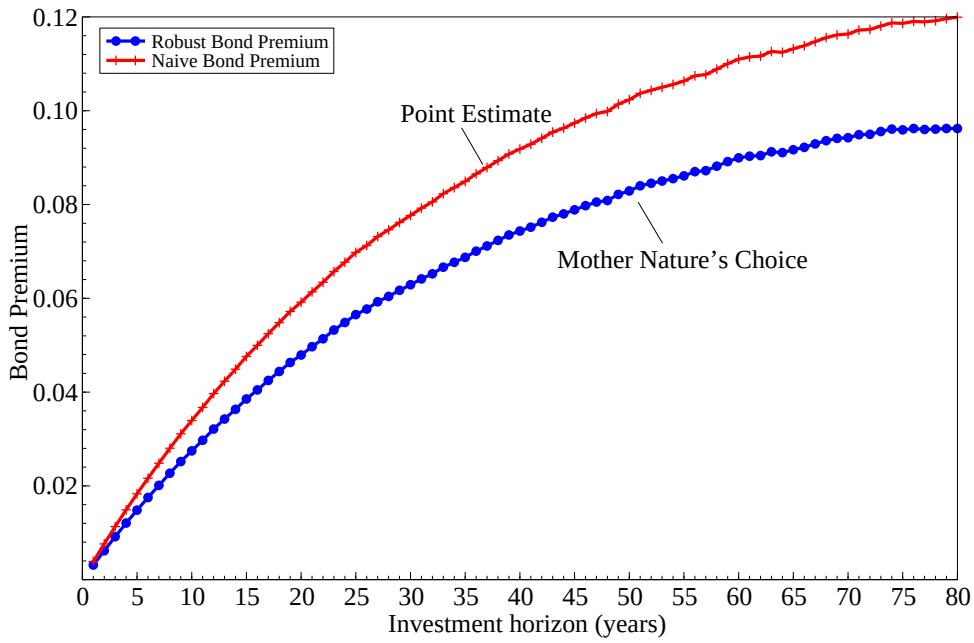
where $\hat{\Lambda} = (\hat{\Lambda}_0, \hat{\Lambda}_1)'$, $c_t = (c_0, c_1)'$, $\alpha_t = (1, r_t)$ $A = \frac{\Omega}{\gamma^2}$. We can easily find the optimal solution for c_t:

$$c_t^* = \frac{A^{-1}\alpha_t}{\sqrt{\alpha_t' A^{-1} \alpha_t}}$$

Therefore, mother natures' decision at each rebalancing time step t, depends on r_t only. The min-max problem is now simplified to minimization problem only.

Figure 2 displays the robust optimal bond premium as a function of bond maturity against the point estimate. Mother nature intends to minimize the bond premium so as to maximize the expected shortfall at time T, therefore, the robust bond premium of any different maturity is always lower then the point estimate.

Figure 2: **Bond Premium.** The figure presents Mother nature's choice of bond risk premium against the naive estimate $-B(\tau)\sigma\hat{\lambda}$ under different bond maturities. We set spot rate equal to 2%



In second part of the algorithm, we use backward least square monte carlo (LSMC) method to solve for optimal portfolio. We summarize the LSMC algorithm into three steps.

Step 1 At time step t, i.e. start from period $T - \Delta t$ and iterating to period 0, construct realized loss $V_T = (1 - X_T)^+$ for all simulated path using following information

- Robust optimal portfolio from previous time steps $w_s^*(s = t + \Delta t, \cdots, T - \Delta t)$
- Optimal mother natures decision from previous steps λ_s^* (this is analytically solved)
- A small grid of testing portfolio w_h, $h = 1, \cdots, H$
- Spot rate $r_{i,s}$ with $i = 1, \cdots, N$
- Current wealth $X_{j,t}$, $j = 1, \cdots, M$

so we have a cross-section size of $N \times M \times H$.

Step 2a Run a cross-sectional regression by approximating the realized objective function $V_{T,ijh}$ calculated from Step 1 on trajectories of state variables as well as the testing portfolio w_h on a second-order polynomial expansion (including cross term) at a certain point in time.

$$V_{ijh,T}(\mathcal{F}_t) = \beta' f\left(X_{j,t}, r_{i,t}, w_h\right) + \epsilon_{ijh,t} \tag{10}$$

Step 2b Then we parameterize the approximate conditional value function as a quadratic function of portfolio weights such that we can analytically calculate the optimal portfolio.

For a naive investor, first part of the algorithm can be ignored since Mother nature does not play a role in native hedging framework. Appendix A elaborate the LSMC algorithm in more details.

6 Long-term Investors and Bond Premia Uncertainty

In this section, we investigate the impact of model uncertainty on bond portfolio allocations. Section 6.1 analyzes the robust optimal portfolio. Section 6.2 discuss the property of robust yield curve.

6.1 Optimal Portfolio Choice

Figure 3 plots the optimal portfolio of 20-year nominal bond as a function of the investment horizon based on three different hedging policies, namely naive policy, robust policy and delta hedging policy.

Delta Hedging The delta of a long dated liability is defined as the rate of change of the price of liability with respect to the price of the underlying τ_2- year bond. The delta of long-term liability is

$$\Delta = \frac{\partial P(0,T)}{\partial P(0,\tau_2)} = \frac{B(T)}{B(\tau_2)}$$

We use Delta hedge as a benchmark to investigate the optimal portfolio choice as a function of solvency condition. We use funding ratio, a fraction of current wealth level X_0 and the hypothetical price of liability $P(0,T)$, to measure the solvency position of a fund. If funding ratio is lower than one, then the fund has more expected value of liability than its asset which implies a underfunded solvency condition.

Suppose we have a funding ratio bigger than one, and the underlying model is specified, then Delta position is indeed the optimal portfolio to hedge the long-dated liability. It is verified by Panel (b) that if the current funding ratio is above one, a delta neutral position is comparable to the optimized policies.

Figure 3 tells that bond allocations highly depends on hedging horizon and the present solvency condition for both naive and robust policies. When funding ratio is low (Panel (a)), both hedging methods are remarkable identical when horizon ($T \leq 20$), because market completeness rules out the model uncertainty. Also, both policies suggest a risker position than the Delta hedge. When funding ratio is low, following the Delta strategy may hedge the long-term interest rate risk while does not help to meet the long-dated commitment due to insufficient current wealth. Therefore, we need a risker portfolio in order to obtain higher expected excess return so as to gamble out of trouble.

At long-term investment horizon, a robust investor takes more risky position than a naive investor when the present funding ratio is low. This is because, on the one hand, a robust investor worries that mother nature might choose even lower bond premia than the model estimated, and on the other hand, she faces a non-stochastic long-dated commitment, therefore the robust method is even more aggressive.

Our robust policy suggests an opposite altitude towards Maenhout (2004) and Shen et al. (2013)'s conclusion. Both studies concludes that the robust policy is more conservative compared with the naive one. Further, they both find that a robust investor worries about an under-estimated risk premium which is, however, inline with our finding (see Figure 2). In Maenhout (2004)'s model, an invest aims to maximize her terminal wealth utility function. The preference for robustness drives down the beliefs of risk premium. Without liability constraint, a robust investor will therefore take less risk. Shen et al. (2013)'s setup include stochastic liability constraint. A robust agent worries about an over estimated stock return as well as an under estimated expected liability return at the same time. Therefore, she prefers a safer asset allocation strategy. In short, we conclude from these studies that, robust policy is not always conservative, it depends on the feature

of liability constraint.

We now illustrate how the optimal policies responds to changes in funding ratio and spot rate. Figure 4 displays the optimal asset allocation to the 20-year bond under naive policy (panel (a)) and robust policy (panel (b)) for a reasonable range of funding ratio (FR). These figures are constructed by regressing the optimal policy along all trajectories of state variables used in simulated-based method at certain point in time.

Figure 5 summarizes the insight of Figure 4. The lower the funding ratio, the more risk position is required so as to hedging against the shortfall risk. This property holds for both policies. However, the robust policy is risker due to the fear for underestimated bond premia. Figure 3 also leads us to the same conclusion.

Figure 6 presents the optimal asset allocation under different spot rate when funding ratio is 80%. It holds for both policies that the optimal allocation increases with spot rate, for spot rate is positively related to bond premia. A higher bond premia leads to a higher risk exposure on long-term bonds. The robust policy (panel (b)) differs from the naive one (panel (a)) in two ways. First, the robust policy is less sensitive to spot rate. This is an important feature of the robust policy that we desire to. Second, the difference between the two policies gets larger for a lower value of spot rate.

Figure 7 presents the optimal expected shortfall value as a function of funding ratio and hedging horizon. Figure 8 summarizes the insights the Figure 7. First, without concerning model uncertainty, the expected shortfall value is dramatically under priced when funding ratio is low. Second the mispricing error increases with investment horizon. The increasing amount of mispricing is caused by an accumulative fear for the underestimated pond premia.

6.2 Robust Term Structure

An investor is interested in the optimal strategy that guarantees the liability with the lowest required initial wealth X^*, which is called the super-hedging strategy. Once we have determined the minimum wealth X^*, we can define the implicit $T-$ period discount rate as

$$y(T) = -\frac{1}{T} \ln X^* \tag{11}$$

For most models of the term structure the super-hedging strategy will be extremely costly. In our model, there will not exist a super-hedging strategy since there will always be a small probability of underfunding because our interest rate process is Gaussian. We define approximately risk free rates through the minimum assets required to have an expected shortfall less than S,

$$y(T) = -\frac{1}{T} \ln X_S^*, \qquad \text{with} \quad S = \mathbb{E}_0 \left[(1 - X_T)^+ \mid X_0 = X_S^* \right] \tag{12}$$

Expected shortfall is a coherent risk measure and does not reward any upside. The lower S is, the more initial wealth is required to hedge against the targeting shortfall risk, hence the lower $y(T)$ will be. The lowest level of $y(T)$ is simply the underlying term structure model that by definition guarantees for zero expected shortfall.

Figure 9 presents the optimal yield curves under different hedging policies at different targeting shortfall level and initial spot rate. As a benchmark, we plot the two hedging policy based yields together against the Vasicek yield curve. We highlight two crucial insights from Figure 9. First, the robust yield is always lower than the naive yields regardless of current spot rate or shortfall target. Due to the concern of an underestimated bond premia, a robust investor requires more initial wealth to meet the shortfall target. Second, both policy-based yield curves are higher than the Vasicek curve when spot rate is low (panel (a), (b)). When spot rate is low, the present value of liability will be higher, hence the required initial wealth would be higher as well in order to meet the long-term commitment. Vasicek model is based on zero shortfall requirement, hence has a lower yield curve compared with the other two curves which allow for positive expected shortfall. We also find that the Vasicek yield curve overlaps the naive curve when spot rate is high if S is sufficiently small (panel (c)).

7 Conclusion

We explore the impact of model uncertainty on the optimal demand for long-term bonds to hedge a very long dated liability. Our aim is to design a robust hedging policy that is less sensitive to parameter uncertainty and to price the additional cost of this additional risk. The first important characteristic of robust policy is that, it requires more exposure to the long-term bonds. Second, the robust yield curve is lower than the naive one due to an additional initial wealth requirement.

Our study can be extended in several directions. First, I would like to explore the influence of parameter uncertainty on the optimal life-cycle investment decision. As an individual life-cycle investor, one would not only worries about misspecification of the financial models, but also her career uncertainty, health condition uncertainty etc. Therefore, it is interesting to investigate how does the robust portfolio look like for different age cohort under different career categories. Second, instead of considering uncertainty on drift terms of Geometric Brownian Motion only, we shall also study the impact of uncertainty on mean reversion.

A Robust LSMC algorithm

In this section, we elaborate the numerical method used in this paper. We show how the regression-based method works in our robust hedging problem and we also analyze the

accuracy of our algorithm. This section proceeds as follows, we first introduce the naive LSMC algorithm without considering model uncertainty. Second, we show how mother nature's choice can be solved analytically. Third, we explain the robust LSMC algorithm. Last, we analyze the accuracy of our algorithm.

A.1 Naive LSMC

A.1.1 Grid Generation

The hedging period is from 0 to T. We partition $[0, T]$ into m subintervals of length $\Delta t = \frac{T}{m}$. Hence the bond portfolio is rebalanced every Δt unit of time. We start by simulating N trajectories of m time periods of spot rate under $\mathbb{P}$ measure using discrete Euler approximation.

$$r_{t+\Delta t} = r_t + \kappa\left(\theta - r_t\right)\Delta t + \sigma\sqrt{\Delta t}Z$$

with Z a standard normal random variable. We indicate the spot rate at time t in trajectory i by $r_{i,t}$, $i = 1, \cdots, N$, $t = 0, \Delta t, \cdots, T - \Delta t, T$. At step 0, r_0 is also assumed stochastic surrounding the mean parameter θ with volatility $\frac{\sigma}{\sqrt{2\kappa}}$.

Next, we need to generate an M dimensional grid of funding ratio surrounding one. Funding ratio at time t is defined as a fraction of instantaneous wealth against the hypothetical bond price maturing at T, $P(t, T)$. The funding ratio grids are indicated by FR_j, $j = 1, \cdots M$. The reason for choosing funding ratio grid instead of wealth grid will be explained later in LSMC algorithm.

We also generate a small grid of testing portfolio, denoting w_h, $h = 1, \cdots, H$. We assume the portfolio grid is bounded between 0 and twice of delta hedge $\Delta = \frac{B(T)}{B(\tau_2)}$, $w_h \in [0, 2\Delta]$.

As a benchmark, we choose $\Delta t = 0.25$, $N = 10,000$, $M = 40$ and $H = 5$.

A.1.2 LSMC

The problem is solved by means of simulation based dynamic programming. We outline the general recursion. We first explain the naive case. In addition to the generated grids of the state variables, we also generate N paths of gross bond returns based on point estimates of bond premia $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$,

$$R_{t+\Delta t} = 1 + \left(r_t(1 - B(\tau_2)\sigma\hat{\Lambda}_1) - B(\tau_2)\sigma\hat{\Lambda}_0\right)\Delta t - B(\tau_2)\sigma\sqrt{\Delta t}Z$$

The gross bond return at time t in trajectory i is denoted by $R_{i,t+\Delta t}$.

Time $T-\Delta t$ The problem at time step $T-\Delta t$ can be summarized by

$$\min_{w_{T-\Delta t}} \mathbb{E}\left[(1-X_T)^+ \mid \mathcal{F}_{T-\Delta t}\right]$$

We first generate a time-dynamic wealth grid denoting $X_{j,T-\Delta} = \mathrm{FR}_j \times P(T-\Delta t, T)$, where $P(T-\Delta t, T)$ is the hypothetical bond price at time $T-\Delta$ maturating at T.

The reason we choose a fixed funding ratio grid instead of a fixed wealth grid is because our value function depends on terminal wealth X_T, keeping wealth grid fixed cannot guarantee us a reasonable range of terminal wealth, typically for long hedging horizon and when recursion approaches to time step 0.

Next we construct the realized terminal wealth under each simulated path (i,j),

$$X_{ijh,T}(\mathcal{F}_{T-\Delta t}) = X_{j,T-\Delta}\left((1+r_{i,T-\Delta})(1-w_h) + R_{i,T}w_h\right)$$

The realized objective function is $V_{ijh,T}(\mathcal{F}_{T-\Delta t}) = (1-X_{T,ijh}(\mathcal{F}_{T-\Delta t}))^+$. Next, we regress the realized value function with a polynomial expansion of the state variables.

$$\begin{aligned} V_{ijh,T}\left(\mathcal{F}_{T-\Delta t}\right) = &\left(a_0 + a_1 r_{i,T-\Delta t} + a_2\mathrm{FR}_j + a_3\mathrm{IFR}_j + a_4\mathrm{FR}_j\mathrm{IFR}_j\right) \\ &+ \left(b_0 + b_1 r_{i,T-\Delta t} + b_2\mathrm{FR}_j + b_3\mathrm{IFR}_j + a_4\mathrm{FR}_j\mathrm{IFR}_j\right) w_h \\ &+ \left(c_0 + c_1 r_{i,T-\Delta t} + c_2\mathrm{FR}_j + c_3\mathrm{IFR}_j + a_4\mathrm{FR}_j\mathrm{IFR}_j\right) w_h^2 + \epsilon_{T-\Delta t} \end{aligned} \tag{13}$$

where IFR is an indicator function, with $\mathrm{IFR}_j = (1-\mathrm{FR}_j)^+$.

It is noticed that, along each path, the conditional variables are known hence the conditional expectation of the approximation regression is a quadratic function of the portfolio weight. Therefore, minimizing the conditional expectation boils down to minimizing a quadratic function of portfolio. We rewrite the conditional expectation of the value function as follows

$$\mathbb{E}\left[V_{ijh,T} \mid \mathcal{F}_{T-\Delta t}\right] = a_{ij} + b_{ij}w_{T-\Delta t} + c_{ij}w_{T-\Delta t}^2 \tag{14}$$

where

$$\begin{aligned} a_{ij}(\mathcal{F}_{T-\Delta t}) &= a_0 + a_1 r_{i,T-\Delta t} + a_2\mathrm{FR}_j + a_3\mathrm{IFR}_j + a_4\mathrm{FR}_j\mathrm{IFR}_j \\ b_{ij}(\mathcal{F}_{T-\Delta t}) &= b_0 + b_1 r_{i,T-\Delta t} + b_2\mathrm{FR}_j + b_3\mathrm{IFR}_j + b_4\mathrm{FR}_j\mathrm{IFR}_j \\ c_{ij}(\mathcal{F}_{T-\Delta t}) &= c_0 + c_1 r_{i,T-\Delta t} + c_2\mathrm{FR}_j + c_3\mathrm{IFR}_j + c_4\mathrm{FR}_j\mathrm{IFR}_j \end{aligned}$$

The optimization problem boils down to solving for the root of Eq.(14), $w^*_{ij,T-\Delta t} = -\frac{b_{ij}}{2c_{ij}}$ if $c_{ij} > 0$.

Time $t = T - 2\Delta t, \cdots, 0$ We now discuss the general recursion of all any other point in time. Suppose we have optimized the hedging policy as of time $t + \Delta t$ onwards. The realized value function at time t is given by

$$V_{ijh,T}(\mathcal{F}_t) = \left[1 - X_{j,t}\left((1 + r_{i,t})(1 - w_h) + R_{i,t+\Delta t} w_h\right) \prod_{s=t+\Delta t}^{T-\Delta t} \left((1 + r_{i,s})(1 - w_s^*) + R_{i,s+\Delta t} w_s^*\right)\right]^+$$

where $X_{j,t} = \mathrm{FR}_j \times P(t, T)$. Next, we approximate conditional expectations $[V_{ijh,T} \mid \mathcal{F}_t]$ by functions of state variables and the testing portfolio

$$\mathbb{E}\left[V_{ijh,T} \mid \mathcal{F}_t\right] = \beta' f(X_{j,t}, r_{j,t}, w_h)$$

Last, we rewrite the conditional expectation of each path as a quadratic function of portfolio w_t and solving for the root.

$$\min_{w_t} \mathbb{E}\left[V_{ijh,T} \mid \mathcal{F}_t\right] = \min_{w_t} f(w_t)$$

We need to re-calculate the dynamic allocation $\frac{T}{\Delta t}$ times to retrieve the optimal decision now w_0 that we are interested in. Along this way, we have also obtained all other optimal portfolios for different hedging horizons $\tau < T$.

A.2 Closed-Form Mother Nature's Choice

Next, we consider the robust case when bond premia are misspecified. A straight forward but more complicated method is to repeat the naive algorithm over a set of testing bond premia taking into uncertainty set constraint. Then we obtain a set of naive optimal policies under each path conditional on a testing bond premium. Then we could find the optimal mother nature's choice under each path either by grid search or using regression based method. However, neither methods are efficient nor accurate. If we use grid search method, we need to generate a fine grid of λ_0 and λ_1. This costs a huge computational memory. Unlike the portfolio weight, the quadratic approximation does not work in the mother nature's choice (λ_0, λ_1) due to low R^2.

A more efficient and accurate method we proposed is to approximate the expected shortfall by a function of bond return. Remark: this only works because we have simplified the problem so much that only Λ is uncertainty. We find that value function is monotonically decreasing with bond return. The approximation is sufficiently accurate since R^2 is nearly one. Therefore, we transform a maximization problem to a minimizing bond return problem, which can be analytically solved by linear programming.

To test the validity of the new methods, we first generate L pairs of testing market price of risk (MPR) denoting $(\lambda_{0,l}, \lambda_{1,l})$, with $l = 1, \cdots, L$ and $(\lambda_{0,l}, \lambda_{1,l}) \in \mathbf{S}$. Under each

pair of testing MPR, we simulate N paths of gross bond returns over m periods. The gross return at time t in path i under testing MPR $(\lambda_{0,l}, \lambda_{1,l})$ is denoted by $R_{il,t+\Delta t}$

At step $T - \Delta t$, we construct realized value function $V_{ijl,T}$ based on mul-specification of bond returns $R_{il,T}$ using a random fixed portfolio weight, then we approximate the conditional value function as a function of $R_{il,T}$ and fund ratio

$$\mathbb{E}\left[V_{ijl,T}\left(w^*_{T-\Delta t}\right) \mid \mathcal{F}_{T-\Delta t}\right] = \beta_0 + \beta_1 R_{il,T} + \beta_2 R^2_{il,T} + \beta_3 FR_j + \beta_4 IFR_j$$

The goodness of fit is larger than 0.99 regardless of the portfolio weight we choose. We find that the conditional expectation of value function is a strict downward sloping convex function of $R_{il,t}$. This property holds for the entire hedging horizon if we recurse the algorithm backward till step 0.

Therefore, the global maximization problem boils down to minimizing bond return which is equivalent to minimizing bond premia since mother nature can only control over the drift term of bond diffusion process. Further, the ellipsoid uncertainty set $\mathbf{S}$ is convex, hence the minimum bond premium should locates on the ellipsoid.

The resulting mother nature's optimal decision depends only on the instantaneous spot rate r_t.

A.3 Robust LSMC

We can follow the naive LSMC algorithm to calculate the robust optimal portfolio expect that we first need to analytically solve for mother nature's choice at each backward step in time. Hence at time t, realized value function contains both optimal portfolios as well as mother nature's choice of bond premia from steps onwards

$$V_{ijh,T} = \left[1 - X_{j,t}\left((1+r_{i,t})(1-w_h) + R2^*_{i,t+\Delta t} w_h\right) \prod_{s=t+\Delta t}^{T-\Delta t} \left((1+r_{i,s})(1-w^*_s) + R^*_{i,s+\Delta t} w^*_s\right)\right]^+$$

where $R^*_{i,t+\Delta}$ indicates the optimal bond return at time t on path i with optimal MPR $\lambda^*_{0,t} + \lambda^*_{1,t} r_{i,t}$

A.4 Goodness of Fit

We investigate the accuracy of our algorithm by means of R^2. The R^2s of parametrization regression at step $T - \Delta t$ are higher than 0.995 for both naive and robust case. The goodness of fit by construction, has to decay backwards of time, because we are accumulating cross sectional information over each time period onward. The quality of the global quadratic approximation depends on R^2 at first approximation step $T - \Delta t$ and the speed of decaying. The longer the hedging horizon is, the lower R^2 will be at time 0. i.e. If $T \leq 20$ years, R^2 at time 0 is higher than 95%. If we set investment horizon

extremely long, $T = 80$ years, R^2 drops to 0.83 at time step 0. This is still reasonably high, since the last step cross-sectional regression contains 320 time steps of cross-section information. Grid sizes or rebalancing frequency do not influence the goodness of fit.

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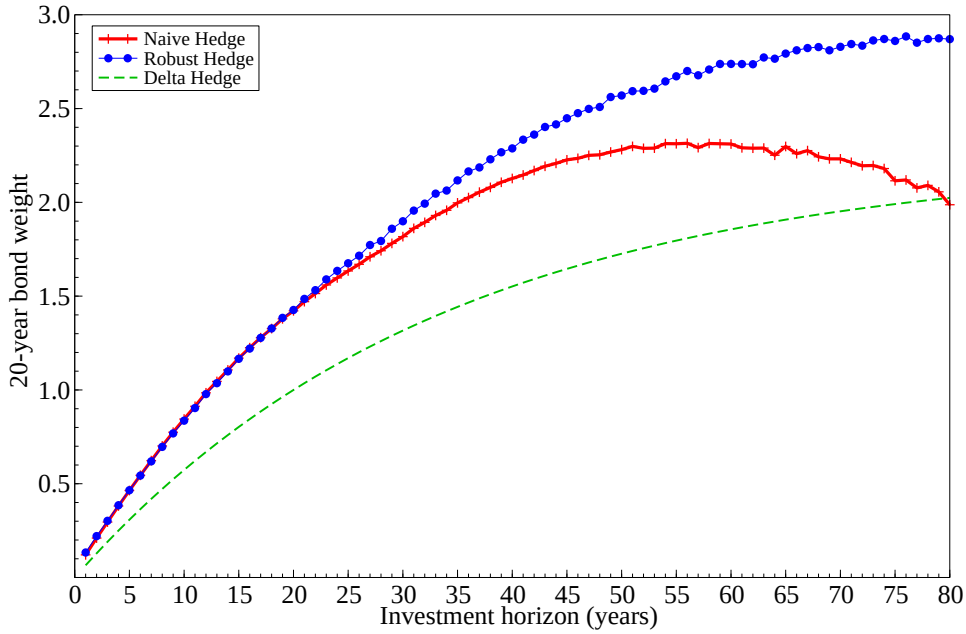
Figure 3: **Optimal Bond Portfolio for Different Horizons**
The figure plots 20-year bond weights against different investment horizons when current funding ratio is 80% (panel (a)) and 110% (panel (b)) for different hedging policies. The present spot rate is 2%. The weights are based on three different hedging policies: naive hedge, robust hedge and delta hedge. Results are based on 10,000 draws of predictive bond distribution.

(a) 80% Funding Ratio



(b) 110% Funding Ratio

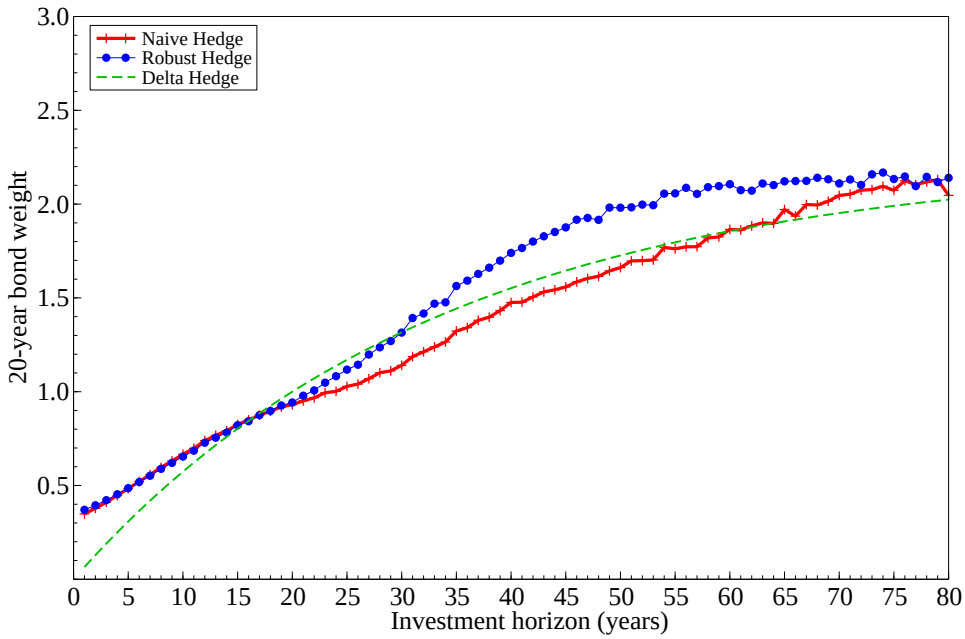
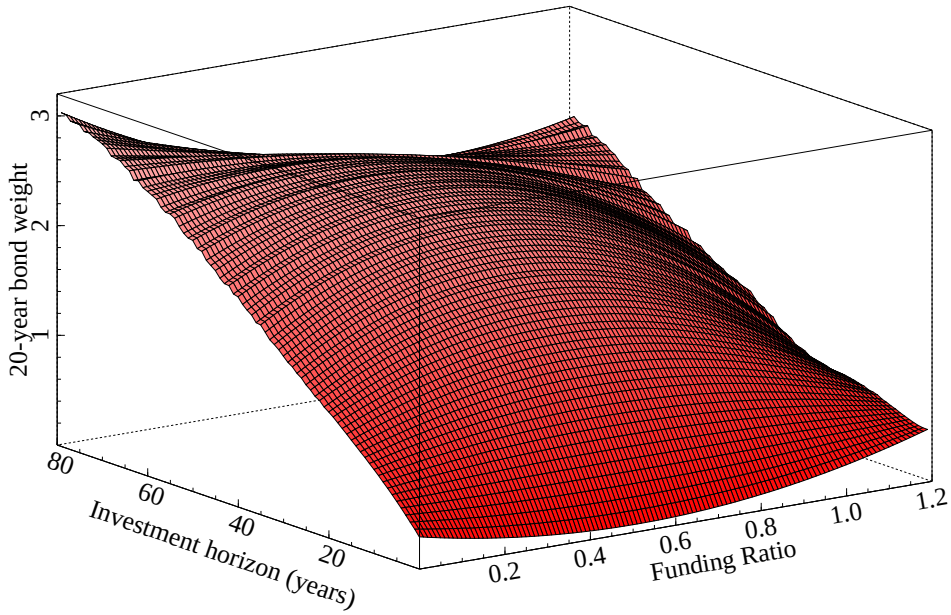


Figure 4: **Optimal Portfolio under Different Funding Ratio (1)** The figure plots the optimal portfolio of 20-year bond as a function of investment horizon and current funding ratio with spot rate equal to 2%.

(a) Naive Policy



(b) Robust Policy

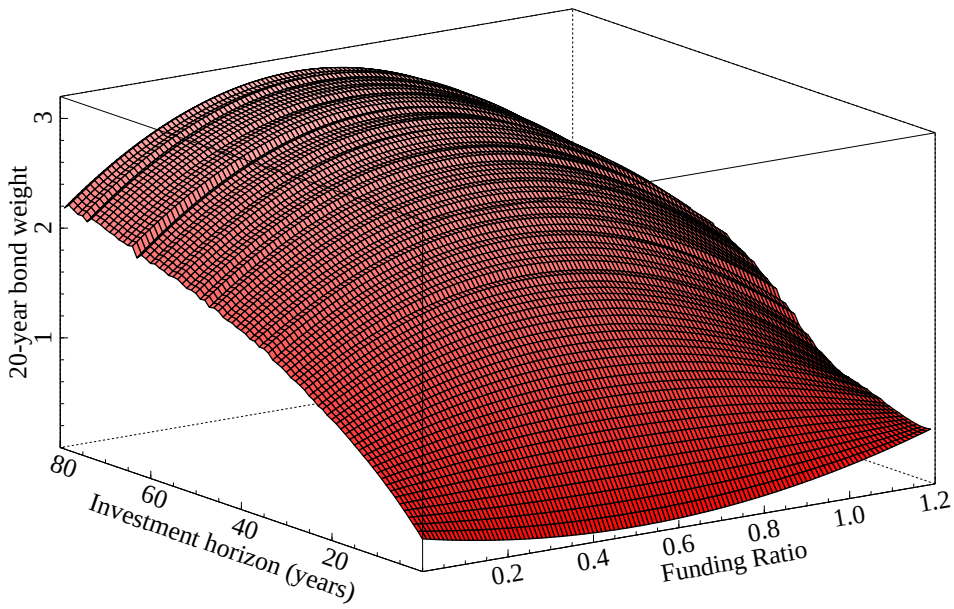
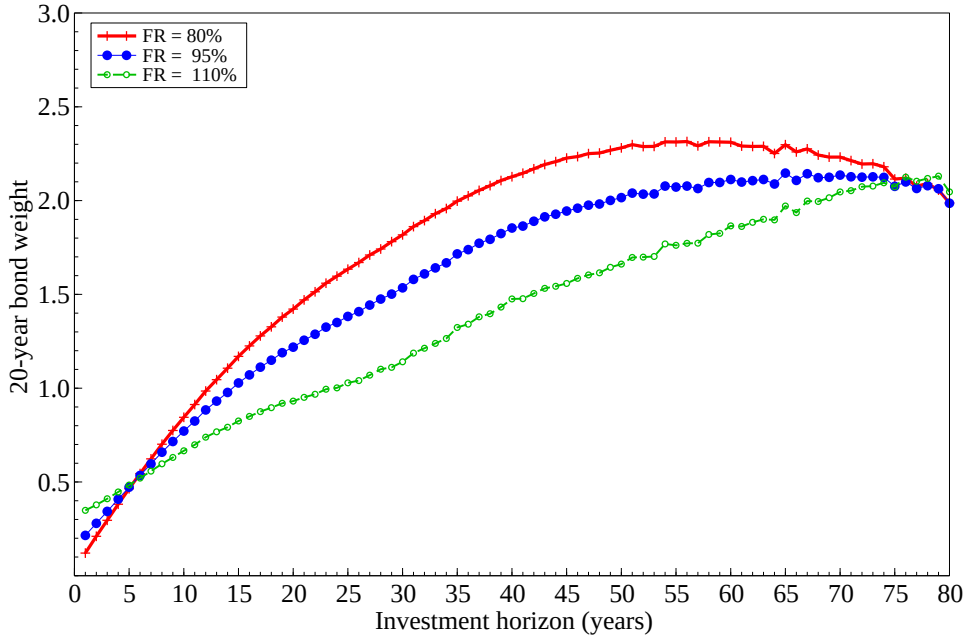


Figure 5: **Optimal Portfolio under Different Funding Ratio (2)** The figure summarize the key insight of Figure 4 under three reasonable funding ratio levels.

(a) Naive Policy



(b) Robust Policy

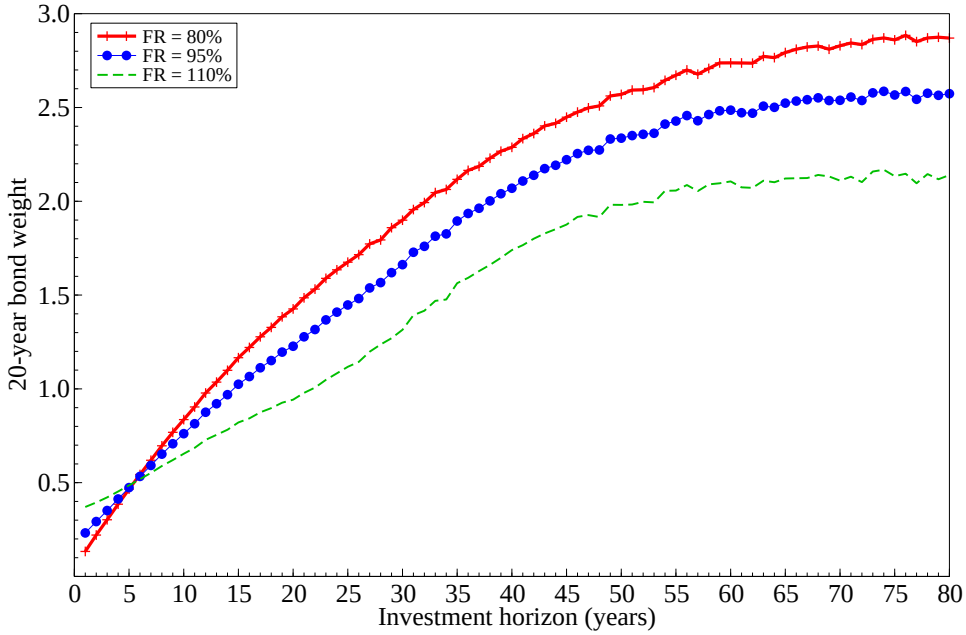
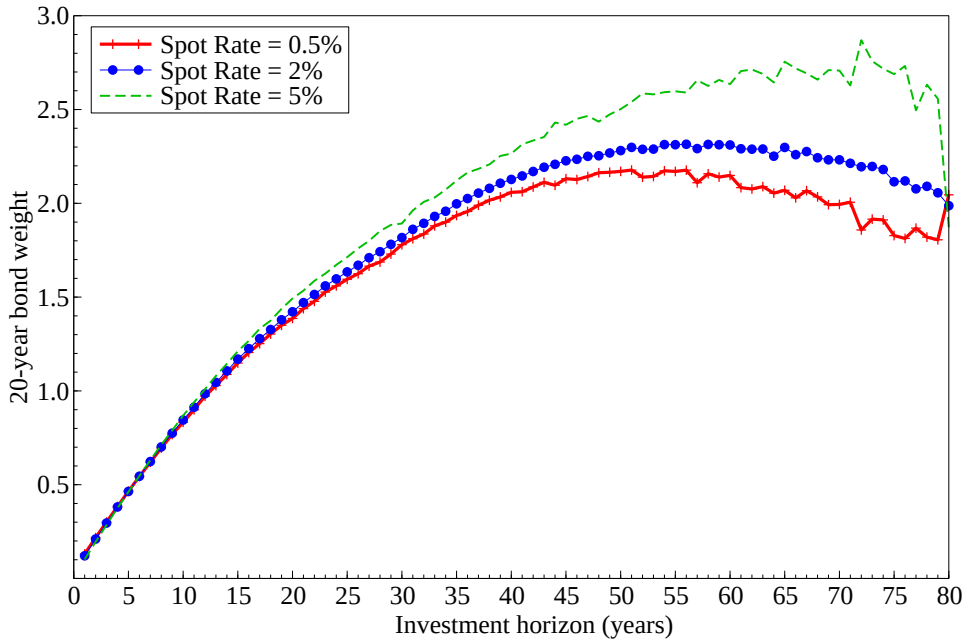


Figure 6: **Robust Policy under Different Spot Rate**
The figure plots naive (panel (a)) and robust (panel(b)) optimal bond portfolio as a function of spot rate and hedging horizon when current funding ratio is 80%.

(a) Naive Policy



(b) Robust Policy

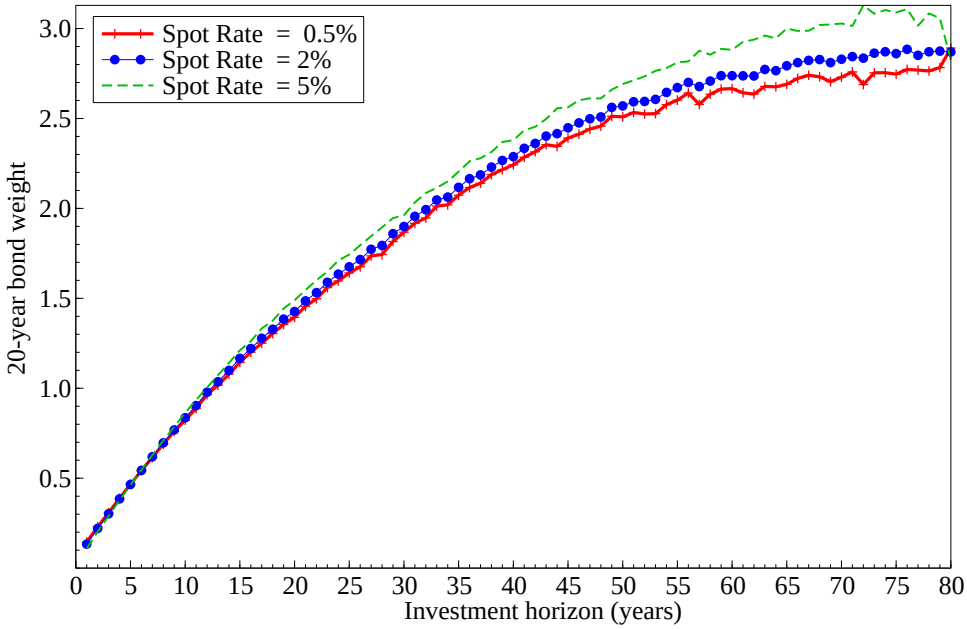
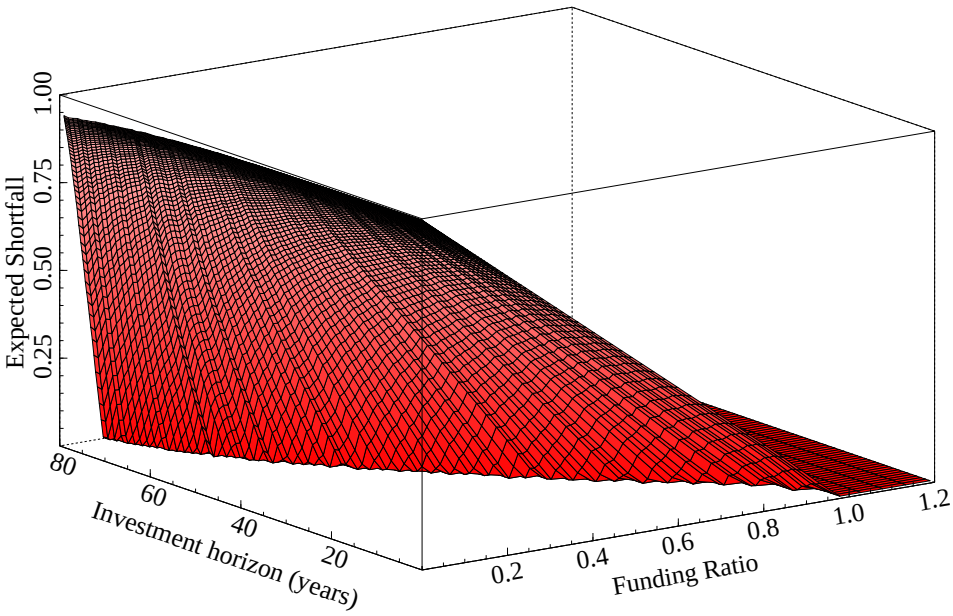


Figure 7: **Optimal Expected Shortfall (1).**
The figure plots the expected shortfall as a function of hedging horizon and funding ratio under naive (panel (a)) and robust (panel(b)) policies when spot rate is 2%.

(a) Naive Expected Shortfall



(b) Robust Expected Shortfall

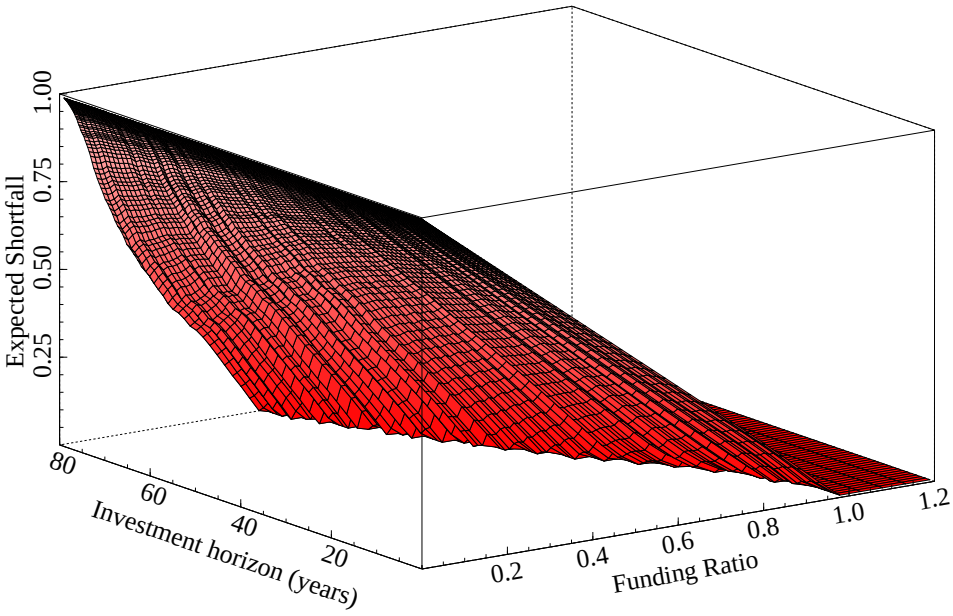
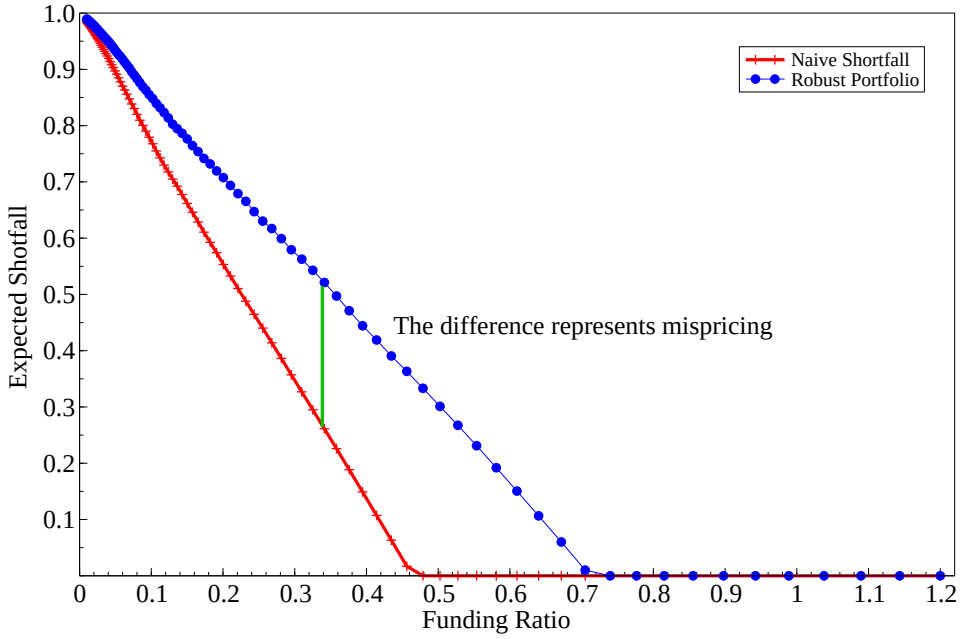


Figure 8: **Optimal Expected Shortfall (2).**
The figure plots the robust expected shortfall against the naive one as a function of funding ratio when hedging horizon is 40 years (panel (a)) and when the horizon is 60 years (panel(b)).

(a) T = 40



(b) T = 60

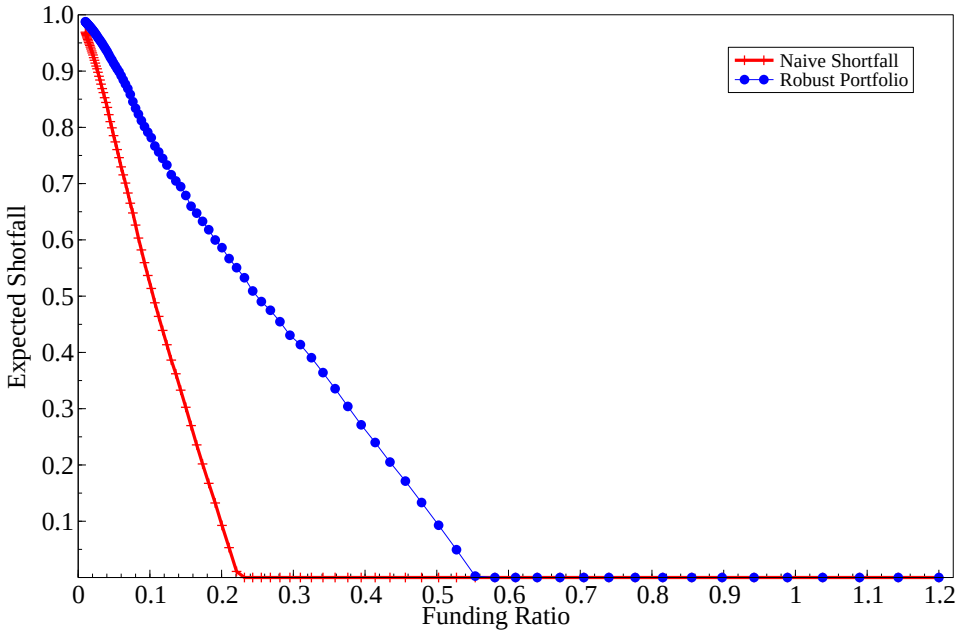


Figure 9: **Robust Yield Curve**
The figure plots policy-based yield curve again the benchmark Vasicek yield curve under different spot rate and shortfall target. The benchmark yield curve by definition has zero expected shortfall.