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Risk Models with Jumps and Time-Varying Second Moments

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Risk models with jumps and time-varying second moments *

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Abstract

In this paper, we propose a new risk model to better address events like the recent credit crisis. First, the possible start of a crisis is modeled by including a low-probability jump process. Second, the risk characteristics of the crisis are captured by allowing for time-varying volatilities and correlations. Time variation in correlations is due to the changing importance of two sources: monetary shocks leading to a positive stock–bond correlation, and risk aversion (or "flight to safety") shocks leading to a negative stock–bond correlation. The model stays within the essentially affine class, thereby allowing for closed-form solutions for arbitrage-free nominal and real bond prices of all maturities. Moreover, equity options and swaption prices are included in the estimation procedure to enhance the proper modeling of the volatility on the equity and interest rate markets. The model captures a large part of the time variation in financial risks for pension funds due to both changing volatilities and correlations.

Keywords: essentially affine macro-finance term structure model, time-varying volatilities and correlations, jumps, options, swaptions

JEL codes: E43, G13, G17

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1 Introduction

The 2008 credit crisis caught most financial practitioners and academics by surprise. Although some analysts were aware of problems in the US subprime mortgage market, the financial disorder and economic collapse after the default of Lehman Brothers were unprecedented. Moreover, even for those who did expect a severe correction, the timing of such an event was impossible to predict. Risk models widely used by the financial industry regarded such events as highly unlikely. Moreover, these models generally considered volatilities and correlations to be constant, whereas during the crisis both volatilities and correlations between asset classes became much more extreme. Although time variation in correlation had been documented before (see, e.g., Campbell and Ammer (1993), Ang and Bekaert (2004), Campbell, Sunderam, and Viceira (2009), and Baele, Bekaert, and Inghelbrecht (2010)), the events in 2008 imply that the benefits of diversification are sometimes much smaller than anticipated by standard models. The positive correlation between risky assets grew much higher as all went down simultaneously. At the same time, the correlation between stocks and treasuries changed from positive to negative. As a consequence, the usefulness of risk models came into question.

To remedy part of these flaws of traditional risk models, we add two features. First, stochastic jumps are introduced. These jumps represent a sudden loss of confidence of the market leading to a significant drop in the stock market, lower risk-free interest rates, and a severe increase in credit spreads. These abrupt changes in sentiment can be caused by an economic environment that is suddenly perceived to be untenable (a bursting bubble), as well as by events outside the economy, such as a terrorist attack. Since the start of a crisis is, almost by definition, unpredictable and since crises can occur in any economic environment, we assume the probability of a jump to be constant.

Second, to accommodate changes in volatilities and correlations, we implement a time-varying covariance matrix for the normally distributed shocks in our scenario model. We assume there are two dominant time-varying sources of uncertainty in the economy: monetary (inflationary) uncertainty and real uncertainty (affecting risk aversion). The importance of these sources of risk is assumed to depend on the level of some of the driving variables in the system. The monetary volatility factor depends positively on the current level of inflation and the short-term interest rate. The risk aversion volatility factor increases in the credit spread and the dividend yield, whereas it decreases in the stock market return. To help identify the second moments, we include equity option and swaption prices in the estimation procedure. The correlation structure differs between the monetary and real uncertainty environments. Consequently, changing correlations is a direct result of the changing importance over time of the two sources of risk. The time-varying second moments also generate asymmetry in interest rates, inflation, and credit spreads. Apart from the fact that this pattern is confirmed in the historical data, the asymmetry is convenient in simulation exercises as negative interest rates becomes less likely since volatility decreases in such a model if interest rates are declining. Especially in a low interest rate environment, this is an important advantage. As Rudebusch (2010) states, "In the future, developing versions of the affine arbitrage-free model that prevent interest rates from going negative will be a priority" (p. 25). Our paper is one of the first attempts at this in the macro-finance context.

Specifically, for pension funds, the time-varying correlations between stocks and bonds are important, since stocks are much more attractive if they correlate positively with bonds and thereby the liabilities of pension funds. We assume positive correlation is primarily caused by monetary uncertainty, whereas negative correlation has to do with risk aversion shocks. Although the negative impact of (expected) inflation on stock returns has been documented many times, starting with Fama and Schwert (1977), it is not uncontroversial, since stocks can be seen as a claim on real cash flows. Notwithstanding the real aspects of stocks, inflation will have a negative impact in the short run if (expectations of) higher inflation leads to higher real interest rates. Yang, Zhou, and Wang (2009) indeed find that stock–bond correlations in the US and UK are especially high if short rates are high and (to a lesser extent) if inflation is high. The risk aversion impact is especially apparent during crisis times, when "flight to safety" leads to negative correlation (see Connolly, Stivers, and Sun (2005, 2007)).

Our model also includes the complete nominal and real term structures based on an essentially affine specification (see Duffee (2002)). Although the jumps complicate the analytical expressions for the term structure parameters somewhat, closed-form solutions can still be obtained as long as the jump process is unrelated to the state variables. Arbitrage opportunities are ruled out and the model generates reasonable results under both the physical measure and the risk-neutral measure. Modeling the whole term structure (up to maturities of even 90 years) has the advantage that nominal or real liabilities can be calculated exactly this way, instead of via a duration approximation. The arbitrage-free structure also allows for closed-form solutions for short-term stock option prices and approximate solutions for swaptions.

We use German and Eurozone data on inflation and interest rates to estimate the model. Since medium-term inflation expectations are lower today than during the high-inflation episode of the 1970s, we include a time-varying inflation target in our model. For this purpose, we use the so-called medium-term price assumption, which was the (implicit) inflation target the Bundesbank published starting in 1975 (see Gerberding, Worms, and Seitz (2004)).

The rest of this paper is organized as follows. Section 2 discusses previous literature. Section 3 describes the model. First, the dynamics of the main driving variables are shown, including the variance and jump specifications. Second, the specification of the price of risk and the resulting term structures of nominal and real interest rates are given. Third, the closed-form solutions for options and swaptions are provided. Section 4 describes the estimation results, with its implications on volatilities, correlations, and term structures. Section 5 shows the model's simulation results. It is demonstrated that the term structure of risks depends crucially on the starting values of the state variables. The main results are summarized in Section 6. Appendices A and B describe the Feller conditions (necessary to prevent negative variances) and the data sources, respectively.

2 Related Literature

As mentioned in the introduction, there is evidence of significant time variation in the stock–bond correlation, that is, the correlation between stock and government bond returns. While the correlation has, on average, been positive over the past four decades, it frequently drops and can become negative in times of turbulence. In fact, the stock–bond correlation in the new millennium has, on average, been slightly negative. The time variation in correlations may have large effects on optimal portfolio choice.

Previous authors have studied the time variation in stock–bond correlations from various angles.

Campbell and Ammer (1993) attribute time-varying correlations in the US to a number of underlying factors. The authors find that changes in expected inflation impact the returns on stocks and bonds in opposite directions and thereby induce a negative correlation: An increase in expected inflation is bad news for bonds but good news for stocks, and vice versa. Changes in the real interest rate, however, are found to move stocks and bonds in the same direction, leading to positive correlation. A positive correlation also results from stocks and bonds reacting similarly to changes in their respective expected returns, as in Fama and French (1989). Campbell and Ammer (1993) do not allow for negative correlations other than through changes in expected inflation, and their sample period ends in 1987. However, uncertainty about expected inflation has greatly decreased since the mid-1980s, ushering in the era referred to as the Great Moderation, while the correlation has turned negative since 2000. This suggests that factors other than expected inflation changes are at work.

Ilmanen (2003) analyzes stock–bond correlations in various economic environments, using data since 1926. The author finds that inflation introduces positive correlation due to the common discount rate effect, while deflation leads to negative correlation, since it pushes up equity risk premiums and lowers bond risk premiums. Volatility shocks (potentially associated with a flight to quality) and growth shocks are found to have opposite effects on stock and bond returns, leading to a decrease in the correlation.

Several studies analyze the correlations using GARCH models. For one, Scrugs and Glabadanis (2003) strongly reject models that impose a constant correlation restriction on the covariance matrix of stock and bond returns. Stock volatility is found to react asymmetrically to shocks in stock and bond returns, while bond volatility reacts symmetrically to bond return shocks. Yang, Zhou, and Wang (2009) find differences in the stock–bond correlation over the business cycle using a century and a half's worth of data for the US and the UK. For the US, the correlation tends to be higher during expansions than during recessions. The UK data, however, lead to the opposite conclusion. The authors specify the conditional correlation as a function of macroeconomic variables. For both countries, they find that correlation reacts positively to higher short rates and higher inflation rates, particularly for the period after 1923. Kim, Moshirian, and Wu (2006) report a downward trend in the stock–bond correlations within Europe, Japan, and the US, whereas international stock–stock and bond–bond correlations have increased.

Gulko (2002) and Connolly, Stivers, and Sun (2005, 2007) use market uncertainty as an explanatory factor. They find that a large drop in the stock market and an increase in implied volatilities or stock market turnover cause a shift from stocks to bonds, which are considered to be a safe haven. This flight-to-quality behavior results in lower correlations (see also Ilmanen (2003)). This effect is observed for the US, as well as for European countries.

Another strand of literature uses regime-switching models to explain time-varying volatilities and correlations. Ang and Bekaert (2004) and Guidolin and Timmermann (2006) identify (low-persistence) regimes characterized by stock–stock correlations and volatilities that increase in bad times. Ang and Bekaert (2004) allow the transition probabilities to depend on the short rate. A high short rate increases the probability of a transition to a volatile regime, as well as the probability of staying in that regime. Guidolin and Timmermann (2006) identify four regimes, two of which (a crash state and a recovery state) are characterized by high volatility and low or even negative stock–bond correlation. Baele, Bekaert, and Inghelbrecht (2010) use a regime-switching factor model with time-varying exposures. The authors find that fundamental factors explain bond returns, while non-macro factors such as illiquidity and the variance premium (an options data-implied risk premium) explain relatively more of the stock returns. However, they do not find a good fit of the correlation. The fit improves if the factor exposures can depend on the variance premium, although the timing and magnitude are still not perfect. Connolly, Stivers, and Sun (2005, 2007) analyze a regime-switching model where the probability of switching depends on the implied volatility or the detrended stock turnover. The probability of transition to a more volatile regime increases with the implied volatility.

Finally, some studies are based on affine pricing models for stocks and bonds. Campbell, Sunderam, and Viceira (2009) use a latent variable to model the changing correlation between the inflation and real variables. The model results in negative correlations between stocks and bonds since the end of the 1990s. D'Addona and Kind (2006) find that, for the G-7 countries, the volatility of the real rates increases the stock–bond correlation. Inflation shocks decrease the correlation, although this is partly driven by the modeling assumption that stocks provide complete protection against future inflation. A higher variability of the dividend yield decreases the correlation. Li (2002) analyzes the correlation for the G-7 countries also and concludes that uncertainty about the long-term expected inflation and real rates is the driver of an increase in the stock–bond correlation, whereas uncertainty about stock-specific returns reduces the correlation. The effect of unexpected inflation is ambiguous.

Although time-varying second moments are important, for the proper modeling of stock market returns especially, jumps are probably important as well (see, e.g., Bakshi, Cao, and Chen (1997) and Andersen, Benzoni, and Lund (2002)). The jumps capture the extreme events in the data due to unexpected news. Starting with the jump-diffusion model of Merton (1976), the literature on stochastic jumps has focused almost entirely on the Poisson distribution (see, e.g., Ball and Torous (1985), Jorion (1988), Chan and Maheu (2002), Eraker, Johanners, and Polson (2003), Liu and Pan (2003) and Maheu and McCurdy (2004)). A characteristic of the Poisson process is that the number of jumps per unit of time is not bounded. For empirical purposes, the number of jumps has to be truncated at some point. Since extreme news events are rare, the jump probability is usually very small and the probability of more than one jump is already negligible. An alternative is to start from a Bernoulli distribution, which postulates either zero or one jump per unit of time. Vlaar and Palm (1993) compare the Bernoulli and Poisson jump processes and find that the Bernoulli process is easier to estimate (less dependent on starting values), whereas the models are almost equivalent in terms of likelihood.

3 The Model

3.1 The Dynamics of the State Variables

The core model comprises six stochastic state variables and four deterministic ones. The stochastic state variables are the log of the annual inflation in the Eurozone (π_t), the continuously compounded three-month Euribor ($y_t^{(1)}$), the quarterly log excess return on the stock market (xs_t), the dividend yield in logit form (dy_t),[1] the credit spread between US Baa bonds and treasuries (cs_t, also in log percentages), and an unobserved variable we call maturity preference (mp_t). Maturity preference measures time-varying influences on bond prices that are unrelated to the other state variables. These can, for instance, be related to changes in the supply and demand of long-term bonds. Of the observed state variables, the nominal short interest rate, the dividend yield, and the credit spread are known to help predict excess returns on stocks and bonds (see, e.g., Campbell and Shiller (1988), Fama and French (1989), Campbell, Chan, and Viceira (2003), Campbell and Viceira (2005), Brandt and Santa-Clara (2006) and Hoevenaars, Molenaar, Schotman, and Steenkamp (2008)). The deterministic state variables are the medium-term price assumption ($\bar{\pi}_t$, also referred to as the inflation target) and quarterly inflation (π_t^q) lagged one, two, and three quarters. The medium-term price assumption is taken from the Bundesbank (see Gerberding, Worms, and Seitz (2004)). The Bundesbank started publishing an inflation target in 1975, which was 5% at the time. We assume the same number for 1973 and 1974. After 1975, it gradually decreased, reaching 2% in 1985. Since the number has not changed since and was later adopted by the European Central Bank as its inflation target, we take the variable to be exogenous.

The dynamics of the stochastic state variables are modeled in a quarterly vector autoregressive system with time-varying second moments and subject to stochastic jumps:

$$\mathbf{x}_{t+1} = \begin{bmatrix} \pi_{t+1} \\ y_{t+1}^{(1)} \\ xs_{t+1} \\ dy_{t+1} \\ cs_{t+1} \\ mp_{t+1} \end{bmatrix} = \mathbf{c}_t + \mathbf{\Gamma}\mathbf{x}_t + J_{t+1}\nu + \mathbf{\Sigma}\mathbf{S}_t^{1/2}\zeta_{t+1} \quad (1)$$

$$\mathbf{c}_t = (\mathbf{I}_6 - \mathbf{\Gamma})(\mu_0 + \mu_{\bar{\pi}}\bar{\pi}_t) - \mathbf{p}\nu \quad (2)$$

$$\zeta_{t+1} \sim \mathbb{N}(\mathbf{0}, \mathbf{I}_6) \quad (3)$$

The intercepts depend linearly on the medium-term inflation target through a vector of slope coefficients $\mu_{\bar{\pi}}$.[2] Only the equilibrium values for π_t, $y_t^{(1)}$, and dy_t are assumed to depend on the inflation target (see, e.g., Bekaert and Engstrom (2010) for the impact of $\bar{\pi}_t$ on dy_t). To alleviate identification problems for the latent factor, we rule out lagged dependence between mp_t and the other state variables, which is common in the term structure literature.

The small chance of sudden panic in the market is modeled by means of stochastic jumps. With probability $\mathbf{p}$, the jump indicator J_{t+1} equals one; otherwise, it is zero. The impact of the jumps is measured by the vector of mean jump sizes (ν). The jump represents a sudden change in sentiment in the market. Although there may have been early warning indicators that signaled the economic crises

[1] Positivity of the dividend yield is guaranteed by means of the following modified logit transformation: $dy = \ln(\text{dividend yield}/(10 - \text{dividend yield}))$, which restricts the dividend yield between 0 and 10%. The correlation between our measure and ln(dividend yield) is 99.66%. The simulation properties of the log specification were less favorable, however, since unrealistically high dividend yields sometimes resulted.

[2] We subtract $\mathbf{p}\nu$ in Equation (2) to make the equilibrium value for the state variables, given the inflation target, equal to $\mu_0 + \mu_{\bar{\pi}}\bar{\pi}_t$.

from the past, we assume the probability of a jump to be constant, since the next economic crisis may have a completely new unexpected reason. Besides, from a technical point of view, this assumption is necessary to facilitate analytical expressions for term structures and options.

The time variation in volatility is captured by the diagonal matrix $\mathbf{S}_t$, which represents the variances of the normally distributed shocks to the state variables:

$$s_{m,t} = \alpha_m + \pi_t + \beta_{my} y_t^{(1)} , \qquad \beta_{my} \geq 0 \quad (4)$$

$$s_{r,t} = \alpha_r - \beta_{rx} xs_t + \beta_{rd} dy_t + cs_t , \qquad \beta_{rx}, \beta_{rd} \geq 0 \quad (5)$$

$$\text{Diagonal}(\mathbf{S}_t) = \alpha + \beta \mathbf{x}_t = \begin{bmatrix} s_{m,t} \\ s_{r,t} \\ 1.01 - \omega_m + \omega_m s_{m,t} \\ 1.01 - \omega_r + \omega_r s_{r,t} \\ 1.01 - \omega_{1v} + \omega_{1v}(\omega_{1m} s_{m,t} + (1 - \omega_{1m}) s_{r,t}) \\ 1.01 - \omega_{2v} + \omega_{2v}(\omega_{2m} s_{m,t} + (1 - \omega_{2m}) s_{r,t}) \end{bmatrix} , \quad 0 \leq \omega_i \leq 1 \quad (6)$$

Two volatility factors are identified: a monetary factor and a risk aversion factor. The monetary factor ($s_{m,t}$) measures the uncertainty in monetary policy. The factor depends on inflation (used for normalization) and the short-term interest rate.[3] The risk aversion factor ($s_{r,t}$) depends on excess returns on the stock market, the dividend yield, and the credit spread (used for normalization).[4] The other elements on the diagonal of $\mathbf{S}_t$ are linear combinations of these two volatility factors, subject to two conditions. First, to prevent these volatilities from approaching zero, a small positive intercept is added. This requirement allows a much more flexible price of risk specification for these shocks in the essentially affine model (see Duffee (2002)). Second, each element on the diagonal of $\mathbf{S}_t$ is distinct. This condition helps to identify $\mathbf{\Sigma}$ from the time-varying conditional covariances of the state variables ($\mathbf{\Sigma}\mathbf{S}_t\mathbf{\Sigma}^\top + \mathbf{p}(1-\mathbf{p})\nu\nu^\top$). Consequently, we do not need identification restrictions on the error loading matrix $\mathbf{\Sigma}$, and the ordering of the elements in $\mathbf{S}_t$ is unrelated to the ordering of the variables in $\mathbf{x}_t$.[5] A switch in the ordering in $\mathbf{S}_t$ simply leads to an equivalent switch in the columns of $\mathbf{\Sigma}$. Restrictions on $\mathbf{\Sigma}$ (and $\mathbf{\Gamma}$) are still necessary, however, to avoid negative variances. Therefore, we impose the Feller conditions on the model (see, e.g., Duffie and Kan (1996), Spreij, Veerman, and Vlaar (2011), and Appendix A).

Combining the jumps with the normally distributed shocks results in a Bernoulli mixture of normal distributions (see Vlaar and Palm (1993)). Conditional on no jump, the shocks are normally distributed with expectation $-\mathbf{p}\nu$ and variance $\mathbf{\Sigma}\mathbf{S}_t\mathbf{\Sigma}^\top$. If the market panics, the shocks are still normally distributed but their mean size is ν higher. The Bernoulli mixture of normals can be seen as a special case of a regime-switching model (see Ang, Bekaert, and Wei (2008)). The difference is

[3] We also investigated the impact of the inflation target but found no influence.

[4] Yang, Zhou, and Wang (2009) use the implied volatility from equity index options (the VIX) and detrended stock turnover to measure market uncertainty. We do not use these variables because they are not available over the whole sample period. Moreover, the VIX cannot easily be incorporated consistently in our model, given the links between the VIX and excess stock returns. Our credit spread variable is highly correlated (about 0.6) with the VIX, however.

[5] We still include one zero restriction in the last column of $\mathbf{\Sigma}$, since the last two elements on the diagonal of $\mathbf{S_t}$ may be similar.

that, in a regime-switching model, the probability of ending up in a certain state in the next period depends on the current state, whereas, in a jump model, the current state does not make a difference. The persistence in volatility and correlations in our model results directly from the persistence in the state variables (apart from stock returns) affecting the volatility factors. In a regime-switching model, this is incorporated by the persistence of the regimes.

3.2 Term Structures

Arbitrage-free term structures of interest rates can be derived by means of a pricing kernel. The pricing kernel M_{t+1} relates the current price of an asset to its future payoff. For a zero-coupon bond, this means

$$P_t^{(n)} = \mathbf{E}_t\left[M_{t+1}P_{t+1}^{(n-1)}\right] \quad (7)$$

$$= (1-\mathbf{p})\mathbf{E}_{t,0}\left[M_{t+1}P_{t+1}^{(n-1)}\right] + \mathbf{p}\,\mathbf{E}_{t,1}\left[M_{t+1}P_{t+1}^{(n-1)}\right], \quad (8)$$

where $P_t^{(n)}$ is the price at time t of a risk-free n-period zero-coupon bond paying €1 at time $t+n$.[6] The time t expectation in (7) can be decomposed into two conditional expressions (8), one for normal times (denoted $\mathbf{E}_{t,0}$) and one conditional on a jump at time $t+1$ (denoted $\mathbf{E}_{t,1}$). We assume in this paper that asset prices and the kernel are jointly lognormal conditional on the jump status. Thus, we can rewrite (8) as

$$P_t^{(n)} = (1-\mathbf{p})\exp\left(\mathbf{E}_{t,0}\left[m_{t+1}+p_{t+1}^{(n-1)}\right] + \frac{1}{2}\mathbf{var}_{t,0}\left[m_{t+1}+p_{t+1}^{(n-1)}\right]\right) + \mathbf{p}\exp\left(\mathbf{E}_{t,1}\left[m_{t+1}+p_{t+1}^{(n-1)}\right] + \frac{1}{2}\mathbf{var}_{t,1}\left[m_{t+1}+p_{t+1}^{(n-1)}\right]\right) \quad (9)$$

where lowercase notation is used to denote log values and $\mathbf{var}_{t,0}$ and $\mathbf{var}_{t,1}$ represent the variance given time t information and the time $t+1$ jump status, respectively. Separating the impact on the price into a no-jump result and the impact of jumps, the log pricing equation can be expressed as

$$p_t^{(n)} = \mathbf{E}_{t,0}\left[m_{t+1}+p_{t+1}^{(n-1)}\right] + \frac{1}{2}\mathbf{var}_{t,0}\left[m_{t+1}+p_{t+1}^{(n-1)}\right] + \ln\left(1-\mathbf{p}+\mathbf{p}\exp\left(\mathbf{E}_{t,1}\left[m_{t+1}+p_{t+1}^{(n-1)}\right] - \mathbf{E}_{t,0}\left[m_{t+1}+p_{t+1}^{(n-1)}\right] + \frac{1}{2}\left(\mathbf{var}_{t,1}\left[m_{t+1}+p_{t+1}^{(n-1)}\right] - \mathbf{var}_{t,0}\left[m_{t+1}+p_{t+1}^{(n-1)}\right]\right)\right)\right) \quad (10)$$

It is assumed that the jumps have a fixed additive effect on the mean of the distribution of asset returns or the pricing kernel.[7] The yield of the n-period zero-coupon bond is given by $y_t^{(n)} = -400\, p_t^{(n)}/n$, where the 400 comes from the fact that we express yields in annual percentages. Since a one-period zero-coupon bond is riskless, its log expected payoff and variance are both zero. If we denote the

[6] The same pricing relation holds for all other assets in the economy. We refer to Campbell (2000) and Cochrane (2001) for a further description of the pricing kernel.

[7] We also investigated the impact of stochastic jump sizes but found no substantial improvement from this modification.

impact of jumps on the log pricing kernel by the additive factor $-J_{t+1}\phi$, for $n = 1$ (10) leads to

$$-\frac{y_t^{(1)}}{400} = \mathbf{E}_{t,0}\left[m_{t+1}\right] + \frac{1}{2}\mathbf{var}_{t,0}\left[m_{t+1}\right] + \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi}\right) \tag{11}$$

Based on this result, we specify the log pricing kernel as[8]

$$-m_{t+1} = \frac{y_t^{(1)}}{400} + \frac{1}{2}\lambda_t^\top\lambda_t + \lambda_t^\top\zeta_{t+1} + \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi}\right) + J_{t+1}\phi \tag{12}$$

where[9]

$$\lambda_t = \mathbf{S}_t^{-1/2}\left(\lambda_0 + \mathbf{\Lambda}_1\mathbf{x}_t\right) \tag{13}$$

Since the volatility of some of the shocks can approach zero arbitrarily closely, without restrictions the market price of risk can become arbitrarily large (though finite). Since it is not clear whether this is justified, assuming no arbitrage, restrictions are usually imposed on λ_0 and $\mathbf{\Lambda}_1$ in Equation (13). In accordance with the essentially affine model (see Duffee (2002)), we restrict the first two rows of $[\lambda_0\ \mathbf{\Lambda}_1]$ to be proportional to the first two rows of $[\alpha\ \beta]$.[10] As a result, the prices of risk for the first two shocks are proportional to their volatility, whereby λ_t goes to zero if $\mathbf{S}_t$ approaches zero.[11]

Combining (10) and (11), it follows that the expected excess return of bonds (and all other assets) conditional on no jump equals

$$\begin{aligned}\mathbf{E}_{t,0}[p_{t+1}^{(n-1)}]-p_t^{(n)}-\frac{y_t^{(1)}}{400} = -\frac{1}{2}\mathbf{var}_{t,0}[p_{t+1}^{(n-1)}] - \mathbf{cov}_{t,0}[m_{t+1},p_{t+1}^{(n-1)}] + \ln\Big(1-\mathbf{p}+\mathbf{p}e^{-\phi}\Big) - \\ \ln\Big(1-\mathbf{p}+\mathbf{p}\exp\Big(-\phi + \mathbf{E}_{t,1}[p_{t+1}^{(n-1)}]-\mathbf{E}_{t,0}[p_{t+1}^{(n-1)}]\Big)\Big)\end{aligned} \tag{14}$$

Assets that co-vary negatively with the pricing kernel tend to have low returns whenever the marginal utility is high. In equilibrium, such assets must have a high expected return to compensate for the low returns in bad states of the world. The last term contains the incremental means of the pricing kernel and asset returns due to the jumps. Since the jump sizes are assumed to be deterministic, the variances and covariances conditional on a jump are equal to those in the absence of a jump. The price of risk related to the excess stock returns is completely determined by this no-arbitrage condition,

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[8]This specification implies that closed-form solutions for bond prices can be obtained, provided that both the jump probability and the jump size are independent of the state variables. In the regime-switching literature, it is usually assumed that $\mathbf{E}_{t,0}[M_{t+1}] = \mathbf{E}_{t,1}[M_{t+1}] = \exp(-y_t^{(1)}/400)$. In that case, closed-form solutions for bond prices are only possible if either a log-linear approximation is used, as in Bansal and Zhou (2002), or if the prices of risk do not hinge on regime-dependent state variables, as in Ang, Bekaert, and Wei (2008).

[9]For notational convenience, we include the exogenous variables also in $\mathbf{x}_t$. We assume the pricing kernel is affected by the inflation target via $\mathbf{\Lambda}_1$, but not by lagged quarterly inflation (the last three columns of $\mathbf{\Lambda}_1$ contain only zeros).

[10]Duffee (2002) expresses (13) as $\lambda_t = \mathbf{S}_t^{1/2}\xi_1 + \mathbf{S}_t^{-}\Xi_2\mathbf{x}_t$, where $\mathbf{S}_{t_{(i,i)}}^{-} = \begin{cases}\mathbf{S}_{t_{(i,i)}}^{-1/2}, & \text{if } \inf(\mathbf{S}_{t_{(i,i)}}) > 0\\ 0 & \text{otherwise}\end{cases}$. In our case, only the first two volatilities can approach zero, and we have $[\lambda_0\ \mathbf{\Lambda}_1]_{[12]} = [\alpha\ \beta]_{[12]} \odot \xi_{1_{[12]}}$, where the Hadamard product $\odot$ denotes entrywise multiplication. Similarly, $A^{\odot 2}$ denotes that each element in the matrix A is raised to the second power.

[11]Cheridito, Filipović, and Kimmel (2007) show this restriction is not strictly necessary to avoid arbitrage opportunities. As long as the process for the variances of the shocks is strictly nonnegative under both the physical and risk-neutral measures, arbitrage opportunities are ruled out.

leading to

$$\begin{aligned} l_{xs}^{\top}\mathbf{\Sigma}\lambda_0 &= l_{xs}^{\top}\mathbf{c} + \frac{1}{2}\alpha^{\top}\left(\mathbf{\Sigma}^{\top}l_{xs}\right)^{\odot 2} - \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi}\right) + \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi+l_{xs}^{\top}\nu}\right) \\ l_{xs}^{\top}\mathbf{\Sigma}\mathbf{\Lambda}_1 &= l_{xs}^{\top}\mathbf{\Gamma} + \frac{1}{2}\beta^{\top}\left(\mathbf{\Sigma}^{\top}l_{xs}\right)^{\odot 2} \end{aligned}$$

where l_{xs} denotes a selection vector.

The assumed dynamics of the state variables and the pricing kernel imply that log bond prices are affine functions of the state variables (see, e.g., Campbell, Lo, and MacKinlay (1997) and Cochrane and Piazzesi (2006)):

$$-p_t^{(n)} = A_n + \mathbf{B}_n^{\top}\mathbf{x}_t \tag{15}$$

Since $y_t^{(1)}$ is included in the state variables, we can express $y_t^{(1)}/400 = l_y^{\top}\mathbf{x}_t$. The pricing equation (14) results in

$$\begin{aligned} A_n + \mathbf{B}_n^{\top}\mathbf{x}_t = l_y^{\top}\mathbf{x}_t + A_{n-1} + \mathbf{B}_{n-1}^{\top}\left(\mathbf{c} + \mathbf{\Gamma}\mathbf{x}_t\right) - \frac{1}{2}\mathbf{var}_{t,0}\left[\mathbf{B}_{n-1}^{\top}\mathbf{\Sigma}\mathbf{S}_t^{1/2}\zeta_{t+1}\right] - \\ \mathbf{cov}_{t,0}\left[\lambda_t^{\top}\zeta_{t+1}, \mathbf{B}_{n-1}^{\top}\mathbf{\Sigma}\mathbf{S}_t^{1/2}\zeta_{t+1}\right] + \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi}\right) - \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi-\mathbf{B}_{n-1}^{\top}\nu}\right) \end{aligned}$$

Filling in $\mathbf{S}_t$ and λ_t (i.e., (6) and (13)) and matching coefficients, we have

$$\begin{aligned} A_n &= A_{n-1} + \left(\mathbf{c} - \mathbf{\Sigma}\lambda_0\right)^{\top}\mathbf{B}_{n-1} - \frac{1}{2}\alpha^{\top}\left(\mathbf{\Sigma}^{\top}\mathbf{B}_{n-1}\right)^{\odot 2} + \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi}\right) - \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi-\mathbf{B}_{n-1}^{\top}\nu}\right) \\ \mathbf{B}_n &= l_y + \left(\mathbf{\Gamma} - \mathbf{\Sigma}\mathbf{\Lambda}_1\right)^{\top}\mathbf{B}_{n-1} - \frac{1}{2}\beta^{\top}\left(\mathbf{\Sigma}^{\top}\mathbf{B}_{n-1}\right)^{\odot 2} \end{aligned} \tag{16}$$

with $A_0 = 0$ and $\mathbf{B}_0 = \mathbf{0}$. If we express annualized yields as $y_t^{(n)} = a_n + \mathbf{b}_n\mathbf{x}_t$, it follows that $a_n = 400\ A_n/n$, and $\mathbf{b}_n = 400\ \mathbf{B}_n/n$.

A similar derivation can be used for the real term structure. Let $\widetilde{P}_{s,t}^{(n)}$ denote the price at time t of a real zero-coupon bond that has been issued at time s and matures at time $t+n$. At time $t+1$, a real bond issued at time t (its price is $\widetilde{P}_{t,t+1}^{(n-1)}$) will pay at maturity an amount equal to Π_{t+n-1}/Π_{t-1}, with Π_t the price index at time t.[12] The minus one in the timing of the price indices accounts for the fact that the prices of index-linked bonds are based on one-quarter–lagged inflation. Since the payment at maturity of a real bond, which is issued at time $t+1$ and matures at time $t+n$, equals Π_{t+n-1}/Π_t, the no-arbitrage argument requires that the price of the bond issued at time t be a factor Π_t/Π_{t-1} higher than that of the bond issued at time $t+1$, that is, $\widetilde{p}_{t,t+1}^{(n-1)} = \pi_t^q + \widetilde{p}_{t+1,t+1}^{(n-1)}$. The price of the real bond $\widetilde{p}_{t,t}^{(n)}$ is also an affine function of the state variables: $-\widetilde{p}_{t,t}^{(n)} = \widetilde{A}_n + \widetilde{\mathbf{B}}_n^{\top}\mathbf{x}_t$. Since $\widetilde{p}_{t,t}^{(0)} = 0$, it follows that $\widetilde{A}_0 = 0$ and $\widetilde{\mathbf{B}}_0 = \mathbf{0}$. Quarterly inflation is a linear function of the extended state variable set, that is, $\pi_t^q = l_{\pi}^{\top}\mathbf{x}_t$. Since the general pricing relation (14) also holds for real bonds,

[12] Note that this formulation does not include the option that the payment of the principal at time $t+n$ will be at least 100% even if the cumulative inflation is negative (i.e., $\Pi_{t+n-1} < \Pi_{t-1}$).

straightforward substitutions lead to the following recursion for the real term structure parameters:

$$\begin{aligned}\widetilde{A}_n &= \widetilde{A}_{n-1} + (\mathbf{c} - \mathbf{\Sigma}\lambda_0)^\top \widetilde{\mathbf{B}}_{n-1} - \frac{1}{2}\alpha^\top \left(\mathbf{\Sigma}^\top \widetilde{\mathbf{B}}_{n-1}\right)^{\odot 2} + \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi}\right) - \ln\left(1-\mathbf{p}+\mathbf{p}e^{-\phi-\widetilde{\mathbf{B}}_{n-1}^\top \nu}\right) \\ \widetilde{\mathbf{B}}_n &= l_y - l_\pi + (\mathbf{\Gamma} - \mathbf{\Sigma}\mathbf{\Lambda}_1)^\top \widetilde{\mathbf{B}}_{n-1} - \frac{1}{2}\beta^\top \left(\mathbf{\Sigma}^\top \widetilde{\mathbf{B}}_{n-1}\right)^{\odot 2} \end{aligned} \quad (17)$$

For real yields (expressed in annual percentages), we have $\widetilde{y}_t^{(n)} = \widetilde{a}_n + \widetilde{\mathbf{b}}_n \mathbf{x}_t$, with $\widetilde{a}_n = 400\ \widetilde{A}_n/n$ and $\widetilde{\mathbf{b}}_n = 400\ \widetilde{\mathbf{B}}_n/n$.

3.3 Derivatives

We also include equity options and swaptions in the estimation procedure. These derivatives give the right, but not the obligation, to buy or sell equities or interest rate swaps. This right becomes more valuable when the future is more uncertain. Hence, the prices of these products can be viewed as a market price for volatility and contain useful information.

Applying risk-neutral valuation, European call and put options are priced as

$$Call_t^{(K,n)} = P_t^{(n)} \mathbf{E}^{\mathbb{Q}} \left[max(Z_{t+n} - K, 0)\right] \quad (18)$$

$$Put_t^{(K,n)} = P_t^{(n)} \mathbf{E}^{\mathbb{Q}} \left[max(K - Z_{t+n}, 0)\right] \quad (19)$$

where $P_t^{(n)}$ $(= e^{-y_t^{(n)} n/400})$ is the price of a n-period risk-free bond, $\mathbb{Q}$ denotes expectation in the risk-neutral $\mathbb{Q}$-measure, Z_{t+n} is stock price, and K is strike price.

To price these derivatives, we need to find the dynamics of the state variables under the risk-neutral measure. To arrive at the $\mathbb{Q}$ measure, we aim to find a different probability space for the stochastic shocks and jumps $(\zeta^{\mathbb{Q}}, J^{\mathbb{Q}})$ in which minus the log price deflator $-m^{\mathbb{Q}}$ is simply the risk-free rate $y^{(1)}/400$. The risk-neutral measure dynamics becomes (see Lin and Vlaar (2011) for details).

$$\mathbf{x}_{t+1} = \mathbf{c}_t^{\mathbb{Q}} + \mathbf{\Gamma}^{\mathbb{Q}} \mathbf{x}_t + J_{t+1}^{\mathbb{Q}} \nu + \mathbf{\Sigma} \mathbf{S}_t^{1/2} \zeta_{t+1}^{\mathbb{Q}} \quad (20)$$

where $\mathbf{c}_t^{\mathbb{Q}} = \mathbf{c}_t - \mathbf{\Sigma}\lambda_0$, $\mathbf{\Gamma}^{\mathbb{Q}} = \mathbf{\Gamma} - \mathbf{\Sigma}\mathbf{\Lambda}_1$, $J_{t+1}^{\mathbb{Q}}$ is Bernoulli distributed with probability $\mathbf{p}^{\mathbb{Q}} = \frac{\mathbf{p}e^{-\phi}}{1-\mathbf{p}+\mathbf{p}e^{-\phi}}$, and $\zeta_{t+1}^{\mathbb{Q}}$ is standard normally distributed.

Given the dynamics under $\mathbb{Q}$, option prices can be simulated using (18) and (19). For an estimation procedure, this is not feasible, since new Monte Carlo simulations have to be run for every parameter set. Closed-form solutions can be found, however, to price options with a maturity of a single period, one quarter in this case:[13]

$$\begin{aligned} Call_t^{(K,1)} &= \left(1-\mathbf{p}^{\mathbb{Q}}\right)\left(e^{l_{xs}^\top(\mathbf{c}_t^{\mathbb{Q}}+\mathbf{\Gamma}^{\mathbb{Q}}\mathbf{x}_t)+\sigma_{xs}^2/2}\mathbb{N}(d_{xs}+\sigma_{xs}) - P_t^{(1)} K \mathbb{N}(d_{xs})\right) \\ &+ \mathbf{p}^{\mathbb{Q}}\left(e^{l_{xs}^\top(\mathbf{c}_t^{\mathbb{Q}}+\mathbf{\Gamma}^{\mathbb{Q}}\mathbf{x}_t+\nu)+\sigma_{xs}^2/2}\mathbb{N}(d_{xs}+\frac{l_{xs}^\top \nu}{\sigma_{xs}}+\sigma_{xs}) - P_t^{(1)} K \mathbb{N}(d_{xs}+\frac{l_{xs}^\top \nu}{\sigma_{xs}})\right) \end{aligned} \quad (21)$$

where $\sigma_{xs} = \sqrt{l_{xs}^\top \mathbf{\Sigma}\mathbf{S}_t \mathbf{\Sigma}^\top l_{xs}}$, $d_{xs} = \frac{-\ln(K) - p_t^{(1)} + l_{xs}^\top(\mathbf{c}_t^{\mathbb{Q}}+\mathbf{\Gamma}^{\mathbb{Q}}\mathbf{x}_t)}{\sigma_{xs}}$ (both depend on t), and $\mathbb{N}(d_{xs})$ denotes

[13] If the jump probability is zero, the no-arbitrage conditions imply that $l_{xs}^\top(c_t^{\mathbb{Q}} + \Gamma^{\mathbb{Q}} x_t) + \sigma_{xs}^2/2 = 0$, and (21) reduces to the Black Scholes formula.

the value of the cumulative normal distribution function at d_{xs}. Similarly, for puts we have

$$
\begin{aligned}
Put_t^{(K,1)} &= \left(1-\mathbf{p}^{\mathbb{Q}}\right)\left(P_t^{(1)}K\mathbb{N}(-d_{xs}) - e^{l_{xs}^\top(\mathbf{c}_t^{\mathbb{Q}}+\mathbf{\Gamma}^{\mathbb{Q}}\mathbf{x}_t)+\sigma_{xs}^2/2}\mathbb{N}(-d_{xs}-\sigma_{xs})\right) \\
&+ \mathbf{p}^{\mathbb{Q}}\left(P_t^{(1)}K\mathbb{N}(-d_{xs}-\frac{l_{xs}^\top\nu}{\sigma_{xs}}) - e^{l_{xs}^\top(\mathbf{c}_t^{\mathbb{Q}}+\mathbf{\Gamma}^{\mathbb{Q}}\mathbf{x}_t+\nu)+\sigma_{xs}^2/2}\mathbb{N}(-d_{xs}-\frac{l_{xs}^\top\nu}{\sigma_{xs}}-\sigma_{xs})\right) \quad (22)
\end{aligned}
$$

For options with longer maturities, the stochastic volatility prevents an exact analytical solution. In the estimation, we include both at-the-money call options and calls and puts that are 10% out of the money, all with a maturity of one quarter.

For swaptions, an approximation is needed to derive an analytical expression, even for the single-period swaption. The sum of discounted cash flows of a swap is approximated by a single cash flow via a first-order Taylor expansion of the log cash flows around the current state. This approximation can be interpreted as replacing a coupon-bearing bond by the most representative zero-coupon bond. For a receiver swaption with strike X, tenor N, and maturity n, this means replacing

$$
Receiver_t^{(X,N,n)} = P_t^{(n)}\mathbf{E}^{\mathbb{Q}}\left[max(X\sum_{i=1}^{N}e^{-A_i-\mathbf{B}_i^\top\mathbf{x}_{t+n}} + e^{-A_N-\mathbf{B}_N^\top\mathbf{x}_{t+n}} - 1, 0)\right] \quad (23)
$$

by

$$
Rec_t^{(X,N,n)} = P_t^{(n)}\mathbf{E}^{\mathbb{Q}}\left[max(R_t e^{-D_t^\top\mathbf{x}_{t+n}} - 1, 0)\right] \quad (24)
$$

where

$$
D_t = \frac{\sum_{i=1}^{N}G_i\mathbf{B}_i}{\sum_{i=1}^{N}G_i}
$$

$$
R_t = \sum_{i=1}^{N}G_i e^{D_t^\top\mathbf{x}_t}
$$

and G_i is defined as

$$
G_i = \begin{cases} Xe^{-A_i-\mathbf{B}_i^\top\mathbf{x}_t} & \text{if } i = 1,2,...,N-1 \\ (X+1)e^{-A_N-\mathbf{B}_N^\top\mathbf{x}_t} & \text{if } i = N \end{cases}
$$

and D_t, R_t, and G_i are functions of the initial state $\mathbf{x}_t$, the tenor of the underlying swap N, and the strike of the swaption X. For $n = 1$ (swaptions maturing next quarter), a closed-form solution for (24) can be found:

$$
\begin{aligned}
Rec_t^{(X,N,1)} &= P_t^{(1)}\left(\left(1-\mathbf{p}^{\mathbb{Q}}\right)\left(R_t e^{-D_t^\top(c^{\mathbb{Q}}+\Gamma^{\mathbb{Q}}\mathbf{x}_t)+\sigma_y^2/2}\mathbb{N}(d_y+\sigma_y) - \mathbb{N}(d_y)\right)\right. \\
&+ \left.\mathbf{p}^{\mathbb{Q}}\left(R_t e^{-D_t^\top(c^{\mathbb{Q}}+\Gamma^{\mathbb{Q}}\mathbf{x}_t+\nu)+\sigma_y^2/2}\mathbb{N}(d_y-\frac{D_t^\top\nu}{\sigma_y}+\sigma_y) - \mathbb{N}(d_y-\frac{D_t^\top\nu}{\sigma_y})\right)\right) \quad (25)
\end{aligned}
$$

where $\sigma_y = \sqrt{D_t^\top\mathbf{\Sigma}\mathbf{S}_t\mathbf{\Sigma}^\top D_t}$ and $d_y = \frac{\ln(R_t)-D_t^\top(\mathbf{c}_t^{\mathbb{Q}}+\mathbf{\Gamma}^{\mathbb{Q}}\mathbf{x}_t)}{\sigma_y}$ are both time varying. For the payer swaption we have

$$
\begin{aligned}
Pay_t^{(X,N,1)} &= P_t^{(1)} \Big(\big(1-\mathbf{p}^{\mathbb{Q}}\big) \Big(\mathbb{N}(-d_y) - R_t e^{-D_t^\top (c^{\mathbb{Q}}+\Gamma^{\mathbb{Q}}\mathbf{x}_t)+\sigma_y^2/2} \mathbb{N}(-d_y-\sigma_y) \Big) \\
&+ \mathbf{p}^{\mathbb{Q}} \Big(\mathbb{N}(-d_y + \frac{D_t^\top \nu}{\sigma_y}) - R_t e^{-D_t^\top (c^{\mathbb{Q}}+\Gamma^{\mathbb{Q}}\mathbf{x}_t+\nu)+\sigma_y^2/2} \mathbb{N}(-d_y + \frac{D_t^\top \nu}{\sigma_y} - \sigma_y) \Big) \Big) \quad (26)
\end{aligned}
$$

We include payer and receiver swaptions with tenors of 10, 20, and 30 years and a maturity of one quarter. With respect to the strike, both at the money and 50 basis points out of the money receiver swaptions are included, whereas for payers only 50 basis points out of the money swaptions are taken into account.

4 Estimation Results

Since the maturity preference series is not observable, all parameters are estimated in a one-step approach by means of maximum likelihood. The maturity preference is identified by assuming that the 10-year nominal rate is observed without measurement error. Consequently, we do not need the Kalman filter. The one-step approach fully exploits the restrictions imposed by the term structure model and is therefore expected to give more accurate estimates of the dynamics of the term structure (see, e.g., De Jong (2000)). Our analysis runs from 1973 to 2010.[14] Table 1 contains summary statistics for the observable state variables and the 10-year nominal rate.

Table 1: Sample Statistics (1973:I–2010:IV)

Variable	Average	Std. Deviation	Minimum	Maximum
π_t	2.90	1.81	-1.03	7.65
$y_t^{(1)}$	5.19	2.76	0.63	13.10
xs_t	0.53	8.80	-27.72	19.79
dy_t	-0.91	0.55	-1.93	0.28
cs_t	2.07	0.96	0.67	7.17
$y_t^{(40)}$	6.44	1.83	2.60	10.63

For many of the yields, data only became available later on. This is especially the case for the real rates. Index-linked bonds were not traded in the Eurozone until 1998. Moreover, liquidity in this market was limited in the first years, which makes the results hard to interpret (see, e.g., D'Amico, Kim, and Wei (2008)). Consequently, reliable real interest rate data are only available since 2004. Very long maturity nominal rates (50-year swaps) have been traded only since August 2001. A 60-year yield is calculated by the Dutch supervisor (De Nederlandsche Bank), based on swap market data, assuming constant forward rates from 40 years on. Pension funds in the Netherlands are obliged to calculate their liabilities based on this nominal term structure. We use the same method to construct very long-term yields. That is, for bond prices with maturities above 50 years, Equations (16) are

[14]Before 1973, no data were available for the credit spread, one of the state variables.

replaced by

$$
\begin{aligned}
A_n &= A_{n-1} + (A_{200} - A_{160})/40 \\
\mathbf{B}_n &= \mathbf{B}_{n-1} + (\mathbf{B}_{200} - \mathbf{B}_{160})/40
\end{aligned}
\tag{27}
$$

4.1 Mean Dynamics

The dynamics of state variables show that most series are primarily affected by their own history (see Table 2). Only the diagonal elements of Γ are statistically significantly different from zero. In economic terms, some of the interactions may be important though. The stock market series, for instance, is hardly affected by its own past, but it is affected to some extent by the lagged dividend yield and lagged short-term rate. Despite the insignificant t-values, 10% of the variance of excess stock returns is explained by the model. Also, as expected with a credible central bank, the short-term interest rate is positively affected by inflation.

Table 2: Mean Parameters and Resulting Fit State Variables

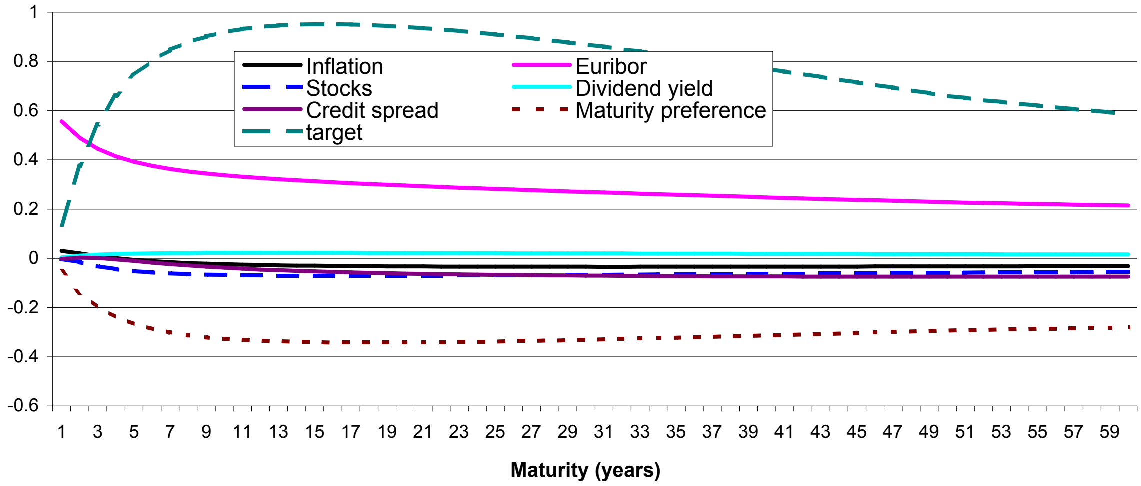
Variable	Γ						μ_0	$\mu_{\bar{\pi}}$	R^2
π_t	0.84	0.03	0.01	0.03	0.05	0	0	1	0.89
	(15.4)	(0.3)	(1.2)	(0.2)	(1.2)	(-)	(-)	(-)	
$y_t^{(1)}$	0.09	0.92	-0.01	-0.03	-0.05	0	1.42	1.04	0.93
	(1.6)	(27.8)	(1.2)	(0.2)	(1.3)	(-)	(0.5)	(1.2)	
xs_t	0.03	-0.50	0.02	4.19	-0.22	0	0.91	0	0.10
	(0.1)	(0.9)	(0.1)	(1.3)	(0.1)	(-)	(0.6)	(-)	
dy_t	0.01	0.00	0.00	0.95	-0.02	0	-2.41	0.51	0.95
	(0.7)	(0.0)	(0.0)	(21.9)	(0.6)	(-)	(2.1)	(1.4)	
cs_t	0.00	0.00	-0.01	0.02	0.87	0	1.94	0	0.75
	(0.2)	(0.7)	(1.5)	(0.1)	(11.0)	(-)	(2.6)	(-)	
mp_t	0	0	0	0	0	0.89	0	0	0.81
	(-)	(-)	(-)	(-)	(-)	(17.5)	(-)	(-)	

Note: The t-values are shown in parentheses.

The equilibrium values of the state variables are in accordance with historical realizations. The impact of the inflation target on the short-term interest rate is about one to one, whereas the impact on the dividend yield (in level terms) is just above 80%. This less than proportional impact may be due to the fact that the dividend yield refers to global equities, whereas the inflation target is only for the Eurozone. At an inflation target of 2%, the equilibrium dividend yield is also about 2%. With respect to the model's fit, we consider the results quite acceptable, especially since the estimation procedure is not entirely focused on maximizing R^2 values (as is the case in a homoskedastic model without jumps).

Since the individual prices of risk parameters are hard to interpret, we summarize their impact graphically and in terms of fit. Figure 1 presents the partial impact of the state variables on the nominal term structure. It shows the effect of a shock of one standard deviation in one state variable, assuming no change in the other variables. For the inflation target, a change of one percentage point is modeled. The nominal term structure is dominated by three state variables. For short nominal maturities, the current short rate is by far the most important variable. Nominal long-term interest rates are generally believed to depend primarily on inflation expectations (see, e.g., Buraschi and

Figure 1: Impact of a shock of one standard deviation in one state variable on nominal term structure.



Jiltsov (2005)). In our model, the inflation target and maturity preference dominate for maturities of five years or longer. An increase in the inflation target by one percentage point leads to an increase in long-term rates of 50 to 95 basis points. This less than proportional impact on (very) long yields may reflect the fact that the inflation target reflects the intermediate horizon, and not an indefinite one. In the very long run, the 2% inflation target of the European Central Bank seems credible. The impact of inflation, the stock market, the dividend yield, and the credit spread are all negligible. The minor impact of actual inflation does not, however, mean inflation is unimportant for nominal rates. For short maturities, the inflation impact is already reflected in the current short rate, whereas the inflation target measures the impact for long rates.

Figure 2 shows the impact of the state variables on the real term structure. The effects on the real yields are generally smaller, when compared to the nominal yields, leading to slightly less volatile real rates than nominal ones. The impact of inflation is now negative, since real bonds protect against inflation increases.

Table 3 presents the fit of the model for the yields and derivatives included in the estimation

Figure 2: Impact of a shock of one standard deviation in one state variable on real term structure.

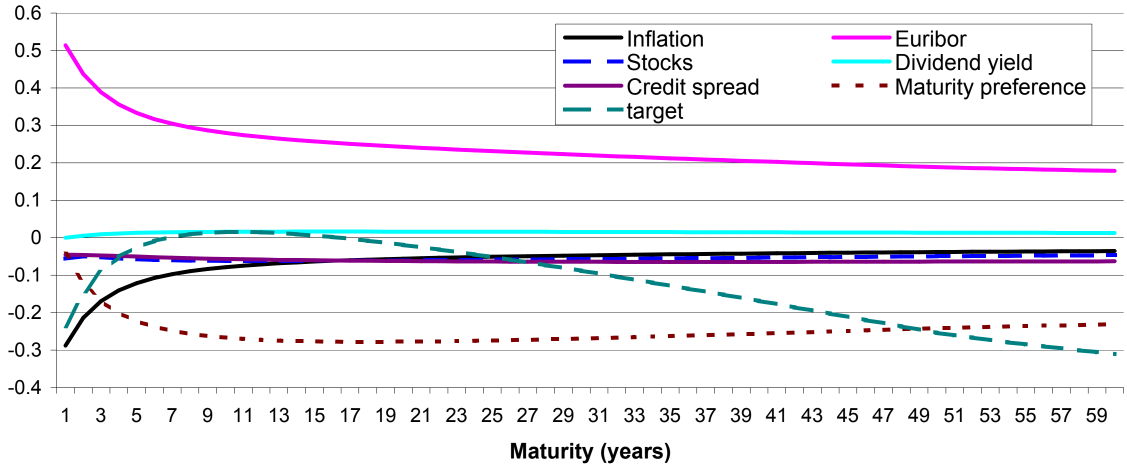


Table 3: R^2 Values Included Yields and Derivatives

Nominal yields			Real yields			Derivatives		
$y_t^{(4)}$	0.96	(1973:II)	$\widetilde{y}_t^{(4)}$	0.76	(2004:II)	$Put_t^{(0.9,1)}$	0.32	(1994:III)
$y_t^{(8)}$	0.95	(1973:II)	$\widetilde{y}_t^{(8)}$	0.82	(2003:IV)	$Call_t^{(1,1)}$	0.49	(1994:III)
$y_t^{(12)}$	0.96	(1973:II)	$\widetilde{y}_t^{(12)}$	0.82	(2004:II)	$Call_t^{(1.1,1)}$	0.63	(1994:III)
$y_t^{(16)}$	0.97	(1973:II)	$\widetilde{y}_t^{(16)}$	0.83	(2004:II)	$Rec_t^{(0,10,1)}$	0.65	(2002:IV)
$y_t^{(20)}$	0.98	(1973:II)	$\widetilde{y}_t^{(20)}$	0.83	(2004:II)	$Rec_t^{(0,20,1)}$	0.61	(2002:IV)
$y_t^{(28)}$	0.99	(1973:II)	$\widetilde{y}_t^{(28)}$	0.83	(2004:II)	$Rec_t^{(0,30,1)}$	0.48	(2002:IV)
$y_t^{(40)}$	0.95	(1973:II)	$\widetilde{y}_t^{(40)}$	0.77	(2004:II)	$Rec_t^{(-50,10,1)}$	0.60	(2004:I)
$y_t^{(60)}$	0.98	(1986:II)	$\widetilde{y}_t^{(60)}$	0.60	(2004:II)	$Rec_t^{(-50,20,1)}$	0.42	(2004:I)
$y_t^{(80)}$	0.98	(1996:I)	$\widetilde{y}_t^{(80)}$	0.71	(2004:II)	$Rec_t^{(-50,30,1)}$	0.25	(2004:I)
$y_t^{(100)}$	0.96	(1996:I)	$\widetilde{y}_t^{(100)}$	0.81	(2004:II)	$Pay_t^{(50,10,1)}$	0.61	(2004:I)
$y_t^{(120)}$	0.94	(1996:I)	$\widetilde{y}_t^{(120)}$	0.66	(2004:II)	$Pay_t^{(50,20,1)}$	0.52	(2004:I)
$y_t^{(160)}$	0.79	(2001:III)	$\widetilde{y}_t^{(160)}$	0.39	(2004:II)	$Pay_t^{(50,30,1)}$	0.39	(2004:I)
$y_t^{(200)}$	0.79	(2001:III)	$\widetilde{y}_t^{(200)}$	0.20	(2007:I)			

Note: The first observation is shown in parentheses.

process. With the exception of the 10-year nominal yield, the residuals for these series are based on the realized state variables. The fit for the nominal yields is very good for maturities of up to 30 years. After that, the measurement errors become somewhat larger. For real yields, measurement errors are bigger, but for maturities up to 30 years still at least 60% of the variance of the yields is explained by the model. For equity options, the model explains 32% (out-of-the-money puts) to 63% (out-of-the-money calls) of the price. With respect to swaptions, the fit declines somewhat with maturity. For at-the-money receivers (for which the strike equals the forward rate), 65% (10 year maturity) to 48% (30 year maturity) of the variance in the price is explained. For the 50-basis-point out-of-the-money receivers and payers, these percentages are somewhat lower. A nice feature of the model is that it accounts for the so-called volatility skew in options and swaptions, where implied volatilities decrease monotonically with increasing strikes (see Andersen and Andreasen (2000)).

It should be emphasized that these individual R^2 values can be improved upon if one does not care about a worse fit for other categories. In the current setting, the same parameters determine the nominal and real yields of all maturities and the prices of the derivatives. If, for instance, nominal yields above 10 years, real yields, and derivatives were ignored, as is the case in the majority of the term structure literature, a better fit would be obtained for the remaining nominal yields.

4.2 Jump Dynamics

The parameters of the jump process are given in Table 4. Although the t-values suggest that the statistical significance of jumps is limited, the value of the likelihood function for this model is more than 50 points higher than for a similar model without jumps. From a statistical point of view, this difference justifies these eight extra parameters by all standards. The low t-values may be due to the difficult identification of jumps, since the number of extreme market moves is limited and their characteristics are not always the same. The economic relevance of the jumps is substantial. Each quarter, there is a 4.2% probability of a jump. Although this probability may seem small, in the longer run the likelihood of experiencing them becomes far from negligible. For instance, over a 15-

Table 4: Jump Parameters

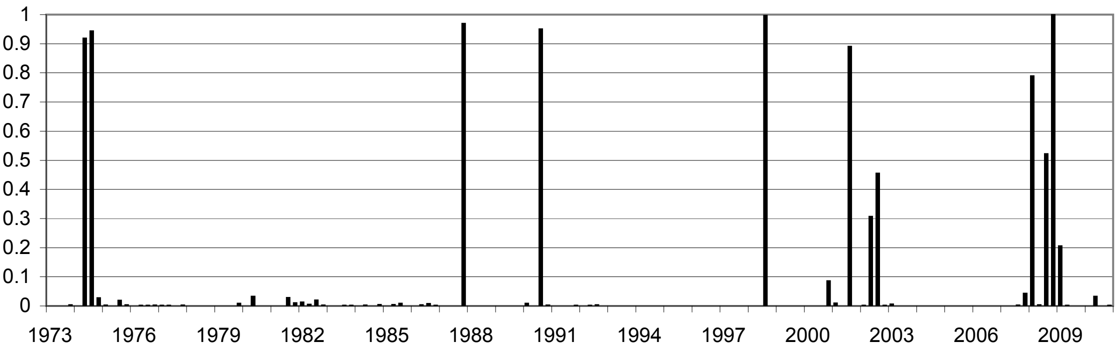
Probability ($\mathbf{p}$)	0.042					
	(0.7)					
	π_t	$y_t^{(1)}$	xs_t	dy_t	cs_t	mp_t
Jump size ($\nu^\top$)	-0.02	-0.30	-20.77	0.26	0.82	0.46
	(0.0)	(0.2)	(1.6)	(1.6)	(1.3)	(0.2)
Price of risk (ϕ)	-1.25					
	(0.5)					

Note: The t-values are in parentheses.

year horizon, the probability of not being hit by at least one jump is just 7.7%. Given their impact, jumps should not be neglected by long-run investors. A jump results, on average, in a drop in the stock market of almost 21% and an increase in credit spreads of 0.82%, and short-term interest rates decline by 30 basis points. This can be attributed to a reaction of central banks to market panic. Maturity preference also increases, thereby depressing long-term rates as well (by about 25 basis points for nominal rates and 20 basis points for real ones). This is a clear manifestation of the flight-to-safety effect. In case of sudden market panic, the demand for risk-free assets (and hedge demands by pension funds and life insurers) increases, thereby depressing long-term rates (see, e.g., Gulko (2002)).

Figure 3 shows the posterior jump probabilities. The credit crisis is the clearest example of an extreme period, with signs of jumps in the first, third, and fourth quarters of 2008 and the first quarter of 2009. There are also some signs of panic during the gradual market downturn over the period 2000 to 2003. Other clear examples are the aftermath of the first oil crisis in 1974, Black Monday in October 1987, and the Russian crisis in 1998.

Figure 3: Posterior jump probabilities.



4.3 Variance Dynamics

With respect to the variance specifications, the short-term interest rate is the main source of variation for the monetary variance. The coefficient on the short rate is slightly above one, and the equilibrium volatility of the short rate (0.63) is also somewhat higher than that of inflation (0.55). Assuming that positive correlation between stocks and bonds is indeed caused by high monetary uncertainty, we find this result to be in accordance with Yang, Zhou, and Wang (2009), who find that stock–bond correlations in the US and UK are especially high if short rates are high and (to a lesser extent) if inflation is high. Risk aversion variance is dominated by the credit spread, though large stock market moves also have a significant temporary impact:

$$
\begin{aligned}
s_{m,t} &= \underset{(2.4)}{1.475} + \pi_t + \underset{(2.5)}{1.079}\ y_t^{(1)} \\
s_{r,t} &= -\underset{(0.1)}{0.024} - \underset{(1.4)}{0.009}\ xs_t + \underset{(0.9)}{0.222}\ dy_t + cs_t
\end{aligned}
$$

Figure 4 shows the resulting volatilities. In the monetary factor, a clear downward trend is visible, due to relatively high inflation rates in the 1970s. Since 2009, monetary volatility has been historically low, since both inflation and short-term rates have been low. This low level of the monetary factor may seem at odds with the high uncertainty regarding future inflation. Indeed, the dispersion in opinion on future inflation is very high at the moment, probably because the high current fiscal deficits may induce governments to allow for higher inflation. In our model, this possibility of rising inflation can be captured by (exogenously) changing the inflation target. The monetary factor shows primarily the uncertainty regarding monetary policy rates (and inflation) in the short run. Irrespective of the future balance of power between central banks and governments, monetary policy is likely to remain loose for quite a while. The low monetary factor reflects this low uncertainty. The risk aversion factor does not show a clear trend. The recent extreme credit spreads (over seven percentage points) resulted in an all-time high of the risk aversion volatility in the last quarter of 2008.

Regarding correlations, both the jumps and the stochastic volatilities cause correlations to vary over time. Table 5 shows the impact of both elements: The top three matrices show the unconditional correlation (Panel (a)), as well as those conditional on the jump status (Panels (b) and (c)). For these correlations, the state variables are assumed to be at their equilibrium level, given an inflation target of 2%. The bottom three matrices in Table 5 show the dependence on the state variables. Panel (d) gives the resulting correlation if both the monetary and the risk aversion volatility factors approach zero. This correlation is based on the last four columns of $\mathbf{\Sigma}$, with weights based on the additional intercepts on the last four diagonal elements of $\mathbf{S}_t$, and on the jumps. If monetary volatility dominates completely, the correlation in Panel (e) results. This correlation is based on the first, third, fifth, and sixth columns of $\mathbf{\Sigma}$, depending on the weights of the monetary variance in the corresponding elements of $\mathbf{S}_t$. Similarly, risk aversion dominance in Panel (f) is based on columns 2, 4, 5, and 6 of $\mathbf{\Sigma}$. Finally, the weights shown in the bottom row of Table 5 give the equilibrium contribution of the constant,

Figure 4: Conditional volatility factors.

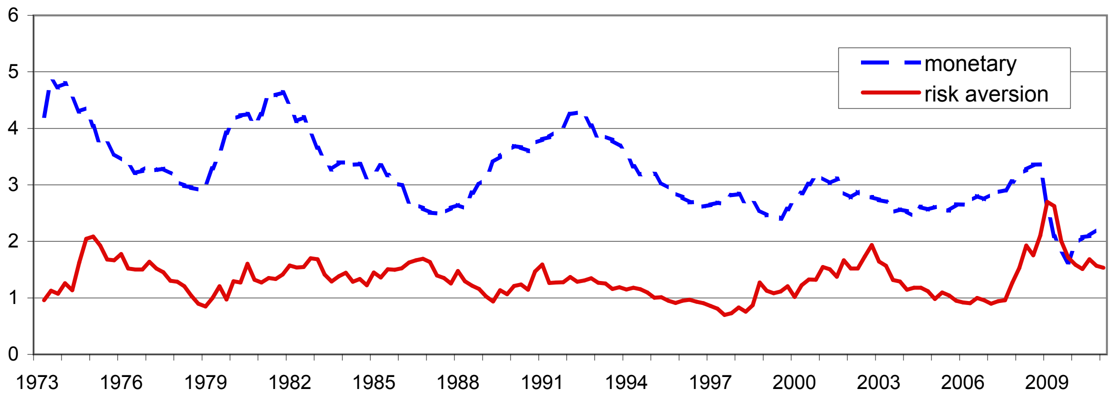


Table 5: Equilibrium and Conditional Correlations

	π_t	xs_t	cs_t	$y_t^{(40)}$	π_t	xs_t	cs_t	$y_t^{(40)}$	π_t	xs_t	cs_t	$y_t^{(40)}$
	(a) Unconditional				(b) Conditional on no jump				(c) Conditional on a jump			
π_t	1				1				1			
xs_t	-0.19	1			-0.22	1			-0.05	1		
cs_t	-0.05	-0.40	1		-0.05	-0.23	1		-0.05	-0.90	1	
$y_t^{(40)}$	0.12	0.08	-0.24	1	0.12	0.03	-0.21	1	0.12	0.49	-0.54	1
	(d) Constant volatility				(e) Monetary volatility				(f) Risk aversion volatility			
π_t	1				1				1			
xs_t	0.03	1			-0.50	1			-0.07	1		
cs_t	-0.04	-0.96	1		-0.51	0.98	1		0.02	-0.32	1	
$y_t^{(40)}$	-0.04	0.16	-0.24	1	0.64	-0.98	-0.97	1	-0.91	0.47	-0.17	1
Variance weight	3%	26%	20%	24%	83%	14%	1%	32%	14%	60%	78%	44%

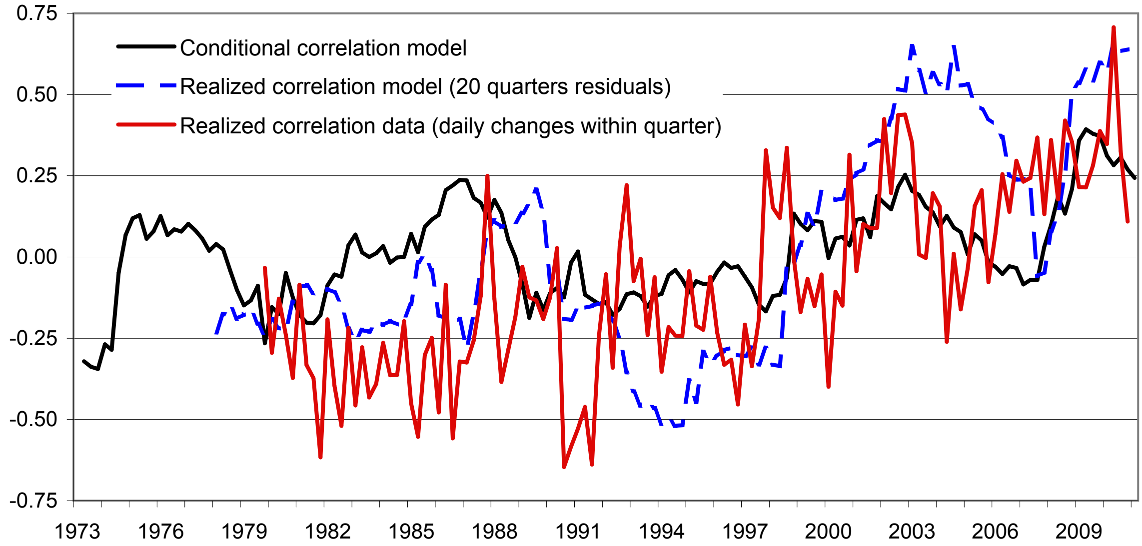
Notes: For the calculation of the correlations conditional on the jump status (Panels (b) and (c)), it is assumed that all state variables are at their equilibrium value, given an inflation target of 2%. The same assumption is used to calculate the contribution of the constant, monetary, and risk aversion volatilities in the total variance of the variables (bottom row).

monetary, and risk aversion volatility factors in the total variance of inflation, stock returns, credit spreads, and long-term interest rates.

With respect to the correlation between equities and long-term interest rates, our premise is confirmed. If monetary volatility completely dominates (Panel (e)), the correlation between the two is highly negative, whereas risk aversion dominance (Panel (f)) cause the correlation to be positive. Conditional on a jump (Panel (c)), the positive correlation is further strengthened due to the simultaneous drop in stock markets and interest rates. Most of the other conditional correlations are in accordance with economic theory as well. Monetary uncertainty causes positive correlation between inflation and interest rates and negative correlation between inflation and stock returns. Higher risk aversion volatility causes negative correlation between stocks and credit spreads, though the impact of jumps is even bigger. Regarding the relevance of the volatility factors, we think the results make perfect sense. The weights indicate that inflation uncertainty is dominated by the monetary volatility factor, stock returns and credit spreads by risk aversion volatility, and long-term interest rates by all three factors about equally. Within the constant volatility factor category, stocks and credit spread volatility are caused by the jumps, whereas jumps are relatively unimportant for inflation and interest rates.

The conditional correlations for the different volatility regimes represent the extremes. In practice, the conditional correlation will be in between these extremes, since the volatility factors will never be zero or infinite. Figure 5 shows the historical conditional as well as realized correlations between equities and long-term interest rates. Due to the declining trend in monetary volatility, we see an upward trend for the conditional correlation in our model. The same is confirmed in the realized data. Based on daily interest and equity price changes, we observe mainly negative correlations until the end of the 1990s, whereas positive correlations dominate thereafter. We find a comparable pattern in our 20-quarter model residuals. The conditional correlation is calculated based on ex ante information, and hence without prior knowledge of a jump. Conditional on the occurrence of a jump, the correlation

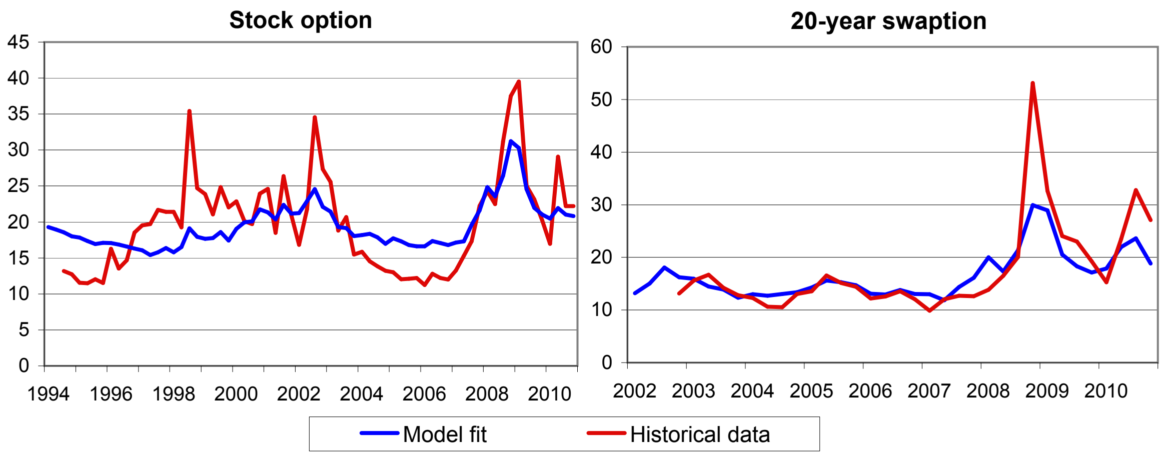
Figure 5: Correlation of stocks with 10-year interest rates.



between equities and the long-term interest rates is around 0.5 (see Table 5c). The relatively high correlation in the residuals over the recent period can therefore be justified by the high number of jumps in the most recent period.

The fact that the monetary volatility factor is low at the moment does not mean that the conditional volatility of interest rates is low as well (see Figure 6). Indeed, both implied and realized interest rate volatilities have been extremely high lately, and our model mimics this uncertainty to some extent. Although the extreme implied volatilities of both the stock and interest rate market are not reproduced, the correlation between the model-implied volatility and the market volatility is high. Consequently, in terms of risk characteristics, the model is likely to give a better representation than the traditional homoskedastic models.

Figure 6: Implied volatility of at-the-money derivatives maturing next quarter.



5 Simulation Results

In our Asset Liability Management studies, we want to evaluate the impact of investment decisions in both the short and the long run (15 to 20 years). Consequently, the simulation results should also be reliable for long forecast horizons. One necessary precondition for this is that the eigenvalues of the system must not be too close to one to prevent ever-diverging simulation paths. In our case, the biggest eigenvalue of Γ is 0.957. This implies a half-time of shocks of about four years.

Another important characteristic is that variances, nominal interest rates, and credit spreads cannot become negative. There is no automatic guarantee for this. The probability of this happening is reduced, however, by imposing the Feller conditions (see Appendix A). The Feller conditions guarantee the positivity of the volatility factors in a continuous-time model. In our discrete-time model with normal innovations, a positive probability of negative variances remains due to possible extreme outliers in the innovations. Nevertheless, a solution that always works in a continuous-time model is likely to work reasonably well in a discrete-time approximation as well (see Spreij, Veerman, and Vlaar (2011)). Positivity of interest rates and credit spreads is not guaranteed by the Feller conditions (even in a continuous-time model).[15] This latter problem is smaller in a heteroskedastic model than in a homoskedastic one, since already low values of the state variables result in lower volatilities. The Feller conditions further reduce this probability, since they avoid future negative volatility factors, and thereby also restrict the possibility of negative state variables (e.g., negative short-term interest rates are only possible if inflation is, at the same time, relatively high).

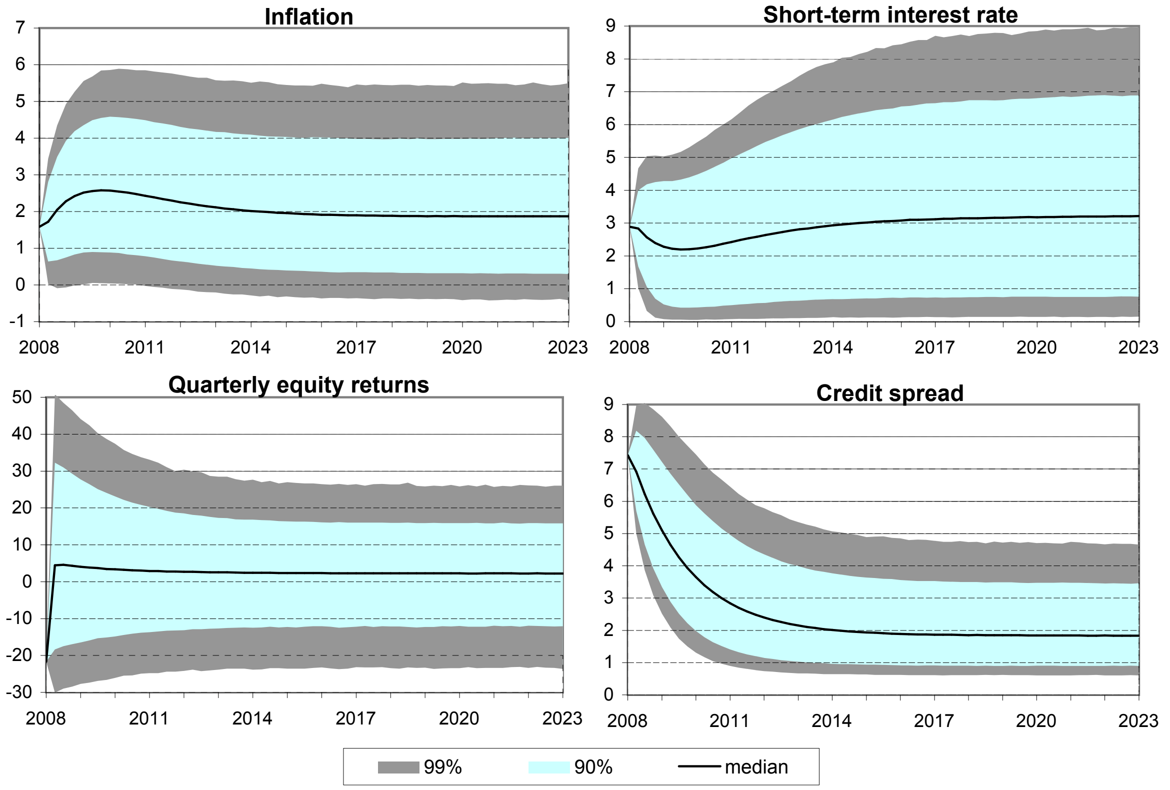
The way we deal with inadmissible scenarios is to draw a new vector of normal disturbances ζ_{it} for every scenario i that delivers inadmissible negative values at time t. The jump status is not adjusted. Moreover, we use antithetic variates (see, e.g., Glasserman (2004)). This means that every innovation vector ζ_{it} and jump indicator J_{it} is used for two scenarios, one using ζ_{it} and J_{it} and the other using $-\zeta_{it}$ and J_{it}. This way, the fact that we redraw certain innovations does not affect the mean of the simulated series. If in the first instance there are n scenarios with inadmissible values at time t, we calculate at most $2n$ new scenarios for this period. These $2n$ scenarios are rechecked, and new innovations are drawn for the inadmissible scenarios (and their mirror images) up to the point where all scenarios are admissible. Subsequently the scenarios for time $t + 1$ are generated. In practice the non-negativity problem is not severe. Usually at most five rounds are necessary to get rid of undesirable negative values.

5.1 The State Variables

Figure 7 presents the distribution of possible outcomes for 60 subsequent quarters for inflation, short-term interest rates, equity returns, and the credit spread, using the extreme situation of December 2008 as the starting point. Specifically, it shows the median and the 90% as well as 99% confidence intervals. For inflation, short-term interest rates, and the credit spread, we clearly observe an asymmetric distribution where very high values are more likely than very low values. Equity behavior is more symmetric. Due to the severe consequences of jumps on equity returns, negative outliers are even

[15] Of course, positivity is guaranteed for an interest rate that coincides with a volatility factor.

Figure 7: Simulation paths for state variables starting from 2008:IV.



slightly more likely than positive ones. Directly after the crash, we see high uncertainty for equity, but expected returns are somewhat higher. Credit spreads higher than 7%, as observed during the credit crisis, remain very unlikely, despite the asymmetry.

5.2 Equilibrium Term Structures

Figure 8 displays some percentiles for the nominal and real term structures at the end of a 50-year simulation horizon. The general pattern seems well in accordance with the stylized facts. As seen in

Figure 8: Long-term distribution of the nominal and real term structure.

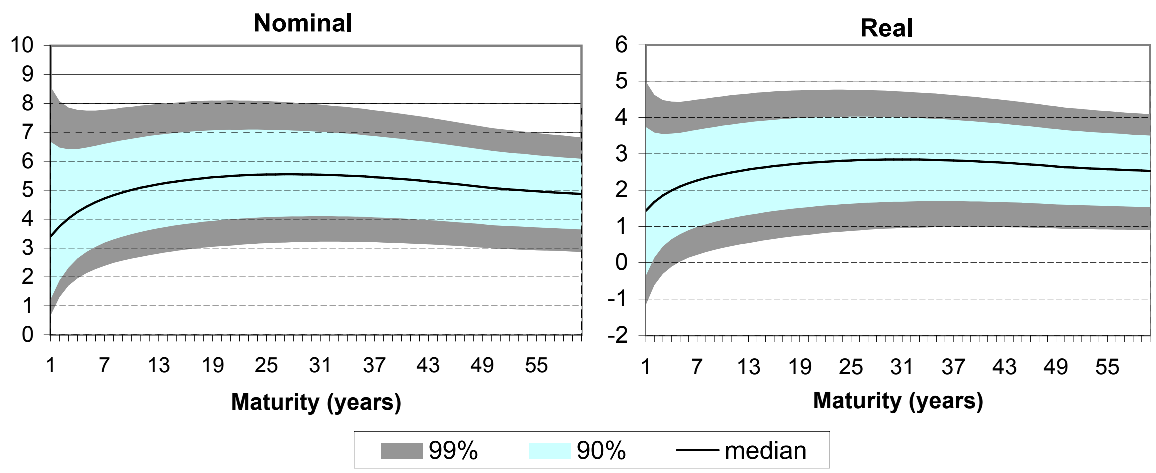
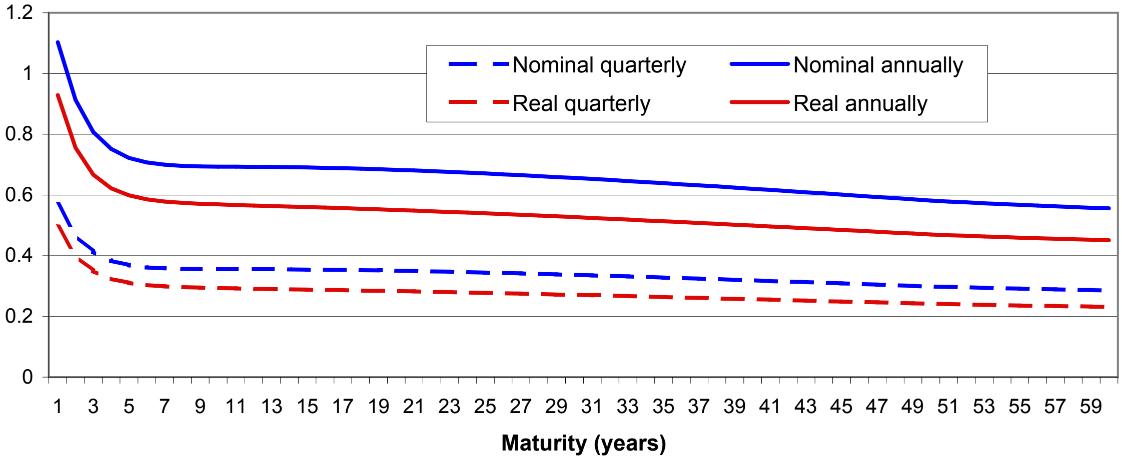


Figure 9: Equilibrium volatility of yield changes.



the data, the nominal term structure is upward sloping, up to maturities of 20 to 30 years, after which it is slightly downward sloping. The dispersion of possible outcomes is quite large, even for maturities of 60 years. For short maturities, the asymmetry is apparent, with low interest rates bounded by zero but allowing for very high rates if inflation picks up. The zero lower bound is never a big problem in the simulations. For maturities of, say, 10 years or more, the asymmetry is only visible in the extremes. This limited asymmetry is due to the assumed fixed (exogenous) inflation target.

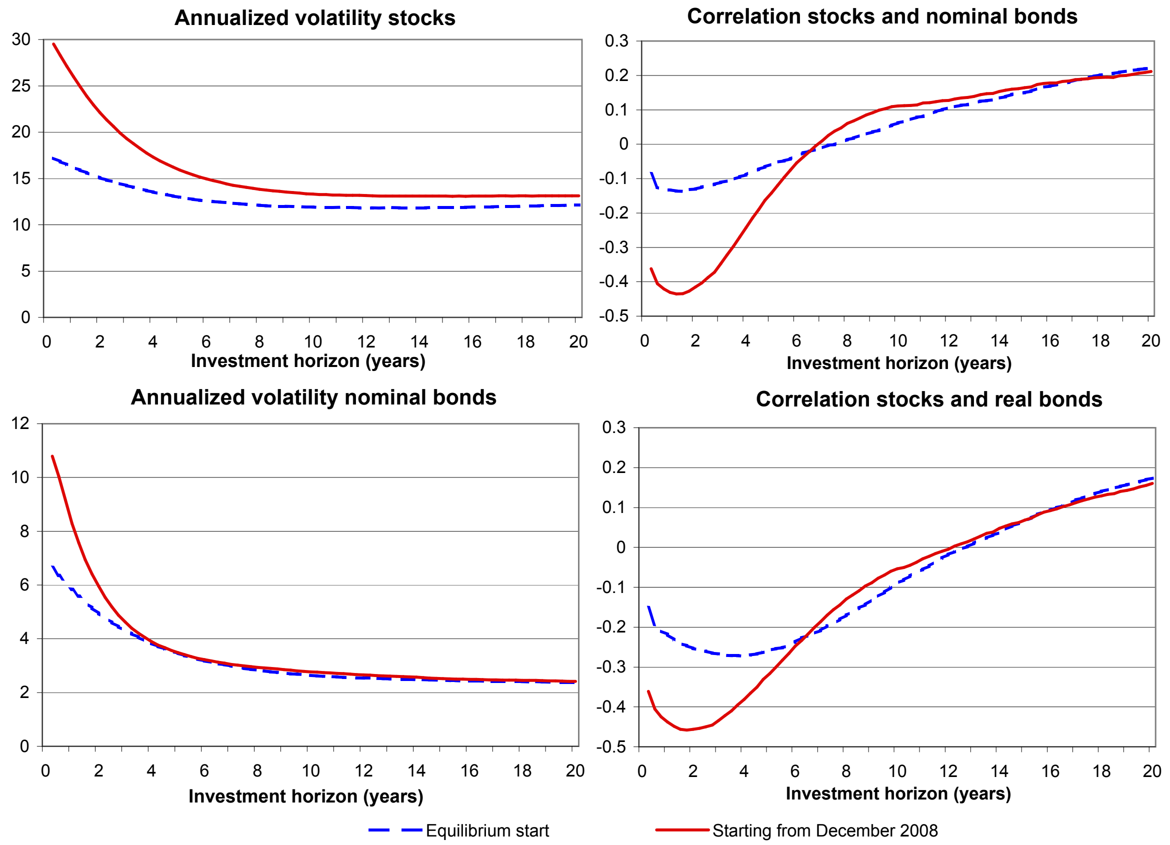
The real term structure shows slightly less dispersion than the nominal one. However, real interest rates are far from constant themselves. This is also confirmed in Figure 9, which shows the volatility of quarterly and annual yield changes for both the nominal and the real term structure. Volatilities of real yields are somewhat lower than for nominal ones, but still far from negligible. Over the maturities, we see the volatility is gradually decreasing, but both nominal and real yield volatilities do not go to zero, even for very long maturities. This pattern is also apparent in the (limited) historical data.

5.3 Term Structure of the Risk–Return Trade-Off

An important feature for a long-term investor is the so-called term structure of the risk–return trade-off (see Campbell and Viceira (2005)). Since risks and returns of different assets change over time in predictable ways, an optimal asset portfolio can also change, depending on the investment horizon. An important element in this respect is whether assets are mean reverting or not. Assets that are strongly mean reverting have a higher expected future return if the current return is low. In the long run, part of the current negative results are expected to be compensated, making the risk in the long run smaller. Figure 10 shows the model-implied volatility of nominal returns on equity and nominal bonds over investment horizons of up to 20 years, as well as the correlation between the two. Two different starting points are shown. The dotted lines show the annualized volatility if all state variables are at their equilibrium value (assuming an inflation target of 2%) at the start of the simulation. The straight lines take the extreme circumstances of December 2008 as a starting point.

The stock market shows signs of mean reversion. In equilibrium, the annualized volatility reduces from 17% at short horizons to 12% for holding periods of 15 years or more. The mean reversion is due to both the dividend yield and, to a lesser extent, the credit spread. Both variables correlate negatively

Figure 10: The term structure of the risk–return trade-off.



with stock returns, whereas higher dividend yields or credit spreads induce higher expected stock returns. The amount of mean reversion is somewhat smaller than in Campbell and Viceira (2005) and Hoevenaars, Molenaar, Schotman, and Steenkamp (2008). In these models the long horizon volatility is only about half of the short-term volatility. Starting from December 2008, volatility reduces from 30% to 13% over the forecast horizon. The main reason for this reduction is not time diversification, however, but the gradual decline in volatility to equilibrium levels. Nominal bonds also show mean reversion, since initial losses after an interest rate increase are compensated by higher yields afterward. Despite the low monetary volatility in December 2008, bond volatility was also higher than average due to the high risk aversion volatility factor.

Figure 10 clearly shows the impact of the starting position on the risks of a pension fund. Not only are the volatilities of both equities and bonds (and therefore pension liabilities) significantly higher in a period of stress, the correlation between the two also becomes much more negative. The combination of higher volatilities and negative correlation increases the mismatch risks of equities versus bonds and makes the solvency ratio even more volatile. The effects of the higher mismatch risk could be witnessed during the credit crisis as the solvency ratios of Dutch pension funds decreased to unprecedented numbers. Only after four to six years does the effect of the starting position become negligible. Real bonds, which can be viewed as a proxy for real pension liabilities, are always negatively correlated with equities up to 12 years. In a crisis, this correlation becomes increasingly negative as well.

6 Conclusions

Traditional risk models have hardly been able to foresee either the start of the 2008 credit crisis or its impact on volatilities and correlations. To fill this gap in the existing models, this paper adds two new features. First, uncertainty is driven not only by normally distributed shocks, but also by stochastic jumps. These jumps represent a sudden unexpected loss in confidence of the market, leading to lower interest rates, higher credit spreads, and lower stock markets. Since the start of a crisis is, almost by definition, unpredictable and crises can occur in any economic environment, we assume the probability of a jump to be constant. Second, the model incorporates time-varying second moments to accommodate the specific characteristics associated with a crisis. The two main building blocks of the time-varying volatility in the model are inflation-related monetary uncertainty, on the one hand, and changes in risk aversion, on the other, measured primarily by changes in the credit spread. The importance of these two sources of risk is assumed to depend on the level of the driving variables in the system. Since the correlation structure differs between the monetary and real uncertainty environments, changing correlations is a direct result of the changing importance over time of the two sources of risk.

The model includes an essentially affine term structure model for both nominal and real interest rates. Since the model is arbitrage free, we can generate scenarios under both the physical and risk-neutral measures. Therefore, the model can be used to value various assets consistently. To enhance the modeling of volatilities, we include option prices and swaptions in the estimation procedure. A nice feature of the jump process is that one can account for the so-called volatility skew, where implied volatilities decrease monotonically with increasing strikes (see Andersen and Andreasen (2000)). Closed-form solutions can be obtained for options maturing the next period. For single-period swaptions, we can obtain analytical expressions, provided the stream of cash flows of the swap is approximated by a single cash flow.

The results of the model are promising. As a result of a jump, the stock market drops considerably, the yield curves show a uniform decrease over the whole curve, while at the same time the credit spread increases, much in line with the events in 2008. Moreover, the conditional covariance captures changes in volatility and correlations quite well. Monetary uncertainty induces a negative correlation between equity and interest rates as higher monetary policy rates increase long-term rates but depress stock returns. Increasing risk aversion volatility, which arises during times of crisis, results in a strongly positive correlation between stocks and interest rates due to flight-to-safety effects. At the same time, the model produces a reasonable fit for both the nominal and real term structure for maturities of up to even 90 years, and for stock and interest rate market derivatives. The simulation results of the model have plausible characteristics. The dispersion of possible outcomes seems realistic, with inflation, credit spreads, and short-term interest rates showing asymmetric behavior. The volatility of yield changes is also in accordance with the data: The volatility slightly decreases with maturity, without going to zero, and real volatilities are somewhat lower than nominal ones. Finally, the implied term structure of the risk–return trade-off is shown to vary considerably over time, thereby demonstrating the relevance of these modifications for risk analysis. The time-varying risk is due to both changing volatilities and correlations.

Appendix A. Feller Conditions

To reduce the probability of negative variances, interest rates, or credit spreads, we impose the Feller conditions. In a continuous-time model, these conditions guarantee the non-negativity of the variances (Duffie and Kan 1996). In the model without jumps, these conditions imply the following:

$$\beta_{[i]}\mathbf{\Sigma}^{[j]} = 0 \quad \text{or} \quad s_i = s_j + z \qquad \text{for some } z \geq 0 \tag{A1}$$

$$\beta_{[i]}\left(\mathbf{c}_t + (\mathbf{\Gamma} - \mathbf{I}_6)\mathbf{x}\right) > \frac{1}{2}\beta_{[i]}\mathbf{\Sigma}\mathbf{\Sigma}^\top\beta_{[i]}^\top \qquad \text{for all } \{\mathbf{x} \in \mathbb{R}^6 : s_i = 0, s_j \geq 0 \; \forall \; j \neq i\} \tag{A2}$$

where s_i is the ith diagonal element of $\mathbf{S}_t$, $\beta_{[i]}$ represents ith row of β, and $\mathbf{\Sigma}^{[j]}$ is the jth column of $\mathbf{\Sigma}$.[16] The basic idea behind these conditions is that whenever a variance is zero, it will always return to the positive area. The first condition implies that when a volatility factor is zero, it cannot be shocked. The second condition ensures that the deterministic process is such that the zero variance bound is escaped in the positive direction, and with enough speed. In our model, only the first two rows of β are relevant for the Feller conditions, since only s_1 and s_2 can approach zero. The Feller conditions are usually imposed by estimating a model in canonical form. Since our volatilities are unknown linear functions of the state variables, this is not possible in our case. Instead, we impose the conditions by restricting the $\mathbf{\Sigma}$ and $\mathbf{\Gamma}$ matrices.

The first Feller condition can easily be imposed:

$$\mathbf{\Sigma}_{[1]}^{[23456]} = -\beta_{my}\mathbf{\Sigma}_{[2]}^{[23456]} \tag{A3a}$$

$$\mathbf{\Sigma}_{[5]}^{[13456]} = \beta_{rx}\mathbf{\Sigma}_{[3]}^{[13456]} - \beta_{rd}\mathbf{\Sigma}_{[4]}^{[13456]} \tag{A3b}$$

The second condition is more complicated. For the monetary factor (i=1), it implies

$$\beta_{[i]} = \begin{bmatrix} 1 & \beta_{my} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A4a}$$

$$x_1 = -\alpha_m - \beta_{my}x_2 \tag{A4b}$$

$$x_5 = -\alpha_r + \beta_{rx}x_3 - \beta_{rd}x_4 + k^2 \tag{A4c}$$

$$x_2, x_3, x_4, x_6, k \in \mathbb{R}^5 \tag{A4d}$$

[16] Duffie and Kan (1996) assume that $z = 0$, but it is not hard to see that we can take $z > 0$ as well. Since the authors' model is expressed in "error-correction" form, their feedback parameter a equals $(\mathbf{\Gamma} - \mathbf{I}_6)$ in our case. Alternatively, we can rewrite (A2) as $\alpha_{[i]} + \beta_{[i]}\left(\mathbf{c}_t + \mathbf{\Gamma}\mathbf{x}\right) > \frac{1}{2}\beta_{[i]}\mathbf{\Sigma}\mathbf{\Sigma}^\top\beta_{[i]}^\top$, since $\beta_{[i]}\mathbf{x} = -\alpha_{[i]}$ if $s_i = 0$.

Combining (A4) and (A2) and solving for all possible values of x and k results in

$$\mathbf{\Gamma}_{(1,2)} = -\beta_{my}\mathbf{\Gamma}_{(2,2)} + \beta_{my}\beta_{[1]}\mathbf{\Gamma}^{[1]} \tag{A5a}$$
$$\mathbf{\Gamma}_{(1,3)} = -\beta_{my}\mathbf{\Gamma}_{(2,3)} - \beta_{rx}\beta_{[1]}\mathbf{\Gamma}^{[5]} \tag{A5b}$$
$$\mathbf{\Gamma}_{(1,4)} = -\beta_{my}\mathbf{\Gamma}_{(2,4)} + \beta_{rd}\beta_{[1]}\mathbf{\Gamma}^{[5]} \tag{A5c}$$
$$\mathbf{\Gamma}_{(1,5)} \geq -\beta_{my}\mathbf{\Gamma}_{(2,5)} \tag{A5d}$$
$$\mathbf{\Gamma}_{(1,6)} = -\beta_{my}\mathbf{\Gamma}_{(2,6)} \tag{A5e}$$
$$\beta_{[1]}\mathbf{c}_t > \frac{1}{2}(\beta_{[1]}\mathbf{\Sigma})^{\odot 2}\iota_6 + \beta_{[1]}\mathbf{\Gamma}^{[15]}\alpha_{[12]} - \alpha_m \tag{A5f}$$

where ι_6 represents a vector of ones. Since these conditions should be fulfilled for all possible values of $\mathbf{c}_t$, we check inequality (A5f) for both the maximum historical value for $\bar{\pi}_t$ (5%) and its minimum value (2%), both conditional on a jump and in the absence of jumps.

Similarly, for the risk aversion factor (i=2), we obtain

$$\beta_{[i]} = \begin{bmatrix} 0 & 0 & -\beta_{rx} & \beta_{rd} & 1 & 0 \end{bmatrix} \tag{A6a}$$
$$x_1 = -\alpha_m - \beta_{my}x_2 + k^2 \tag{A6b}$$
$$x_5 = -\alpha_r + \beta_{rx}x_3 - \beta_{rd}x_4 \tag{A6c}$$
$$x_2, x_3, x_4, x_6, k \in \mathbb{R}^5 \tag{A6d}$$

and

$$\mathbf{\Gamma}_{(5,1)} \geq \beta_{rx}\mathbf{\Gamma}_{(3,1)} - \beta_{rd}\mathbf{\Gamma}_{(4,1)} \tag{A7a}$$
$$\mathbf{\Gamma}_{(5,2)} = \beta_{rx}\mathbf{\Gamma}_{(3,2)} - \beta_{rd}\mathbf{\Gamma}_{(4,2)} + \beta_{my}\beta_{[2]}\mathbf{\Gamma}^{[1]} \tag{A7b}$$
$$\mathbf{\Gamma}_{(5,3)} = \beta_{rx}\mathbf{\Gamma}_{(3,3)} - \beta_{rd}\mathbf{\Gamma}_{(4,3)} - \beta_{rx}\beta_{[2]}\mathbf{\Gamma}^{[5]} \tag{A7c}$$
$$\mathbf{\Gamma}_{(5,4)} = \beta_{rx}\mathbf{\Gamma}_{(3,4)} - \beta_{rd}\mathbf{\Gamma}_{(4,4)} + \beta_{rd}\beta_{[2]}\mathbf{\Gamma}^{[5]} \tag{A7d}$$
$$\mathbf{\Gamma}_{(5,6)} = \beta_{rx}\mathbf{\Gamma}_{(3,6)} - \beta_{rd}\mathbf{\Gamma}_{(4,6)} \tag{A7e}$$
$$\beta_{[2]}\mathbf{c}_t > \frac{1}{2}(\beta_{[2]}\mathbf{\Sigma})^{\odot 2}\iota_6 + \beta_{[2]}\mathbf{\Gamma}^{[15]}\alpha_{[12]} - \alpha_r \tag{A7f}$$

where the last inequality is again checked for both $\bar{\pi}_t = 2\%$ and $\bar{\pi}_t = 5\%$, both with and without jumps.

Appendix B. Data Sources

- Inflation: From 1999 on, the Harmonized Index of Consumer Prices for the euro area from the European Central Bank data website (http://sdw.ecb.europa.eu) is used. Before then, German (Western German until 1990) consumer price index figures published by the International Financial Statistics of the International Monetary Fund are included.

- Euribor: Three-month money market rates are taken from the Bundesbank (www.bundesbank.de). For the period 1973:I to 1990:II, end-of-quarter money market rates reported by Frankfurt banks are taken, whereas thereafter three-month Frankfurt Interbank Offered Rates are included.

- Stock returns: MSCI index from FactSet. Returns are in euros (Deutschmark before 1999) and hedged for US dollar exposure.

- Dividend yield: MSCI index from Datastream, code MSWRLD$(MSDY). To guarantee positive dividend yields in simulation, we include $dy = \ln(\text{dividend yield}/(10 - \text{dividend yield}))$, thereby restricting the dividend yield between 0 and 10%.

- Credit spread: Barclays US aggregate, redemption yield on Baa credits minus US Treasury. Datastream codes LHUABAA(RY) and LHUSTRY(RY).

- Long nominal yields: From 1987:IV on, zero-coupon rates are constructed from swap rates published by De Nederlandsche Bank (www.dnb.nl). For the period 1973:I to 1987:III, zero-coupon yields with maturities of one to 15 years (from the Bundesbank website) based on government bonds were used as well (15-year rates start in June 1986). No adjustments were made to correct for possible differences in the credit risk of swaps, on the one hand, and German bonds, on the other. The biggest difference in yield between the two term structures (for the two-year yield) in 1987:IV was only 12 basis points.

- Real yields: Inflation swap data provided by Bloomberg, starting from 2003:IV, combined with zero-coupon nominal rates from De Nederlandsche Bank.

- Option volatilities: Bloomberg. The series represent the three-month implied volatility on the Standard & Poor's 500 Index.

We use swap rates for both nominal and real yields. Swap yields are more relevant for pension funds than treasuries, because their liabilities are calculated against the swap rate curve and asset duration policies are also executed with swaps. Moreover, for longer maturities, treasuries are hardly available, and the treasury market in Europe is less liquid than the swap market. The swap rates are also more in line with the available short rate (three-month Euribor). Since we want to link the term structures to monetary policy, we use the inflation rate in the Eurozone. Before 1999, German inflation and interest rates are used. Since the swap market did not exist before the end of 1987, we use German treasury rates before this date.

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