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Kamil Korhan Nazliben

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To my mother

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Chapter 1

Introduction

The dissertation consists of three chapters that represent separate papers in the area of asset pricing. The first chapter studies investors optimal asset allocation problem in which mean reversion in stock prices is captured by explicitly modeling transitory and permanent shocks. The second chapter focuses on option pricing with stochastic dividend yield. In this paper, we present an option formula which does not depend on the dividend yield risk premium. In the final chapter, we work on commodity derivative pricing under the existence of stochastic convenience yield. In this paper, we discuss a Gaussian complete market model of commodity prices in which the stochastic convenience yield is assumed to be an affine function of a weighted average of past commodity price changes. All chapters are joint works with Juan Carlos Rodriguez.

In chapter one, we study portfolio selection problem of an investor when stock price is decomposed into temporary and permanent components. In our setting, the permanent component of the stock price is a random walk with drift and the transitory component is an autoregressive process of order one. The portfolio model is formulated in continuous time framework. We investigate two cases: complete information, in which investors are able to distinguish between shocks, and incomplete information, in which investors are not. Accordingly, the model generates a small hedging demand that becomes flat at relatively short investment horizons. Interestingly, the hedging demand is smallest under incomplete information.

In this paper, we show that standard models forecast a large allocation to stocks since they implicitly assume that transitory shocks dominate the stock price dynamics. Moreover, we discuss that a stock price model with dominant permanent shocks will produce asset allocations more in line with empirical observations. We find that our model generates a smaller and less horizon-dependent allocation to stocks under both complete information (investors can distinguish transitory from permanent shocks) and incomplete information (investors cannot). We estimate the model using the Kalman filter, avoiding in this way the use of proxies, and find that it captures the time variation in expected returns, even though the permanent component dominates the dynamics of the stock price. We calibrate the model to stock price data and show that it generates asset allocations that are smaller and less dependent on the investment horizon.

In chapter 2, we study option pricing when dividend yield is stochastic. We presented a simple framework that renders option formulas not depending on the dividend yield risk premium. These formulas can be applied to derivatives written on an index in complete markets, and can be extended to incomplete markets. We assume that shocks orthogonal to the returns on the index are not priced. Given that indexes are broad portfolios of stocks, this assumption is equivalent to the CAPM assertion that only systematic risk (covariance with the returns on the index) is priced. In this case it is possible to obtain valid pricing formulas in complete and in incomplete markets for which no risk premia has to be estimated.

We postulate a regression model in which changes in dividend yield are linearly related to the dividend yield level and to the index return, the regression error being pure dividend yield risk. The model restricts the mean of the dividend yield to be a function of the index expected return, and we exploit this fact, at the time of risk-neutralizing the model, to extract the index risk premium from the mean dividend yield. We show that, when the market is complete, this is sufficient to obtain option prices in which no risk premium has to be estimated. When the market is incomplete we still need to deal with the risk premium on pure dividend yield risk.

We showed that neglecting the randomness in the dividend yield leads to signicant mispricing stemming from two main sources. These are mispecied dividend yield and mispecied volatility. Consequently, we show that the standard Black-Scholes model underprices options at all maturities. It is observed that the underpricing is economically signicant, especially for out of the money options. Furthermore, our results have also consequences for hedging. We computed the greeks of European calls and puts from our model and show the they are different from the ones implied by the Black-Scholes model with constant dividend yield. In particular, the delta of a call is larger in our model, and it can even be larger than one. The main reason is that the option seller must hedge not only index price but also dividend yield risk, which is mostly explained by index price risk.

In chapter 3, we study commodity derivative pricing under the existence of stochastic convenience yield. In this paper, we present a complete market model of commodity prices that exhibits price nonstationarity and mean reversion under the risk neutral measure, and, as a consequence, it is able to fit a slowly decaying term structure of futures return volatilities. The model has strong mean reversion and geometric Brownian motion as special cases, and renders formulas for the prices of futures contracts and European options for which no risk premium must be estimated. Our model is parsimonious and provides a useful benchmark to value complex contracts for which no closed form solutions are known. From this point of view, it can be seen as a good alternative to widely used one-factor models.

In our model, in particular, the stochastic convenience yield is assumed to be an affine function of a weighted average of past commodity price changes. This assumption captures the dependence of the convenience yield on the state of the market, and generalizes the Ornstein-Uhlenbeck (O-U) process, which can be interpreted as one in which the convenience yield is a linear function of the spot price. We provide an empirical assessment of the model on a sample of oil futures prices. It is found that the model outperforms the O-U process both in terms of model fit and in terms of pricing errors

Chapter 2

Permanent Shocks, Signal Extraction, and Portfolio Selection

Abstract
Recent empirical research in portfolio selection shows that investor's allocation to risky assets is low at young ages and that it does not exhibit a clear pattern of change as investors grow old. We show that standard models in the current literature predict a large allocation to stocks because they implicitly assume that transitory shocks dominate the stock price dynamics, and study a portfolio selection model in which the stock price is driven by a transitory and a dominant permanent component. The model captures the time variation in expected returns and generates asset allocations that are small relative to the ones obtained in the current literature, and less dependent on the investor's horizon. We investigate our model under complete and incomplete information, and find that under incomplete information our results are stronger.

2.1 Introduction

Recent empirical research in portfolio selection shows that investor's allocation to risky assets is low at young ages and that it does not exhibit a clear pattern of change as investors grow old: it may increase or exhibit a hump-shaped pattern, depending on the study [1]. These findings contradict the theoretical results in the academic literature [2]. When calibrated to historical values of the equity premium and stock market return volatility, standard academic models predict that reasonably risk averse young investors must allocate more than 100% of their wealth to risky assets, and that this allocation must decrease as they grow old (for a survey on this literature, see Campbell and Viceira (1999) and Brandt (1999)).

Investors form their portfolios by investing in two "funds". The first fund is the tangency portfolio, aimed to provide optimal diversification. The second fund is the hedging portfolio, aimed to hedge adverse movements in the investment opportunity set (see Ingersoll (1987)). The hedging portfolio, whose purpose is to minimize consumption volatility, explains the ultimate size and shape of the investor's allocation to the risky asset. It is stylized fact that stock prices exhibit some degree of mean reversion, and this leads to a positive (long the stock) hedging portfolio that increases with the investment horizon. Positive, however, does not necessarily mean large.

In this paper we show that standard models predict a large allocation to stocks because they implicitly assume that transitory shocks dominate the stock price dynamics. Next, we argue that a stock price model with dominant permanent shocks will generate asset allocations more in line with empirical results. We set up such a model, take it to data, and find that it indeed generates a smaller and less horizon-dependent allocation to stocks under both complete information (investors can distinguish transitory from permanent shocks) and incomplete information (investors cannot).

Standard models[3] capture mean reversion through a stochastic expected return whose changes are negatively correlated to realized stocks returns[4]. Because the expected return is unobservable, the models must be calibrated to the parameters of a proxy -typically, the dividend yield. For instance, Wachter (2002) formulates the optimal consumption and asset allocation problem in which the time varying expected return is proxied by the dividend yield. In this paper we show, however, that to every model with a time-varying expected return there is an associated transitory-permanent component model with correlated components, so there is no clear way out from transitory shocks when describing mean reversion. To complicate matters, the properties of the proxy used to characterize the unobservable expected return -typically, the dividend yield- may end up inflating the importance of

[1] Bertaut and Haliassos (1995) states that majority of household investors do not have stocks. On the other hand, Ameriks and Zeldes (2000) documents several empirical findings of hump-shaped investment pattern. Furthermore, Heaton and Lucas(2000) indicates that investing in stocks becomes less important for middle-aged households who mainly prefers private businesses activities.

[2] Campbell and Viceira (2002) presents a comprehensive survey on life-cycle portfolio choice.

[3] For example Wachter (2002), Kim and Omberg (1996)

[4] The finance literature widely discusses the risk factors affecting the dynamics of the expected return. The sources of risk can be linked to business cycles and general macroeconomic environment (See for example Fama and French (1989, 1993), for the further discussion)

transitory shocks. Moreover, we show that standard models predict a large hedging portfolio because they implicitly assume a large transitory component in stock prices. We find that in Wachter (2002), for example, 84% of the stock price variation is explained by the transitory component.

In this paper we propose a model to capture mean reversion in stock prices by explicitly modeling transitory and permanent shocks[5]. That is, we implement a transitory-permanent component model in which the transitory component is stationary and the permanent component is a random walk. In this setting, the permanent component reflects the fundamental stock price level such that any shock occurred to the permanent component shifts the stock price level to another equilibrium price level. The transitory component captures cyclical price variations in stock return. In a transitory-permanent component model, the transitory component explains stock return mean reversion. Our paper is the first to use a transitory-permanent framework in the asset allocation literature.

Next, we explore the asset allocation consequences of assuming that the stock price is explicitly driven by transitory and permanent shocks. The permanent component of the stock price is a random walk with drift and the transitory component is an autorregressive process of order one. We estimate the model using the Kalman filter, avoiding in this way the use of proxies, and find that it captures the time variation in expected returns, even though the permanent component dominated the dynamics of the stock price. We calibrate the model to stock price data and show that it generates asset allocations that are smaller and less dependent on the investment horizon. We investigate two cases: complete information, in which investors are able to distinguish between shocks, and incomplete information, in which investors are not. The model generates a small hedging demand that becomes flat at relatively short investment horizons. Interestingly, the hedging demand is smallest under incomplete information.

Summers (1986) was the first to use the transitory-permanent component model to describe mean reversion in stock prices (see also Poterba and Summers (1988) and Fama and French (1988)). Cochrane (1994) finds that even though permanent shocks dominate the dynamics of stock prices, there is still a substantial transitory component. More recently, Gonzalo (2008) reports that the transitory component is sizable but much smaller than Cochrane's estimates. These results suggest that a model with transitory and dominant permanent shocks provides a plausible description of stock prices. Our own empirical results (see Section 6) add evidence in support of the model.

Filtered expected return has been discussed by the several authors in the asset pricing literature. Conrad and Kaul (1988) and Khil and Lee (2002) estimated expected returns out of realized return data with the Kalman filter. They focus on the time series properties of the filtered expected return. More recently, Binsbergen and Koijen (2010) (see also Rytchkov, 2012) exploit present value relations to estimate simultaneously the expected returns and the expected dividend growth on an index. As these two variables are unobservable to the econometrician, they filter them out from observable data using a state space framework and the Kalman filter. Binsbergen and Koijen (2010) take the dividend yield and dividend growth as observables; Rytchkov (2012), the realized return and dividend growth.

[5]See Summers (1986), Poterba and Summers (1988) and Fama and French (1988)

They both find that expected returns and expected dividend growth are time varying and stochastic.

We study the asset allocation problem under two hypotheses: complete information (investor can distinguish transitory from permanent shocks) and incomplete information (investors cannot). Our discussion of incomplete information is based on Dothan and Feldman (1986), Feldman (1986), and Gennotte (1986). These authors introduced in the finance literature the concept of a partially observable economy and the tools of non-linear filtering. Gennotte (1986) studies portfolio selection and shows that uncertainty about expected returns reduces the position that a risk averse investor takes in risky assets. Feldman (1986) investigates the term structure of interest rates and finds that incomplete information gives rise to richer term structure curves. Dothan and Feldman (1986) point out that estimation risk does not necessarily mean higher volatility of the spot rate relative to an economy with complete information. In contrast, low volatility of the interest rate might be related to low learning ability about changes in the investment opportunity set. We contribute to this literature by studying asset allocation when there is a signal extraction problem in which the investor cannot distinguish transitory from permanent shocks to the stock price.

Our results can be summarized as follows: we estimate the two-component model using the Kalman filter and find that both transitory and permanent shocks are important for the stock price dynamics, but the permanent component dominates: 68% of the total stock price variation is explained by the permanent component. The transitory component is less persistent than what is implied by the dividend yield as a proxy for expected returns, with a half life of 1.07, much lower than, for example, Wachter's model half life of 3.07. These two results lead to a hedging demand that is small and less dependent on the investment horizon than the hedging demand obtained in the extant literature. For example, an investor responding to our model, with a 10-year investment horizon, a risk aversion level of 10, and complete information, allocates 27.54% of her wealth to the stock with a positive hedging demand of 9.23%. The same investor, responding to the standard model, and with Wachter's parameters, would allocate to the hedging portfolio 60% of her wealth. Also, in our model the hedging demand becomes flat at relatively short investment horizons. Risk averse investors with a risk aversion level of 10 and with investment horizons longer than 10 years have essentially the same hedging demands. With incomplete information the hedging demand is smaller and becomes flat at an even shorter investment horizon.

We make the following contributions to the asset allocation literature. First, we show that to every model with a time-varying expected return there is an associated transitory-permanent component model with correlated components. Second, we are the first to state and solve the optimal portfolio selection problem with transitory and permanent shocks under complete and incomplete information. Third, we show that when the model is calibrated to data, it generates a low less investment horizon dependent hedging demand, and that this result is stronger with incomplete information. The direct relation between incomplete information and lower hedging demand is already discussed by Gennotte(1986) (see also Xia(2001), Veronesi(2000)), but we provide a new, and perhaps surprising, rationale for it: under incomplete information, the Bayesian updating rule makes the transitory component smaller, and so shrinks the hedging demand.

The paper is organized as follows. Section-2 explains the standard model and transitory shocks. Section-3 documents our basic permanent-temporary component model. In the next section, Section-4, there are two subsections. The first subsection investigates investor's optimal portfolio problem with complete information, and the second subsection is about the optimal investment problem with incomplete information. In section-5 and section-6, we present parameter estimations and empirical results respectively. The asset allocation problem is discussed in section 7, and the paper concludes at the final section. The technical details are documented in the appendix.

2.2 The standard model and transitory shocks

Two seminal papers investigate the portfolio problem with stochastic expected return: Kim and Omberg(1996) and Wachter (2002). Both papers model time varying risk premium with Ornstein-Uhlenbeck process in continuous time. Kim and Omberg (1996) obtain the optimal stock allocation for investor who aims to maximize only terminal wealth, while Wachter (2002) extends the optimal portfolio and consumption problem to an investor with utility over consumption. Both papers find that the optimal risky asset weight increases with investors investment horizon due to the hedging demand induced by the stochastic expected return.

In this section we present the standard stock price-stochastic expected return model as it is described in Kim and Omberg (1996) and Wachter (2002), and show that it can be expressed as a transitory-permanent component model with correlated components. We provide conditions under which the standard model is driven only by transitory shocks and show that under Wachter's parametrization, transitory shocks dominate the stock price dynamics.

We assume that s_t, the log of the stock price, follows arithmetic Brownian motion with a mean-reverting drift:

$$ds_t = (\mu_t - \frac{1}{2}\sigma_s^2)dt + \sigma_s dW_t, \tag{2.1}$$

$$d\mu_t = -\kappa\left(\mu_t - \mu\right)dt + \sigma_\mu dB_t, \tag{2.2}$$

where σ_s is the instantaneous return volatility on the stock, μ_t is the instantaneous expected return, κ its mean reversion speed, σ_μ its instantaneous volatility, and μ is the long run expected return. There are two sources of risk in the economy: W_t and B_t, with $dW_t \times dB_t = \rho dt$,and ρ denotes the instantaneous correlation between dW_t and dB_t. Both are standard Wiener processes defined on a filtered probability space $(\Omega, , \Pi)$. Equation (1) can also be written as:

$$ds_t = (\mu_t - \frac{1}{2}\sigma_s^2)dt + \sigma_1 dB_t + \sigma_2 dZ_t, \tag{2.3}$$

where

$$\sigma_1 = \sigma_s \rho$$
$$\sigma_2 = \sigma_s \sqrt{1-\rho^2}$$

and $dZ_t \times dB_t = 0$.

Define the "transitory component" as:

$$u_t = \frac{\mu - \mu_t}{\kappa}. \tag{2.4}$$

From equation (2), the dynamics of the transitory component can be written as:

$$du_t = -\kappa u_t dt - \frac{\sigma_\mu}{\kappa} dB_t. \tag{2.5}$$

Now we define a new constant ϵ such that:

$$\sigma_1 = -\frac{\sigma_\mu}{\kappa} + \epsilon. \tag{2.6}$$

Finally, we introduce the "permanent component" q_t as a random walk with drift satisfying the following SDE:

$$dq_t = (\mu - \frac{1}{2}\sigma_s^2)dt + \epsilon dB_t + \sigma_2 dZ. \tag{2.7}$$

Replacing equations (2.4) and (2.7) in (2.3), we get:

$$ds_t = (\mu - \frac{1}{2}\sigma_s^2)dt - \kappa u_t dt - \frac{\sigma_\mu}{\kappa} dB_t + \epsilon dB_t + \sigma_2 dZ_t \tag{2.8}$$
$$= dq_t + du_t. \tag{2.9}$$

That is, we have decomposed the log stock price into a transitory and a permanent components, where the components are correlated. In particular:

$$dq_t \times du_t = -\epsilon \frac{\sigma_\mu}{\kappa} dt.$$

The fraction f_u of the total stock price variation explained by the transitory component is:

$$f_u = \frac{\frac{\sigma_\mu}{\kappa}\left(\frac{\sigma_\mu}{\kappa} - \epsilon\right)}{\sigma_s^2}$$

In Wachter's (2002) parametrization,

$$\rho = -1$$
$$\sigma_s = 0.0436$$
$$\sigma_\mu = 8.24 \times 10^{-4}$$
$$\kappa = 0.0226.$$

From equation (2.6), $\epsilon = -0.0075$ and so $f_u = 0.84$. That is, in Wachter's model the transitory component explains 84% of the total variation in the log stock price.

Note that in Wachter's parametrization, $\rho = -1$, so $\sigma_2 = 0$. There is only one shock, dB_t, affecting both the transitory and the permanent components. Because $\epsilon < 0$, the two components are positively correlated: a shock to q_t (the fundamentals) is associated to a simultaneous larger shock (because $\frac{\sigma_\mu}{\kappa} > -\epsilon$) of the same sign to the transitory component (an overreaction) that will fade away as time passes. Therefore, Wachter's model can be interpreted as a model of investor's overreaction.

Given σ_s, the size of the transitory component depends on σ_μ, κ, and ρ. The larger σ_μ, the smaller κ, and the closer ρ to -1, the larger the transitory component. In the standard literature, a widely chosen proxy for the expected return is the dividend yield, which is very persistent (small κ) and whose changes are highly negatively correlated to actual returns (ρ close to -1). These values lead to a large implied transitory component, and explain why asset allocation models predict such a large hedging demand.

A model with a time-varying expected return provides a way to capture stock return mean reversion, which is usually proxyed in the literature by a variable such as dividend yield (as in Wachter(2002)). However, we just showed that to every model with time-varying expected return there is an associated transitory-permanent component model. To complicate matters, the proxy used to describe the unobservable expected return may end up inflating the importance of the transitory component. In the next section we propose an explicit model of the transitory and permanent components, and explore its consequences for asset allocation under complete and incomplete information.

2.3 The Model

2.3.1 Basic Settings

We model the log stock price as the sum of a *permanent* and a *temporary* components. The temporary component is mean-reverting and can be interpreted as capturing deviations of the stock price from its fundamental path, as in Poterba and Summers (1988), or as capturing time variation due to fundamental forces themselves (for example, in the form of a stochastic expected return), as in Fama and French (1988). The permanent component represents the persistent stochastic behavior of the stock price. Such a stock price decomposition has been extensively studied in the asset pricing[6] and the macroeconomics literature[7], but it has not been discussed in the portfolio literature so far.

Let us denote S_t as the price and $s_t = \log(S_t)$ as the log price of a risky asset at time t. We model the log price as sum of two factors:

$$s_t = q_t + u_t \ , \tag{2.10}$$

where q_t and u_t are the permanent and temporary price components, respectively.

The permanent component characterizes the stochastic trend and is assumed to follow a standard geometric Brownian motion:

$$dq_t = \mu_q dt + \sigma_q dZ_t^q \ . \tag{2.11}$$

where the constants μ_q and σ_q are the drift and the diffusion term respectively.

Equation(2.11) can be solved explicitly as:

$$q_t = q_{t_0} + \mu_q(t - t_0) + \sigma(Z_t - Z_{t_0}) \ . \tag{2.12}$$

The temporary component u_t follows an Ornstein-Uhlenbeck process and satisfies the following stochastic differential equation:

$$du_t = -\kappa u_t dt + \sigma_u dZ_t^u \ . \tag{2.13}$$

where σ_u is the instantaneous constant volatility and dZ_t^q and dZ_t^u are changes in Wienner processes that are assumed uncorrelated[8], with associated filtration F_t on probability space (Ω, P, F). The parameter κ indicates the speed of mean reversion. It determines how long a transitory shock affects the stock price. A large κ implies that transitory shocks die down fast; a small κ implies that they die down more slowly.

The explicit solution of the temporary component in equation (2.13):

$$u_t = u_{t_0} e^{-\kappa(t-t_0)} + \sigma_u \int_{t_0}^{t} e^{-\kappa(t-v)} dZ_v^u. \tag{2.14}$$

Finally, note that the variance of log price changes is $\bar{\sigma}^2 = \sigma_q^2 + \sigma_u^2$ and μ_q =$\bar{\mu} - \frac{1}{2}\bar{\sigma}^2$ where $\bar{\mu}$ is the long run expected log return.

Combining equation (2.11) and equation (2.13), we reach the following expression for the log

[6]For example see Fama and French (1988) discuss the idea of permanent and temporary price components of the stock return in a discrete time setting. Besides, Schwartz and Smith (2001) decompose commodity prices in continuous time which is technically similar to our settings.

[7]Decomposition of macroeconomic variables have been used in several analysis in macroeconomic literature. For example, real GNP and GDP, the unemployment rate or consumption are examined by Clark(1987) and Nelson and Plosser (1982).

[8]$E[dZ_u dZ_q] = \rho dt = 0$

price change:

$$ds_t = (\mu_q - \kappa u_t)dt + \sigma_q dZ_t^q + \sigma_u dZ_t^u \ . \tag{2.15}$$

Integrating both sides of equation (2.15), we get

$$s_t = q_{t_0} + \mu_q(t - t_0) + \sigma(W_t - W_{t_0}) \ + u_{t_0} e^{-\kappa(t-t_0)} + \sigma_u \int_{t_0}^{t} e^{-\kappa(t-v)} dZ_v^u,$$

where

$$\mu_q \Delta t + \sigma_q(Z_t^q - Z_{t-\Delta t}^q) = q_t - q_{t-\Delta t} \tag{2.16}$$

and

$$-\kappa \int_{t-\Delta t}^{t} u_\nu d\nu + \sigma_u(Z_t^u - Z_{t-\Delta t}^u) = u_t - u_{t-\Delta t}. \tag{2.17}$$

Thus, the expectation and the variance of the s_t process:

$$E[s_t] = q_0 + \mu_q t + e^{-\kappa t} u_0 \ , \tag{2.18}$$

$$Var[s_t] = (1 - e^{-2\kappa t}) \frac{\sigma_u^2}{2\kappa} + \sigma_q^2 t \ , \tag{2.19}$$

where q_0 and u_0 are the initial values, assumed constant from now on.

Finally, the covariance matrix [9] of the transitory and permanent components is:

$$\Sigma = Cov[q_t, u_t] = \begin{bmatrix} \sigma_q^2 t & 0 \\ 0 & (1 - e^{-2\kappa t}) \frac{\sigma_u^2}{2\kappa} \end{bmatrix}. \tag{2.20}$$

2.3.2 Expected return

It is not difficult to recast the model of the previous section as a stock price-stochastic expected return model, as in the standard literature. Define

$$\mu_t = \mu_q - \kappa u_t \tag{2.21}$$

as the expected return on the log stock price. Therefore,

$$d\mu_t = -\kappa \left(\mu_t - \mu_q\right) dt + \sigma_\mu dZ_t^u,$$

where:

$$\sigma_\mu = -\kappa \sigma_u. \tag{2.22}$$

[9] See Schwartz and Smith (2000) for the details of the derivation.

Denote by ϕ the instantaneous correlation between ds_t and $d\mu_t$.That is:

$$ds_t \times d\mu_t = \phi\sigma_s\sigma_\mu dt, \tag{2.23}$$

where:

$$\begin{aligned}\phi &= \frac{\kappa\sigma_u^2}{\kappa\sigma_u\sqrt{\sigma_q^2+\sigma_u^2}} \\ &= \frac{\sigma_u}{\sqrt{\sigma_q^2+\sigma_u^2}}.\end{aligned} \tag{2.24}$$

From equation (2.23) we get immediately that $\phi = -1$, as in Wachter(2002), implies

$$\sqrt{\sigma_q^2+\sigma_u^2} = \sigma_u,$$

but because $\sqrt{\sigma_q^2+\sigma_u^2} = \sigma_s$, from equation (2.22) we get,

$$\sigma_u = -\frac{\sigma_\mu}{\kappa}.$$

We obtain an even stronger result. From equation (2.24) and $|\phi| < 1$ we get:

$$\sigma_u^2 = \frac{\phi^2}{1-\phi^2}\sigma_q^2.$$

This equation shows that in the simplified model, when ϕ is close to minus one, transitory shocks dominate the dynamics of the stock price. For example, if $\phi = -0.85$,

$$\sigma_u^2 = 2.6\sigma_q^2,$$

that is, the variance of the transitory component is almost three times the variance of the permanent component.

2.4 The Investor's Problem

In a strategic asset allocation problem, a rational investor decides her intertemporal consumption plan and the allocation of her wealth across different asset classes to maximize her expected utility over a given time horizon. If the investor is also risk-averse, she aims to diversify her asset holdings to minimize the risk of her portfolio and to smooth her consumption over the investment cycle[10]. When stock returns are normally distributed, the investor cares only about returns mean and variance if the

[10]For example, see Markowitz(1952) from the early literature or the text book by Campbell and Viciera(2002) from the recent literature.

investment opportunity set is constant, and also engages in market timing if the investment opportunity set is time-varying.

The investor solves the portfolio selection problem applying Dynamic Programming. There is a vast literature in finance using this technique (see Merton (1971), Brennan et al. (1997) and Xia (2001) for representative examples).

We assume a simple portfolio problem with one risky stock with price S and one risk-free bond with price B. We formulate the state dependent continuous stochastic dynamics as[11]

$$\frac{dS_t}{S_t} = \mu_S(X,t)dt + \sigma_S(X,t)dZ_S. \tag{2.25}$$

$$\frac{dB_t}{B_t} = rdt \tag{2.26}$$

$$dX_t = \mu_X(X,t)dt + \sigma_X(X,t)dZ_X \tag{2.27}$$

where $\mu_S(X,t)$ and $\sigma_S(X,t)$ are the state and time dependent drift and volatility terms, respectively. For simplicity we assume that r, the interest rate, is constant, but it is not difficult to make it state dependent as well (see Munk and Sorensen (2004)). We denote by X_t the state variable, whose evolution is described by equation (2.27). Finally, Z_S and Z_X are Wiener processes defined on a filtered probability space $(\Omega, , \Pi)$, with correlated changes: $E[dZ_S dZ_X] = \rho dt$, where ρ is the correlation coefficient.

As it is stated by Merton(1971)**,** an investor with a T-year investment horizon solves the following optimization problem:

$$\max_{\alpha_t, c_t} E[\int_0^T U(c_t,t)dt + U_{Beq}(W_T,T)|F_0^I] \tag{2.28}$$

subject to the budget constraint

$$dW_t = [(\alpha(\mu_S - r) + r)W_t - c_t]dt + \alpha W_t \sigma_S dZ_S \tag{2.29}$$

where F_0^I is the investor's filtration containing all information of investor at $t = 0$, $W_t > 0$ is accumulated wealth, $U(.)$ is the time separable strictly concave utility function, and U_{Beq} is the bequest function which is also assumed to be strictly concave. Finally, α is the fraction of wealth allocated to the stock, and c_t is the positive consumption rate.

The investor chooses α and c_t optimally using Dynamic Programming. For details on the solution applied to the case in which the investor has a CRRA utility function, we refer the reader to the appendix.

The optimal allocation to stocks satisfies the following equation:

[11] See Campbell and Viciera(2002) comprehensive technical review on this standard problem.

$$\alpha^* = -\frac{J_W}{WJ_{WW}}\frac{(\mu_S - r)}{\sigma_S^2} - \frac{J_{WX}}{WJ_{WW}}\frac{\rho\sigma_X^2}{\sigma_S^2}, \tag{2.30}$$

where the first term is the "myopic demand", a portfolio implemented to achieve optimal diversification, and the second term is the "hedging demand", a portfolio implemented to hedge adverse movements in the investment opportunity set (see Ingersoll (1987)). We discuss this optimal allocation in the next section.

2.4.1 The Investor's Portfolio Problem Under Complete Information

In this section we solve the investor's optimal portfolio problem for the case of complete information, where the investors can perfectly disentangle the stochastic processes u_t and q_t.

Let's assume an economy with two securities, one risky and one risk-free. The risky security is a non-dividend paying stock with price S_t; a risk-free is a bond with price B_t. The dynamics of the stock price is as described in the previous section. The bond pays a constant interest per period equal to r.

The investor has CRRA utility

$$U(W,t) = \frac{W_t^{1-\gamma}}{1-\gamma},$$

where γ is the constant coefficient of relative risk aversion, and trades continuously in a frictionless market.

For simplicity, and without loss of generality, we assume away intermediate consumption. The investor, therefore, aims to determine the proportions of the stock and the risk free assets in her portfolio to maximize her terminal wealth.

Let us denote the stock weight in the investor's portfolio at time t, as α_t. The wealth process W_t can be written as

$$\frac{dW_t}{W_t} = (\alpha_t(\mu_q - \kappa u_t - r) + r)dt + \alpha_t(\sigma_q dZ_q + \sigma_u dZ_u). \tag{2.31}$$

The investor chooses α_t to maximize

$$E[\frac{W_T^{1-\gamma}}{1-\gamma} \mid F_t^I]\,, \tag{2.32}$$

subject to equation(2.31), where F_t^I is the filtration containing all information available to the investor up to time t.

In the optimization procedure, we follow the Hamilton-Jacobbi-Bellman (HJB) approach[12]. We define the value function as

[12]This standard problem is originally solved by Merton(1971) for power utility and finite horizon. It is also studied by Kim and Omberg's (1996) and Brennan, Schwartz and Lagnado (1997). In Duffie (1996), the problem is solved by backwardation in a discrete time setting. For a comprehensive textbook on strategic asset allocation, the book by Campbell and Vicieira (2002) can be suggested.

$$J(W, X, t) = \max_{\alpha_t} E[U(W_T, T) \mid F_t^I] \ , \tag{2.33}$$

which implies

$$0 = \max_{\alpha_t}[\frac{1}{dt}E[dJ(W, X, t)] \mid F_t^I]. \tag{2.34}$$

After solving the problem[13], we obtain the proportion of the risky asset in the investor's portfolio:

$$\alpha^* = -\frac{J_W}{WJ_{WW}}\frac{(\mu_q - \kappa u_t - r)}{\sigma_q^2 + \sigma_u^2} - \frac{J_{Wu}}{WJ_{WW}}\frac{\sigma_u^2}{\sigma_q^2 + \sigma_u^2}. \tag{2.35}$$

The first term in equation (2.35) is the myopic demand, which can be interpreted as the allocation to obtain optimal diversification. The myopic demand can be decomposed as

$$\frac{1}{\gamma}[\frac{(\mu - r)}{\sigma_q^2 + \sigma_u^2} - \frac{\kappa u_t}{\sigma_q^2 + \sigma_u^2}] = \frac{1}{\gamma}[\frac{\lambda}{\sqrt{\sigma_q^2 + \sigma_u^2}} - \frac{\kappa u_t}{\sigma_q^2 + \sigma_u^2}] \tag{2.36}$$

The first component is the allocation to stocks corresponding to the case in which the investment opportunity set is constant. It is proportional to the risk premium (λ) and inversely proportional to the stock return volatility and the risk aversion coefficient. The second component is a "market timing" portfolio that depends negatively on u_t. When u_t is positive (that is, above its long run mean of zero), this portfolio becomes negative, reducing the total myopic demand. This is because, due to the mean reversion of the transitory component, a positive u_t reduces the stock's expected return, as the investor expects that the transitory component reverts to its mean. When u_t is negative, the market timing portfolio becomes positive through the same mechanism. In this way, the investor in our model behaves as a contrarian trader.

The second component is the hedging demand, which can be described as the portfolio aimed to hedge adverse changes in the investment opportunity set (Merton, (1973)). Interestingly, the hedging demand is proportional to the fraction of the variance of stock returns explained by the transitory component. The more important the transitory component, the larger the hedging demand.

2.4.2 The Investor's Portfolio Problem Under Incomplete Information

In this part, we solve the investor's optimal portfolio selection problem when she cannot distinguish transitory from permanent components. This means that now and are not observable. Recalling equation (2.15):

$$ds_t = (\mu_q - \kappa u_t)dt + \sigma_q dZ_t^q + \sigma_u dZ_t^u \ , \tag{2.37}$$

it is clear that the unobservability of makes the expected rate of growth of the log endowment unobservable, no matter that consumers know the long run expected rate of growth μ_q.

[13] See appendix for the details of the solution

As in Dothan and Feldman (1986), the representative consumer is assumed to use a nonlinear filtering algorithm to estimate the unobservable variables. The equation describing the dynamics of the innovations process is:

$$d\nu = \frac{ds - (\mu - \kappa \hat{u}_t)dt}{\sqrt{\sigma_q^2 + \sigma_u^2}} \tag{2.38}$$

$$= \frac{ds_t - E_t(ds_t)}{\sqrt{\sigma_q^2 + \sigma_u^2}} \, , \tag{2.39}$$

where E_t is the operator expectation, conditined on information observed up to time t.

The innovations process is Brownian motion with respect to the σ-algebra generated by the observations s_t (see Dothan and Feldman (1986) and references therein). In the partially observable economy neither Z_t^q nor Z_t^u are observable. The innovations process is defined as the normalized deviation of the growth rate from its conditional mean and is therefore observable. This fact shows an important aspect of the partially observable economy, which was pointed out by Feldman (1986): the inference process reduces the martingale multiplicity of the economy, because the innovations process is measurable with respect to the observations.

The estimates of the transitory and permanent components are, respectively:

$$d\hat{u}_t = -\kappa \hat{u}_t dt + \frac{\sigma_u^2 - \xi_t \kappa}{\sqrt{\sigma_q^2 + \sigma_u^2}} d\nu_t \, , \tag{2.40}$$

$$d\hat{q}_t = ds - d\hat{u}_t \tag{2.41}$$

$$= \mu_q dt + \frac{\sigma_q^2 + \xi_t \kappa}{\sqrt{\sigma_q^2 + \sigma_u^2}} d\nu_t \, , \tag{2.42}$$

where ξ_t is the estimation error -a measure of the precision of the estimates. Note that equations (2.40) and (2.42) can be rewritten, in terms of the estimation errors, as:

$$d\hat{u}_t = -\kappa \hat{u}_t dt + \frac{\sigma_u^2 - \xi_t \kappa}{\sigma_q^2 + \sigma_u^2} \left[ds_t - E_t(ds_t) \right] \, , \tag{2.43}$$

$$d\hat{q}_t = \mu dt + \frac{\sigma_q^2 + \xi_t \kappa}{\sigma_q^2 + \sigma_u^2} \left[ds_t - E_t(ds_t) \right] . \tag{2.44}$$

In the partially observable economy, the representative consumer faces a signal extraction problem. She must distinguish transitory from permanent shocks based on the observations. In equation (2.38), the stationary component appears as an adjustment to the long run rate of growth, to take into account cyclical variation. This information is used by the estimates in equation (2.40) and (2.42) to distinguish components. As equation (2.40) and (2.42) show, random shocks have been replaced by estimation errors. In this context, a positive (negative) shock means that the rate of growth has been higher (lower) than expected.

The fraction of a positive (negative) innovation assigned to the transitory component can be divided into two parts: the first part reflects the proportion of the total variance explained by the transitory component: $\frac{\sigma_u^2}{\sigma_q^2+\sigma_u^2}$. The second part, $\frac{-\xi_t\kappa}{\sigma_q^2+\sigma_u^2}$, reflects that the transitory component is adjusted down (up), because a positive (negative) estimation error, due to the positive (negative) transitory shock, means that the stock rate of return has been higher (lower) than expected. By the same mechanism, the fraction of an innovation assigned to the permanent component will reflect the proportion of the total variance explained by the permanent component, $\frac{\sigma_q^2}{\sigma_q^2+\sigma_u^2}$, and the adjustment, $\frac{\xi_t\kappa}{\sigma_q^2+\sigma_u^2}$, to reflect revisions in the stock rate of return. By reducing the importance of the transitory component relative to the perfect information case, this updating rule lowers the hedging demand in the portfolio selection problem.

The path of the estimation error, which measures the precision of the estimates, is governed by the following differential equation of the Ricatti type:

$$\frac{d\xi_t}{dt} = \sigma_u^2 - 2\xi_t\kappa - \frac{(\sigma_u^2 - \kappa\xi_t)^2}{\sigma_q^2 + \sigma_u^2} \; . \tag{2.45}$$

The estimation error is a deterministic function of time. As $t \to \infty$, the estimation error approaches the constant ξ_∞, where

$$\xi_\infty = \frac{1}{\kappa}\left(\frac{\sigma_u^2}{1+\sqrt{1+\frac{\sigma_u^2}{\sigma_q^2}}}\right) . \tag{2.46}$$

From equation (2.38) we can write the stock log return as:

$$ds = d\hat{u} + d\hat{q} = (\mu - \kappa\hat{u}_t)dt + \sqrt{\sigma_q^2 + \sigma_u^2}d\nu \; . \tag{2.47}$$

Therefore, under incomplete information the wealth process evolves as:

$$\frac{dW_t}{W_t} = (\alpha(\mu_q - \kappa\hat{u}_t - r) + r)dt + \alpha\sqrt{\sigma_u^2 + \sigma_q^2}dv. \tag{2.48}$$

Following the same argument used in the previous section, we obtain the optimal allocation to stock:

$$\alpha = -\frac{J_W}{WJ_{WW}}\frac{\mu - \kappa\hat{u} - r}{\sigma_u^2 + \sigma_q^2} - \frac{J_{Wu}}{WJ_{WW}}\frac{\sigma_u^2 - \xi_\infty\kappa}{\sigma_u^2 + \sigma_q^2} \tag{2.49}$$

Incomplete information reduces the importance of the hedging component, and so it changes the hedging demand. The higher the estimation error, or the higher the mean reversion speed of the transitory component, the smaller the hedging demand.

2.5 Parameter Estimation

To estimate our model we use quarterly returns on the value-weighted index from the CRSP data base. Our estimation period ranges from December 1946 to December 2007. As the expected return in our model depends on the transitory component, which is unobservable, we estimate the model parameters by means of the Kalman filter. However, in contrast to other papers employing the same estimation framework (Binsbergen and Koijen (2010), Rytchkov (2012)), we avoid noisy aggregate dividends and use only realized returns in our estimations. Time-varying expected returns do not necessarily imply that the market is inefficient, and in a nearly efficient market realized returns must have most relevant information about conditional expected returns; besides, realized returns are the best quality data. The main advantage of using only returns is that we can work at the quarterly frequency, which increases efficiency. In contrast, Binsbergen and Koijen (2010) and Rytchkov (2012) must work at the annual frequency to avoid modeling the seasonal pattern in aggregate dividends.

Finally, for parameter identification, we assume that the transitory and permanent components are uncorrelated. This assumption is common in the literature (Zivot et al (2003)), and we also provide a detailed explanation in appendix 2.

The endowment's components admit an exact discretization, which correspond to an autoregressive process of order 1, with autoregressive parameter $\varphi = e^{-\kappa}$, and a random walk with drift, respectively:

$$u_t = e^{-\kappa\Delta} u_{t-\Delta} + \sigma_u \sqrt{\frac{1 - e^{-2\kappa\Delta}}{2\kappa}} \varepsilon_t^u$$
$$q_t = \mu_q \Delta + q_{t-\Delta} + \sigma_q \varepsilon_t^q,$$

where $\Delta = \frac{1}{4}$ and ε_t^i $(i = u, q)$ is a sequence of random variables iid, normally distributed with 0 mean and unit variance.

The transitory-permanent component can be written in state-space form as:

$$\log(s_t) = [1 \;\; 1] \begin{bmatrix} u_t \\ q_t \end{bmatrix}$$
$$\begin{bmatrix} u_t \\ q_t \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_q \Delta \end{bmatrix} + \begin{bmatrix} e^{-\kappa\Delta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{t-\Delta} \\ q_{t-\Delta} \end{bmatrix} + \begin{bmatrix} \sigma_u \sqrt{\frac{1-e^{-2\kappa\Delta}}{2\kappa}} & 0 \\ 0 & \sigma_q \end{bmatrix} \begin{bmatrix} \varepsilon_t^u \\ \varepsilon_t^q \end{bmatrix}.$$

Based on Clark (1987), we use state space methods to find the likelihood function of the sample $\log(s_t)$. Define $P_{t|t}$ the as the variance-covariance matrix The Kalman prediction and updating equations are as follows:

i) Initialization:

$$H = [1 \;\; 1]$$
$$F = \begin{bmatrix} e^{-\kappa} & 0 \\ 0 & 1 \end{bmatrix}$$
$$\beta_0 = \begin{bmatrix} u_0 \\ q_0 \end{bmatrix} - \begin{bmatrix} 0 \\ s_0 \end{bmatrix}$$
$$P_0 = \begin{bmatrix} \sigma_u \sqrt{\frac{1-e^{-2\kappa}}{2\kappa}} & 0 \\ 0 & 10^6 \end{bmatrix}$$

ii) Prediction equations:

$$\beta_{t|t-1} = \mu + F\beta_{t-1|t-1} \tag{2.50}$$
$$P_{t|t-1} = FP_{t-1|t-1}F' + Q \tag{2.51}$$
$$\eta_{t|t-1} = y_t - H_t\beta_{t|t-1} \tag{2.52}$$
$$f_{t|t-1} = H_tP_{t|t-1}H_t' + R \tag{2.53}$$

iii) Updating equations:

$$\beta_{t|t} = \beta_{t|t-1} + K_t\eta_{t|t-1} \tag{2.54}$$
$$P_{t|t} = P_{t|t-1} - K_tH_tP_{t|t-1} \tag{2.55}$$

where $K_t = P_{t|t-1}H_t'f_{t|t-1}^{-1}$ known as Kalman gain which determines how new information contained in the prediction error alters the β vector.

The likelihood function can be written as:

$$\ln L = -\frac{1}{2}\sum_{t=0}^{T}\ln(2\pi f_{t|t-1}) - \frac{1}{2}\sum_{t=1}^{T}\eta_{t|t-1}'f_{t|t-1}^{-1}\eta_{t|t-1} \tag{2.56}$$

A nonlinear algorithm that searches the parameter space maximizes this likelihood function. We show our estimation results in the next section.

2.6 Empirical Results

Empirical results are depicted in Table 1. The instantaneous volatilities of the permanent and transitory components are 0.1199 and 0.0817, respectively, both significantly different from zero. The values show that the transitory component is important in explaining the total variation in stock prices, even though it is not its dominant force. The transitory component explains 32% of the log price changes variance, while the permanent component explains the remaining 68%. These results are consistent with Cochrane (1994) (see also Gonzalo (2008)).

The estimated mean reversion speed of the transitory component, κ, is 0.65, also significantly different from zero. This value may seem too high, given that this parameter is also the mean reversion speed of the expected return (see equation (2.15)). In the Wachter's parameterization[14] of the standard model, the mean reversion speed of the expected return is 0.2712 (on an annual basis). It must be noted, however, that the two models are no equivalent. In the standard model, a low κ makes the expected return very persistent without affecting its standard volatility, and leads to a large hedging demand. In the transitory-permanent model, instead, a low κ reduces both the mean reversion speed and the instantaneous volatility of the expected return. In the limit, as $\kappa \to 0$,the expected return becomes a constant, and the hedging demand shrinks to zero. For this reason, the expected return is much less sensitive to κ in the transitory-permanent model than in the standard model. Both models - Wachter's and ours-, however, estimate the long run volatility of the annualized expected return almost identically: 0.0465 and 0.0466, respectively.

If our estimate of the transitory component makes sense, equation (2.21):

$$\mu_t = \mu_q - \kappa u_t \tag{2.57}$$

should estimate the expected return on the stock. According to the present value restriction (See Campbell and Shiller (1988)), the one-period return, the dividend yield, and dividend growth are not independent. If dividend growth is nearly unpredictable, returns must be predictable (see Cochrane (2006)) and, moreover, the expected return must look like the dividend yield. Figure-1 shows filtered annual expected returns computed from equation (2.1) and (2.2) against an estimate of expected returns obtained from a regression of realized returns on the lagged dividend yield. The filtered annual expected return is constructed by taking all December filtered expected returns from the quarterly estimates. The two series look strikingly similar, even though our filtered estimates are obtained from return data (capital gains) alone, suggesting that our model indeed captures the existing time variation in expected returns.

In the next section we explore the asset allocations implied by our model.

2.7 Asset Allocations

In this section, we investigate the implications of our model for strategic asset allocation. We examine the term structure of the hedging demand for both cases: complete and incomplete information. For simplicity we assume $\xi_t = \xi_\infty$, that is, there is no learning by the investor (for a model with learning, see Xia (2001)).

Figure-2 and Figure-3 depicts how the optimal stock allocation and the hedging demand vary

[14]Note that Wachters parameterization is based on the estimations of Barberis ((2000), Table 2)

with the investors horizon. We also show the optimal investment strategies numerically for different risk aversion levels. Table-2 portrays the optimal stock allocations, myopic demands and hedging demands with perfect information for different investment horizons and risk aversion coefficients.

In table-2, the second row block represents the myopic demand of the investor as in Markowitzs mean variance portfolio paradigm. The myopic stock allocation mainly depends on investment risk appetite, assets risk premium and volatility, but not investments horizon. The more (less) risk averse investor allocates smaller (higher) myopic demand. For instance, for γ=10, myopic demand is 0.1831, whereas for γ=3 it is 0.6102. Furthermore, the myopic stock allocation also depends on the current value of the transitory component. When u(t) is larger(smaller), the stock price becomes higher(lower) than its long run equilibrium level, and so the investor reduces (increases) her expected return, consequently reducing (increasing) the myopic demand in her portfolio. Thus, our investors myopic demand exhibits a contrarian investment style. For example, when u(t)=0.02, the myopic demand becomes 25 percent for an investor with risk aversion level γ=5; when u(t)=-0.02 the myopic becomes 50 percent.

The third row block in table-3 indicates the hedging demand of the investor for different risk aversion and time horizon level in complete information. The hedging demand represents the investors incentive to hedge her portfolio against adverse changes in the investment opportunity set. The source of adverse movements in our model is captured by the transitory variations in stock price. In our model the hedging demand is positive because the transitory component induces mean reversion in stock returns, so the allocation to the risky asset must be larger than in the random walk (constant investment opportunity set) case. Also due to mean reversion, accordingly, the hedging demand increases monotonically with the investment horizon. However, the hedging demand we obtain from our model is small relative to the levels obtained in the extant literature.

In table-2, for example, an investor responding to our model, with a 10-year investment horizon and a risk aversion level of γ = 10, allocates 27.54 percent of her wealth to the stock with a positive hedging demand of 9.23 percent. The same investor, responding to the standard model, and with Wachters parameters, would allocate to the hedging portfolio 60 percent of her wealth. Also, in our model the hedging demand becomes at at relatively short investment horizons. Risk averse investors with γ = 10 and with investment horizons longer than 10 years have essentially the same hedging demands (see Figure-2)

With incomplete information, as discussed in section-(4.2), the impact of transitory shocks is seen as weaker by the investor, and hedging demands become even smaller. Table-2.3 summarizes the results. For example, the investor of the previous paragraph, but now with incomplete information, would allocate just 4% of her wealth to the hedging portfolio. This happens because with incomplete information, permanent shocks are perceived as more important by the investor, and these are precisely the shocks that do not generate any hedging demand. Also, the hedging demand becomes flat at an even shorter investment horizon.

2.8 Conclusion

Recent empirical research in portfolio selection shows that investor's allocation to risky assets is low at young ages and that it does not exhibit a clear pattern of change as investors grow old. In this paper we showed that standard models predict a large allocation to stocks because they implicitly assume that transitory shocks dominate the stock price dynamics. Next, we investigated a stock price model with dominant permanent shocks and found that it generates asset allocations more in line with empirical results: smaller and less dependent on the investor's horizon than in the current literature. Our results are stronger under incomplete information.

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2.9 Appendix

2.9.1 The Optimal Investment Problem

The Complete Information Case

Let us consider the simple investment problem with one risky stock and the risk free asset. The wealth process can be formulated as

$$\frac{dW_t}{W_t} = (\alpha(\mu_q - \kappa u_t - r) + r)dt + \alpha(\sigma_q dZ_q + \sigma_u dZ_u). \tag{2.58}$$

where $\mu_q = \bar{\mu} - \frac{1}{2}(\sigma_q^2 + \sigma_u^2)$ and the Z_q and Z_u are the uncorrelated Wienner processes. Basically, the portfolio problem is based on maximizing individual's utility over investment period such that

$$\max_{\alpha_t} E[\frac{W_T^{1-\gamma}}{1-\gamma} \mid F_t^I] \tag{2.59}$$

where γ is risk aversion coefficient, α_t is the optimal stock weight at time t, and F_t^I is the investor's filtration. The optimization problem can be solved with continuous Bellman's dynamic programing approach. We formulate the value function as

$$J(W, u, t) = \max_{\alpha_t} E[U(W_T, T) \mid F_t^I] \tag{2.60}$$

implying that

$$0 = \max_{\alpha_t}[\frac{1}{dt}E[dJ(W, u, t)] \mid F_t^I]. \tag{2.61}$$

Then, we obtain the Hamilton–Jacobi–Bellman (HJB) equation:

$$\begin{aligned} 0 = \max_{\alpha}\{&J_W W(\alpha(\mu_q - \kappa u_t - r) + r) - J_u \kappa u_t + \frac{\partial J}{\partial t} \\ &+ \frac{1}{2}J_{WW}W^2\alpha^2(\sigma_q^2 + \sigma_u^2) + J_{Wu}\alpha W\sigma_u^2 + \frac{1}{2}J_{\mathrm{uu}}\sigma_u^2\}. \end{aligned} \tag{2.62}$$

Applying the first-order-condition with respect to α, we can derive the optimal portfolio rule as follows:

$$\alpha^* = \frac{1}{W}[-\frac{J_W}{J_{WW}}\frac{(\mu_q - \kappa u_t - r)}{\sigma_q^2 + \sigma_u^2} - \frac{J_{Wu}}{J_{WW}}\frac{\sigma_u^2}{\sigma_q^2 + \sigma_u^2}] \tag{2.63}$$

Since we deal with CRRA time additive utility in preferences, the value function $J(W, u, t)$ can be defined as the multiplication of two functions. Therefore, we can reduce the three dimensional partial differential equation (in equation (2.11)) into two dimensional one which is much easier to deal with (See Liu (1999) for the technical details).

Let us define the value function as $J(W,u,t) = \frac{g(u,t)^{\gamma} W^{1-\gamma}}{1-\gamma}$. Plugging this expression and its corresponding derivatives[15] into the HJB, we can reformulate the partial differential equation and the optimal portfolio rule as follows:

$$\begin{aligned} 0 = \frac{g_u}{g}\left[(\mu_q - \kappa u_t - r)\left(\frac{\sigma_u^2}{\sigma_u^2+\sigma_q^2}\right) + \left(\frac{u_t\gamma\kappa}{\gamma-1}\right)\right] + (\frac{g_u}{g})^2\left[-\frac{\gamma}{2}\frac{(\sigma_q\sigma_u)^2}{\sigma_u^2+\sigma_q^2}\right] \\ + \frac{g_{\mathrm{u\,u}}}{g}\left(\frac{\gamma\sigma_u^2}{2(1-\gamma)}\right) + \frac{g_t}{g}\left(\frac{\gamma}{1-\gamma}\right) + \frac{(\mu_q-\kappa u_t - r)^2}{2\gamma(\sigma_q^2+\sigma_u^2)} + r \end{aligned} \tag{2.64}$$

where

$$\alpha_t = \frac{1}{\gamma}\left(\frac{\mu_q - \kappa u_t - r}{\sigma_q^2+\sigma_u^2}\right) + \left(\frac{\sigma_u^2}{\sigma_q^2+\sigma_u^2}\right)\frac{g_u(u,t)}{g(u,t)} \tag{2.65}$$

We solve this partial differential equation (PDE) by using the method of undetermined coefficients by first assigning a guess analytical solution, then reducing the PDE into system of ordinary nonlinear differential equations (Brennan, Schwartz and Lagnado(1997)). In particular, it is assumed that the solution of this PDE has quadratic representation such that

$$g(u,t) = \exp\{-\frac{\delta}{\gamma}(T-t) + \frac{1-\gamma}{\gamma}A_1(T-t) + \frac{1-\gamma}{\gamma}A_2(T-t)u + \frac{1-\gamma}{2\gamma}A_3(T-t)u^2\} \tag{2.66}$$

whose partial derivatives are

$$\begin{aligned} g_u(u,t) &= \frac{1-\gamma}{\gamma}(A_2(T-t) + A_3(T-t)u)g(u,t) \\ g_{\mathrm{u}\,u}(u,t) &= \frac{1-\gamma}{\gamma}(A_3(T-t) + \frac{1-\gamma}{\gamma}[A_2(T-t)+A_3(T-t)u]^2)g(u,t) \\ \frac{\partial g}{\partial t}(u,t) &= (\frac{\delta}{\gamma} - \frac{1-\gamma}{\gamma}A_1'(T-t) - \frac{1-\gamma}{\gamma}A_2'(T-t)u \\ &\quad - \frac{1-\gamma}{2\gamma}A_3'(T-t)u^2)g(u,t) \end{aligned}$$

We can express the PDE in a quadratic polynomial form. Since the coefficients of this polynomial must be equal to zero, the original problem can be transformed into the following system of differential equations with the boundary conditions $A_1(0) = A_2(0) = A_3(0) = 0$,

$$\frac{dA_3(\tau)}{d\tau} = k_1 A_3^2(\tau) + k_2 A_3(\tau) + k_3 \tag{2.67}$$

[15] $J_W = g^{\gamma}W^{-\gamma}$; $J_{WW} = -\gamma g^{\gamma} W^{-\gamma-1}$; $J_{\mathrm{u}} = \gamma g^{\gamma-1} g_u \frac{W^{1-\gamma}}{1-\gamma}$; $J_{\mathrm{u\,u}} = \gamma(\gamma-1)g^{\gamma-2}g_u^2\frac{W^{1-\gamma}}{1-\gamma} + \gamma g^{\gamma-1}g_{\mathrm{u\,u}}\frac{W^{1-\gamma}}{1-\gamma}$; $J_{Wu} = \gamma g^{\gamma-1}g_u W^{-\gamma}$; $\frac{\partial J}{\partial t} = \gamma g^{\gamma-1} g_t \frac{W^{1-\gamma}}{1-\gamma}$

$$\frac{dA_2(\tau)}{d\tau} = k_4 A_3(\tau) + \frac{k_2}{2} A_2(\tau) + k_1 A_2(\tau) A_3(\tau) + k_5 \tag{2.68}$$

$$\frac{dA_1(\tau)}{d\tau} = \frac{k_1}{2} A_2^2(\tau) + k_4 A_2(\tau) + \frac{\sigma_u^2}{2} A_3(\tau) + k_6 \tag{2.69}$$

with the coefficients

$$k_1 = -\frac{\sigma_u^2(\gamma-1)((\sigma_u^2+\sigma_q^2)\gamma - (\gamma-1)\sigma_u^2)}{(\sigma_u^2+\sigma_q^2)\gamma} \tag{2.70}$$

$$k_2 = \frac{-2\kappa\left((\sigma_u^2+\sigma_q^2)\gamma + \sigma_u^2(1-\gamma)\right)}{(\sigma_u^2+\sigma_q^2)\gamma} \tag{2.71}$$

$$k_3 = \frac{\kappa^2}{(\sigma_u^2+\sigma_q^2)\gamma} \tag{2.72}$$

$$k_4 = \frac{(\gamma-1)(r-\mu)\omega}{(\sigma_u^2+\sigma_q^2)\gamma} \tag{2.73}$$

$$k_5 = \frac{\kappa(r-\mu)}{(\sigma_u^2+\sigma_q^2)\gamma} \tag{2.74}$$

$$k_6 = -\frac{\delta}{\gamma-1} + \frac{(\mu-r)^2}{2(\sigma_u^2+\sigma_q^2)\gamma} + r \tag{2.75}$$

where the time parameter τ is time to maturity ($\tau = T - t$). Consequently, the analytical solution of this system of differential equation has a recursive representation such that

$$A_3(\tau) = \frac{1}{2k_1}\left[-k_2 + \Delta Tan[\frac{1}{2}(\Delta\tau \pm 2Arc\cos(\zeta))]\right] \tag{2.76}$$

$$A_2(\tau) = \frac{2(A_3(\tau)k_4 + k_5)[-1 + \exp(\frac{1}{2}(2A_3(\tau)k_1 + k_2))\tau]}{2A_3(\tau)k_1 + k_2} \tag{2.77}$$

$$A_1(\tau) = \frac{1}{2}\left[k_1 A_2^2(\tau) + 2k_4 A_2(\tau) + 2k_6 + \sigma_u^2 A_3(\tau)\right]\tau \tag{2.78}$$

where $\Delta = \sqrt{-k_2^2 + 4k_1k_3}$ and $\zeta = -\frac{\sqrt{-\frac{k_2^2}{k_1} + 4k_3}}{2\sqrt{k_3}}$.

The Incomplete Information Case

In the case of incomplete information, the HJB equation is formulated as follows

$$0 = \max\{J_W W(\alpha(\mu_q - \kappa\hat{u} - r)) + J_u(-\kappa\hat{u}) + \frac{\partial J}{\partial t} + \frac{1}{2} J_{WW} W^2 \alpha^2 (\sigma_q^2 + \sigma_u^2)$$
$$+ J_{Wu}\alpha W(-\xi_\infty\kappa + \sigma_u^2) + \frac{1}{2} J_{\mathrm{uu}} \frac{(-\xi_\infty\kappa + \sigma_u^2)^2}{\sigma_q^2 + \sigma_u^2}\} \tag{2.79}$$

where ξ_∞ is the variance of the estimation when t$\rightarrow \infty$, such that

$$\xi_\infty = \frac{1}{\kappa}\left(\frac{\sigma_u^2}{1+\sqrt{1+(\sigma_u^2/\sigma_q^2)}}\right) \tag{2.80}$$

Following the same procedure, we can formulate the optimal stock allocation problem of an investor:

$$\alpha^* = -\frac{1}{W}\frac{J_W}{J_{WW}}\frac{(\mu_q-\kappa\hat{u}-r)}{\sigma_q^2+\sigma_u^2} - \frac{1}{W}\frac{J_{Wu}}{J_{WW}}\left(\frac{-\xi_\infty\kappa+\sigma_u^2}{\sigma_q^2+\sigma_u^2}\right) \tag{2.81}$$

Considering the same guess function ($J(W,u,t) = \frac{g(u,t)^\gamma W^{1-\gamma}}{1-\gamma}$), the optimal portfolio rule becomes

$$\alpha^* = -\frac{(\mu_q-\kappa u_t-r)}{\gamma(\sigma_q^2+\sigma_u^2)} + \left(\frac{-\xi_\infty\kappa+\sigma_u^2}{\sigma_q^2+\sigma_u^2}\right)\frac{g_u}{g} \tag{2.82}$$

$$= \frac{1}{\gamma}\left(\frac{\mu_q-\kappa u_t-r}{\sigma_q^2+\sigma_u^2}\right) + \left(\frac{-\xi_\infty\kappa+\sigma_u^2}{\sigma_q^2+\sigma_u^2}\right)(A_2(\tau)+A_3(\tau)u_t). \tag{2.83}$$

The PDE becomes

$$0 = \frac{g_u}{g}\left(\frac{\gamma u\kappa}{\gamma-1}\right) + \frac{g_{uu}}{g}\frac{\gamma(-\xi_\infty\kappa+\sigma_u^2)^2}{2(1-\gamma)(\sigma_q^2+\sigma_u^2)} + \frac{g_t}{g}(-\frac{\gamma}{\gamma-1}) + \frac{(\mu-\kappa u-r)^2}{2(\sigma_q^2+\sigma_u^2)\gamma} + r. \tag{2.84a}$$

The system of differential equations in case of incomplete information:

$$\frac{dA_3(\tau)}{d\tau} = c_1A_3^2(\tau) + c_2A_3(\tau) + c_3 \tag{2.85}$$

$$\frac{dA_2(\tau)}{d\tau} = c_4A_3(\tau) + \frac{c_2}{2}A_2(\tau) + c_1A_2(\tau)A_3(\tau) + c_5 \tag{2.86}$$

$$\frac{dA_1(\tau)}{d\tau} = \frac{c_1}{2}A_2^2(\tau) + c_4A_2(\tau) + \frac{(-\xi_\infty\kappa+\sigma_u^2)^2}{2\sqrt{(\sigma_q^2+\sigma_u^2)}}A_3(\tau) + c_6 \tag{2.87}$$

with the coefficients

$$c_1 = -\frac{(\gamma-1)(1+(\sqrt{(\sigma_q^2+\sigma_u^2)}-1)\gamma)(-\xi_\infty\kappa+\sigma_u^2)^2}{(\sigma_q^2+\sigma_u^2)\gamma} \tag{2.88}$$

$$c_2 = -\frac{2\kappa((\sigma_q^2+\sigma_u^2)\gamma+(-\xi_\infty\kappa+\sigma_u^2)(1-\gamma))}{(\sigma_q^2+\sigma_u^2)\gamma} \tag{2.89}$$

$$c_3 = \frac{\kappa^2}{(\sigma_q^2+\sigma_u^2)\gamma} \tag{2.90}$$

$$c_4 = \frac{(-1+\gamma)(r-\mu)(-\xi_\infty\kappa+\sigma_u^2)}{(\sigma_q^2+\sigma_u^2)\gamma} \tag{2.91}$$

$$c_5 = \frac{\kappa(r-\mu)}{(\sigma_q^2+\sigma_u^2)\gamma} \tag{2.92}$$

$$c_6 = \frac{(\mu-r)^2}{2(\sigma_q^2+\sigma_u^2)\gamma} - \frac{\delta}{(-1+\gamma)} + r. \tag{2.93}$$

The closed form solution is

$$A_3(\tau) = \frac{1}{2k_1}\left[-c_2 + \Delta Tan[\frac{1}{2}(\Delta\tau \pm 2Arc\cos(\zeta))]\right] \tag{2.94}$$

$$A_2(\tau) = \frac{2(A_3(\tau)c_4+c_5)[-1+\exp(\frac{1}{2}(2A_3(\tau)c_1+c_2))\tau]}{2A_3(\tau)c_1+c_2} \tag{2.95}$$

$$A_1(\tau) = \frac{1}{2}\left[c_1A_2^2(\tau)+2c_4A_2(\tau)+2c_6+\frac{(-\xi_\infty\kappa+\sigma_u^2)^2}{\sqrt{(\sigma_q^2+\sigma_u^2)}}A_3(\tau)\right]\tau \tag{2.96}$$

where $\Delta = \sqrt{-c_2^2+4c_1c_3}$ and $\zeta = -\frac{\sqrt{-\frac{c_2^2}{c_1}+4c_3}}{2\sqrt{c_3}}$; and the boundary conditions $A_3(0) = 0,\ A_2(0) =$ 0 and $A_2(0) = 0$.

2.9.2 Identifying Restrictions of Trend and Cycle Decomposition Models

Several authors within the econometrics literature have discussed the identifying restriction issue of trend & cycle decomposition models. A remarkable work by Zivot et al(2003) examines the identifying restrictions of the unobserved component models in discrete time framework. In particular, they focus on the differences between two well known decomposition approach Beveridge- Nelson (BN)decomposition (Beveridge and Nelson (1981)) and the unobserved-components (UC) models (Harvey(1985) and Clark(1987)). Once they work with the macroeconomic data (GDP), it is shown that both models are identical if they have same autocovariance structure and joint distribution. Nevertheless, in practice, they usually exhibit different trend and cycle behaviors due to particular underlying empirical factors. For example, once we impose zero identifying restriction, one can observe that BN model yields more dominant trend but noisy and smaller cyclical behavior. On the other hand, UC models with zero covariance assumption exhibit dominance of cyclical component, and very smooth persistent behavior for each component on contrary to BN model.

It is well documented that the empirical differences between these two models do not stem from the fundamental structure of the models, but mainly from the empirical implementations (see Zivot et al.(2003)). In most of the trend and cycle models, for example, the correlation between the stochastic variations of the components is assumed to be zero in order to overcome a possible identification problem (for example Proietti (2002)).

The most fundamental technical distinction between these two models is that the unobserved component models (UC) are typically represented in state space framework while Beveridge- Nelson (BN)decomposition[16] is based on discrete time integrated autoregressive representation (ARIMA).

Let us reconsider our trend&cycle decomposition in discrete time framework:

$$s_t = q_t + u_t \tag{2.97}$$

where s_t, q_t and u_t are the observed series, unobserved trend (permanent) and cyclical (temporary) components respectively. Explicitly, we can formulate the model within an $ARMA(P, Q)$ representation:

$$q_t = q_{t-1} + \mu + \eta_t \tag{2.98}$$

$$\phi_P(L)u_t = \theta_Q(L)\epsilon_t \tag{2.99}$$

with the distributional properties $\eta_t \sim N(0, \sigma_q^2)$ and $\epsilon_t \sim N(0, \sigma_u^2)$; and u_t is stationary and ergodic; the covariance between stochastic components is assumed to be zero ($\sigma_{qu} = 0$). For avoiding any confusion, let us call such unobserved component model as $UC - ARMA(P, Q)$ model. P and Q terms represent the autoregresive and moving average lags respectively [17].

It is widely discussed in the literature that UC models can be represented in an equivalent $ARIMA$ process[18]. Following conventional literature[19], the canonical $ARIMA(P, d, Q)$ representation of s_t with the first difference ($d = 1$) can be written as follows:

$$\phi_P(L)(1 - L)s_t = \phi_p(1)\mu + \phi_P(L)\eta_t \; + \theta_Q(L)(1 - L)\epsilon_t \tag{2.100}$$

By using Granger Lemma (Granger and Newbold (1986)), the equivalent $ARIMA$ representation of equation (2.100) is:

$$\phi_P(L)(1 - L)s_t = \mu^* + \theta_{Q^*}(L)k_t \tag{2.101}$$

while $k_t \sim iid\, N(0, \sigma_k^2)$ and $Q^* = \max(P, Q + 1)$.

[16]See Beveridge and Nelson (1981)

[17]In our notation, one should notice that small p_t and q_t are used for the price components while the capital P and Q is used for the lag of the ARMA process

[18]For example Cochrane(1988) points out that one can formulate an ARIMA process with at least one UC representation.

[19]Nerlove, Grether and Carvalho(1979) shows $UC - ARMA(P, Q)$ representation in a canonical form of an $ARIMA(P, d, Q)$ representation of the observed series.

The order condition for parameter identification is $P \geq Q + 2$. In this case, there will be at least as many nonzero autocovariances as number of parameters.

Let us consider $UC - ARMA(1, 0)$ process which is discrete time equivalence of our continuous time stock price process. The reduced form ARIMA form of $UC - ARMA(1, 0)$ model can be derived as follow:

Take the first difference of s_t process:

$$\Delta s_t = (1 - L)q_t + (1 - L)u_t \tag{2.102}$$

$$= \mu + \eta_t + (1 - L)(1 - \kappa L)^{-1}\epsilon_t \ , \tag{2.103}$$

and then multiplying the both sides with $(1 - \phi L)$

$$(1 - \kappa L)\Delta s_t = \mu^* + \eta_t - \kappa\eta_t + \epsilon_t - \epsilon_{t-1} \tag{2.104}$$

$$= \mu^* + k_t + \theta_1^* k_{t-1}, \tag{2.105}$$

it can be seen that the right hand side of the equation is $MA(1)$ process that indicates the maximum length of such process is one. Also the equivalent $ARIMA$ process of s_t becomes $ARIMA(1, 1, 1)$ such that

$$\phi(L)(1 - L)s_t = \mu^* + k_t + \theta_1 k_{t-1} \tag{2.106}$$

$k_t \sim iid\, N(0, \sigma_k^2)$ and $Q^* = \max(P, Q + 1) = 1$.

Then, we can formulate the autocovariances of $(1 - \phi L)\Delta s_t$ as follows

$$\Gamma_0 = \sigma_q^2(1 + \kappa^2) + 2\sigma_{qu} + 2\sigma_u^2 \tag{2.107}$$

$$\Gamma_1 = -\kappa\sigma_q^2 - (\kappa + 1)\sigma_{qu} - \sigma_u^2 \tag{2.108}$$

$$\Gamma_j = 0 \text{ for } \ j \geq 2. \tag{2.109}$$

In matrix representation

$$\begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} (1+\kappa)^2 & 2 & 2 \\ -\phi & -1 & -(\kappa+1) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_q^2 \\ \sigma_u^2 \\ \sigma_{qu} \end{bmatrix} \tag{2.110}$$

or in compact form $\mathbf{\Gamma} = \mathbf{\Phi\Sigma}$ (we assume that Φ is invertible).

Also we can drive the autocovariance of right hand side of the equation in terms of the reduced form ARIMA(1,1,1) parameters:

$$\Gamma_0 = \sigma_k^2 + \theta^2 \sigma_k^2$$
$$\Gamma_1 = \theta \sigma_k^2$$
$$\Gamma_j = 0 \text{ for } j \geq 2$$

As it is seen from the equation(2.107)-(2.108)-(2.109), we have two non-zero autocovariance relations with three unknown parameters such as $\sigma_q, \sigma_{qu},$ and σ_u. Although the autocovariances can be calculated from time series, there exist infinitely many solution for the covariance of the innovations. In this case, MA(1) is insufficient to identify all parameters, the process does not satisfy the order condition ($P \geq Q + 2$,). In order to overcome the identification problem, we can consider adding one more autoregressive term or imposing zero-covariance restriction.

2.9.3 A Summary of Nonlinear Filtering Theory

In this section, we present the main results of nonlinear filtering theory which is utilized in our paper[20].This section is based on David(1977), Krishnan(1984) and Oksendal(2000). Let us define (Ω,F,P) as a complete probability space. Also define F_t as the filtration of the probability space (Ω, F, P) satisfying[21]

$$\sigma \{X_0, X_s, W_s, Y_s, V_s, s \leq t\} \subset F_t$$

$X_t \in R^n$ is the (unobservable) state vector with dynamics described by the stochastic differential equation:

$$dX_t = a(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.111}$$

where $a : R^{n+1} \to R^n$, $\sigma : R^{n+1} \to R^{n \times p}$ satisfy standard measurability, Lipschitz and growth conditions See Oksendal(2001) for details, and W_t is p-dimensional Brownian motion [22] defined with with respect to F_t and with $\sigma \{W_t - W_s, s < t\}$ independent from $\{F_s, s < t\}$.

$Y_t \in R^m$ is defined as the observations process, with dynamics described by

$$dY_t = b(t, X_t)dt + \vartheta dV_t \tag{2.112}$$

where $b : R^{n+a} \to R^m$ satisfies also standard conditions, ϑ is an $R^{m \times r}$ vector of constants[23], and Y_t is r-dimensional Brownian motion defined with respect to F_t, and with $\sigma \{V_t - V_s, s < t\}$ independent from $\{F_t, s < t\}$ and $\sigma \{Y_s, 0 \leq s \leq t\}$.

Finally, X_0 is the initial condition for equation(2.111), with $E|X_0|^2 < \infty$ and independent of $\sigma \{Y_s, 0 \leq s \leq t\}$, $\sigma \{W_s, 0 \leq s \leq t\}$ and $\sigma \{V_s, 0 \leq s \leq t\}$

The filtering problem can be stated as follows:

Given the observations $\{Y_s, 0 \leq s \leq t\}$ is that, find the *best* estimate $\hat{X}_t$ being based on the observations.

The precise meaning of $\hat{X}_t$ being based on the observations $\{Y_s, 0 \leq s \leq t\}$ is that $\hat{X}_t$ must be Γ_t measurable, where Γ_t is the σ -algebra generated by $\{Y_s, 0 \leq s \leq t\}$. Also, $\hat{X}_t$ is the *best* estimate in the sense that it minimizes mean square error:

$$E\left[|X_t - \hat{X}_t|^2\right] = inf\left\{|X_t - M_t|^2; M \in K\right\},$$

where E is the expectation operator with respect to **P**, and:

[20]Main results on non-linear filtering theory can be found in Lipster and Shyriayev (2001), Davis(1977), Krishnan(1984) and Oksendal(2000)

[21]σ_H is the σ algebra generated by H.

[22]Restricting W_t to be a Brownian motion is not necessary to get the results. W_t can be defined as a general right-continuous L^2 martingale with increments independent of F_t.

[23]The assumption of a vector of constants is unnecessarily restrictive and is made for simplicity. All results are valid also if the vector is a function of the observations which satisfies standard regularity conditions.

$$K = \left\{ M : \Omega \to R^n; M \in L^2(P) \; and \; M \; is \; \Gamma_t - measurable \right\}.$$

Once the filtering problem has been formulated in this way, it can be shown [24] that:

$$\hat{X}_t = E\left[X_t|\Gamma_t\right],$$

where $E\left[A|B\right]$ refers to the expectation of A, conditional on B. Therefore, from the investor's point point of view, the filtering problem reduces to compute the expectation of X_t based on the information generated by the observations $\{Y_s, 0 \leq s \leq t\}$. The nonlinear filtering algorithm provides a means to calculate recursively this conditional expectation, so that the estimate is updated as new information unfolds.

Define $\hat{b}(t, X_t)$ as the expectation of $b(t, X_t)$ conditional on the σ-field generated by the observations. That is:

$$\hat{b}(t, X_t) = E\left[b(t, X_t)|\Gamma_t\right].$$

The *innovation process* ν is given by:

$$d\nu_t = \Theta^{-1/2}(dY_t - \hat{b}(t, X_t)), \tag{2.113}$$

where Θ is the variance-covariance matrix of the changes in Y. Equation(2.113) shows that the innovation process is the new information that arrives to the system, normalized by the variance-covariance matrix. This can be seen more clearly noting that

$$\hat{b}(t, X_t) = E\left[dY_t|\Gamma_t\right].$$

Therefore:

$$\begin{aligned} d\nu_t &= \Theta^{-1/2}(Y_{t+dt} - Y_t - E[Y_{t+dt} - Y_t|\Gamma_t]) && (2.114) \\ &= \Theta^{-1/2}(Y_{t+dt} - E[Y_{t+dt}|\Gamma_t]) \; . && (2.115) \end{aligned}$$

The following is a fundamental results in non-linear filtering theory:

Theorem: The innovation process ν_t is a Γ_t measurable Brownian motion.

Proof: The proof is based on Krishnan(1984), Theorem 8.1.1. The idea of the proof is to show that the characteristic function of the innovations process is identical to that of an independent increment, Gaussian process, i.e. a Brownian motion. The results then follows from the uniqueness of the characteristic function.

Note that from equation(2.114), as the observations and $\hat{b}(t, X_t)$ are Γ_t-measurable, the innovation

[24] See Oksendal(2001), Theorem 6.1.2 for a proof.

process is also Γ_t-measurable.

To prove that the innovations process is Brownian motion, rewrite equation(2.114) as [25]

$$d\nu_t = \frac{b(t, X) - \hat{b}(t, X_t))dt}{\vartheta} + dV_t$$

and define $J = e^{iu\nu_t}$ as a twice continuously differentiable function of the innovations process. i is the imaginary unit [26]. Then applying Ito's lemma to J:

$$de^{iu\nu_t} = iu\nu_t = iue^{iu\nu_t}(\frac{b_t - \hat{b}_t}{\vartheta})dt + iue^{iu\nu_t}dV_t - \frac{u^2}{2}e^{iu\nu_t}dt.$$

Taking conditional expectations on both sides with respect to $\Gamma_s, (s \leq t)$, it follows that:

$$dE^{\Gamma_t}e^{iu\nu_t} = -\frac{u^2}{2}E^{\Gamma_s}e^{iu\nu_t}dt. \tag{2.116}$$

This results follows since:

$$\begin{aligned} E^{\Gamma_s}e^{iu\nu_t}\left(\frac{b_t - \hat{b}_t}{\vartheta}\right)dt &= E^{\Gamma_s}E^{\Gamma_t}e^{iu\nu_t}(\frac{b_t - \hat{b}_t}{\vartheta})dt \\ &= E^{\Gamma_s}e^{iu\nu_t}E^{\Gamma_t}(\frac{b_t - \hat{b}_t}{\vartheta})dt \\ &= 0 \end{aligned}$$

which results from properties of the conditional expectations operator[27].

Also, as V_t is a martingale and V_t, V_s independent of $\Gamma_t, \forall s < t$.

$$\begin{aligned} E^{\Gamma_s}e^{iu\nu_t}dV_t &= E^{\Gamma_s}E^{\Gamma_t}e^{iu\nu_t}dV_t \\ &= E^{\Gamma_s}E^{\Gamma_t}e^{iu\nu_t}E^{\Gamma_t}dV_t \\ &= 0 \end{aligned}$$

Expression(2.116) defines a differential equation in the conditional expectation of J. Integrating this differential equation between s and t yields the result:

$$E^{\Gamma_s}e^{iu(\nu_t - \nu_s)} = e^{(-1/2)u^2(t-s)}$$

This is the characteristic function of a Brownian motion process. Therefore, ν_t is Brownian motion process.

In the partially observable economy, unobservable random shocks are replaced by the innovation

[25] To simplify notation it is assumed that ϑ is a 1-dimensional process. All results carry to dimensions higher than one.
[26] $i^2 = -1$
[27] For an account of properties of the conditional expectations operator, see Williams(1991)

process, which means that the uncertainty is now generated by estimation errors. Moreover, it is possible to show that the innovations process is equivalent to the σ-algebra generated by the innovations process is equivalent to the σ algebra generated by the observations [28].

In what follows, we will state that the Nonlinear Filtering Theorem. As the proof is very involved, we will only present the essentials of it. The details can be found in Krishnan(1984), Theorem 8.4.2., pp. 231-239, and Lipster and Shiryaev (2001), Theorem 8.1, pp 318-326.

Theorem (Nonlinear Filtering Theorem)

Let the probability space and the processes X_t and Y_t be as described above. Then the estimate $\hat{X}_t$ is given by

$$d\hat{X}_t = \hat{b}_t dt + \frac{1}{\vartheta}[\frac{d}{dt} < W, V >_t + E^{\Gamma_t}(X_t b_t) - E^{\Gamma_t} X_t E^{\Gamma_t} b_t] d\nu_t$$

where $< W, V >_t$ is the quadratic covariance process between W_t and V_t.

Proof: (based on Krishnan (1984)): I present only a sketch. The key of the proof is to show that the process:

$$m_t = \hat{X}_t - EX_0 - \int_0^t E^{\Gamma_u} a_u du \tag{2.117}$$

is a square integrable martingale. A martingale Representation Theorem [29] can than be invoked to express m_t as a stochastic integral with respect to the innovation process:

$$m_t = \int_0^t \varphi_u d\nu_u, \tag{2.118}$$

where ν_t is a Γ_t-adapted predictable process [30].

The quadratic covariance between m and ν is:

$$\left\langle m, \int_0^t \vartheta d\nu_u \right\rangle_t = \left\langle \int \varphi_u d\nu_u, \int_0^t \vartheta d\nu_u \right\rangle_t = \int_0^t \varphi_u \vartheta du.$$

Therefore, nu can be given in terms of $\langle m, \nu \rangle_t$ as:

$$\varphi_t = \frac{1}{\vartheta} \frac{d}{dt} \left\langle m, \int_0^t \vartheta d\nu_u \right\rangle_t. \tag{2.119}$$

Using (4) and (5), the estimate $\hat{X}_t$ can be written as:

$$\hat{X}_t = EX_0 + \int_0^t \hat{a}_u du + \int_0^t \frac{1}{\vartheta} \frac{d}{du} \left\langle m, \int_0^t \vartheta d\nu_u \right\rangle_u d\nu_u \tag{2.120}$$

To finish the proof it is necessary to determine the quadratic covariance between m and ν. Defining

[28] A proof of this result can be found in Lipster and Shiryayev(2001), Theorem 12.5

[29] See Krishnan (1984), Section 6.6

[30] The process ν_t satisfies the regularity conditions described in Krishnan (1984), 6.1.3

$\tilde{X}_t = \hat{X}_t - X_t$ and $\tilde{b}_t = \hat{b}_t - b_t$, it is possible to show that:

$$\langle m, \nu \rangle_t = \int_0^t [E^{\Gamma_u}(\tilde{X}_u \tilde{b}_u) + \frac{d}{du} \langle W, V \rangle_u] du. \quad (2.121)$$

Replacing (2.121) in (2.120), and noting that

$$E^{\Gamma_s}[(\hat{X}_t - X_t)\hat{b}_t - b_t] = E^{\Gamma_s} X_t \hat{b}_t - E^{\Gamma_s} X_t E^{\Gamma_s} \hat{b}_t, \quad (2.122)$$

it follows that:

$$\hat{X}_t = EX_0 + \int_0^t \hat{a}_u du + \int_0^t \frac{1}{\vartheta}[E^{\Gamma_u}(\tilde{X}_u \tilde{b}_u) + \frac{d}{du} \langle W, V \rangle_u] d\nu_u, \quad (2.123)$$

which completes the proof.

Equation(2.123) gives a closed form *expression*, but not a closed form solution for the estimate. The first order moment (the conditional expectation) depends on the second order moment (the covariance inside of the second integral), which depends on the third order moment, and so on, because of the nonlinearity of b. In contrast, when a and b are linear, the first moment depends on the second, which is independent of higher order moments. This is the case of the continuous Kalman filter. Only two equation is needed in the linear case to provide an estimator of the state variable: an equation for the estimate itself and an equation for the estimation error. In the nonlinear case, the system of equations needed is infinite-dimensional.

In the complete information economy, the investor's program -i.e., maximizing the utility functional subject to restrictions that are expressed as functions of the state variables - is Markovian. Controls are chosen as functions of the current values of the state variables, which summarize all past behavior of the economy. The partially observable economy looses the Markovian property. The solution of the investor's program no longer can be expressed as function of the state variables. The Seperation Principle Theorem shows that it is optimal for the consumer to estimate first the states using a filtering algorithm, and then solve the optimization problem. The dynamic programming scheme uses now as state variables the observations, the estimates of the states, and the estimation errors. Note that in the linear case this is the only information needed to update the estimates of the states as new information unfolds.

2.9.4 Figures and Tables

The Parameter Estimations			
Parameter Description	Notation	Values	Standard Error
volatility of the permanent component	σ_q	0.1199	0.0253
volatility of temporary component	σ_u	0.0817	0.0413
Long run mean of the stock return	μ	0.0845	0.0268
Mean reversion coefficient	κ	0.6540	0.0790

Table 2.1: **Parameters.** This table reports parameter definitions, notations and their values. In our estimations, we use quarterly return on the value weighted index from CRSP database. Our observation period ranges from December 1946 to December 2007. The parameters are estimated by Kalman filtering algorithm which uses maximum likelihood.

Optimal Stock Allocation (Complete Information Case)						
Risk Aversion/Maturity	1month	1-year	3-year	5-year	10-year	30-year
Hedging Demand						
$\gamma = 3$	0.0088	0.0889	0.1709	0.1960	0.2068	0.2074
$\gamma = 5$	0.0064	0.0658	0.1286	0.1484	0.1572	0.1577
$\gamma = 10$	0.0036	0.0378	0.0748	0.0868	0.0923	0.0926
Myopic Demand						
$\gamma = 3$	0.6102	0.6102	0.6102	0.6102	0.6102	0.6102
$\gamma = 5$	0.3661	0.3661	0.3661	0.3661	0.3661	0.3661
$\gamma = 10$	0.1831	0.1831	0.1831	0.1831	0.1831	0.1831
Stock Weight						
$\gamma = 3$	0.6190	0.6991	0.7811	0.8062	0.8170	0.8176
$\gamma = 5$	0.3725	0.4319	0.4947	0.5145	0.5233	0.5238
$\gamma = 10$	0.1867	0.2208	0.2579	0.2699	0.2754	0.2757

Table 2.2: **The Horizon Effect on Optimal Stock Allocation with Complete Information** This table presents the optimal stock allocation, the myopic stock allocation and hedging demand at different investment horizons for different risk aversion levels when the information is assumed to be complete. The parameters are estimated by Kalman filtering technique (see Table-2.1) by using quarterly observations on the value weighted index from CRSP database.The results are obtained from closed form solution. The columns in the graph show the hedging demands, myopic demands and stock weights when the investor maximizes the expected utility of terminal wealth. Accordingly, the proportion of the risky asset in portfolio and the investor's hedging demand are positive and monotonically increasing under complete information case.

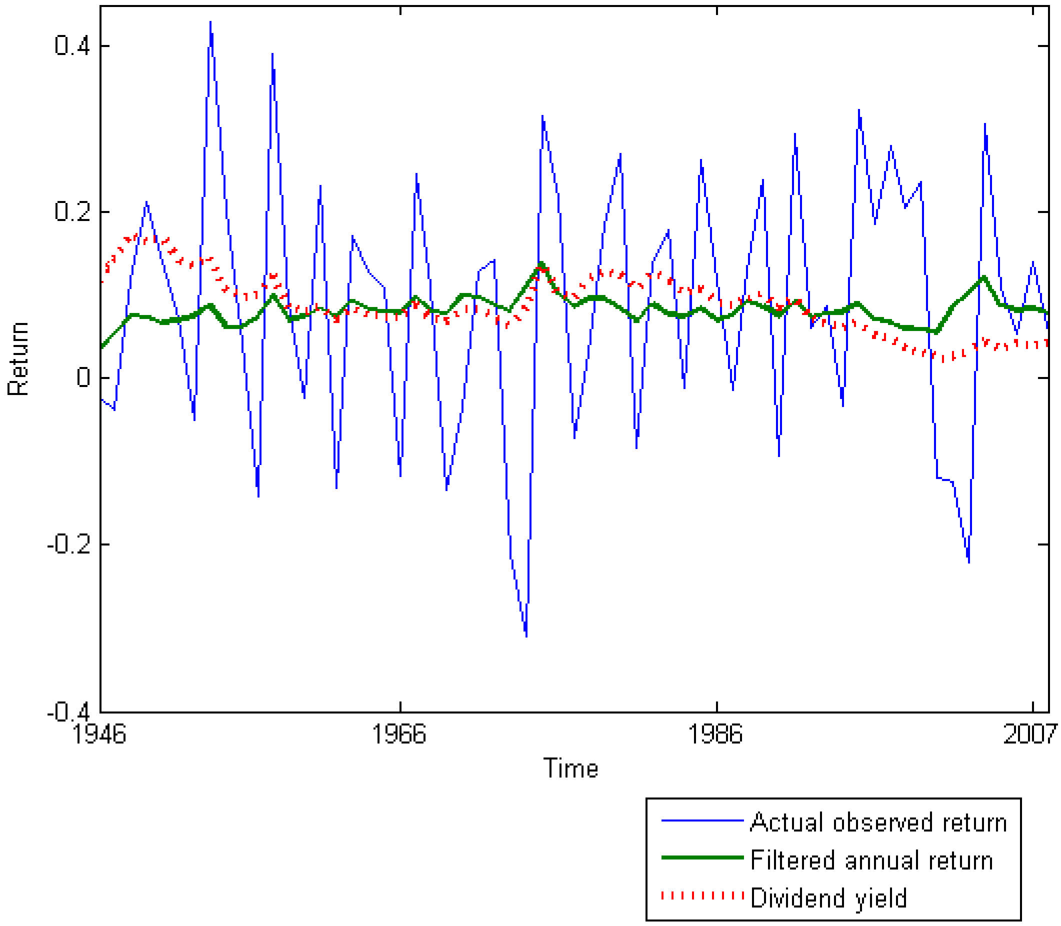

Figure 2.1: **The filtered annualized expected return on CRSP value weighted index:** This figure represents comparative picture of the filtered annual expected return and the dividend-based return. The observation is based on CRSP value weighted quarterly return index from the period of 01/1946 - 12/2007. The x-axis and y-axis show time horizon and the filtered expected return values respectively. Here, the steady line(blue), the bold steady line(green) and the dotted-line(red) are respectively realized annual return, the filtered annual return and the dividend yield estimates.

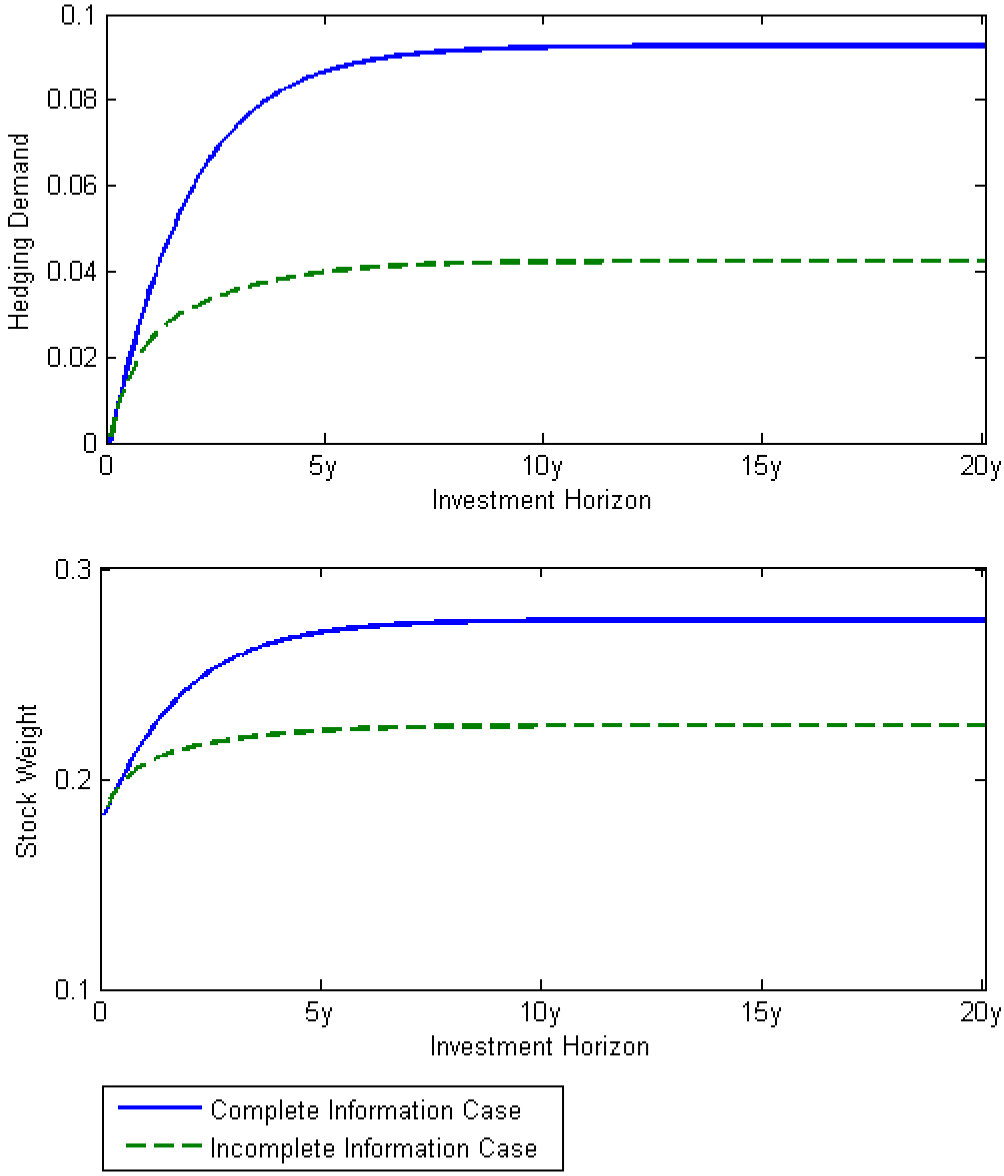

Figure 2.2: **Term Structure of Hedging Demand and Optimal Stock Allocation($\gamma = 10$,).** This graph shows the term structure of the optimal hedging demand and stock allocation. The risk aversion level is ten, and the initial level of the temporary component is zero. The other parameters are based on the *Table-2.1*. The dotted and solid line describes the incomplete information and complete information cases respectively. With complete information, the investor has higher hedging demand than the investor with incomplete information. In each case, the optimal stock allocation increases monotonically with respect to time to maturity.

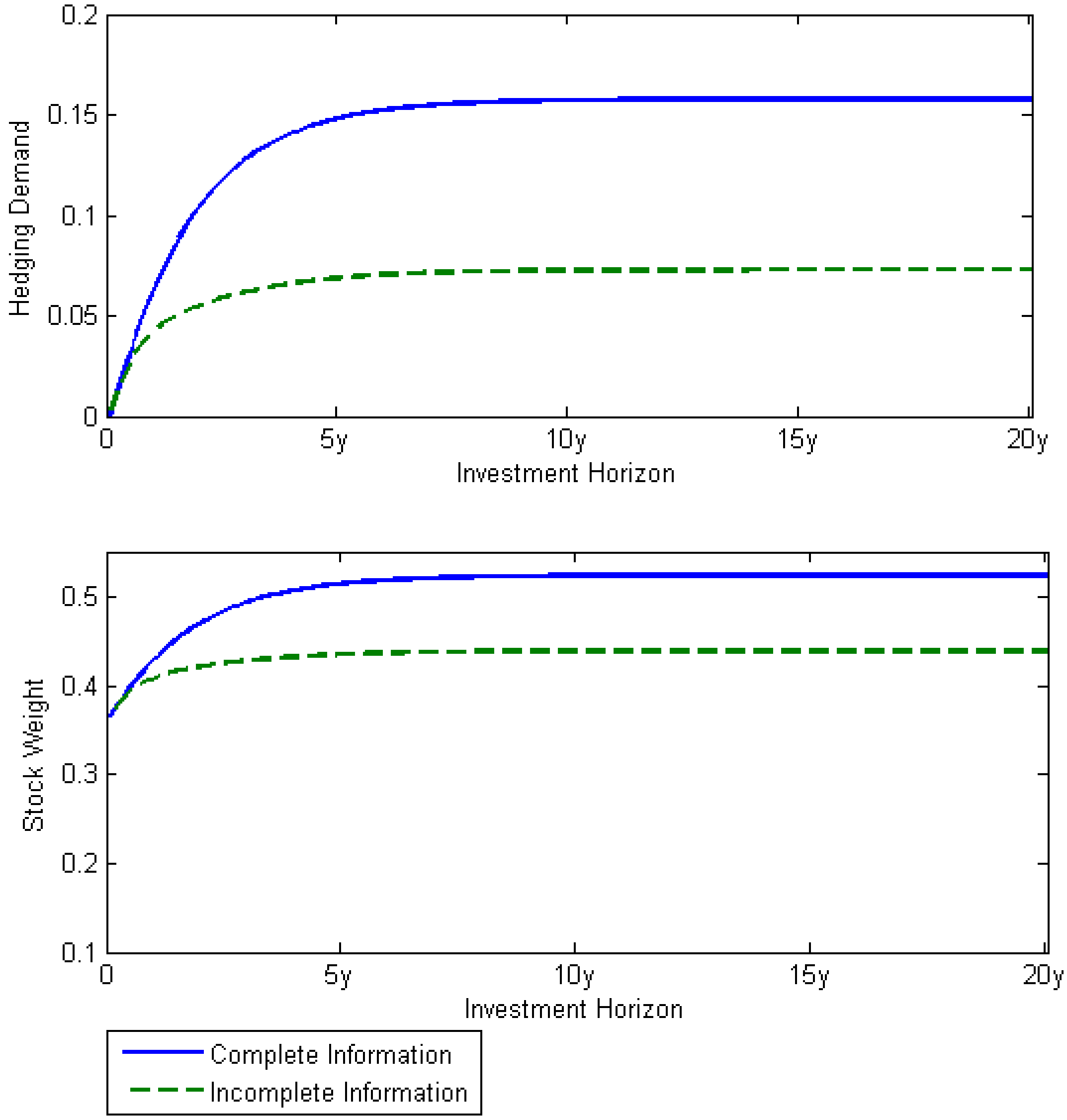

Figure 2.3: **Term Structure of Hedging Demand and Optimal Stock Allocation**($\gamma = 5,$ **).** This graph shows the term structure of the optimal hedging demand and stock allocation. The risk aversion level is five, and the initial level of the temporary component is zero. The other parameters are based on the *Table-2.1*. The dotted and solid line describes the incomplete information and complete information cases respectively. With complete information, the investor has higher hedging demand than the investor with incomplete information. In each case, the optimal stock allocation increases monotonically with respect to time to maturity.

Optimal Stock Allocation (Incomplete Information Case)						
Risk Aversion/Maturity	1month	1-year	3-year	5-year	10-year	30-year
Hedging Demand						
$\gamma = 3$	0.0094	0.0590	0.0829	0.0917	0.0964	0.0967
$\gamma = 5$	0.0066	0.0432	0.0619	0.0688	0.0725	0.0728
$\gamma = 10$	0.0037	0.0247	0.0358	0.0399	0.0422	0.0424
Myopic Demand						
$\gamma = 3$	0.6102	0.6102	0.6102	0.6102	0.6102	0.6102
$\gamma = 5$	0.3661	0.3661	0.3661	0.3661	0.3661	0.3661
$\gamma = 10$	0.1831	0.1831	0.1831	0.1831	0.1831	0.1831
Stock Weight						
$\gamma = 3$	0.6196	0.6692	0.6931	0.7019	0.7066	0.7069
$\gamma = 5$	0.3727	0.4093	0.4280	0.4349	0.4386	0.4389
$\gamma = 10$	0.1868	0.2078	0.2189	0.2230	0.2253	0.2255

Table 2.3: **The Horizon Effect on Optimal Stock Allocation with incomplete information** This table presents the optimal stock allocation, the myopic stock allocation and hedging demand at different investment horizons for different risk aversion levels when the information is assumed to be incomplete. The parameters are estimated by Kalman filtering technique (see Table-2.1) by using quarterly observations on the value weighted index from CRSP database.The results are obtained from closed form solution. The columns in the graph show the hedging demands, myopic demands and stock weights when the investor maximizes the expected utility of terminal wealth. Accordingly, the proportion of the risky asset in portfolio and the investor's hedging demand are positive and monotonically increasing under complete information. But stock weight with complete information is significantly lower than the complete information case

Chapter 3

Revisiting option pricing with stochastic dividend yield

Abstract
We present a simple framework that renders option formulas on an underlying with stochastic dividend yield, in which no risk premium has to be estimated. Our formulas apply to derivatives written on an index in complete markets, and can be extended to incomplete markets under the assumption that the dividend yield risk uncorrelated to index risk is not priced. Given that indexes are broad portfolios of stocks, this assumption is equivalent to the CAPM assertion that only systematic risk (covariance with the returns on the index) is priced.

3.1 Introduction

Stock indexes are typically modeled as paying a continuous dividend yield, and futures and options on indexes are among the most traded derivatives[1], so it is important to find easily implementable formulas, or algorithms, to value them. Traditionally, traders priced these derivatives under the simplifying assumption of a constant dividend yield. Harvey and Whaley (1992) show, however, that this assumption leads to large pricing errors.

What makes valuation challenging in this case, is that the dividend yield is non-tradable and its changes are imperfectly correlated with the stock return, all of which induces market incompleteness. Pricing formulas must include the dividend yield risk premium, which cannot be estimated with precision. This difficulty has hindered the development of models that take the randomness of the dividend yield explicitly into account.

In an early contribution, Geske (1978) proposed a valuation model, but to sort out the dividend yield risk premium issue and obtain option formulas he relied on an equilibrium argument based on the CAPM. Chance, Kumar, and Rich (2002) priced options on an underlying that pays a stochastic dividend; however, they assumed that there is a forward contract written on the present value of all future dividends, an assumption that makes the market effectively complete. More recently, Lioui (2006) argued that we cannot avoid computing the risk premium on the dividend yield, even when the market is complete.

In this paper we present a simple framework that renders option formulas not depending on the dividend yield risk premium. This formula can be applied to derivatives written on an index in complete markets, and can be extended to incomplete markets under the assumption that dividend yield risk orthogonal to index risk is not priced.

Empirical evidence shows that dividend yield changes and index returns are contemporaneously correlated, with correlation close to minus one, suggesting that most dividend yield risk is actually index price risk. We postulate a regression model in which dividend yield changes are linearly related to the dividend yield level and to the index return, the regression error being pure dividend yield risk. The model restricts the mean of the dividend yield to be a function of the index expected return, and we exploit this fact, at the time of risk-neutralizing the model, to extract the index risk premium from the mean dividend yield. We show that, in contrast to Lioui's results, when the market is complete, this is enough to obtain prices in which no risk premium has to be estimated. When the market is incomplete we still need to deal with the risk premium on pure dividend yield risk. We assume that shocks orthogonal to the returns on the index are not priced. Given that indexes are broad portfolios of stocks, this assumption is equivalent to the CAPM assertion that only systematic risk (covariance with the returns on the index) is priced. In this way we obtain formulas -valid in complete and in incomplete markets- for which no risk premia has to be estimated.

Our formulas have more than a theoretical interest. In derivatives pricing, dividend yields are

[1] According to the Futures Industry Association, derivatives on stock indexes explain more than one quarter of the global futures and options volume over the last ten years.

usually modeled as constant. We show that ignoring the randomness in the dividend yield leads to significant mispricing. Mispricing comes from two sources: a mispecified dividend yield, and a mispecified volatility. The dividend yield affects the stock drift under the risk-neutral distribution, so using the wrong yield has a direct impact on derivative prices. A mispecified volatility has a more subtle effect. Under the assumption of a constant dividend yield, the log stock price follows a random walk, and the stock return variance increases linearly with the investment horizon. Due to the negative correlation between changes in the dividend yield and returns, a stochastic dividend yield induces return continuation, or momentum, which implies that the variance of stock returns is larger than the variance corresponding to a random walk at all horizons. As a consequence, the Black-Scholes model underprices options at all maturities. The underpricing is economically significant, especially for out of the money options.

For example, suppose that the current index price is 100, and that the strike price is 110. The dividend yield is stochastic and currently equal to its mean of 3.43%. The instantaneous volatility of the index return is 14.46%, and the long term dividend yield volatility is 1.32%. The interest rate is constant and equal to 2.08%. Under these assumptions[2], the price of a 3-month call option, using our model, is 0.3028. The price of the same call assuming a constant dividend yield of 3.43% and calibrating the instantaneous volatility of the random walk to match the weekly volatility of index returns generated by the random dividend yield model, is 0.2997, a difference of 1% over three-months. This difference is mainly due to the volatility effect and increases with option maturity.

Our results have also consequences for hedging. We compute the "Greeks" of European calls and puts from our model and show that they are different from the ones implied by the Black-Scholes model with constant dividend yield. In particular, the delta of a call is larger in our model, and it can even be larger than one. We show in Section 4 that this result is due to the autocorrelation of stock returns induced by the dividend yield under the risk-neutral measure.

The literature on option pricing with stochastic dividend yield is scarce. Geske (1978) was the first to derive an option pricing formula when the underlying pays a stochastic dividend yield. Due to market incompleteness (he assumes that stock returns and dividend yield changes are imperfectly correlated) Geske (1978) had to rely on an equilibrium argument (Rubinstein (1976)) to obtain the option formula, which, in the end, depends on the CAPM market price of risk. We propose a dividend yield model that renders derivative prices for which no market price of risk has to be computed. Geske also suggests that a major channel through which stochastic dividends may affect the option price is their impact on the variance of stock returns. We further explore this issue, and show that when we calibrate our model to indexes on which derivatives are typically written, the variance effect leads to significant mispricing.

More recently, Lioui (2006) discusses derivative valuation on an underlying paying a stochastic dividend yield under complete markets. One of Lioui's (2006) points is that, even under market completeness, the stochastic dividend yield complicates the implementation of option formulas, because

[2]These values correspond to the CRSP value-weighted index over the period 1-1946, 12-2007, which we take as reference in our calibration exercise.

it is necessary to compute a risk premium. We show that our dividend yield model, which can be interpreted as a reparameterization of Lioui's, renders option prices for which no risk premium must be estimated, suggesting that Lioui's (2006) results are not general.

Our paper is also related to the literature on option pricing with autocorrelated returns. In an early contribution, Lo and Wang (1995) study option pricing when stock returns are predictable. They argue that if returns are predictable, the estimate of the instantaneous variance can be mispecified if computed under the wrong assumption that the stock price follows a random. In particular, the variance will be underestimated when stock returns are negatively autocorrelated, and overestimated when stock returns are positively autocorrelated. Note that this implies that Black-Scholes underprices options when returns exhibit mean reversion, and overprices options when returns exhibit momentum. Lo and Wang (1995) assume a nondividend paying stock. In contrast, in our model predictability is induced by a stochastic dividend yield and it affects capital gains, not the total return. We obtain that, in contrast to Lo and Wang (1995), but consistent with Geske's intuition, returns (capital gains) continuation implies that Black-Scholes underprices options at all maturities.

Following Lioui (2006), we assume that the dividend yield is Gaussian to obtain closed form expressions for forward and option prices. This assumption has the negative implication that there is a positive probability that the dividend yield may become negative, even though this probability can be made negligible by a careful choice of parameters. We also show that the model can be extended to the more realistic assumption that the dividend yield is lognormal (as does Geske (1978)), and that the same results are obtained: that under complete markets there exists a formula to value derivative securities with stochastic dividend yield, in which no risk premium has to be estimated, and that this results also apply to the case of incomplete markets under the assumption that shocks orthogonal to stocks returns are not priced. With a lognormal dividend yield, however, no explicit option formulas can be derived.

The structure of the paper is as follows. In Section 2 we present the stock price dynamics. In Section 3 we discuss the stock price under the risk neutral measure, and derive conditional and unconditional moments of returns. We also extend our discussion to the case on lognormal dividends. In Section 4 we present derivative formulas and discuss hedging. In Section 5 we calibrate the model and show its pricing implication. In Section 6 we conclude.

3.2 Stock price dynamics

Let's assume a frictionless financial market in which trading is continuous. The stock price S_t satisfies the following differential equation:

$$\frac{dS_t}{S_t} = (\mu_t - \delta_t)\, dt + \sigma dW_t, \tag{3.1}$$

where μ_t is the total instantaneous expected return on the stock, δ_t is the stochastic dividend yield, and σ is the instantaneous return volatility. Although we do not model it explicitly, we assume that the expected return is stochastic.

There are two sources of risk in the economy: W_t, which affects both the stock price and the dividend yield, and Z_t, which affects only the dividend yield and is uncorrelated with W_t (see below). Both are standard Wiener processes defined on a filtered probability space $(\Omega, , \Pi)$. There is also a risk-free bond with dynamics:

$$\frac{dB_t}{B_t} = rdt, \tag{3.2}$$

where r, the instantaneous interest rate, is constant.

Assume now that changes in the dividend yield depend on its past level and on changes in the stock through the following equation:

$$d\delta_t = -\omega\left(\delta_t - \alpha\right)dt + \phi\frac{dS_t}{S_t} + \upsilon dZ_t, \tag{3.3}$$

where $\alpha, \omega \geq 0$. The constant ϕ is assumed to be negative to capture the empirical fact that changes in the dividend yield and stock returns are negatively correlated. Equation (3) can be interpreted as a regression of changes in the dividend yield against its own past, a constant $\alpha\omega$, and stock returns[3]. The regression constant is expressed as $\alpha\omega$ for simplicity and without loss of generality. The regression error is dZ_t, so $\frac{dS_t}{S_t} \times dZ_t = 0$. This means that Z_t is pure dividend yield risk [4].

Solving equation (3) gives:

$$\delta_t = \alpha + \left(\delta_0 - \alpha\right)e^{-\omega t} + \phi\int_0^t e^{-\omega(t-u)}\frac{dS_u}{S_u} + \upsilon\int_0^t e^{-\omega(t-u)}dZ_u, \tag{3.4}$$

that is, the dividend yield is the sum of a deterministic function, a weighted average of past stock returns, and a weighted average of errors.

It is straightforward to show that equations (1) and (2) are consistent with Lioui's (2006) model, in which the dividend yield follows an Ornstein-Uhlenbeck process. The main difference is that, in the case of model (1)-(3), the long run mean of the dividend yield is a weighted average of α and of the total expected return on stock μ, while in Lioui's (2006) it is just an arbitrary constant. This is shown in Proposition 1:

The dividend yield is a stationary process satisfying the following stochastic differential equation:

$$d\delta_t = -\kappa\left(\delta_t - \Theta_t\right)dt + \sigma_\delta dW_t + \upsilon dZ_t, \tag{3.5}$$

[3]Binsbergen and Koijen(2010) also use a regression model for the dividend growth rate in which the expected dividend growth is regressed to its own past. However, in their setting, the regression equation is defined as a transition equation which is embedded into a state-space structure. By using the Kalman filtering algorithm, they filter-out the expected dividend growth from the real observations of price dividend ratio and dividend growth.

[4]From technical point of view, our assumption - that dividend yield risk uncorrelated to index risk is not priced - 'resembles' to Merton(1976)'s orthogonality assumption on jumps. However, the economic motivation between these two models is different. In Merton' s paper, he particularly focused on a single stock, while we focus on an index.

where:

$$\begin{aligned}\kappa &= \omega + \phi \\ \Theta &= \frac{\omega}{\kappa}\alpha + \frac{\phi}{\kappa}\mu_t \\ \sigma_\delta &= \phi\sigma\end{aligned} \tag{3.6}$$

Plugging equation (2) in (3), we get:

$$\begin{aligned}d\delta_t &= \omega\alpha dt - \omega\delta_t dt + \phi\left[(\mu_t - \delta_t)\, dt + \sigma dW_t\right] + \upsilon dZ_t \\ &= -(\omega + \phi)\left(\delta_t - \frac{\omega\alpha + \phi\mu_t}{\omega + \phi}\right) dt + \phi\sigma dW_t + \upsilon dZ_t.\end{aligned} \tag{3.7}$$

The last equality proves the proposition.

Equation (5) restricts the long run mean of the dividend yield to be a function of the total expected return on the stock (and other variables). This restriction comes from equation (3), and makes it possible to extract the risk premium on the stock from both equations (1) and (3) when, later, we risk-neutralize the model.

3.3 The price process under the Q-measure

By parametrization the dividend yield as in equation (3), only the risk premium representing compensation for bearing pure dividend yield risk enters the derivative formulas. If the market is complete, or if this risk is not priced, the model renders prices for which no risk premium has to be estimated. This is a key advantage of using equation (3) to describe the dynamics of the dividend yield. This equation restricts the long run mean of the dividend yield to be a function of the expected return on the stock price. Should the mean dividend yield be left unrestricted, as in Lioui (2006), then the risk premium on the stock would enter the dividend yield formula, and then both risk premia would have to be estimated.

Equation (1) defines μ_t as the total expected return on the stock. Define now r as the constant[5] instantaneous risk-free interest rate, and λ_t as the stock risk premium. Then, the total expected return can be decomposed as:

$$\mu_t \equiv r + \lambda_t. \tag{3.8}$$

Plugging equation (8) back in (1) gives the risk-neutralized commodity price process:

$$\begin{aligned}\frac{dS_t}{S_t} &= (r - \delta_t)\, dt + \sigma\left(\frac{\lambda_t}{\sigma} dt + dW_t\right) \\ &= (r - \delta_t)\, dt + \sigma dW_t^*,\end{aligned} \tag{3.9}$$

[5]We assume for simplicity that the risk-free rate is constant. The model can be straightforwardly extended to accommodate time-varying interest rates.

where, by Girsanov's theorem, $W_t^* = \int_0^t \frac{\lambda_s}{\sigma} ds + W_t$ is a Brownian motion under the risk neutral measure Q. Thus, the total expected return on the stock under Q is r. Interestingly, the risk-neutralized process for δ_t does not depend on the stock risk premium either, even though the dividend yield is affected by stock price risk. To see this, plug (9) in (3) to get:

$$\begin{aligned} d\delta_t &= -\kappa \left(\delta_t - \Theta^*\right) dt + \sigma_\delta \left(\frac{\lambda_t}{\sigma} dt + dW_t\right) + \upsilon \left(\frac{\lambda_\delta}{\upsilon} dt + dZ_t\right) \\ &= -\kappa \left(\delta_t - \Theta^*\right) dt + \sigma_\delta dW_t^* + \upsilon dZ_t^* \end{aligned} \tag{3.10}$$

where λ_δ is the risk premium on pure dividend yield risk (assumed constant for simplicity), $Z_t^* = \frac{\lambda_\delta}{\sigma} t + Z_t$ is a Brownian motion under the risk neutral measure Q. and:

$$\Theta^* = \frac{\omega\alpha - \lambda_\delta}{\kappa} + \frac{\phi}{\kappa} r. \tag{3.11}$$

So neither S_t nor δ_t depend on λ_t under Q. Moreover, if the market is complete $(\upsilon \equiv 0)$, or if pure dividend yield risk is not priced $(\lambda_\delta \equiv 0)$, no risk premium will affect the derivatives formulas.

3.3.1 A taxonomy of shocks

In this section we show that, with constant interest rates, $\phi < 0$ is a sufficient condition for stock returns to exhibit momentum under the the risk-neutral measure. Note that this is true even if a time varying risk premium induces return mean-reversion under the statistical measure. In this subsection we present a taxonomy of shocks and discuss the conditional variance of returns. In the next subsection we discuss unconditional second moments in general.

Solving equation (10) we obtain the dividend yield process under the risk-neutral measure:

$$\delta_t = \Theta^* + \left(\delta_t - \Theta^*\right) e^{-\kappa t} + \sigma_\delta \int_0^t e^{-\kappa(u-t)} dW_t^* + \upsilon \int_0^t e^{-\kappa(u-t)} dZ_t^* \tag{3.12}$$

Define $s_t = \log\left(S_t\right)$ .Then:

$$ds_t = \left(r - \frac{1}{2}\sigma^2 - \delta_t\right) dt + \sigma dW_t^*, \tag{3.13}$$

We now integrate equation (13) to get the log index return:

$$s_{t+\tau} - s_t = \Omega_\tau + \frac{\sigma}{\kappa} \int_t^{t+\tau} \left(\omega + \phi e^{-\kappa(t+\tau-u)}\right) dW_t^* - \frac{\upsilon}{\kappa} \int_0^t \left(1 - e^{-\kappa(t+\tau-u)}\right) dZ_t^*, \tag{3.14}$$

where:

$$\Omega_\tau = \left(r - \frac{1}{2}\sigma^2 - \Theta^*\right) \tau - \left(\delta_t - \Theta^*\right) \frac{\left(1 - e^{-\kappa\tau}\right)}{\kappa}, \tag{3.15}$$

W_t^* has direct and indirect (through the dividend yield) effects on the index log return. Z_t^* has only indirect effects. The parameter $\phi < 0$ implies that direct shocks to the index tend to propagate in the long run. To see this, note that, in equation (14), the expressions in the integrals inside the parentheses give the "term structure of shocks". A direct shock of mean zero and variance σ^2, that occurred at t, has a residual impact on $s_{t+\tau}$ of $1 - \frac{\phi}{\kappa}\left(1 - e^{-\kappa\tau}\right) = \frac{\omega}{\kappa} + \frac{\phi}{\kappa}e^{-\kappa\tau}$. As τ grows without bound, this residual impact converges to:

$$\frac{\omega}{\kappa} \geq 1. \tag{3.16}$$

To see this, let us assume $\omega > 0$. When $\phi = 0$, W_t^*-shocks have a residual impact of exactly 1. In contrast, when $\phi < 0$, we have $0 < \kappa = \omega + \phi < \omega$, and the residual impact of a shock experienced at t, as τ grows without bound, is $\frac{\omega}{\kappa} > 1$. This means that W_t^*-shocks further propagate in the long run.

Z_t^*-shocks have a residual impact of: $\frac{1}{\kappa}$

As $0 < \kappa < 1$, Z_t^* shocks also further propagate in the log run.

Therefore, under Q, the logarithm of S_T is normally distributed, with conditional mean Ω_τ and conditional variance Σ_τ, where:

$$\Sigma_\tau = \left(\frac{\sigma^2}{\kappa^2}\omega^2 + \frac{\upsilon^2}{\kappa^2}\right)\tau + 2\left(\frac{\sigma^2}{\kappa^2}\phi\omega - \frac{\upsilon^2}{\kappa^2}\right)\frac{1 - e^{-\kappa\tau}}{\kappa} + \left(\frac{\sigma^2}{\kappa^2}\phi^2 + \frac{\upsilon^2}{\kappa^2}\right)\frac{1 - e^{-2\kappa\tau}}{2\kappa} \tag{3.17}$$

where $\tau = T - t$; that is, the moments are calculated conditional on information up to time t.

Note that if $\phi = \upsilon = 0$, $\Sigma_\tau = \sigma^2\tau$. That is, the variance grows linearly with time to maturity, which corresponds to the random walk case. If $\phi < 0$, we can show that $\Sigma_\tau > \sigma^2\tau$. To see this, write:

$$\Sigma_\tau = \sigma^2\tau\left(\frac{\kappa^2 - 2\phi\omega c_1 - \phi^2 c_2}{\kappa^2}\right) + \upsilon^2\tau\left(\frac{2c_1 - c_2}{\kappa^2}\right), \tag{3.18}$$

where:

$$c_1 = 1 - \frac{1 - e^{-\kappa\tau}}{\kappa\tau},$$

and:

$$c_2 = 1 - \frac{1 - e^{-2\kappa\tau}}{2\kappa\tau},$$

It can be shown that for $\kappa > 0$ and $\tau > 0$, $2c_1 > c_2$ (see Appendix). Then, it follows from $\omega + \phi > 0$ that:

$$-2\phi\omega c_1 - \phi^2 c_2 > \phi^2\left(2c_1 - c_2\right) > 0,$$

so $\left(\frac{\kappa^2 - 2\phi\omega c_1 - \phi^2 c_2}{\kappa^2}\right) > 1$,which means that the stock return variation due to pure stock price risk is larger than $\sigma^2\tau$.

Note also that for large values of τ,

$$2c_1 - c_2 \approx 1,$$

so the term multiplying $\upsilon^2\tau$ in equation (18) will be close to $\frac{1}{\kappa^2} > 1$ under our assumptions. This means that the contribution of pure dividend yield variation to the variance of stock returns is larger than $\upsilon^2\tau$.

3.3.2 Unconditional moments

In this subsection we show that stock price changes will exhibit continuation, or positive autocorrelation, when they are negatively correlated to changes in the dividend yield.

The argument below shows this formally. First, define the τ-period price change as:

$$r_{t+\tau} = s_{t+\tau} - s_t.$$

Then, integrating equation (13) gives:

$$r_{t+\tau} = \Omega_\tau + \frac{\sigma}{\kappa}\int_t^{t+\tau}\left(\omega + \phi e^{-\kappa(t+\tau-u)}\right)dW_t^* - \frac{\upsilon}{\kappa}\int_0^t\left(1 - e^{-\kappa(t+\tau-u)}\right)dZ_t^* \tag{3.19}$$

From equation (19) it is possible to calculate the unconditional variance of $r_{t+\tau}$:

$$Var\left(r_{t+\tau}\right) = \frac{\sigma^2}{\kappa^2}\left[\omega^2\tau + \frac{2\phi}{\kappa}\left(\omega + \frac{\phi}{2}\right)\left(1 - e^{-\kappa\tau}\right)\right] + \frac{\upsilon^2}{\kappa^2}\left[\tau - \frac{\left(1 - e^{-\kappa\tau}\right)}{\kappa}\right], \tag{3.20}$$

and the covariance between r_t and $r_{t+\tau}$ (see the Appendix for details on the derivations of these two equations):

$$\begin{aligned} Cov\left(r_t, r_{t+\tau}\right) &= -\frac{\sigma^2}{\kappa^2}\frac{\phi}{\kappa}\left(\omega + \frac{\phi}{2}\right)\left(1 - e^{-\kappa\tau}\right)^2 + \frac{\upsilon^2}{\kappa^2}\frac{\left(1 - e^{-\kappa\tau}\right)^2}{2\kappa} \\ &= -\left[\frac{\sigma^2}{\kappa^2}2\phi\left(\omega + \frac{\phi}{2}\right) - \frac{\upsilon^2}{\kappa^2}\right]\frac{\left(1 - e^{-\kappa\tau}\right)^2}{2\kappa}. \end{aligned} \tag{3.21}$$

Therefore, the first autocorrelation of τ-period price changes can be expressed as:

$$\rho\left(r_t, r_{t+\tau}\right) = \frac{-\frac{\sigma^2}{\kappa^2}\frac{\phi}{\kappa}\left(\omega + \frac{\phi}{2}\right)\left(1 - e^{-\kappa\tau}\right)^2 + \frac{\upsilon^2}{\kappa^2}\frac{\left(1 - e^{-\kappa\tau}\right)^2}{2\kappa}}{\frac{\sigma^2}{\kappa^2}\left[\omega^2\tau + \frac{2\phi}{\kappa}\left(\omega + \frac{\phi}{2}\right)\left(1 - e^{-\kappa\tau}\right)\right] + \frac{\upsilon^2}{\kappa^2}\left[\tau - \frac{\left(1 - e^{-\kappa\tau}\right)}{\kappa}\right]}. \tag{3.22}$$

The denominator in equation (22) is the unconditional variance of stock returns, so it is positive. The second term in the numerator is also positive. In the first term we have (remember that $\phi < 0$):

$$0 < \kappa = \omega + \phi < \omega + \frac{\phi}{2}$$

So $\phi < 0$ is a sufficient condition for momentum ($\rho\left(r_t, r_{t+\tau}\right) > 0$).

This result applies to the risk-neutral dynamics of stock price changes. Under the statistical dynamics, price changes may be mean reverting (mean reversion may be induced by a time-varying risk premium, for example). What the previous result shows is that they will exhibit momentum under Q if the interest rate is constant and $\phi < 0$.

3.3.3 Lognormal dividends

A normally distributed dividend yield may become negative with positive probability, although this probability can be made as small as desired by a judicious choice of parameters. Geske (1978), studied the more realistic case in which the dividend yield is lognormal distributed, but he relied on an equilibrium argument (Rubinstein (1976)) to obtain option formulas, which, in the end, depended on the CAPM market price of risk. In this subsection we show that our method of extracting the index risk premium still applies when the dividend yield is lognormal distributed, and so cannot become negative.

Let δ_t^L be the natural logarithm of the dividend yield, and V_t the total return process on the index:

$$\frac{dV_t}{V_t} = \frac{dS_t}{S_t} + \delta_t^L dt$$

Assume now the dynamics of the log dividend is described by the regression:

$$d\delta_t^L = -\kappa\left(\delta_t^L - \alpha^L\right)dt + \phi^L \frac{dV_t}{V_t} + \upsilon^L dZ_t, \tag{3.23}$$

Replacing the total return with (23), we get:

$$d\delta_t^L = -\kappa\left(\delta_t^L - \alpha^L - \frac{\phi^L \mu_t}{\kappa}\right)dt + \phi^L \sigma dW_t + \upsilon^L dZ_t, \tag{3.24}$$

Then, we define, as before, $W_t^* = \int_0^t \frac{\lambda_s}{\sigma} ds + W_t$ and $Z_t^* = \frac{\lambda_\delta}{\sigma} t + Z_t$, where λ_t and λ_δ are the index and pure (log) dividend yield risk premia. By Girsanov's theorem[6], W_t^* and Z_t^* are Brownian motions under the risk neutral measure Q. Again, the risk-neutralized process for δ_t^L does not depend on the stock risk premium. To see this, plug (9) in (23) to get:

$$\begin{aligned} d\delta_t^L &= -\kappa\left(\delta_t^L - \Theta^*\right)dt + \phi^L \sigma \left(\frac{\lambda_t}{\sigma} dt + dW_t\right) + \upsilon\left(\frac{\lambda_\delta}{\upsilon} dt + dZ_t\right) \\ &= -\kappa\left(\delta_t^L - \Theta^*\right)dt + \phi^L \sigma dW_t^* + \upsilon dZ_t^* \end{aligned} \tag{3.25}$$

where λ_δ is the risk premium on pure dividend yield risk, $Z_t^* = \frac{\lambda_\delta}{\sigma} t + Z_t$ is a Brownian motion under the risk neutral measure Q and:

$$\Theta^* = \alpha - \lambda_\delta + \frac{\phi^L}{\kappa} r. \tag{3.26}$$

Neither S_t nor δ_t^L depend on λ_t under Q. Moreover, if the market is complete $(\upsilon \equiv 0)$, or if pure

[6]There are technical conditions to be met in order to apply Girsanov's theorem, but they are automatically satisfied when the risk premium is constant. See Karatzas and Shreve (1991).

dividend yield risk is not priced $(\lambda_\delta \equiv 0)$, no risk premium will affect the derivatives formulas.

3.4 Derivative Formulas

In this section we derive derivative formulas under the assumption that $\lambda_\delta = 0$, that is, pure dividend yield risk is not priced. This assumption is equivalent to choosing a given probability measure Q, equivalent to Π, such that the discounted prices of the stock (cum dividend) and of other traded assets are martingales under Q (Harrison and Kreps (1979)).

In the last section we obtained the stock price process under the Q-measure; now we derive formulas for futures and European option prices. These formulas allow us to price derivative contracts without the need to estimate any risk premia.The futures price[7] for delivery of one share of the stock τ periods ahead is the expected stock price under the risk-neutral measure. Given the normality of log (S_t) under Q, the futures price is easily obtained in closed form:

$$\begin{aligned} F_\tau &= E_t^Q (S_T) \\ &= S_t \times \exp\left(\Omega_\tau + \frac{1}{2}\Sigma_\tau\right). \end{aligned} \tag{3.27}$$

The price of a European call option written on the stock, with maturity T and strike K, is the expectation under Q of its payoff at maturity, discounted by the risk-free rate:

$$C_t = e^{-r\tau} E_t^Q \left[S_T \times \mathbf{1}_{\{S_T > K\}}\right] - e^{-r\tau} K P^Q (S_T > K), \tag{3.28}$$

where $\mathbf{1}_{\{S_T>K\}}$ is the indicator function of the event $\{S_T > K\}$, $E_t^Q \left[(S_T\ ) \times \mathbf{1}_{\{S_T>K\}}\right]$ is the Q-expected value of the stock at maturity, conditioned on the event that the option will be exercised at maturity, and $P^Q (S_T > K)$ is the probability under Q of this event. Due to the normality of $\log(S_t)$, the expectation in the first term of equation (29) can be solved as:

$$E_t^Q \left[S_T \times \mathbf{1}_{\{S_T>K\}}\right] = S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N(d_1), \tag{3.29}$$

where $N(d_1)$ is the value of the Normal cumulative distribution function at d_1, and:

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \Omega_\tau + \Sigma_\tau}{\sqrt{\Sigma_\tau}}. \tag{3.30}$$

The probability of the option finishing in -the-money is $P^Q (S_T > K) = N(d_2)$, where: $d_2 = d_1 - \sqrt{\Sigma_\tau}$. So:

$$C_t = \left[S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N(d_1) - K N(d_2)\right] e^{-r\tau}. \tag{3.31}$$

The price of a European put on the same index can be found using put-call parity. That is, because

[7]Note that the words ”futures price” and ”forward price” can be used interchangeably in this context, because they are equal under the current assumption of a constant risk-free rate.

buying a call and shorting a put, both with maturity T and strike K, is equivalent to having a long position in a forward contract with maturity T and forward price K, we can express the put price as:

$$P_t = C_t - \left[E_t^Q\left(S_T\right) - K\right] e^{-r\tau} \tag{3.32}$$

Plugging (30) and (32) in (33) we get:

$$P_t = \left[KN\left(-d_2\right) - S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N\left(-d_1\right)\right] e^{-r\tau}. \tag{3.33}$$

3.4.1 Hedging index risk and delta

The financial market in this paper is incomplete, because there are two sources of risk, one of them nontradable. It is not possible to construct a riskless hedge by continuously trading in the stock and a riskless bond: the investor cannot avoid bearing pure dividend yield risk, for which she demands compensation in the form of a risk premium. The investor can, however, completely eliminate index risk, but to do that she must choose a delta differente from the BS delta. This section shows how to construct such a hedge.

Assume that a call has been written on the stock and that a hedging portfolio is started consisting on the shorted call and a long position in the underlying stock. The initial value of the portfolio is:

$$\Pi_t = \Delta S_t - C\left(S_t, \delta_t, t\right). \tag{3.34}$$

where Δ is the number of long units on the stock. The change in the value of the portfolio over the next period is:

$$\begin{aligned} d\Pi_t &= \Delta dS_t + \Delta\delta_t S_t dt \\ &- \frac{\partial C}{\partial S} dS_t - \frac{\partial C}{\partial \delta}\left[-\omega\left(\delta_t - \alpha\right)dt + \phi\frac{dS_t}{S_t} + \upsilon dZ_t\right] \\ &- \frac{\partial C}{\partial t}dt - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 dt - \frac{1}{2}\frac{\partial^2 C}{\partial \delta^2}\left(\sigma_\delta^2 + \upsilon^2\right)dt - \frac{\partial^2 C}{\partial S \partial_\delta}\sigma\sigma_\delta dt, \end{aligned} \tag{3.35}$$

where in the second term of the second equation line we replace $d\delta_t$ with the right-hand side of equation (3).

The index risk in the portfolio comes from its exposure to W_t. To eliminate this risk, choose:

$$\Delta = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial \delta}\frac{\phi}{S_t}. \tag{3.36}$$

The option delta has two components: the traditional delta of the BS formula ($\frac{\partial C}{\partial S}$), and a second component needed to hedge the dividend yield exposure to index risk ($\frac{\partial C}{\partial \delta}\frac{\phi}{S_t}$). Given the option type (call or put), the sign of this last component depends on ϕ, the parameter capturing the correlation between dividend yield changes and stock returns. In the call case, $\frac{\partial C}{\partial \delta}$ is negative. As ϕ is also negative, due to the negative correlation between changes in the dividend yield and stock returns, the call delta is larger than the BS delta, and, as we show later, can become even larger than one when the option is deep in the money.

Plugging (36) in (35) cancels the portfolio's overall exposure to W_t. To preclude arbitrage, the portfolio must earn an average rate that compensates the investor for waiting and for bearing pure dividend yield risk:

$$\left(\frac{\partial C}{\partial S}+\frac{\partial C}{\partial \delta}\frac{\phi}{S_t}\right)\delta_t S_t+\frac{\partial C}{\partial \delta}\omega\left(\delta_t-\alpha\right)-A\left(t\right)=\left(r+\lambda_\delta\right)\Pi_t, \tag{3.37}$$

where:

$$A\left(t\right)=\frac{\partial C}{\partial t}dt+\frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2+\frac{1}{2}\frac{\partial^2 C}{\partial \delta^2}\left(\sigma_\delta^2+\upsilon^2\right)+\frac{\partial^2 C}{\partial S\partial \delta}\sigma\sigma_\delta. \tag{3.38}$$

Operating on (33) we get:

$$\frac{\partial C}{\partial S}S_t\left(r+\lambda_\delta-\delta_t\right)+\frac{\partial C}{\partial \delta}\left[\phi\left(r+\lambda_\delta\right)+\omega\left(\delta_t-\alpha\right)\right]+A\left(t\right)-\left(r+\lambda_\delta\right)C=0, \tag{3.39}$$

Equation (35) is the fundamental partial differential equation that all contingent claims written on the stock must satisfy. The nature of the derivative at hand will be determined by the boundary conditions.

In Proposition-2 we compute the delta of the call:

The delta of a European call option is:

$$\begin{aligned}\Delta &= \frac{\partial C}{\partial S}+\frac{\partial C}{\partial \delta}\frac{\phi}{S} &(3.40)\\ &= \frac{\partial C}{\partial S}\left[1-\frac{\phi}{\kappa}\left(1-e^{-\kappa\tau}\right)\right]. &(3.41)\end{aligned}$$

See Appendix.

Note that $\Delta \geq 0$. Even though as expected, $\Delta \to \left\{\begin{smallmatrix}1 \text{ if } S_t>K\\ 0 \text{ if } S_t\leq K\end{smallmatrix}\right.$ as $\tau \to 0$, Δ can be above 1 if $\phi < 0$, that is, if changes in the dividend yield and stock returns are negatively correlated. In this case, changes in the dividend yield induce positive autocorrelation in stock returns under the risk neutral distribution, which lead stock returns over discrete time intervals to be more volatile than when the dividend yield is constant, and so delta is larger accordingly. This result depends on the sign of ϕ and not on the fact that now the option seller must use the stock to hedge also part of the dividend yield risk. Should ϕ be positive, inducing negative correlation of stock returns over discrete time intervals, delta would be always below 1, and even below the Black-Scholes delta.

3.5 Pricing implications

In this section we investigate the empirical consequences of our model. We take as a reference the CRSP value-weighted index and calibrate the model in equation (8) to reproduce its first and second moments over the period 01-1946, 12-2007. Although no options are written on this index, its behavior is similar to other indexes on which options are written, such as the SP100 or the SP500. Then, we compare our pricing results to Black-Scholes prices under the constant dividend yield hypothesis.

The moments to match are summarized in Table 1:

Table 1: Moments

Monthly Volatility of $\frac{dS}{S}$	0.0418
Volatility of δ	0.0132
Average δ	0.0343
Correlation ($\frac{dS}{S}$, $d\delta$)	-0.850
Mean rev speed of δ (year)	0.10

The stock has total average return $\mu = 0.0808$, and instantaneous volatility $\sigma = 0.1446$. The instantaneous volatility of the dividend yield is $\sqrt{\sigma_\delta^2 + \nu^2} = 0.0059$, and κ, the annualized mean reversion speed is given a value of 0.10, which is the actual annualized mean reversion speed of the index dividend yield. These values imply an unconditional dividend yield volatility of:

$$\frac{\sqrt{\sigma_\delta^2 + \nu^2}}{\sqrt{2\kappa}} = 0.0132,$$

also in line with data.

The parameter κ is decomposed as:

$$\kappa = \omega + \phi,$$

where $\phi = -0.0347$, and $\omega = 0.1347$. We obtain ϕ by solving the equation:

$$\phi = \frac{\rho\sqrt{\sigma_\delta^2 + \nu^2}}{\sigma},$$

where $\rho = -0.85$ is the contemporaneous correlation between dividend yield changes and index returns found in the data. Finally, we obtain ν as

$$\nu = 0.0059\sqrt{1 - \rho^2}.$$

These parameters imply an $\alpha = 0.0463$. The annual risk-free interest rate is assumed $r = 0.0208$, to obtain an equity premium equal to 6%.

We plug these parameters in equation (21) to match exactly the monthly unconditional return volatility of 0.0418. The implied annual return volatility is 0.1476, sightly higher than the volatility obtained by multiplying 0.0418 by $\sqrt{12}$, which is what we would do if the index price were geometric Brownian motion. The difference in the volatilitites, due to the momentum induced by the dividend yield under the Q measure, will partly explain why Black-Scholes missprices options in our model.

Results should not depend on an artificially inflated momentum, or on a dividend yield likely to become negative, so a main is to guarantee that the first autocorrelation of returns and the probability of a negative dividend yield are sufficiently low. The monthly autocorrelation of stock returns implied by our parameters is 0.0035, lower than the data (2.2%). With our parameterization, the probability of a negative dividend yield is 0.45% (that is, we will observe a negative dividend once every 213 years). These values seem low enough to conduct the exercise. The values of the parameters are summarized in Table 2.

Table 2: Parameter values

Parameters	
μ	0.0808
$\overline{\delta}$	0.0343
σ	0.1446
υ	0.0031
ϕ	-0.0347
ω	0.1347
r	0.0208

As noted above, the benchmark case is the Black-Scholes price, computed under the assumption that the stock is a random walk. In the benchmark case, the dividend yield is constant and equal to 3.43%. Also, the instantaneous volatility of the benchmark case is $\sigma_{BS} = 0.1448$. We obtain this value by assuming a trader who estimates the volatility on a weekly basis and, ignoring time variation in the dividend yield, extrapolates to longer horizons using the rule of the square root.

Table 3 compares Black-Scholes prices and prices obtained from equation (31) and (33) for various holding periods (one week to one year) and strikes, and for three different values of the of the dividend yield δ_t: the mean dividend yield, and the mean plus and minus one standard deviation.

There are two forces explaining the differences between Black-Scholes prices and prices obtained from equations (31) and (33) reported in Table 3. On the one hand, there is the volatility effect, arising from the fact that $Var\left(r_{t+\tau}\right) > \sigma^2\tau$. On the other hand there is a level effect, stemming from the influence of current state of the dividend yield. Note that the level effect is not affected by the risk premium, because formulas (32) and (34) do not include it. The volatility effect increases the prices of options relative to Black-Scholes prices for all maturities and across all strikes, although this effect is relatively more pronounced for out-of-the money options. The level effect increases the prices of calls and reduces the prices of puts when the dividend yield is relatively low (for example, after a stock price rally), and reduces the prices of calls and increases the prices of puts when the dividend yield is relatively high. The level effect applies also for all maturities and across strikes.

Results reported in table 3 can be summarized as follows. Call prices are higher than the corresponding Black-Scholes prices when $\delta_t = \overline{\delta} - vol$, and decrease with δ_t and eventually become lower than Black-Scholes prices as $\overline{\delta} + vol$. Interestingly, they are still higher that their Black-Scholes counterparts when $\delta_t = \overline{\delta}$, which shows that the pure volatility effect is strong, especially for longer maturities. As an example, the price of a 3-month out-of-the-money call struck at 110 is 11% higher than the Black-Scholes price when $\delta_t = \overline{\delta} - vol$, while it is 1% higher when $\delta_t = \overline{\delta}$, and 8.5% lower when $\delta_t = \overline{\delta} + vol$. Put prices are lower than the corresponding Black-Scholes prices when $\delta_t = \overline{\delta} - vol$, and increase with δ_t and eventually become higher than Black-Scholes prices as $\delta_t = \overline{\delta} + vol$. The pure volatility effect also works for puts. As a final example, the price of a 3-month out-of-the-money put struck at 90 is 8% lower than the Black-Scholes price when $\delta_t = \overline{\delta} - vol$, while it is 1% higher when $\delta_t = \overline{\delta}$, and 11% higher when $\delta_t = \overline{\delta} + vol$.

The economic significance of our results is centered on stock momentum which is induced by stochastic dividend yield. Momentum amplifies the volatility effect relative to the case in which dividend yield is assumed constant. Ignorance of momentum not only leads the investor to an economic loss due to mispricing but also to imperfect hedging.

Table 3: Call and Put Prices

Table 3 compares Black-Scholes call (BS call) and put (BS put) option prices under geometric Brownian motion to call and put prices from equations (31) and (33). Parameters are as in table 1. The stock on which the options are written has a current value of $100. In the case of Black-Scholes, the dividend yield is assumed constant and equal to 4%. In the case of equations (31) and (33), the average dividend yield is 4%. Prices are compared for three values of the state variable δ_t corresponding to negative, constant, and positive performance of the stock, respectively. Please find the table in the next page.

Table 3: Call and Put Prices

Strike	BS Call	Call (Eq. 31) $\delta_t=\bar{\delta}-vol$	$\delta_t=\bar{\delta}$	$\delta_t=\bar{\delta}+vol$	BS Put	Put (Eq. 33) $\delta_t=\bar{\delta}-vol$	$\delta_t=\bar{\delta}$	$\delta_t=\bar{\delta}+vol$
Time to maturity: 7 days ($T-t=7/3604$)								
80	19.9661	19.9914	19.9660	19.9407	0.0000	0.0000	0.0000	0.0000
90	9.9701	9.9954	9.9700	9.9447	0.0000	0.0000	0.0000	0.0000
100	0.7877	0.7996	0.7869	0.7743	0.8137	0.8002	0.8129	0.8256
110	0.0000	0.0000	0.0000	0.0000	10.0219	9.9966	10.0220	10.0473
120	0.0000	0.0000	0.0000	0.0000	20.0179	19.9926	20.0180	20.0433
Time to maturity: 91 days ($T-t=91/364$)								
80	19.5633	19.8825	19.5592	19.2370	0.0022	0.0019	0.0022	0.0026
90	9.8582	10.1578	9.8573	9.5600	0.2453	0.2253	0.2486	0.2738
100	2.7034	2.8731	2.7097	2.5527	3.0386	2.8888	3.0491	3.2146
110	0.2997	0.3340	0.3028	0.2740	10.5830	10.2979	10.5902	10.8841
120	0.0129	0.0152	0.0132	0.0114	20.2443	19.9272	20.2488	20.5696
Time to maturity 182 days ($T-t=182/364$)								
80	19.1835	19.7957	19.1703	18.5504	0.0562	0.0500	0.0593	0,0701
90	10.0860	10.6292	10.0895	9.5627	0.8552	0.7799	0.8750	0.9790
100	3.7030	4.0451	3.7237	3.4201	4.3687	4.0924	4.4058	4.7329
110	0.8871	1.0182	0.9028	0.7982	11.4494	10.9621	11.4814	12.0075
120	0.1402	0.1707	0.1454	0.1234	20.5990	20.0111	20.6206	21.2293
Time to maturity: 273 days ($T-t=273/364$)								
80	18.8999	19.7757	18,8773	17,9935	0.2013	0.1794	0.2147	0.2557
90	10.3554	11.1153	10.3665	9.6448	1.5020	1.3643	1.5491	1.7522
100	4.4173	4.9286	4.4550	4.0135	5.4091	5.0228	5.4828	5.9662
110	1.4387	1.6882	1.4718	1.2781	12.2757	11.6276	12.3449	13.0759
120	0.3637	0.4529	0.3801	0.3176	21.0459	20.2375	21.0983	21.9606
Time to maturity: 364 days ($T-t=364/364$)								
80	18.6896	19.8035	18.6587	17.5416	0.4147	0.3707	0.4466	0.5349
90	10.6084	11.5674	10.6284	9.7334	2.1276	1.9288	2.2105	2.5209
100	4.9826	5.6583	5.0382	4.4669	6.2959	5.8138	6.4144	7.0485
110	1.9301	2.3068	1.9832	1.6965	13.0376	12.2564	13.1536	14.0723
120	0.6267	0.7948	0.6589	0.5432	21.5283	20.5386	21.6234	22.7131

3.6 Conclusions

We presented a simple framework that renders option formulas not depending on the dividend yield risk premium. These formulas can be applied to derivatives written on an index in complete markets, and can be extended to incomplete markets under the assumption that dividend yield risk uncorrelated to the index is not priced. In this case, we assume that shocks orthogonal to the returns on the index are not priced. Given that indexes are broad portfolios of stocks, this assumption is equivalent to the CAPM assertion that only systematic risk (covariance with the returns on the index) is priced. In this way we were able to obtain formulas valid in complete and in incomplete markets- for which no risk premia has to be estimated.

Our formulas have more than a theoretical interest. We showed that ignoring the randomness in the dividend yield leads to significant mispricing stemming from two sources: a mispecified dividend yield, and a mispecified volatility. The underpricing is economically significant, especially for out of the money options.

Our results have also consequences for hedging. We computed the ”greeks” of European calls and puts from our model and show the they are different from the ones implied by the Black-Scholes model with constant dividend yield. In particular, the delta of a call is larger in our model, and it can even be larger than one. This is because the option seller must hedge not only index price but also dividend yield risk, which is mostly explained by index price risk.

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3.7 Appendix

In this Appendix I provide an overview of the derivation of second moments of returns, and show how to obtain the delta of a call.

3.7.1 Proof of $2c_1 - c_2 > 0$.

Proposition: For $\tau > 0$, $2k_1 - k_2 > 0$.

First I prove the following lemma:

Lemma: Define $f(\tau) = \kappa\tau$, and $g(\tau) = \frac{3-4e^{-\kappa\tau}+e^{-2\kappa\tau}}{2}$. Then, for $\tau > 0$:

$$f(\tau) > g(\tau)$$

Proof: First note that:

$$f(0) = g(0) = 0,$$

and that:

$$f'(\tau) = \kappa.$$

Also:

$$g'(\tau) = 2\kappa e^{-\kappa\tau} - \kappa e^{-2\kappa\tau}.$$

Adding and subtracting κ, this last equation can be written as:

$$g'(\tau) = \kappa\left[1 - \left(1 - e^{-\kappa\tau}\right)^2\right] < \kappa.$$

Now define:

$$h(\tau) = f(\tau) - g(\tau).$$

Then:

$$h(0) = f(0) - g(0) = 0,$$

and that for $\tau > 0$:

$$h'(\tau) = f'(\tau) - g'(\tau) > 0,$$

which implies $h(\tau) > 0$. Therefore, it must be that:

$$f(\tau) > g(\tau),$$

for $\tau > 0$, and the lemma is proved.

Proof of the proposition:

Assume, on the contrary, that $2c_1 - c_2 \leq 0$. Then:

$$2\left(1 - \frac{1-e^{-\kappa\tau}}{\kappa\tau}\right) \leq 1 - \frac{1-e^{-2\kappa\tau}}{2\kappa\tau}.$$

Operating on both sides:

$$\frac{2\kappa\tau - 2\left(1-e^{-\kappa\tau}\right)}{\kappa\tau} \leq \frac{2\kappa\tau - \left(1-e^{-2\kappa\tau}\right)}{2\kappa\tau}$$

.

Multiplying both sides by $\kappa\tau$:

$$2\kappa\tau - 2\left(1-e^{-\kappa\tau}\right) \leq \frac{2\kappa\tau - \left(1-e^{-2\kappa\tau}\right)}{2}$$

.

Operating again:

$$\kappa\tau \leq \frac{3 - 4e^{-\kappa\tau} + e^{-2\kappa\tau}}{2}.$$

But this contradicts the previous lemma. So it must be that:

$$2c_1 - c_2 > 0,$$

completing the proof.

3.7.2 Second moments of price changes

First, define $q_t = m_t - \theta$ and $q_{t-\tau} = m_{t-\tau} - \theta$. Then, from equation (7) in the main text we have:

$$q_t = q_{t-\tau}e^{-\kappa\tau} + \int_{t-\tau}^{t} e^{-\kappa(t-u)}dW_u,$$

and:

$$E\left(q_t q_{t-\tau}\right) = e^{-\kappa\tau}Var\left(q_{t-\tau}\right) = e^{-\kappa\tau}\frac{\sigma_\delta^2 + \nu^2}{2\kappa}. \tag{3.42}$$

From equation (8), the unconditional variance of price changes is:

$$Var\left(r_{t+\tau}\right) = E\left(\begin{array}{c} \frac{\phi}{\kappa}q_t\left(1-e^{-\kappa\tau}\right)+... \\ +\sigma\int_t^{t+\tau}\left[1+\frac{\phi}{\kappa}\left(1-e^{-\kappa(t+\tau-u)}\right)\right]dW_u-\frac{\upsilon}{\kappa}\int_0^t\left(1-e^{-\kappa(t+\tau-u)}\right)dZ_t^* \end{array}\right)^2 \tag{3.43}$$

$$= \left(\frac{\phi}{\kappa}\right)^2\frac{\sigma^2}{2\kappa}\left(1-e^{-\kappa\tau}\right)^2+... \tag{3.44}$$

$$+\frac{\sigma^2}{\kappa^2}\int_t^{t+\tau}\left[1-\phi e^{-\kappa(t+\tau-u)}\right]^2du+\frac{\upsilon^2}{\kappa^2}\int_0^t\left(1-e^{-\kappa(t+\tau-u)}\right)^2du. \tag{3.45}$$

Solving the integral, and after some messy algebra, we get equation (21). The difference between (21) and (18) is that (18) is a conditional variance, so only the second and third terms in (18) is used in the computation.

The formula for $Cov\left(r_t, r_{t+\tau}\right)$ is calculated in the same way, using now equation (20) and taking care that the cross-products overlap.

3.7.3 Derivation of delta

The following lemma will be useful in the derivation of delta:

Lemma: Define $F_\tau = S_t e^{\Omega_\tau+\frac{1}{2}\Sigma_\tau}$. Then:

$$F_\tau N'\left(d_1\right) - KN'\left(d_2\right) = 0,$$

where d_1 and d_2 are as in equations (23) and (25).

Proof: Recall that:

$$N'\left(x\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

and write $d_1 = d_2 + \sqrt{\Sigma_\tau}$. Then:

$$\begin{aligned} N'\left(d_1\right) &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_2^2+2d_2\sqrt{\Sigma_\tau}+\Sigma_\tau}{2}\right) \\ &= N'\left(d_2\right)\exp\left(-d_2\sqrt{\Sigma_\tau}-\frac{1}{2}\Sigma_\tau\right) \\ &= N'\left(d_2\right)\frac{K}{F_\tau}. \end{aligned}$$

So:

$$F_\tau N'\left(d_1\right) - KN'\left(d_2\right) = F_\tau N'\left(d_2\right)\frac{K}{F_\tau} - KN'\left(d_2\right) = 0,$$

and the lemma is proved.

Now it is straightforward to derive delta.

Proposition:

$$\Delta = \frac{\partial C}{\partial S}\left[1 - \frac{\phi}{\kappa}\left(1 - e^{-\kappa\tau}\right)\right]$$

Proof: Recall that:

$$\Delta = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial \delta}\frac{\phi}{S_t}.$$

Deriving equation (26) with respect to S_t gives:

$$\frac{\partial C}{\partial S} = \left[\frac{\partial F_\tau}{\partial S} N\left(d_1\right) + F_\tau N'\left(d_1\right)\frac{\partial d_1}{\partial S} - KN'\left(d_2\right)\frac{\partial d_2}{\partial S}\right] e^{-r\tau}.$$

Noting that $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$,

$$\frac{\partial C}{\partial S} = \left[\frac{\partial F_\tau}{\partial S} N\left(d_1\right) + \left[F_\tau N'\left(d_1\right) - KN'\left(d_2\right)\right]\frac{\partial d_2}{\partial S}\right] e^{-r\tau},$$

which, from the previous lemma, is:

$$\frac{\partial C}{\partial S} = \frac{\partial F_\tau}{\partial S} N\left(d_1\right) e^{-r\tau}.$$

Now,

$$\frac{\partial C}{\partial \delta} = \left[\frac{\partial F_\tau}{\partial \delta} N\left(d_1\right) + F_\tau N'\left(d_1\right)\frac{\partial d_1}{\partial S}\frac{\partial S}{\partial \delta} - KN'\left(d_2\right)\frac{\partial d_2}{\partial S}\frac{\partial S}{\partial \delta}\right] e^{-r\tau}.$$

Proceeding as before, we get:

$$\begin{aligned}\frac{\partial C}{\partial \delta} &= \frac{\partial F_\tau}{\partial \delta} N\left(d_1\right) e^{-r\tau} \\ &= -S\frac{\partial C}{\partial S}\frac{1 - e^{-\kappa\tau}}{\kappa},\end{aligned}$$

and the result follows.

Similarly, it is possible to calculate the other "Greeks". In particular:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}\left[1 - \frac{\phi}{\kappa}\left(1 - e^{-\kappa\tau}\right)\right],$$

where:

$$\begin{aligned}\frac{\partial^2 C}{\partial S^2} &= e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N'\left(d_1\right)\frac{\partial d_1}{\partial S} e^{-r\tau} \\ &= e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau}\frac{N'\left(d_1\right)}{S\sqrt{\Sigma_\tau}} e^{-r\tau}\end{aligned}$$

Chapter 4

A generalized mean-reverting model of commodity prices

Abstract
We introduce a tractable model of commodity prices in which the stochastic convenience yield depends on a weighted average of past commodity price changes. Our model preserves market completeness and exhibits mean reversion under the martingale measure, as a consequence of which it is able to fit a slowly decaying term structure of futures return volatilities. The model nests the Ornstein-Uhlenbeck process and geometric Brownian motion, and renders formulas for the prices of futures contracts for which no risk premium must be estimated.

4.1 Introduction

We study a new, Gaussian complete market model of commodity prices in which the stochastic convenience yield is assumed to be an affine function of a weighted average of past commodity price changes. This assumption captures the dependence of the convenience yield on the state of the market, and generalizes the Ornstein-Uhlenbeck (O-U) process, which can be interpreted as one in which the convenience yield is a linear function of the spot price (see Schwartz (1997), model 1). Our model exhibits weak mean reversion[1]under the martingale measure, and, as a consequence, it is able to fit a slowly decaying term structure of futures return volatilities. Also, the model has the O-U process and geometric Brownian motion as special cases, and renders closed form derivative prices that do not depend on the spot risk premium.

Commodity prices have empirical characteristics, such as spikes, seasonality and mean reversion, that distinguish them from the prices of stocks and bonds. Spikes are the result of random shocks in markets in which the supply is relatively fixed in the short run, while seasonal patterns appear as a response of supply and demand to cyclical fluctuations due mainly to changes in weather[2]. Mean reversion arises as free entry and exit in competitive markets forces prices to gravitate towards the minimum average cost of production. As it reflects a phenomenon affecting commodities as a class, mean reversion is probably the most pervasive of all empirical characteristics of commodities. Moreover, Bessembinder, Coghenour, Seguin and Smoller (1996) argue that mean reversion explains the term structure of futures returns volatilities, and, in a more recent paper, Casassus and Dufresne (2005) show that mean reversion is necessary to capture the cross section of commodity futures prices.

There exists a rich array of multifactor models aimed to describe the complex dynamics of commodity prices. Gibson and Schwartz (1990) introduced a model that combines nonstationarity and mean reversion through a stochastic convenience yield[3] (see also Schwartz (1997)). Most of the literature that followed can be seen as an extension of Gibson and Schwartz (1990) seminal paper. To mention just a few representative examples, Hilliard and Reis (1998) add jumps to the spot price through a Poisson component[4]. Sorensen (2002) and Richter and Sorensen (2002) combine seasonal effects and stochastic volatility, and apply the model to the study of agricultural futures markets. Yan (2002) incorporates stochastic volatility and jumps in both the spot price and the spot volatility. In a recent study, using oil, copper, gold and silver data, Casassus and Dufresne (2005) find that three

[1]In this paper we use the expression "mean reversion" to refer to negative autocorrelation of price changes generated by a temporary component in the spot price. This use of words is common in the literature (see Schwartz (1997)). We distinguish situations in which shocks partially vanish in the long run, from situations in which shocks totally vanish in the long run. In the first case, in which the spot price is nonstationary, although it exhibits a tendency to mean-revert, we use the expression "weak mean reversion". In the second case, in which the spot price is stationary, we use the expression "strong mean reversion".

[2]Schwartz and Smith (2000) characterize the oil prices based on the temporary-permanent price component analysis in which transitory component captures mean reversion. Sorensen (2002) investigates seasonality in a similar setting of Schwartz and Smith (2000). He defines logarithmic commodity prices as summation of permanent (non-stationary), temporary (stationary) and a deterministic seasonal component.

[3]The convenience yield is defined (Brennan (1958)) as the benefit, net of storage costs, that accrues to the holder of inventories rather than to the owner of a derivative contract written on the commodity.

[4]See also Cassassus and Dufresne (2005).

factors are needed to describe the dynamics of futures prices. However, and perhaps due to the very complexity that makes them successful in capturing key features of data, multifactor models have been "adopted rather slowly by practitioners"[5].

On the other hand, one-factor models, such as geometric Brownian motion (Black (1976), Brennan and Schwartz (1985)) and the one factor Ornstein-Uhlenbeck (O-U) process (Bjerksund and Ekert (1995), Schwartz (1997), model 1), may look too simple in comparison, but are still popular in the industry. Their popularity is partly explained by the practitioners' tendency to use models as means to extrapolate prices of liquid instruments to prices of illiquid instruments, which creates a strong demand for simple and parsimonious models. Market completeness is another reason that makes one-factor models popular. Under market completeness, unique option prices can be obtained by a straightforward arbitrage argument, and it is also possible to hedge a derivatives position using just the underlying asset (or a futures contract written on it) and a bond. A complete market model may also prove useful in the risk management of a derivatives book. In addition, both Geometric Brownian motion and the O-U process make it possible to obtain closed form solutions for futures prices and European option premia.

But simplicity comes at a cost, and one-factor models are not free of shortcomings. First, they imply that futures prices are perfectly correlated at all maturities, a prediction that is not supported by the data. Futures prices are in general imperfectly correlated, with correlations decreasing steadily with maturity. Second, one-factor models are unable to fit the term structure of futures return volatilities. For commodities, this term structure is negatively sloped, a stylized fact that can be explained by mean reversion (see Bessembinder et al. (1996)). However, although volatilities in the data go down uniformly as maturity increases, they do not seem to converge to zero, which suggests that random shocks to prices are only partially reversed in the long run. One-factor models are not able to capture this stylized fact: geometric Brownian motion implies a flat term structure, while the O-U process exhibits volatilities that converge quickly to zero. However, one-factor models may still be useful for derivatives that do not depend on the correlation between different futures prices, or when the maturities of the futures prices involved are not too far from each other. The inability to fit the term structure of futures return volatilities is more problematic, because most derivatives on commodities use futures as the underlying asset, and so it is important for accurate valuation that models fit this term structure properly.

Our model has several of the features making one-factor models widely used, such as simplicity, complete markets, and availability of closed form solutions. At the same time, it improves on them by being able to fit a slowly decaying term structure of futures return volatilities. The key assumption of the model is that the convenience yield is an affine function of a weighted sum of past commodity price changes[6]. This assumption makes innovations to the convenience yield perfectly correlated with

[5]Cortazar and Schwartz (2003), page 216.

[6]Cassasus and Dufresne (2005) argue that the convenience yield is implied by the equilibrium relation among supply, demand and inventories, and that this dependence provides a rationale to model the convenience yield as a function of the spot price. The model studied in this paper uses a more general measure of performance, based on the history of past price changes. The advantage of this measure of performance is that it allows the effect of random shocks to partially vanish in

spot price changes, and is the source of market completeness in the model. Although strong, perfect correlation is a reasonable approximation for some commodities, like oil and copper, which exhibit strong comovement between convenience yield and spot price changes. For example, using 10 years of weekly data, Schwartz (1997) finds a correlation between 70 and 90% for oil, depending on the subperiod analyzed, and 82% for copper. In a more recent study, Casassus and Dufresne (2005) find 79% and 78% for oil and copper, respectively. On the other hand, the convenience yield can be seen as a construction used to generate weak mean reversion under the risk-neutral measure, and this paper presents a new model that accommodates it in a complete market.

The model also renders derivative prices for which no risk premium must be estimated. The availability of closed form solutions is not an issue in Gaussian models like the one studied in this paper[7]. What is an issue, however, is whether we can obtain formulas, even in complete markets, in which there is no need to compute the risk premium on the spot. This is especially important in commodity pricing models, where the convenience yield is nontradable. Also, some authors have shown that the formulas for the prices of futures contracts for which no risk premium must be estimated are not just a consequence of market completeness. Duan(2001) presents an example of a market in which all contingent claims can be perfectly replicated, but in which the prices of contingent claims are still function of the risk premium on the underlying asset[8]. Based on this example, Duan (2001) concludes that risk-neutral prices do not follow necessarily from the complete market assumption. More recently, Lioui (2006) studies the problem of pricing derivatives in complete markets in which the stock pays a stochastic dividend yield, an shows that, even if there is a single source of uncertainty, the risk premium on the stock will appear in the derivatives formulas as an adjustment to the long run mean of the dividend yield. This paper presents a parameterization of the convenience yield (equivalent to the dividend yield in Lioui) that renders derivatives formulas for which no risk premium must be estimated.

We provide an empirical assessment of the model on a sample of oil futures prices. Oil is one of the most important traded commodities, and it has been widely studied in the literature. It has also been shown to exhibit mean reversion under the martingale measure (see Casassus and Dufresne (2005)). We find that the model outperforms the O-U process both in terms of model fit and in terms of pricing errors.

Practitioners usually estimate volatilities by calculating the volatilities implied by the Black-Scholes formula and the prices of a set of liquid options. These implied volatilities are then used to price less liquid contracts. If the underlying asset exhibits mean reversion under the martingale measure, this procedure will overestimate volatilities -and prices- especially for longer term contracts. Imposing strong mean reversion is a step towards the solution of this problem, but it may lead to the

the log run, which produces spot price weak mean reversion in a complete market setting.

[7]Schwartz (1997), for example, presents a collection of Gaussian models and their closed form solutions.

[8]Duan(2001) proposed an alternative complete market world by using a semi-recombined binomial lattice model - which is nested on the GARCH option pricing model of Kallsen and Taqqu (1998). In this model, he showed that although all contingent claims can be perfectly replicated, risk neutral values are still a function of the risk premium.

underestimation of volatilities when shocks to the underlying do not vanish completely in the long run. This seems to be the case for most commodities. This paper contributes to the literature by introducing a complete market model that exhibits weak mean reversion under the martingale measure, and that is capable to fit the term structure of futures return volatilities. As the model renders formulas for futures and European option prices for which no risk premium must be estimated, it provides a useful benchmark to value more complex contracts for which no closed form solutions are known.

The structure of the paper is as follows. The model is presented in section 2. The price distribution under the martingale measure is obtained in section 3. Futures and option prices are derived in section 4. Section 5 presents empirical results. Finally, section 6 concludes.

4.2 Commodity price dynamics

Let's assume a frictionless financial market in which trading is continuous. The commodity spot price S_t satisfies the following differential equation:

$$\frac{dS_t}{S_t} = (\mu - \delta_t)\, dt + \sigma dW_t, \tag{4.1}$$

where μ is the total instantaneous expected return on the spot, δ_t is the stochastic convenience yield, and σ is the instantaneous return volatility. The only source of risk in the economy is a standard Wiener process, W_t, defined on a filtered probability space $(\Omega, , \Pi)$.

In this model, as it is common in the literature since the seminal Gibson and Schwartz (1991) paper, weak mean reversion in the spot is induced by a stationary convenience yield whose innovations are positively correlated to spot price changes. The convenience yield is implied by the equilibrium relation among supply, demand and inventories[9], in such a way that when the market is tight, with strong demand and raising prices, the convenience yield is large, and when the market is loose, with weak demand and falling prices, the convenience yield is small. As Cassasus and Dufresne (2005) point out, this provides a rationale to model the convenience yield as a function of the spot price. Consistent with this, recent empirical work finds that, for certain commodities, spot price changes and innovations to the convenience yield are highly correlated. For example, Schwartz (1997) and, more recently, Casassus and Dufresne (2005) find that in the case of oil this correlation is about 80%.

Let $s_t = \log(S_t)$. Then, from equation (1):

$$ds_t = \left(\mu - \frac{1}{2}\sigma^2 - \delta_t\right) dt + \sigma dW_t, \tag{4.2}$$

Assume now that changes in the convenience yield depend on its past level and on changes in the log spot price through the following equation:

$$d\delta_t = -\omega\,(\delta_t - \alpha)\, dt + \phi ds_t, \tag{4.3}$$

[9]See Cassasus and Dufresne (2005).

where $\alpha, \omega \geq 0$, and $\phi > 0$. Equation (3) is aimed to capture the relation between changes in the convenience yield and spot price changes implied by the theory of storage and observed in the data. It can be interpreted as a regression of changes in the convenience yield against its own past, a constant $\alpha\omega$, and spot price changes. The complete market assumption implies that the regression error is identically zero, so there is no autonomous convenience yield risk.

Solving equation (3) gives:

$$\delta_t = \alpha + (\delta_0 - \alpha) e^{-\omega t} + \phi \int_0^t e^{-\omega(t-u)} ds_u, \tag{4.4}$$

that is, the convenience yield is the sum of a deterministic function and a weighted average of past (log) commodity price changes. Given $\phi > 0$, two polar cases are of interest[10]: $\omega = 0$, and $\omega = \infty$. When $\omega = 0$ we have:

$$\begin{aligned} \delta_t &= \delta_0 + \phi \int_0^t ds_u \\ &= \delta_0 + \phi (s_t - s_0) . \end{aligned} \tag{4.5}$$

Plugging equation (5) in (2) we obtain that the log spot price follows an O-U process. This shows that the strong mean reverting model is a special case of the model introduced in this paper. The difference between the O-U process and the model introduced in this paper can be seen as follows: in both models the convenience yield depends on the past history of spot price changes, but in the model introduced in this paper the most recent price innovations are given more weight (assuming $\omega > 0$), while in the O-U process all past innovations are equally weighted.

The second polar case arises when $\omega = \infty$. In this case, the constant dividend yield model obtains :

$$\delta_t = \alpha. \tag{4.6}$$

This shows that Geometric Brownian motion is also a special case the model introduced in this paper.

Although equation (4) looks unfamiliar, it is straightforward to show that it is consistent with the dividend yield following an Ornstein-Uhlenbeck process, as it is common in the literature (see Schwartz (1997)). In this case, the long run mean of the convenience yield is a weighted average of α and the expected returns of the log spot price $\mu - \frac{1}{2}\sigma^2$. This is shown in Proposition 1:

The convenience yield is a stationary process satisfying the following stochastic differential equation:

$$d\delta_t = -\kappa (\delta_t - \Theta) dt + \sigma_\delta dW_t, \tag{4.7}$$

[10]We consider only the case $\phi > 0$. because it is the one relevant to commodity prices.

where:

$$\kappa = \omega + \phi$$
$$\Theta = \frac{\omega}{\kappa}\alpha + \frac{\phi}{\kappa}\left(\mu - \frac{1}{2}\sigma^2\right)$$
$$\sigma_\delta = \phi\sigma$$

See Appendix 1.

The log spot price change over a discrete interval of length τ can be found by integrating equation (2):

$$s_{t+\tau} - s_t = \left(\mu - \frac{1}{2}\sigma^2 - \Theta\right)\tau - (\delta_t - \Theta)\frac{1 - e^{-\kappa\tau}}{\kappa} + \sigma\int_t^{t+\tau}\left[1 - \frac{\phi}{\kappa}\left(1 - e^{-\kappa(t+\tau-u)}\right)\right]dW_u. \tag{4.8}$$

The expression in the integral inside the brackets gives the impulse-response function of the model. As it will be shown later, this ”term structure of shocks” plays a key role in the determination of futures return volatilities. As equation (8) shows, a shock that occurred at t has a residual impact on $s_{t+\tau}$ of $1 - \frac{\phi}{\kappa}\left(1 - e^{-\kappa\tau}\right)$. As τ grows without bound, this residual impact converges to:

$$1 - \frac{\phi}{\kappa} = \frac{\omega}{\kappa}.$$

Assuming $\phi > 0$, there are three cases to consider: 1) $\omega = 0$, 2) $\omega > 0$, 3) $\omega = \infty$. When $\omega = 0$, the effect of shocks completely vanish in the long run, so the process exhibits strong mean reversion. When ω is positive, the residual impact of a shock experienced at t, as τ grows without bound[11], is $\frac{\omega}{\kappa} < 1$. Shocks have permanent effects, although a part of any shock vanishes in the long run. This is the case of weak mean reversion. Finally, when $\omega = \infty$, shocks still have permanent effects, but their residual impact is exactly 1. In this case, the spot is a random walk. Similarly, it can be shown that the unconditional variance of spot price changes in case 1 dominates the unconditional variance of spot price changes in cases 2 and 3, and that $\phi > 0$ is sufficient for spot price changes to be negatively autocorrelated (see Appendix 2).

The financial market is naturally complete through the dependence of δ_t on W_t, the spot source of risk. Assume additionally that there are no arbitrage opportunities. Then, there exists a unique probability measure Q, equivalent to Π, such that the discounted prices of the spot (cum dividend) and of other traded assets are martingales under Q (Harrison and Kreps (1979)). In the next section we obtain the spot price process under the Q-measure and derive formulas for futures prices.

[11]Recall that $\kappa = \omega + \phi$.

4.3 The price process under the Q-measure

In this section We derive the risk-neutral spot price process and a closed form solution for futures prices. The main objective of this section is to obtain the term structure of futures returns volatilities as a function of ϕ and ω. We also show that the term structure of futures returns volatilities coincides with the impulse response function of the spot price shocks.

To compute the futures price, we need first the spot price process under the risk-neutral measure. An interesting consequence of the convenience yield parameterization of equation (3) is that the risk premium does not enter the futures price formula. Equation (1) defines μ as the total expected return on the commodity (capital gains plus convenience yield). As in Schwartz (1997), μ is assumed constant. Define now r as the constant[12] instantaneous risk-free interest rate, and λ as the risk premium[13]. Then, the total expected return can be decomposed as:

$$\mu \equiv r + \lambda \tag{4.9}$$

Plugging equation (9) back in (1) gives the risk-neutralized commodity price process:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r - \delta_t)\, dt + \sigma \left(\frac{\lambda}{\sigma} dt + dW_t \right), \\ &= (r - \delta_t)\, dt + \sigma dB_t, \end{aligned} \tag{4.10}$$

where, by Girsanov's theorem[14], $B_t = \frac{\lambda}{\sigma} t + W_t$ is a Brownian motion under the risk neutral measure Q. Thus, the total expected return on the commodity under Q is r. Interestingly, the risk-neutralized process for δ_t does not depend on the spot risk premium either. To see this, plug (9) in (7) to get:

$$\begin{aligned} d\delta_t &= -\kappa \left(\delta_t - \Theta^*\right) dt + \sigma_\delta \left(\frac{\lambda}{\sigma} dt + dW_t \right) \\ &= -\kappa \left(\delta_t - \Theta^*\right) dt + \sigma_\delta dB_t, \end{aligned} \tag{4.11}$$

where now:

$$\Theta^* = \frac{\omega}{\kappa} \alpha + \frac{\phi}{\kappa} \left(r - \frac{1}{2} \sigma^2 \right). \tag{4.12}$$

So neither S_t nor δ_t depend on λ under Q. As a consequence, the model renders formulas for contingent claims for which no risk premium must be estimated.

Under Q, the logarithm of S_T is normally distributed, with conditional mean Ω_τ and conditional variance Σ_τ, where:

[12] We assume for simplicity that the risk free rate is constant. The model cen be straightforwardly extended to accommodate time-varying interest rates along the lines of Schwartz (1997) model 3.

[13] As the risk premium does not enter the derivatives formulas, all results in this and the next sections carry through even if the risk premium is stochastic and time-varying.

[14] There are technical conditions to be met in order to apply Girsanov's theorem, but they are automatically satisfied when the risk premium is constant. See Karatzas and Shreve (1991).

$$\Omega_\tau = \left(r - \frac{1}{2}\sigma^2 - \Theta^*\right)\tau - (\delta_t - \Theta^*)\frac{(1 - e^{-\kappa\tau})}{\kappa}, \tag{4.13}$$

and:

$$\Sigma_\tau = \frac{\sigma^2}{\kappa^2}\left(\omega^2\tau + \frac{2\phi\omega}{\kappa}\left(1 - e^{-\kappa\tau}\right) + \frac{\phi^2}{2\kappa}\left(1 - e^{-2\kappa\tau}\right)\right), \tag{4.14}$$

where $\tau = T - t$; that is, the moments are calculated conditional on information up to time t. Note that $\omega = 0$ implies $\Sigma_\tau = \frac{\sigma^2}{2\phi}\left(1 - e^{-2\phi\tau}\right)$, the variance of a mean reverting process with reversion rate ϕ. On the other hand, if $\omega \to \infty$, $\Sigma_\tau = \sigma^2\tau$. That is, the variance grows linearly with time to maturity, which corresponds to the random walk case.

The futures price[15] for delivery of one unit of the commodity τ periods ahead is the expected commodity price under the risk-neutral measure. Given the normality of $\log(S_t)$ under Q, the futures price is easily obtained in closed form:

$$\begin{aligned} F_\tau &= E_t^Q(S_T) \\ &= S_t \times \exp\left(\Omega_\tau + \frac{1}{2}\Sigma_\tau\right). \end{aligned} \tag{4.15}$$

From equations (13) and (14), this formula does not include the risk premium.

The futures price process has no drift under Q, because no money is paid to enter the contract. The dynamics of the futures price is described, after applying Ito's lemma, by the following differential equation:

$$\frac{dF_\tau}{F_\tau} = \sigma(\tau)\, dB_t,$$

where:

$$\sigma(\tau) = \begin{cases} \sigma e^{-\phi\tau} & if\ \ \omega = 0 \\ \sigma\left[1 - \frac{\phi}{\kappa}\left(1 - e^{-(\omega+\phi)\tau}\right)\right] & if\ \ 0 < \omega < \infty \\ \sigma & if\ \ \omega \to \infty \end{cases} \tag{4.16}$$

The volatility term structure reflects the term structure of shocks: When $\omega \to \infty$, the volatility of the futures return is independent of time to maturity. This is the case in which the spot is a random walk. When $0 < \omega < \infty$, the volatility of the futures return decreases slowly with time to maturity. Finally, when $\omega = 0$, the spot price is stationary, and the volatility collapses to $\sigma e^{-\phi\tau}$. In the long run, the volatility of the random walk stays at σ, while the volatility of the stationary process goes down to 0. In the general case, it converges to:

$$\sigma\left(1 - \frac{\phi}{\kappa}\right) = \frac{\sigma\omega}{\kappa} > 0. \tag{4.17}$$

Figure 1 shows the calibration of equation (16) to the term structure of futures returns volatilities obtained from the data set described in Section 5. While the model introduced in this paper is able

[15] Note that the words "futures price" and "forward price" can be used interchangeably in this context, because they are equal under the current assumption of a constant risk-free rate.

to fit the term structure almost perfectly, the O-U model overestimates the mid-term volatilities, and underestimates the volatilities of the shortest and longest maturities.

The inability to fit the volatility curve is a serious shortcoming of the O-U model, as practitioners may want to use these volatilities to calibrate more complex derivative models to market data. With just one additional parameter (ω), and without the need to relax the assumption of complete markets, the model introduced in this paper produces an almost perfect fit.

The term structure of futures return volatilities is the same as the term structure of shocks (see equation (8)). This is consistent with Bessembinder et al. (1996) explanation of the Samuelson hypothesis. Samuelson (1965) asserted that the volatility of futures price changes should decrease with the maturity time of the contracts. Bessembinder et al. (1996) argue that a sufficient condition for the Samuelson hypothesis to hold is that there is a temporary component in spot price changes such that investors expect that those changes will be at least partially reversed in the long run. This implies mean reversion, but not necessarily stationarity of the spot price.

4.4 Pricing options

The price of a European call option written on the spot, with maturity T and strike K, is the expectation under Q of its payoff at maturity, discounted by the risk-free rate:

$$C_t = e^{-r\tau} E_t^Q \left[Max \ \left(S_T - K, 0 \right) \right]. \tag{4.18}$$

Equation (18) can be written as:

$$C_t = e^{-r\tau} E_t^Q \left[\left(S_T \ \right) \times \mathbf{1}_{\{S_T > K\}} \right] - e^{-r\tau} K P^Q \left(S_T > K \right), \tag{4.19}$$

where $\mathbf{1}_{\{S_T > K\}}$ is the indicator function of the event $\{S_T > K\}$, $E_t^Q \left[\left(S_T \ \right) \times \mathbf{1}_{\{S_T > K\}} \right]$ is the Q-expected value of the spot at maturity, conditioned on the event that the option will be exercised at maturity, and $P^Q \left(S_T > K \right)$ is the probability under Q of this event. Due to the normality of $\ln \left(S_t \right)$, the expectation in the first term of (19) can be solved as:

$$E_t^Q \left[\left(S_T \ \right) \times \mathbf{1}_{\{S_T > K\}} \right] = S_t e^{\Omega_\tau + \frac{1}{2} \Sigma_\tau} N \left(d_1 \right), \tag{4.20}$$

where $N \left(d_1 \right)$ is the value of the Normal cumulative distribution function at d_1, and:

$$d_1 = \frac{\log \left(\frac{S_t}{K} \right) + \Omega_\tau + \Sigma_\tau}{\sqrt{\Sigma_\tau}}. \tag{4.21}$$

The probability of the option finishing in the money is:

$$P^Q \left(S_T > K \right) = N \left(d_2 \right), \tag{4.22}$$

where:

$$d_2 = d_1 - \sqrt{\Sigma_\tau}. \tag{4.23}$$

So:

$$C_t = \left[S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N(d_1) - KN(d_2)\right] e^{-r\tau}. \tag{4.24}$$

It is important to note that this formula, as the formula for the futures price (eq. 16), does not include preference parameters.

The price of a European put on the same commodity can be found using put-call parity. That is, because buying a call and shorting a put, both with maturity T and strike K, is equivalent to having a long position in a forward contract with maturity T and forward price K, we can express the put price as:

$$P_t = C_t - \left[E_t^Q(S_T) - K\right] e^{-r\tau} \tag{4.25}$$

Using equation (15) we get:

$$P_t = \left[KN(-d_2) - S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N(-d_1)\right] e^{-r\tau}. \tag{4.26}$$

In the case of commodities, it is usually easier to observe futures rather than spot prices. It is even the case that in some exchanges the nearest maturity futures price is taken as a proxy for the spot price. For this reason, many options on commodities are not written directly on the spot, but on the futures price, and, as a consequence, it is not uncommon in the literature to deal directly with the pricing of options on futures. It is straightforward to adapt formulas (24) and (26) to price this kind of options.

If the maturity of the option and the maturity of the futures contract are the same, the current futures price $F_\tau = F(t,T)$, where $T = t + \tau$, can be used to price options on the spot. So, as $F(T,T) = S_T$, the call price can be rewritten as:

$$C(t,T) = \left[F_\tau N(d_1) - KN(d_2)\right] e^{-r\tau}, \tag{4.27}$$

with:

$$d_1 = \frac{\log\left(\frac{F_\tau}{K}\right) + \frac{1}{2}\Sigma_\tau}{\sqrt{\Sigma_\tau}}, \tag{4.28}$$

where Σ_τ is as defined in equation (14). On the other hand, suppose that T is the maturity time of the futures contract, and that the option matures at $s < T$. Then, as integration of equation (2) must be done over the life of the option, the variance in equation (14) has to be replaced by:

$$\Sigma_\tau^* = \frac{\sigma^2}{\kappa^2}\left(\omega^2(s-t) + \frac{2\phi\omega}{\kappa}\left(e^{-\kappa(T-s)} - e^{-\kappa(T-t)}\right) + \frac{\phi^2}{2\kappa}\left(e^{-2\kappa(T-s)} - e^{-2\kappa(T-t)}\right)\right) \tag{4.29}$$

4.5 Empirical Results

This section provides an empirical assessment of the model on a data set consisting of weekly observations of futures prices of oil (NYMEX WTI). To avoid repetition, we refer from now on to the model introduced in this paper as the ”our-model”, and to its restricted version, the O-U process, as the ”O.U.-model” [16]. Model estimation is complicated by the fact that the state variables -the spot price and the convenience yield- are unobservable. Schwartz (1997) shows how in these cases the Kalman filter can be used for parameter estimation and the recovery of state variables from futures price data, and this technique is still widely used in the literature. So in what follows we will implement the Kalman filter to investigate the relative performance of the our-model and O.U.-model by means of pricing errors[17].

4.5.1 Data and Estimation

The models are implemented on a data set consisting of weekly observations of futures prices of oil (NYMEX WTI). Daily data was originally obtained from Bloomberg and then transformed into weekly by choosing every Wednesday observation. Eleven maturities were used in the empirical exercises, going from the contract closest to maturity (F1) to the longest term contract (F11). The shortest maturity is about two weeks; the longest maturity, less than two years. For each contract there are 249 observations, starting on March 17, 1999, and ending on December 31, 2003. The interest rate is assumed constant and fixed at 4%. The data is described in Table 1:

Table 1: Oil Data Description

Futures Contract	Mean Price (Standard Error)	Mean Maturity (Standard Error)	Standard Dev. of futures return
F1	27.05 (4.72)	0.043 (0.024)	0.373
F2	26.41 (4.19)	0.210 (0.024)	0.313
F3	25.64 (3.81)	0.377 (0.024)	0.265
F4	25.03 (3.57)	0.544 (0.024)	0.235
F5	24.44 (3.37)	0.711 (0.024)	0.216
F6	23.94 (3.19)	0.878 (0.024)	0.199
F7	23.54 (3.06)	1.045 (0.024)	0.186
F8	23.16 (2.92)	1.212 (0.024)	0.175
F9	22.84 (2.82)	1.379 (0.024)	0.169
F10	22.58 (2.72)	1.546 (0.024)	0.161
F11	22.39 (2.66)	1.713 (0.024)	0.159

[16] We compare only the two models that accommodate mean reversion. Results on Geometric Brownian motion are available upon request.

[17] For details about estimation, see Appendix 3.

The mean prices go down uniformly with maturity. The futures returns are calculated as the difference between the log of the futures prices, and their volatilities also decrease steadily with maturity.

We estimate the parameters of the two models on five maturities: F1, F3, F5, F7, and F9, using the whole set of observations, and reserve the remaining maturities for out of sample testing. Estimation results are presented in Table 2.

Table 2: Estimation Results: our-model and O.U.-model

Parameters	our-Model	Std.Error	O.U.-model	Std.Error
μ	0.5042	(0.1656)	0.3606	(0.1160)
α	0.1336	(0.0051)	-0.5763	(0.0422)
σ	0.3531	(0.0179)	0.2492	(0.0106)
ϕ	0.8482	(0.032)	0.2259	(0.0133)
ω	0.5696	(0.0257)	0.00	
$\sigma_{\varepsilon 1}$	0.0543	(0.0025)	0.0671	(0.0023)
$\sigma_{\varepsilon 2}$	0.0190	(0.0008)	0.0251	(0.0010)
$\sigma_{\varepsilon 3}$	0.00		0.00	
$\sigma_{\varepsilon 4}$	0.0103	(0.0005)	0.0170	(0.0008)
$\sigma_{\varepsilon 5}$	0.0183	(0.0009)	0.0305	(0.0014)
Likelihood	2948.4		2578.3	

For the our-model, the value of the likelihood function is 2948.4. The parameter ϕ is positive and significant, implying that there is mean reversion in the data. The parameter ω, which measures the weight of past spot price changes in the convenience yield, is also positive and significant. As discussed in section 2, a positive ω means that shocks are only partially reversed in the long run, suggesting that the O-U process is not an adequate model for the data. The instantaneous volatility of spot price changes, σ, is also positive and significant. The total return on the spot, μ, is 0.5042, while α is equal to 0.1336. Both parameters are significant. Imposing $\omega = 0$ reduces the value of the likelihood function from 2948.4 to 2580.2. A likelihood ratio test shows that this difference is strongly significant, with negligible p-value. Note that in the O.U.-model the parameter α is actually the vertical intercept of the convenience yield (in s_t, δ_t space): $\delta_0 - \phi s_0$; this explains the negative value of the estimate.

Another way to assess the models' ability to capture essential features of the data is to investigate whether they are able to reproduce the shape of the term structure of futures return volatilities. Note that these volatilities did not play an explicit role in the estimation of the parameters; only futures prices were used. Results are shown in figure 2. The our-model captures the shape quite well; in contrast, the O.U.-model misses the curve of empirical returns volatilities almost completely.

Following Schwartz (1997) we implement two tests to compare the models. The first one is a cross sectional test in which pricing errors are computed on the six maturities not used in the estimation of parameters. The second test requires reestimating the models using a subset of the observations per maturity, and computing price prediction errors on the remaining observations not used in the estimation. As Schwartz (1997) pointed out, although the first test is of interest, because it involves data not used in the estimations, only the second test is a true out of sample exercise.

Table 3 provides a cross-sectional comparison of the pricing errors generated by the three models. Following standard practice in the literature, pricing errors are measured by the average mean square error (RMSE) and the mean error (ME). The pricing errors are calculated on the 6 maturities that were not used in the estimation. The our-model generates the smallest pricing errors, and the O.U.-model , the largest. For both models, pricing errors in percentage are larger at the short and long ends of the price curve. Measured by the RMSE, the pricing errors are on average (across maturities) below 3%.

Table 3: Cross-Section comparison between models
Maturities not used in the estimations

	RMSE		ME	
Model Contract	OU-model	our-model	O.U.-model	our-model
Panel A: In Dollars				
F2	1.172	0.991	-0.385	0.006
F4	0.295	0.247	-0.085	-0.036
F6	0.231	0.168	0.028	0.030
F8	0.536	0.332	-0.077	0.025
F10	0.814	0.484	-0.292	-0.027
F11	1.693	0.544	-0.468	-0.113
All	0.961	0.461	-0.213	-0.019
Panel B: In Percentage				
F2	4.694	3.467	-1.492	-0.088
F4	1.240	0.939	-0.347	-0.154
F6	1.007	0.690	0.120	0.119
F8	2.413	1.434	-0.327	0.064
F10	3.742	2.222	-1.304	-0.219
F11	4.453	2.578	-2.118	-0.628
All	2.925	1.888	-0.911	-0.151

Table 4 shows the results of the out of sample test, in which prediction errors are computed for period $t+1$ using all information up to period t $(200 < t \leq 248)$. Implementing this test requires the reestimation of parameters at every period t. The table shows mean square errors and mean errors of the log prices, as they are obtained from the Kalman filter algorithm. They can be interpreted as approximate percentage pricing errors.

Table 4: Time series comparison between models
Last 49 observations

	RMSE		ME	
Model Contract	O.U-model	our-model	O.U. model	our-model
Out of sample parameter estimation				
F1	0.0674	0.0671	0.0205	0.0235
F3	0.0395	0.0404	0.0013	0.0074
F5	0.0299	0.0298	0.0000	-0.0007
F7	0.0300	0.0272	0.0099	-0.0033
F9	0.0406	0.0303	0.0246	-0.0038
All	0.0415	0.0390	0.0113	0.0046

In Table 4, the O.U-model is again outperformed by the our-model. This time, the O.U.-model generates prediction errors that are on average 6.5% above the prediction errors generated by the our-model.

4.6 Conclusions

This paper presents a complete market model of commodity prices that exhibits price nonstationarity and mean reversion under the martingale measure, and, as a consequence, it is able to fit a slowly decaying term structure of futures return volatilities. The model has strong mean reversion and geometric Brownian motion as special cases, and renders formulas for the prices of futures contracts and European options for which no risk premium must be estimated.

Implemented on a sample of oil futures prices, the model outperforms the strong mean reversion model in term of pricing errors, and is capable of producing a perfect fit of the term structure of futures return volatilities.

The model is parsimonious and provides a useful benchmark to value complex contracts for which no closed form solutions are known. On this regard, it can be seen as a good alternative to widely used one-factor models.

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4.7 Appendices

4.7.1 Proof of Proposition 1

Plugging equation (2) in (3), we get:

$$d\delta_t = -\omega\left(\delta_t - \alpha\right)dt + \phi\left[\left(\mu - \frac{1}{2}\sigma^2 - \delta_t\right)dt + \sigma dW_t\right] \tag{4.30}$$

$$= -\left(\omega + \phi\right)\left(\delta_t - \frac{\omega\alpha + \left(\mu - 1/2\sigma^2\right)}{\omega + \phi}\right)dt + \phi\sigma dW_t. \tag{4.31}$$

The last equality proves the proposition.

4.7.2 Second moments of spot price changes

Lemma 1: Define $r_{t+\tau} = s_{t+\tau} - s_t$, the $\tau-$period log price change. Then, the unconditional variance of $r_{t+\tau}$ can be written as:

$$Var\left(r_{t+\tau}\right) = \frac{\sigma^2}{\left(\omega+\phi\right)^2}\left[\omega^2\tau + \frac{2\phi}{\omega+\phi}\left(\omega + \frac{\phi}{2}\right)\left(1 - e^{-(\omega+\phi)\tau}\right)\right],$$

and the covariance between r_t and $r_{t+\tau}$ is:

$$Cov\left(r_t, r_{t+\tau}\right) = -\frac{\sigma^2}{\left(\omega+\phi\right)^2}\frac{\phi}{\omega+\phi}\left(\omega + \frac{\phi}{2}\right)\left(1 - e^{-(\omega+\phi)\tau}\right)^2.$$

Proof: First, define $\widetilde{\delta}_t = \delta_t - \Theta$ and $\widetilde{\delta}_{t-\tau} = \delta_{t-\tau} - \Theta$. Then, from equation (7) in the main text we have:

$$\widetilde{\delta}_t = \widetilde{\delta}_{t-\tau}e^{-(\omega+\phi)\tau} + \phi\sigma\int_{t-\tau}^{t} e^{-(\omega+\phi)(t-u)}dW_u,$$

and:

$$E\left(\widetilde{\delta}_t\widetilde{\delta}_{t-\tau}\right) = e^{-(\omega+\phi)\tau}Var\left(\widetilde{\delta}_{t-\tau}\right) = e^{-(\omega+\phi)\tau}\frac{\left(\phi\sigma\right)^2}{2\left(\omega+\phi\right)}. \tag{4.32}$$

From equation (8), the unconditional variance of price changes is:

$$Var\left(r_{t+\tau}\right) = E\left(-\widetilde{\delta}_t\frac{1-e^{-(\omega+\phi)\tau}}{\omega+\phi} + \sigma\int_t^{t+\tau}\left[1 + \frac{\phi}{\omega+\phi}\left(1 - e^{-(\omega+\phi)(t+\tau-u)}\right)\right]dW_u\right)^2$$

$$= \frac{\left(\phi\sigma\right)^2}{2\left(\omega+\phi\right)}\left(\frac{1-e^{-(\omega+\phi)\tau}}{\omega+\phi}\right)^2 + \frac{\sigma^2}{\left(\omega+\phi\right)^2}\int_t^{t+\tau}\left[1 + \phi e^{-(\omega+\phi)(t+\tau-u)}\right]^2 du. \tag{4.33}$$

Solving the integral, and after some messy algebra, we get:

$$Var\left(r_{t+\tau}\right)=\frac{\sigma^2}{\left(\omega+\phi\right)^2}\left[\omega^2\tau+\frac{2\phi}{\omega+\phi}\left(\omega+\frac{\phi}{2}\right)\left(1-e^{-(\omega+\phi)\tau}\right)\right]. \tag{4.34}$$

The difference between equation (34) and equation (14) is that (14) is the equation of a conditional variance, so only the second term in (34) is used in the computation of (14).

The formula for $Cov\left(r_t, r_{t+\tau}\right)$ is calculated in the same way, using now equation (40) with $\tau > 0$ and taking care that the cross-products overlap.

From Lemma 1, the first autocorrelation of τ-period log price changes can be expressed as:

$$\rho\left(r_t, r_{t+\tau}\right)=-\frac{\frac{\phi}{\omega+\phi}\left(\omega+\frac{\phi}{2}\right)\left(1-e^{-(\omega+\phi)\tau}\right)^2}{\omega^2\tau+\frac{2\phi}{\omega+\phi}\left(\omega+\frac{\phi}{2}\right)\left(1-e^{-(\omega+\phi)\tau}\right)}. \tag{4.35}$$

The following two lemmas show that $\phi > 0$, that is, positive correlation between innovations to the convenience yield and log price changes is a sufficient condition for mean reversion. Lemma 2 demonstrates that if $\phi > 0$, the unconditional variance of τ-period price changes is lower than the variance corresponding to the random walk ($\phi = 0$) for $\tau > 0$. The third lemma shows that the sign of the first autocorrelation of τ-period log price changes is equal to minus the sign of ϕ.

Lemma 2: If $\phi > 0$, $Var\left(r_{t+\tau}\right) \leq \sigma^2\tau$. The inequality is strict for $\tau > 0$.

Proof: Write:

$$Var\left(r_{t+\tau}\right)=\sigma^2\tau\frac{\omega^2+2\phi\left(\omega+\frac{\phi}{2}\right)\frac{\left(1-e^{-(\omega+\phi)\tau}\right)}{(\omega+\phi)\tau}}{\left(\omega+\phi\right)^2}.$$

If $\tau = 0$, $Var\left(r_{t+\tau}\right) = \sigma^2\tau = 0$. So it is necessary to show that for $\tau > 0$:

$$\frac{\omega^2+2\phi\left(\omega+\frac{\phi}{2}\right)\frac{\left(1-e^{-(\omega+\phi)\tau}\right)}{(\omega+\phi)\tau}}{\left(\omega+\phi\right)^2}<1.$$

Note that this follows from the fact that for $\tau > 0$:

$$\frac{\left(1-e^{-(\omega+\phi)\tau}\right)}{\left(\omega+\phi\right)\tau}<1.$$

Then, for $\phi > 0$:

$$\omega^2+2\phi\left(\omega+\frac{\phi}{2}\right)\frac{\left(1-e^{-(\omega+\phi)\tau}\right)}{\left(\omega+\phi\right)\tau}<\omega^2+2\phi\left(\omega+\frac{\phi}{2}\right)=\left(\omega+\phi\right)^2.$$

Therefore:

$$Var\left(r_{t+\tau}\right)<\sigma^2\tau\frac{\left(\omega+\phi\right)^2}{\left(\omega+\phi\right)^2}=\sigma^2\tau,$$

and the lemma is proved. Note that $Var\left(r_{t+\tau}\right)\sigma^2\tau$ for $\tau 0$, and that for large τ, $Var\left(r_{t+\tau}\right)\sigma^2\tau\left(\frac{\omega}{\omega+\phi}\right)^2 < \sigma^2\tau$.

Lemma 3: $\phi > 0$ implies negative first autocorrelation of τ-period log price changes.
Proof: Note that we have that:

$$sign\ Cov\left(r_t, r_{t+\tau}\right) = -sign\ \phi.$$

Also, from Lemma 1, the denominator in (9) is positive. Therefore:

$$sign\ \rho\left(r_t, r_{t+\tau}\right) = -sign\ \phi,$$

and this completes the proof of the lemma.

4.7.3 Parameter estimation

To estimate the model's parameters by means of the Kalman filter it is necessary first to express the model in state-space form. The measurement equation is:

$$y_t = d_t + Z_t \times \begin{bmatrix} S_t \\ m_t \end{bmatrix} + \varepsilon_t, \quad t = 1, ..., NT$$

where T is the number of observations, N is the number of maturities, and:

$$y_t = \ln\left(F_{\tau_i}\right) \quad i = 1, ..., N$$

is $N \times 1$ vector of observable log futures prices. Also, d_t and Z_t are $N \times 1$ and $N \times 2$ matrices:

$$d_t = \left[\left(r - \Theta^* - \frac{\sigma_\delta^2}{2\kappa^2} - \frac{\sigma\sigma_\delta}{\kappa}\right)\tau_i + \frac{\sigma_\delta^2\left(1 - e^{-2\kappa\tau_i}\right)}{4\kappa^3} + \left(\Theta^*\kappa + \sigma\sigma_\delta - \frac{\sigma_\delta^2}{\kappa}\right)\frac{1 - e^{-\kappa\tau_i}}{\kappa^2}\right]$$

and:

$$Z_t = \left[1, -\frac{1 - e^{-\kappa\tau_i}}{\kappa^2}\right], \quad i = 1, ..., N$$

The vector of observation errors, ε_t, is normally distributed with zero mean and covariance matrix Λ, where:

$$\Lambda_{ij} = \begin{cases} \sigma_{\varepsilon i}^2 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

The transition equations describe the dynamics of the discretized state variables:

$$[S_t, m_t]^T = c_t + Q_t \times \begin{bmatrix} S_{t-1} \\ m_{t-1} \end{bmatrix} + \eta_t,$$

where:

$$c_t = \left[\left(\mu - \frac{1}{2}\sigma^2 \right) \Delta t, \Theta\kappa\Delta t \right],$$

$$Q_t = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 - \kappa\Delta t \end{bmatrix},$$

and η_t is normally distributed with:

$$E(\eta_t) = 0, \;\; Var(\eta_t) = \sigma^2 \mathbf{O}_2,$$

where $\mathbf{O}_2$ is a 2×2 matrix of ones.

4.7.4 Figures

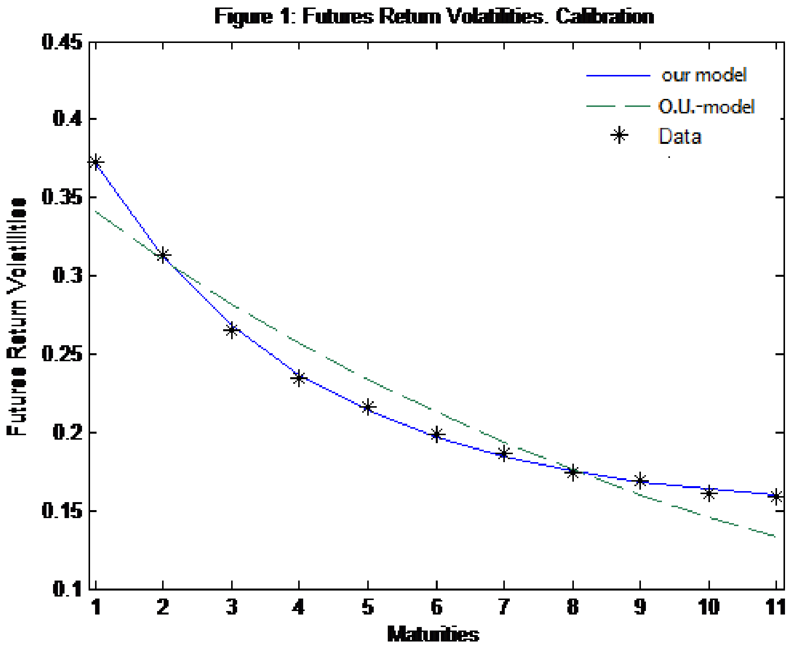

Figure 4.1: This figure shows the calibration of equation (16) to the term structure of oil futures returns volatilities obtained from weekly observations on NYMEX WTI - as described in Section 5. The model introduced in this paper is denoted as "our-model "; and the restricted version with the O-U process is denoted as "O.U.-model ". Accordingly, the model introduced in this paper fits the term structure almost perfectly, while he O-U model overestimates the mid-term volatilities, and underestimates the volatilities of the shortest and longest maturities.

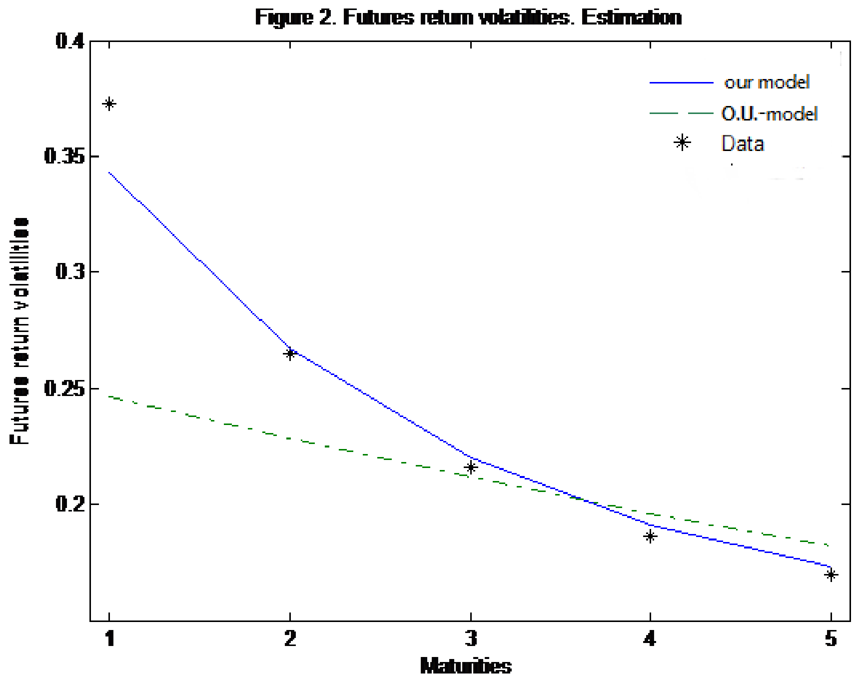

Figure 4.2: This figure shows estimations of the term structure of futures return volatilities of five different maturities. Accordingly, the *our-model* captures the shape of the data much more significantly than the *O.U-model* which misses the curve of empirical returns volatilities almost completely.