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Portfolio and Consumption Choice with Habit Formation under Inflation

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Portfolio and Consumption Choice with Habit formation under inflation*

Preliminary Version: Comments Welcome

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Abstract

We investigate the optimal portfolio and consumption policies for a finite-horizon investor in a life-cycle model with habit formation and inflation risk. We consider two types of habit investors: one forms habit based on real past consumption, while the other on nominal past consumption, which is motivated by money illusion. The optimal strategy is expressed explicitly in terms of the solution to a linear partial differential equation. We find that the effects of inflation on the optimal strategy depend on the type of habit investor, because it determines the risk profile of the hedge portfolio and subsistence portfolio. This dependence is robust to the incompleteness of the financial market.

Keywords: Portfolio and consumption choice; Habit formation; Inflation risk; Money illusion

JEL Codes: D91; G11

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1 Introduction

Time separable utility functions, such as power utility, are common in the optimal portfolio and consumption choice literature. Nonetheless, this time-separability has long been criticized because it is at odds with the empirical evidence that households' consumption depends on their own past consumption and/or the consumption of a reference group[1]. To this end, utility functions with habit formation have been proposed, which prescribe that investors form habit on the basis of their own previous consumption (internal habit formation) or the past history of aggregate consumption (external habit formation) and derive utility only from the consumption in excess of the habit levels. Although some studies[2] on the optimal portfolio choice already employ preferences with internal habit formation, to the best of our knowledge none of them considers inflation uncertainty. However, hedging inflation risk is important for long-term investors, as it substantially increases the volatility of the households' wealth. In particular, the interaction between the need to sustain future minimum consumption and that to hedge inflation risk may have substantial influence on the composition of the optimal portfolio. Moreover, the under-development of the inflation-indexed bonds market in practice makes it difficult for the habit-households to ensure future habit consumption due to the unspanned residual inflation risk. Therefore, incorporating inflation risk in the habit-based life-cycle models will add realism to the analysis of household decision making and enables us to obtain relevant policy implications.

The introduction of inflation risk may have different effects on different types of habit formation. Specifically, we consider two habit formation models, namely real habit formation and nominal habit formation. Under real habit formation, the real habit level is generated directly by past real consumption rates. In contrast, under nominal habit formation, investors form their nominal habit on the basis of previous nominal consumption but derive utility still from consumption in excess of real habit. This mismatch can be referred to as money illusion, because investors mistake nominal consumption for real consumption in forming habit levels. More importantly, it produces two distinctive features of real habit dynamics under nominal habit formation relative to those under real habit formation: the evolution of the real habit level becomes stochastic

[1] See, for example, Heien and Durham (1991), Ferson and Constantinides (1991) Ravina (2005), Korniotis (2010).

[2] See, for example, Detemple and Karatzas (2003), Bodie, Detemple, Otruba, and Walter (2004), Munk (2008).

and subject to the erosion of inflation. These distinctions reveal the bigger role of inflation under nominal habit formation. Therefore, it is of interest to compare the optimal portfolio and consumption strategy under different types of habit formation.

In this paper, we introduce inflation risk to a life-cycle model with habit formation and study the effects of inflation risk on the optimal portfolio and consumption strategy of a representative habit investor. In particular, we compare both the qualitative and quantitative properties of the optimal portfolio and consumption strategy under different habit formation and link the differences to the different roles of inflation. We begin by investigating a complete market case as the benchmark and proceed to a case with a single nominal bond. The analysis of both complete and incomplete market cases is performed under real habit formation and nominal habit formation, respectively. Specifically, the motivations for the incomplete market case are twofold. First, the perfect hedge against both expected inflation risk and interest rate risk by linear combination of two nominal bonds always requires a short position in one of the bonds, which is unrealistic from a practical point of view. Second, it is tempting to exclude the inflation-indexed bond from the asset menu due to the lack of a well-developed inflation-indexed bonds market.

Our main results are as follows: First, consistent with Munk (2008)[3], the optimal portfolio can be explicitly expressed in terms of the solution to a linear partial differential equation and is a combination of three portfolios: (1) a myopic mean-variance portfolio, (2) a hedge portfolio against variation of future investment opportunities in the economy with adjustment of habit formation and (3) a subsistence portfolio ensuring future minimum consumption. Habit formation affects the optimal portfolio strategy through two channels: on the one hand, it induces a subsistence demand and thus reduces the free wealth. This channel is referred to as leverage effect. On the other hand, habit persistence enters the pricing kernel and is a determinant of future investment opportunities. As a result, the hedge portfolio depends on habit strength as well as the risk profile of future habit levels.

Second, the effects of inflation risk on the optimal portfolio strategy differ substantially between the two cases. Under real habit formation, the importance of inflation risk is rather limited because there is no interaction between habit persistence and expected inflation. A direct consequence is the absence of influence of expected inflation on the

[3]Munk (2008) study a life-cycle model of consumption and investment with both habit formation and stochastic investment opportunities.

two habit formation channels; both the hedge portfolio and the subsistence portfolio are affected by inflation risk merely through the hedge against unexpected inflation. In contrast, inflation risk plays a much bigger role in the case of nominal habit formation, because the expected inflation exposure borne by the real habit levels carries over to the hedge portfolio and subsistence portfolio. On the one hand, expected inflation enters the pricing kernel, leading to a sharp increase in the inflation risk exposure of the hedge portfolio. On the other hand, the erosion of the subsistence portfolio caused by inflation leads to weaker leverage effect. Furthermore, the subsistence portfolio becomes substantially exposed to inflation risk. Another distinction between the two cases arises with respect to unexpected inflation hedging: the optimal portfolio takes full insurance against unexpected inflation risk in the case of real habit formation but leaves the subsistence portfolio uninsured in the alternative case because of the perfectly negative correlation between the real habit level and realized inflation.

Third, comparison with Brennan and Xia (2002)[4] identifies the effects of habit formation on the optimal portfolio in the presence of inflation risk: First of all, a new portfolio for sustaining subsistence consumption shows up. Moreover, the optimal portfolio takes lower equity exposure and interest rate risk exposure and the decline is more pronounced for higher habit strength. The inflation exposure is lower in the case of real habit formation, but higher in the case of nominal habit formation. Third, both the equity exposure and inflation risk exposure become dependent on investment horizon and the horizon effect on the interest rate exposure strengthens.

Fourth, in the incomplete market case, the optimal stock investment, optimal bond investment and stock-to-bond ratio decrease with habit strength and initial habit level. The horizon effect is negative for the stock investment and positive for the bond investment. Both the optimal stock investment and bond investment are higher under nominal habit formation than those under real habit formation, but the optimal portfolio leans more towards the bond.

Finally, we examine the expected wealth and expected consumption for three types of investors, namely non-habit investor, real habit investor and nominal habit investor. The wealth decumulation is slowest for the nominal habit investor, modest for the real habit investor and fastest for the non-habit investor. The non-habit investor starts with

[4]Brennan and Xia (2002) study the optimal portfolio and consumption choice of a finite-horizon investor in the presence of inflation risk.

higher consumption but has much lower consumption growth than her counterparts. Within the habit investors, the nominal one has higher wealth and consumption over the whole life-cycle than the real one.

Based on the theoretical analysis, some policy implications can be drawn for long-term investors, particularly for pension funds. There is some empirical evidence that households exhibit strong demand for guarantees for their pension income[5]. A possible explanation for such a demand is habit formation. As pension funds invest on behalf of their members, it is important to take into account the habit persistence of pension participants in making investment decisions: First, to ensure future guaranteed pension payout, there should be a clear separation between the subsistence portfolio and other traditional portfolios proposed in the portfolio choice literature. Moreover, the composition of the hedge portfolio and subsistence portfolio should depend on the type of guarantees offered (real v.s. nominal).

This article builds on the strand of papers on dynamic asset allocation with inflation risk. See, for example, Campbell and Viceira (2001), Brennan and Xia (2002), Sangvinatsos and Wachter (2005), Munk and Sørensen (2004), De Jong (2008), Koijen, Nijman, and Werker (2010) and Van Hemert (2010). In particular, we follow Brennan and Xia (2002) in modeling the asset price dynamics in the presence of inflation risk. On the other hand, this paper also relates to the literature on the optimal portfolio and consumption choice with habit formation in preferences. See, for example, Constantinides (1990), Detemple and Zapatero (1992), Detemple and Karatzas (2003), Bodie, Detemple, Otruba, and Walter (2004) and Munk (2008). Constantinides (1990) derives the optimal portfolio and consumption strategy for an infinitely-lived investor under the assumption of constant investment opportunities. Based on the insightful observation of Schroder and Skiadas (2002) that the model with linear habit formation can be mechanically transformed into an equivalent model without habit formation, Bodie, Detemple, Otruba, and Walter (2004) provide an analysis of optimal portfolio and consumption decision in a more general setting with endogenous labor supply and stochastic wages and Munk (2008) introduces stochastic investment opportunities to the habit-based life-cycle model, which is closest to this paper. We extend Munk's model by incorporating inflation risk and study how inflation influences the optimal strategy of the habit investor and how these effects depend on the type of habit formation.

[5]See, for example, Van Rooij, Kool, and Prast (2007) and Antolín, Payet, Whitehouse, and Yermo (2011).

The remainder of the paper is organized as follows. Section 2 sets up the model by describing the financial markets and preferences. Section 3 presents the solution to the optimization problem. Section 4 calibrates the model and carries out some numerical experiments. Section 5 concludes the paper and Appendix shows all proofs.

2 The Model

2.1 Financial Markets

We follow Brennan and Xia (2002) in modeling the asset price dynamics. There are four variables determining asset prices in the Brennan-Xia model: the nominal stock price S, the instantaneous real interest rate r, the instantaneous expected inflation π and the commodity price level Π. The term structure is characterized with the real interest rate and expected inflation. For simplicity, we assume that the risk premia on sources of uncertainty are constant[6]. The stock price follows a geometric Brownian motion as in the Black and Scholes (1973) model . The real interest rate and expected inflation follow Ornstein-Uhlenbeck processes as in the Vasicek (1977) model. The realized inflation equals the expected inflation plus a random shock. The equations driving the state variables are given by,

$$\frac{\mathrm{d}S_t}{S_t} = (R_t + \sigma_S \lambda_S)\mathrm{d}t + \sigma_S \mathrm{d}z_{St}, \tag{1}$$

$$\mathrm{d}r_t = \kappa(\bar{r} - r_t)\mathrm{d}t + \sigma_r \mathrm{d}z_{rt}, \tag{2}$$

$$\mathrm{d}\pi_t = \theta(\bar{\pi} - \pi_t)\mathrm{d}t + \sigma_\pi \mathrm{d}z_{\pi t}, \tag{3}$$

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = \pi_t \mathrm{d}t + \sigma_\Pi \mathrm{d}z_{\Pi t}, \tag{4}$$

where R is the nominal interest rate, σs capture the volatility, λ_S is the nominal price of equity risk, dzs are changes in standard Brownian motions z, θ and κ are mean reversion parameters, and $\bar{r}$ and $\bar{\pi}$ are unconditional means. Note that throughout this paper, we use uppercase letters for nominal variables and the corresponding lowercase letters for their real counterparts.

[6]Sangvinatsos and Wachter (2005) and Koijen, Nijman, and Werker (2010) allow for time variation in risk premia, but abstract from habit formation in preferences.

We can orthogonalize Equation (4) for unexpected inflation:

$$\begin{aligned}\frac{\mathrm{d}\Pi_t}{\Pi_t} &= \pi_t \mathrm{d}t + \xi_S \mathrm{d}z_{St} + \xi_r \mathrm{d}z_{rt} + \xi_\pi \mathrm{d}z_{\pi t} + \xi_u \mathrm{d}z_{ut} \\ &= \pi_t \mathrm{d}t + \xi' \mathrm{d}z_t, \end{aligned} \tag{5}$$

where $\mathrm{d}z = (\mathrm{d}z_S, \mathrm{d}z_r, \mathrm{d}z_\pi, \mathrm{d}z_u)'$ denotes the vector of innovations in standard Brownian motions with $\mathrm{d}z_u$[7] orthogonal to $\mathrm{d}z_S$, $\mathrm{d}z_r$, and $\mathrm{d}z_\pi$. The correlation matrix of $\mathrm{d}z$ therefore is

$$\rho = \begin{pmatrix} \{\rho_{S,r,\pi}\}_{3\times 3} & 0_{3\times 1} \\ 0_{1\times 3} & 1 \end{pmatrix}. \tag{6}$$

The real pricing kernel of the economy m_t, follows a diffusion process:

$$\begin{aligned}\frac{\mathrm{d}m_t}{m_t} &= -r_t \mathrm{d}t + \phi_S \mathrm{d}z_{St} + \phi_r \mathrm{d}z_{rt} + \phi_\pi \mathrm{d}z_{\pi t} + \phi_u \mathrm{d}z_{ut} \\ &= -r_t \mathrm{d}t + \phi' \mathrm{d}z_t, \end{aligned} \tag{7}$$

where, ϕs represents the constant loadings on the stochastic innovations in the economy and determines the market prices of risk, λ_S, λ_r, λ_π and λ_u, which are associated with innovations $\mathrm{d}z_S$, $\mathrm{d}z_r$, $\mathrm{d}z_\pi$ and $\mathrm{d}z_u$, respectively. Brennan and Xia (2002) show that the nominal short-term risk-free rate R and the vector of nominal market price of risk $\lambda = (\lambda_S, \lambda_r, \lambda_\pi, \lambda_u)'$ are given by

$$\lambda = \rho(\xi - \phi), \tag{8}$$

$$R_t = r_t + \pi_t - \xi' \lambda. \tag{9}$$

The time t nominal price of a nominal zero-coupon bond maturing at T, denoted by $P(t, T)$, evolves as,

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = [R_t - B_r(t,T)\sigma_r \lambda_r - B_\pi(t,T)\sigma_\pi \lambda_\pi]\mathrm{d}t - B_r(t,T)\sigma_r \mathrm{d}z_{rt} - B_\pi(t,T)\sigma_\pi \mathrm{d}z_{\pi t}, \tag{10}$$

[7]Note that in Brennan and Xia (2002), the subscript "u" means unhedgeable. However, as explained below, we consider a complete market in the benchmark model and thus there is no unhedgeable component of inflation risk. We follow this notation for the purpose of comparison.

where

$$B_r(t,T) = \kappa^{-1}(1 - e^{-\kappa(T-t)}), \tag{11}$$

$$B_\pi(t,T) = \theta^{-1}(1 - e^{-\theta(T-t)}). \tag{12}$$

In contrast, the time t real price of an inflation-indexed bond maturing at time T evolves as

$$\frac{\mathrm{d}p(t,T)}{p(t,T)} = [r_t - B_r(t,T)\sigma_r\bar{\lambda}_r]\mathrm{d}t - B_r(t,T)\sigma_r\mathrm{d}z_{rt}, \tag{13}$$

where $\bar{\lambda}_r = -\phi'\rho e_2$ and $e_2 = (0,1,0,0)'$. Applying Itô's Lemma to its nominal value, $P_t^* = \Pi_t p_t$, yields its nominal return,

$$\frac{\mathrm{d}P^*(t,T)}{P^*(t,T)} = [r_t + \pi_t - B_r(t,T)\sigma_r\lambda_r]\mathrm{d}t - B_r(t,T)\sigma_r\mathrm{d}z_{rt} + \xi'\mathrm{d}z_t. \tag{14}$$

Equation (10) shows that nominal bonds have loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$, but no loading on $\mathrm{d}z_u$. Thus, in an economy with only stocks and nominal bonds, the inflation process can not be fully spanned and the market is incomplete, which corresponds to the setting of Brennan and Xia (2002). In this paper, however, we add an inflation-indexed bond to the asset menu in order to complete market, because, as shown in (14), inflation-indexed bonds have non-zero loading on $\mathrm{d}z_u$, which allows the investor to hedge against unexpected inflation risk. It is important to note that the return processes of nominal bonds with different maturities only differ in their loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$. Hence, any desired combination of loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$ can be achieved by positions in any two bonds with different maturities.

In what follows, we consider two financial market settings. In the benchmark model, we assume that the investor can invest in five securities: a nominal instantaneously riskless asset, a stock, two nominal bonds with maturities T_1 and T_2 and an inflation-indexed bond with maturity T_3. Let σ be the factor loadings matrix of the stock and three bonds and Λ be the vector of the nominal risk premia, which are given by,

$$\sigma = \begin{pmatrix} \sigma_S & 0 & 0 & 0 \\ 0 & -B_r(0,T_1)\sigma_r & -B_\pi(0,T_1)\sigma_\pi & 0 \\ 0 & -B_r(0,T_2)\sigma_r & -B_\pi(0,T_2)\sigma_\pi & 0 \\ \xi_S & \xi_r - B_r(0,T_3)\sigma_r & \xi_\pi & \xi_u \end{pmatrix}, \tag{15}$$

and

$$\Lambda = \sigma\lambda = (\sigma_S\lambda_S, -B_r(0,T_1)\sigma_r\lambda_r - B_\pi(0,T_1)\sigma_\pi\lambda_\pi, -B_r(0,T_2)\sigma_r\lambda_r - B_\pi(0,T_2)\sigma_\pi\lambda_\pi,$$
$$-B_r(0,T_3)\sigma_r\lambda_r + \xi'\lambda)' . \tag{16}$$

In the alternative setting, the investor has access to only one nominal bond and stocks and therefore the market is incomplete. The motivations for the incomplete market case are twofold. First, the perfect hedge against both expected inflation risk and interest rate risk by linear combination of two nominal bonds always requires a short position in one of the bonds[8]. However, borrowing constraints prevail for most of market participants, making this combination largely infeasible in practice. Second, it is of interest to exclude the inflation-indexed bond from the asset menu because the inflation-indexed bonds market is less developed. Let σ_I be the factor loadings matrix of the stock and the nominal bond with maturity of T_4 and Λ_I be the vector of the nominal risk premia, which are given by,

$$\sigma_I = \begin{pmatrix} \sigma_S & 0 & 0 & 0 \\ 0 & -B_r(0,T_4)\sigma_r & -B_\pi(0,T_4)\sigma_\pi & 0 \end{pmatrix}, \tag{17}$$

and

$$\Lambda_I = \sigma_I\lambda = (\sigma_S\lambda_S, -B_r(0,T_4)\sigma_r\lambda_r - B_\pi(0,T_4)\sigma_\pi\lambda_\pi)' . \tag{18}$$

2.2 Preferences

We consider an investor with a fixed investment horizon T. The objective of the investor is to maximize over her life-cycle the expected discounted sum of all finite utility which are generated by the difference between real consumption c_t and real habit level h_t. In line with most of the literature, the utility function is assumed to be of the isoelastic form with risk aversion parameter γ. The individual's portfolio and consumption optimization problem can be formulated as

$$\max_{(C,x)\in A} \quad \mathrm{E}\left[\int_0^T \mathrm{e}^{-\delta t}\frac{(\frac{C_t}{\Pi_t} - h_t)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] \tag{19}$$

[8] See, for example, Brennan and Xia (2002) and Sangvinatsos and Wachter (2005).

where δ is the subjective discount factor, C is the nominal consumption rate, h is the real habit level and A is the set of admissible consumption and portfolio strategy. x is the vector of the portfolio weights on the risky assets and $1 - x'\iota$ is the weight on the nominally riskless asset. The investor maximizes her utility by appropriately choosing a nominal consumption process $C = (C_t)$ and a portfolio strategy $x = (x_t)$. The nominal wealth dynamics can be written as,

$$\mathrm{d}W_t = [W_t(R_t + x_t'\Lambda) - C_t]\,\mathrm{d}t + W_t x_t'\sigma\mathrm{d}z_t. \tag{20}$$

The requirement that the future consumption streams must be financeable by the initial wealth of the investor implies a static budget constraint,

$$\mathrm{E}\left[\int_0^T \frac{m_t}{m_0}\frac{C_t}{\Pi_t}\mathrm{d}t\right] \leq \frac{W_0}{\Pi_0}. \tag{21}$$

where W_0 is the nominal initial wealth, Π_0 is the initial price level. Choosing C_τ and x_τ over the period $\tau \in [t, T]$ to maximize utility in the remaining lifetime yields the indirect utility:

$$J_t = \max_{(C,x)\in A} \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\delta(s-t)}\frac{(\frac{C_s}{\Pi_s} - h_s)^{1-\gamma}}{1-\gamma}\mathrm{d}s\right]. \tag{22}$$

As shown in (19), the habit level can be regarded as a subsistence consumption rate, since the consumption rate must exceed the habit level. Note that γ is not the actual level of relative risk aversion, but still an important determinant of it:

$$RRA = \gamma\frac{c_t}{c_t - h_t}. \tag{23}$$

Obviously, the relative risk aversion is no longer constant, but decreasing in the ratio of consumption to habit. In other words, for any given habit level, higher consumption rate leads to lower risk aversion.

We consider two types of internal habit formation. The first one is real habit formation, in which the real habit level is generated by previous *real* consumption rates,

$$h_t = h_0\mathrm{e}^{-\beta t} + \alpha\int_0^t \mathrm{e}^{-\beta(t-s)}c_s\mathrm{d}s, \tag{24}$$

and evolves as,

$$\mathrm{d}h_t = -(\beta h_t - \alpha c_t)\mathrm{d}t. \tag{25}$$

Here c is the real consumption, α is the scaling parameter, β is the persistence parameter and h_0 is the initial real habit level. The real habit level is a weighted average of past consumption rates. The weights are exponentially decreasing so that the recent consumption rates are given higher weights. Following Munk (2008), we require that $\beta > \alpha$ to ensure that the real habit level will decline when her consumption rate coincides with the habit level. Note that when $c_t = h_t$, $\mathrm{d}h_t = -(\beta - \alpha)h_t \mathrm{d}t$. Thus, $(\beta - \alpha)$ can be interpreted as the decay rate of habit level at the minimum consumption and captures habit strength[9].

The alternative is nominal habit formation, in which the nominal habit level is generated by previous *nominal* consumption rates:

$$H_t = H_0 \mathrm{e}^{-\beta t} + \alpha \int_0^t \mathrm{e}^{-\beta(t-s)} C_s \mathrm{d}s. \tag{26}$$

As the investor derives utility from consumption on top of real habit level, but forms habit on the basis of previous nominal consumption, there is a mismatch between utility function and habit formation process. This can considered *money illusion*: the investor mistakes nominal consumption stream for real consumption stream in forming habit levels.

Applying Itô's lemma to the relationship $h_t = H_t/\Pi_t$ yields the dynamics of h_t,

$$\mathrm{d}h_t = -(\beta h_t + \pi_t - \xi'\rho\xi - \alpha c_t)\mathrm{d}t - h_t \xi' \mathrm{d}z_t. \tag{27}$$

Comparison with (25) reveals two noteworthy features of real habit dynamics under nominal habit formation: First, the evolution of the real habit level becomes stochastic because of the uncertainty inherited from unexpected inflation. Second, expected inflation enters the drift term, which implies that the real habit level in this case is eroded by inflation and therefore decays faster than that in the case of real habit formation.

[9] In what follows, we refer to $(\beta - \alpha)$ as habit strength. But, it is important to note that the smaller $(\beta - \alpha)$, the stronger the habit formation preference.

3 Solutions

3.1 Real Habit Formation

Solving the portfolio and consumption optimization problems formulated in Section 2 is far from trivial, because linear habit formation produces strong past dependence and renders the utility function not time separable. We follow Schroder and Skiadas (2002) and Munk (2008) in finding the solutions. Schroder and Skiadas (2002) show that the optimal portfolio choice models with habit formation in a given financial markets is closed linked to the corresponding models without habit formation in a financial market with a habit-adjusted price kernel. Applying this relation, Munk (2008) derives a general characterization of the optimal portfolio and consumption strategy and studies the quantitative effects of habit formation in some concrete settings. Under real habit formation, we extend Munk (2008) by incorporating inflation risk and examining how it affects the optimal portfolio strategy in both complete and incomplete market settings.

We first present two auxiliary processes, f and g, which are used to characterize the solutions under real habit formation. The process f is defined by

$$f_t = \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} \frac{m_s}{m_t} \mathrm{d}s \right] = \int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} p(t,s) \mathrm{d}s. \tag{28}$$

If $c_s = h_s$ for all $s \geq t$, future real habit levels depreciate at a rate of $(\beta - \alpha)$. Hence, f_t can be thought of as the time t market price of a bond paying continuous real coupons which are declining at the decay rate of real habit levels and $h_t f_t$ is the cost of ensuring that future real consumption never falls below the current real habit.

The process g is defined by,

$$g_t = \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-(\delta/\gamma)(s-t)} \left(\frac{m_s}{m_t} \right)^{1-\frac{1}{\gamma}} (1 + \alpha f_s)^{1-\frac{1}{\gamma}} \, \mathrm{d}s \right]. \tag{29}$$

As $(1 + \alpha f)$ can interpreted as the shadow price of one unit of consumption today, g captures the effects of both the habit formation (via f) and the future investment opportunities (via m) on the expected utility. It should be noted that for $\gamma > 1$, both f and g decrease with $(\beta - \alpha)$.

We write the dynamics of f and g as

$$\mathrm{d}f_t = f_t \left[\mu_{ft}\mathrm{d}t + \sigma'_{ft}\mathrm{d}z_t\right], \tag{30}$$

$$\mathrm{d}g_t = g_t \left[\mu_{gt}\mathrm{d}t + \sigma'_{gt}\mathrm{d}z_t\right], \tag{31}$$

where

$$\sigma_{gt} = \left(0, \frac{\partial g/\partial r(r,t)}{g(r,t)}\sigma_r, 0, 0\right)', \tag{32}$$

$$\sigma_{ft} = \left(0, \frac{-\int_t^T B(t,s)\mathrm{e}^{-(\beta-\alpha)(s-t)}p(t,s)\mathrm{d}s}{\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)}p(t,s)\mathrm{d}s}\sigma_r, 0, 0\right)', \tag{33}$$

and μ_f and μ_g are some adapted processes. Equation (33) shows that under real habit formation, the volatilities of f and g are driven solely by the interest rate risk. This stems from the fact that real zero-coupon bonds, which constitute f, only carry exposure to interest rate risk and this exposure is passed on to g through f.

Theorem 1 characterizes the optimal strategy in terms of the solution of a one dimensional, second order PDE for g.

Theorem 1. *Assume that $w_0 \geq h_0 f_0$. The indirect utility is*

$$J_t = \frac{{g_t}^{\gamma}(w_t^* - h_t^* f_t)^{1-\gamma}}{1-\gamma} \tag{34}$$

and $g(r,t)$ solves the PDE,

$$\begin{aligned}\frac{\partial g}{\partial t}(r,t) + \left(\kappa(\bar{r} - r_t) + \left(1 - \frac{1}{\gamma}\right)\sigma_r\phi_r\right)\frac{\partial g}{\partial r}(r,t) + \frac{1}{2}\sigma_r^2\frac{\partial^2 g}{\partial r^2}(r,t)\\ +(1+\alpha f(r,t))^{1-\frac{1}{\gamma}} = \left(\frac{\delta}{\gamma} + \left(1-\frac{1}{\gamma}\right)r_t + \frac{\gamma-1}{2\gamma^2}\phi'\rho\phi\right)g(r,t)\end{aligned} \tag{35}$$

with the terminal condition $g(r_T, T) = 0$. The optimal real consumption strategy is

$$c_t^* = h_t^* + (1+\alpha f_t)^{-\frac{1}{\gamma}}\frac{w_t^* - h_t^* f_t}{g_t}. \tag{36}$$

The optimal portfolio strategy, $x_t^* = (x_{St}^*, x_{Nt}^{T_1 *}, x_{Nt}^{T_2 *}, x_{It}^{T_3 *})'$, is given by

$$
\begin{aligned}
x_t^* &= \frac{w_t^* - h_t^* f_t}{w_t^*} \frac{1}{\gamma} (\sigma')^{-1}(-\phi) + \frac{w_t^* - h_t^* f_t}{w_t^*} (\sigma')^{-1} \sigma_{gt} + \frac{h_t^* f_t}{w_t^*} (\sigma')^{-1} \sigma_{ft} + (\sigma')^{-1} \xi \\
&= \frac{w_t^* - h_t^* f_t}{w_t^*} \frac{1}{\gamma} \Sigma^{-1} \Lambda + \frac{w_t^* - h_t^* f_t}{w_t^*} \left(1 - \frac{1}{\gamma}\right) \Sigma^{-1} \sigma \rho (\hat{\sigma}_{gt} + \xi) + \frac{h_t^* f_t}{w_t^*} \Sigma^{-1} \sigma \rho (\sigma_{ft} + \xi),
\end{aligned}
\tag{37}
$$

where w_t^* is the real wealth process induced by the optimal strategy, and h_t^* is the real habit level induced by the optimal real consumption strategy. $\hat{\sigma}_{gt} = \left(\frac{\gamma}{\gamma - 1}\right) \sigma_{gt}$. $\Sigma = \sigma \rho \sigma'$ is the variance-covariance matrix of the nominal asset returns and $(\sigma')^{-1}\xi$ represents the vector of covariances between the asset returns and inflation.

The condition $w_0 \geq h_0 f_0$ ensures that the initial wealth of the investor can sustain the minimum consumption level in the future. As shown in Appendix A, we first derive the solution of the dual model without habit formation, which is closely related to the model of Brennan and Xia (2002) and then transform it to the solution of the primal model with habit formation by applying the results of Schroder and Skiadas (2002) to the case with inflation risk.

The optimal consumption in (36) contains two components: the current habit level and a time and state-dependent fraction of the free wealth $w_t - h_t f_t$. Since both f and g decrease with $(\beta - \alpha)$ for $\gamma > 1$, the marginal propensity to consume $(1 + \alpha f_t)^{-1/\gamma} / g_t$ and the consumption rate increase with $(\beta - \alpha)$, implying that as the habit strength declines the investor tends to consume more out of her wealth. As f_t and g_t have no loadings on both expected and unexpected inflation risk factors, the optimal consumption strategy is unaffected by inflation risk.

Equation (37) expresses the optimal portfolio as the sum of three portfolios: a myopic portfolio that invests in the nominal mean-variance tangency portfolio represented by $\Sigma^{-1}\Lambda$, a hedge portfolio that provides hedge against variation of future investment opportunities in the economy modified by the presence of habit formation, and a subsistence portfolio that ensures future minimum consumption. As the presence of habit formation induces the investor to set aside a fraction of wealth for future minimum consumption stream, the free wealth is reduced to $w_t - h_t f_t$, which dampens both the myopic demand and the hedge demand. In addition to this leverage effect, habit formation affects the hedge demand also through σ_g. Equation (64) in Appendix A shows that the habit-adjusted pricing kernel, which determines the investment opportunities in the presence of habit formation, involves f. Therefore, the optimal hedge against variations in future investment opportunities must take into account the changes in the cost of ensuring the minimum consumption level.

Comparison with Munk (2008) reveals that under real habit formation, the effects of inflation risk on the optimal portfolio strategy are very small: it only induces a hedge against unexpected inflation, which corresponds to the term $(\sigma')^{-1}\xi$. This is a direct consequence of no interaction between habit persistence and expected inflation risk under real habit formation: since σ_f and σ_g are unaffected by inflation risk, both the hedge portfolio and the subsistence portfolio carry exposure to inflation risk only through the hedge against unexpected inflation, which is consistent with Brennan and Xia (2002).

Turning to the incomplete case with only one nominal bond, we follow the approach taken in De Jong (2008) to derive the optimal portfolio strategy. The rationale behind the approach is to minimize a pre-specified norm of the difference between optimal and feasible wealth dynamics. Theorem 2 characterizes the solution.

Theorem 2. *The optimal portfolio strategy in the case of one nominal bond* $x_t^* = (x_{St}^*, x_{Nt}^{T_4*})'$, *is given by*

$$\begin{aligned} x_t^* &= \frac{w_t^* - h_t^* f_t}{w_t^*}\frac{1}{\gamma}\Sigma_I^{-1}\sigma_I\rho(-\phi) + \frac{w_t^* - h_t^* f_t}{w_t^*}\Sigma_I^{-1}\sigma_I\rho\sigma_{gt} + \frac{h_t^* f_t}{w_t^*}\Sigma_I^{-1}\sigma_I\rho\sigma_{ft} + \Sigma_I^{-1}\sigma_I\rho\xi \\ &= \frac{w_t^* - h_t^* f_t}{w_t^*}\frac{1}{\gamma}\Sigma_I^{-1}\Lambda_I + \frac{w_t^* - h_t^* f_t}{w_t^*}\left(1 - \frac{1}{\gamma}\right)\Sigma_I^{-1}\sigma_I\rho(\hat{\sigma}_{gt} + \xi) + \frac{h_t^* f_t}{w_t^*}\Sigma_I^{-1}\sigma_I\rho(\sigma_{ft} + \xi), \end{aligned} \tag{38}$$

where $\Sigma_I = \sigma_I\rho\sigma_I'$ *is the variance-covariance matrix of the nominal asset returns.*

3.2 Nominal Habit Formation

In this subsection, we turn to nominal habit persistence, which is formed based on the households' previous nominal consumption. The individual's portfolio and consumption optimization problem can be reformulated as

$$\max_{(C,x)\in A} \quad \mathrm{E}\left[\int_0^T \mathrm{e}^{-\delta t}\frac{\left(\frac{C_t - H_t}{\Pi_t}\right)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] \tag{39}$$

where H is the nominal habit level defined by,

$$H_t = H_0 \mathrm{e}^{-\beta t} + \alpha \int_0^t \mathrm{e}^{-\beta(t-s)} C_s \mathrm{d}s \tag{40}$$

Once again, we present the solution in terms of two auxiliary processes denoted by $\hat{f}$ and $\hat{g}$, respectively. The process $\hat{f}$ is defined by

$$\hat{f}_t = \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} \frac{m_s/m_t}{\Pi_s/\Pi_t} \mathrm{d}s \right] = \int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)} P(t,s) \mathrm{d}s. \tag{41}$$

If $C_s = H_s$ for all $s \geq t$, future nominal habit levels depreciate at a rate of $(\beta - \alpha)$. Hence, $\hat{f}_t$ can be thought of as the time t market price of a bond paying continuous nominal coupons which are declining at the decay rate of nominal habit levels and $H_t \hat{f}_t$ is the cost of ensuring that future nominal consumption never falls below the current nominal habit. Comparison between (28) and (41) shows that the habit bond under nominal habit formation is comprised of nominal zero-coupon bonds rather than inflation-indexed zero-coupon bonds. It is worth noting that under the calibrated parameter values shown below, $f_t > \hat{f}_t$ for any $t < T$, which implies that the nominal habit bond is cheaper than the real habit bond. This can be explained by the fact that in the case of nominal habit formation, the real habit level is allowed to be eroded by inflation and depreciates faster. Since the values of future coupons decline, the price of the habit bond drops.

The process $\hat{g}$ is defined by,

$$\hat{g}_t = \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-(\delta/\gamma)(s-t)} \left(\frac{m_s}{m_t} \right)^{1-\frac{1}{\gamma}} \left(1 + \alpha \hat{f}_s \right)^{1-\frac{1}{\gamma}} \mathrm{d}s \right]. \tag{42}$$

$\hat{g}$ captures the effects of both the habit formation (via $\hat{f}$) and the future investment opportunities (via m) on the expected utility. It should be noted that for $\gamma > 1$, both $\hat{f}_t$ and $\hat{g}_t$ decrease with $(\beta - \alpha)$ and $\hat{g}_t < g_t$.

We define the dynamics of $\hat{f}$ and $\hat{g}$ as

$$\mathrm{d}\hat{f}_t = \hat{f}_t \left[\mu_{\hat{f}t} \mathrm{d}t + \sigma'_{\hat{f}t} \mathrm{d}z_t \right] \tag{43}$$

$$\mathrm{d}\hat{g}_t = \hat{g}_t \left[\mu_{\hat{g}t} \mathrm{d}t + \sigma'_{\hat{g}t} \mathrm{d}z_t \right] \tag{44}$$

where

$$\sigma_{\hat{g}t} = \left(0, \frac{\partial \hat{g}/\partial r(r,\pi,t)}{\hat{g}(r,\pi,t)}\sigma_r, \frac{\partial \hat{g}/\partial \pi(r,\pi,t)}{\hat{g}(r,\pi,t)}\sigma_\pi, 0\right)', \tag{45}$$

$$\sigma_{\hat{f}t} = \left(0, \frac{-\int_t^T B(t,s)\mathrm{e}^{-(\beta-\alpha)(s-t)}P(t,s)\mathrm{d}s}{\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)}P(t,s)\mathrm{d}s}\sigma_r, \frac{-\int_t^T D(t,s)\mathrm{e}^{-(\beta-\alpha)(s-t)}P(t,s)\mathrm{d}s}{\int_t^T \mathrm{e}^{-(\beta-\alpha)(s-t)}P(t,s)\mathrm{d}s}\sigma_\pi, 0\right)'. \tag{46}$$

and $\mu_{\hat{f}}$ and $\mu_{\hat{g}}$ are some adapted processes. Equation (46) shows that under nominal habit formation, the volatilities of $\hat{f}$ and $\hat{g}$ are driven by both the interest rate risk and expected inflation risk. This is because nominal zero-coupon bonds, which constitute $\hat{f}$, are exposed to both risk factors and $\hat{g}$ inherit these exposures from $\hat{f}$.

Theorem 3 characterizes the optimal strategy in terms of the solution of a two dimensional, second order PDE for $\hat{g}$.

Theorem 3. *Assume that $w_0 \geq h_0\hat{f}_0$. The indirect utility is*

$$J_t = \frac{\hat{g}_t^\gamma (w_t^* - h_t^*\hat{f}_t)^{1-\gamma}}{1-\gamma} \tag{47}$$

and $\hat{g}(r,\pi,t)$ solves the PDE,

$$\begin{aligned}\left(\frac{\delta}{\gamma} + \left(1-\frac{1}{\gamma}\right)r_t + \frac{\gamma-1}{2\gamma^2}\phi'\rho\phi\right)\hat{g}(r,\pi,t) &= \frac{\partial \hat{g}}{\partial t}(r,\pi,t) + (1+\alpha\hat{f}(r,\pi,t))^{1-\frac{1}{\gamma}} \\ &+ \left(\kappa(\bar{r}-r_t) + \left(1-\frac{1}{\gamma}\right)\sigma_r\phi_r\right)\frac{\partial \hat{g}}{\partial r}(r,\pi,t) + \frac{1}{2}\sigma_r^2\frac{\partial^2 \hat{g}}{\partial r^2}(r,\pi,t) \\ &+ \left(\theta(\bar{\pi}-\pi_t) + \left(1-\frac{1}{\gamma}\right)\sigma_\pi\phi_\pi\right)\frac{\partial \hat{g}}{\partial \pi}(r,\pi,t) + \frac{1}{2}\sigma_\pi^2\frac{\partial^2 \hat{g}}{\partial \pi^2}(r,\pi,t)\end{aligned} \tag{48}$$

with the terminal condition $\hat{g}(r,\pi,T) = 0$. The optimal real consumption strategy is

$$c_t^* = h_t^* + (1+\alpha\hat{f}_t)^{-\frac{1}{\gamma}}\frac{w_t^* - h_t^*\hat{f}_t}{\hat{g}_t}. \tag{49}$$

The optimal portfolio strategy, $x_t^* = (x_{St}^*, x_{Nt}^{T_1*}, x_{Nt}^{T_2*}, x_{It}^{T_3*})'$, *is given by*

$$\begin{aligned} x_t^* &= \frac{W_t^* - H_t^* \hat{f}_t}{W_t^*} \frac{1}{\gamma} (\sigma')^{-1}(-\phi) + \frac{W_t^* - H_t^* \hat{f}_t}{W_t^*} (\sigma')^{-1}(\sigma_{\hat{g}t}) \\ &\quad + \frac{H_t^* \hat{f}_t}{W_t^*} (\sigma')^{-1} \sigma_{\hat{f}t} + \frac{W_t^* - H_t^* \hat{f}_t}{W_t^*} (\sigma')^{-1} \xi \\ &= \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \frac{1}{\gamma} \Sigma^{-1} \Lambda + \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \left(1 - \frac{1}{\gamma}\right) \Sigma^{-1} \sigma \rho (\hat{\sigma}_{\hat{g}t} + \xi) + \frac{h_t^* \hat{f}_t}{w^*} \Sigma^{-1} \sigma \rho \sigma_{\hat{f}t}. \end{aligned} \quad (50)$$

We assume that $\gamma > 1$ and focus on the comparison between the optimal strategy in two cases. The relation $\hat{f}_t < f_t$ implies that the value of the habit bond declines. This is a result of erosion by inflation: as the inflation drives the real habit level to decay faster, the habit bond price goes down and therefore less money is needed to ensure future subsistence consumption. The relation $\hat{g}_t < g_t$, together with $\hat{f}_t < f_t$ implies that the marginal propensity to consume $(1 + \alpha f_t)^{-1/\gamma}/g_t$ and the consumption rate increase.

On the other hand, there are some major changes to the optimal portfolio strategy. First, the reduction in the value of the habit bond leads to weaker leverage effects and lower subsistence demand. Therefore, the speculative portfolio expands while the subsistence portfolio shrinks. However, it is not possible to determine analytically how the hedge portfolio changes between the two cases, because habit persistence influences the hedge portfolio not only through the leverage effect but also through its effect on investment opportunities and the latter effect has to be evaluated numerically. Second, the inflation risk has much larger impact on the optimal portfolio than it does under the real habit formation. The explanation for this bigger effect is that the habit bond $\hat{f}$, which determines not only the risk profile of the subsistence portfolio but also future investment opportunities, is comprised of nominal zero-coupon bonds and therefore bears expected inflation risk. As a result, both σ_f and σ_g become subject to expected inflation risk, thereby substantially increasing the inflation risk exposures of both the hedge portfolio and the subsistence portfolio. Third, the optimal portfolio no longer takes full insurance against unexpected inflation risk; the subsistence portfolio is left uninsured. This is because under nominal habit formation the real habit level is permitted to be reduced by inflation and therefore has a perfectly negative correlation with realized inflation, which is clearly shown in (27).

Turning to the incomplete case with only one nominal bond, we once again follow the approach taken in De Jong (2008) to find the optimal portfolio strategy. Theorem 4 characterizes the solution.

Theorem 4. *The optimal portfolio strategy in the case of one nominal bond* $x_t^* = (x_{St}^*, x_{Nt}^{T_4 *})'$, *is given by*

$$\begin{aligned}
x_t^* &= \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \frac{1}{\gamma} \Sigma_I^{-1} \sigma_I \rho (-\phi) + \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \Sigma_I^{-1} \sigma_I \rho \sigma_{\hat{g}t} \\
&\quad + \frac{h_t^* \hat{f}_t}{w_t^*} \Sigma_I^{-1} \sigma_I \rho \sigma_{\hat{f}t} + \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \Sigma_I^{-1} \sigma_I \rho \xi \\
&= \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \frac{1}{\gamma} \Sigma_I^{-1} \Lambda_I + \frac{w_t^* - h_t^* \hat{f}_t}{w_t^*} \left(1 - \frac{1}{\gamma}\right) \Sigma_I^{-1} \sigma_I \rho (\hat{\sigma}_{\hat{g}t} + \xi) + \frac{h_t^* \hat{f}_t}{w_t^*} \Sigma_I^{-1} \sigma_I \rho \sigma_{\hat{f}t}. \quad (51)
\end{aligned}$$

4 Numerical Illustrations

In this section, we carry out some numerical experiments to compare the effects of inflation risk and habit persistence on the optimal consumption and portfolio strategy under different types of habit formation. In the benchmark case, we consider an investor with risk aversion parameter $\gamma = 3$, a 30-year horizon, and a time preference rate $\delta = 0.02$. Initial wealth, initial habit level and initial price level are set to $W_0 = 10000$, $h_0 = 400$ and $\Pi_0 = 1$, respectively. Habit parameters are taken to be $\alpha = 0.3$ and $\beta = 0.4$. To calibrate the model, we follow the parameter estimates reported in Brennan and Xia (2002), which are shown in Table 1. Note that we assume that unexpected inflation is uncorrelated with stock returns, real interest rate and expected inflation, so that only inflation-index bonds can be used to hedge against unexpected inflation. In the cases with complete market, we assume that the there are three bonds available to the investor, namely an 1-year nominal bond ($T_1 = 1$), an 10-year nominal bond ($T_2 = 10$) and an 1-year inflation-indexed bond ($T_3 = 1$). Results under real habit formation are obtained by solving the one dimensional PDE (35) for g using a Crank-Nicolson finite difference scheme, with 500 real interest rate subintervals and 1000 time steps. In contrast, results under nominal habit formation are obtained by solving the two dimensional PDE (48) for g using an explicit finite difference scheme, with 50 real interest rate subintervals, 50 expected inflation subintervals and 1000 time steps.

We can calculate the loadings on the innovations in different risk factors to decompose

Table 1: Parameter values

Parameter	Value
Stock return process: $\mathrm{d}S/S = (R_f + \lambda_S \sigma_S)\mathrm{d}t + \sigma_S \mathrm{d}z_S$	
σ_S	0.158
λ_S	0.343
Real interest rate process: $\mathrm{d}r = \kappa(\bar{r} - r)\mathrm{d}t + \sigma_r \mathrm{d}z_r$	
$\bar{r}$	0.017
κ	0.105
σ_r	0.013
λ_r	-0.209
Expected inflation process: $\mathrm{d}\pi = \theta(\bar{\pi} - \pi)\mathrm{d}t + \sigma_\pi \mathrm{d}z_\pi$	
$\bar{\pi}$	0.054
θ	0.027
σ_π	0.014
λ_π	-0.105
Realized inflation process: $\frac{\mathrm{d}\Pi}{\Pi} = \pi \mathrm{d}t + \xi_S \mathrm{d}z_S + \xi_r \mathrm{d}z_r + \xi_\pi \mathrm{d}z_\pi + \xi_u \mathrm{d}z_u$	
ξ_S	0
ξ_r	0
$\xi\pi$	0
ξ_u	0.013
Pricing kernel process: $\frac{\mathrm{d}m}{m} = -r\mathrm{d}t + \phi_S \mathrm{d}z_S + \phi_r \mathrm{d}z_r + \phi_\pi \mathrm{d}z_\pi + \phi_u \mathrm{d}z_u$	
ϕ_S	-0.333
ϕ_r	0.170
ϕ_π	0.120
ϕ_u	0
Correlations	
ρ_{Sr}	-0.129
$\rho_{S\pi}$	-0.024
$\rho_{r\pi}$	-0.061

This table shows the parameter values taken from Brennan and Xia (2002).

the risk exposure of the optimal portfolio:

$$L_S = x_S, \tag{52}$$

$$L_r = -x_N^{T_1} B_r(0,T_1) - x_N^{T_2} B_r(0,T_2) - x_I^{T_3} \left(B_\pi(0,T_3) - \frac{\xi_r}{\sigma_r} \right), \tag{53}$$

$$L_\pi = -x_N^{T_1} B_\pi(0,T_1) - x_N^{T_2} B_\pi(0,T_2) + x_I^{T_3} \frac{\xi_r}{\sigma_r}, \tag{54}$$

$$L_u = x_I^{T_3}. \tag{55}$$

The loadings on the innovations in the equity risk and unexpected inflation risk coincide

Table 2: Optimal portfolio strategy in complete market under real habit formation

(a) For different habit strength $(\beta - \alpha)$

$\beta - \alpha$	L_r	L_π	x_S	x_N^1	x_N^{10}	x_I^1
0.1	-9.027	-1.969	0.484	26.502	-2.759	1.000
0.2	-9.541	-2.348	0.581	27.458	-2.818	1.000
0.3	-9.752	-2.501	0.620	27.868	-2.846	1.000
0.4	-9.868	-2.592	0.638	28.110	-2.873	1.000
No habit	-10.454	-2.857	0.703	29.673	-3.015	1.000

(b) For different initial habit level h_0

h_0	L_r	L_π	x_S	x_N^1	x_N^{10}	x_I^1
200	-10.121	-2.438	0.586	29.598	-3.061	1.000
300	-9.571	-2.188	0.535	28.046	-2.910	1.000
400	-9.027	-1.969	0.484	26.502	-2.759	1.000
500	-8.478	-1.754	0.431	23.411	-2.612	1.000
600	-7.928	-1.531	0.384	23.405	-2.459	1.000

(c) For different investment horizon T

T	L_r	L_π	x_S	x_N^1	x_N^{10}	x_I^1
1	-4.542	-2.751	0.678	6.517	-0.421	1.000
5	-5.433	-2.432	0.601	10.974	-0.964	1.000
10	-6.506	-2.203	0.539	15.887	-1.544	1.000
20	-8.061	-2.018	0.501	22.537	-2.308	1.000
30	-9.027	-1.969	0.484	26.502	-2.759	1.000

The table shows the optimal portfolio strategy in complete market under real habit formation. L_r (L_π) is the sensitivity of the optimal portfolio to innovations in r (π). x_S, x_N^1 x_N^{10} and x_I^1 are the fractions of wealth invested in the stock, the 1-year nominal bond, the 10-year nominal bond and the 1-year inflation-indexed bond respectively. The parameter values are as follows: $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 1.

with the optimal stock allocation and optimal inflation-indexed bond allocation, because these two risks are borne solely by the stock and inflation-indexed bond, respectively. Hence, in what follows, we don't report L_S and L_u. It should be noted that the stock is only contained in the myopic portfolio, because it is appropriated neither for hedging purpose nor for ensuring the future subsistence consumption.

Table 2 summarizes the optimal portfolio strategy in complete market under real habit formation. As shown in panel (a), the introduction of habit formation remarkably reduces the equity exposure and expected inflation risk exposure and this effect is more pronounced for stronger habit formation, which is associated with smaller $(\beta-\alpha)$. These lower risk exposures can be attributed to the reduction of the free wealth, because under real habit formation the equity risk and expected inflation risk are only taken by the myopic portfolio. In contrast, habit strength has different effects on the interest risk exposure of different portfolios. While the leverage effect reduces the myopic demand and hedge demand, the expansion of the subsistence demand driven by larger habit strength leads to higher interest rate loadings. Moreover, habit strength can affect the hedge portfolio also by changing the volatility of habit-adjusted investment opportunities. The observation that the interest rate sensitivity is decreasing in habit strength indicates that the leverage effect dominates. The lower interest rate and inflation risk exposures associated with weaker habit persistence reduce the absolute demand for both nominal bonds. Panel (b) shows that as initial habit level rises, the optimal portfolio takes less interest rate risk exposure and inflation risk exposure and reduces the holdings of the stock and the two nominal bonds because of the pure leverage effect.

Panel (c) illustrates the importance of investment horizon. Equity exposure and inflation exposure are decreasing in the investment horizon, since longer horizon substantially increases the price of the habit bond and generates stronger leverage effect. On the contrary, the optimal interest rate loadings rise with investment horizon. The reason is that the volatilities of both the habit bond σ_f and future investment opportunities σ_g increase sharply, which induces much larger subsistence demand and hedge demand and offsets the leverage effect. As a result, the absolute portfolio shares in both nominal bonds are higher for longer horizon. These observations stand in stark contrast to Brennan and Xia (2002), who find limited horizon effect on the optimal interest rate risk exposure (about five years) and no horizon effects on the optimal equity exposure and optimal inflation risk exposure. Finally, the optimal inflation-indexed bond holding is independent of habit parameters and investment horizon, because the optimal portfolio simply takes a full insurance against the unexpected inflation risk, which corresponds to the term $(\sigma')^{-1}\xi$.

Table 3 reports the optimal portfolio strategy with one nominal bond under real habit formation. From panel (a) we can see that the presence of habit persistence in preference drives down the demand for both risky assets because of the leverage effect. While both the stock holding and bond holding increase with habit strength, the whole portfolio tilts towards the bond. This is a result of higher hedge demand and subsistence demand induced by stronger habit persistence. Panel (b) shows that higher initial habit level dampens the risky investment because of the reduction in free wealth and makes

Table 3: Optimal portfolio strategy with one nominal bond under real habit formation

	(a) For different habit strength ($\beta - \alpha$)				
Bond maturity	0.1	0.2	0.3	0.4	No habit
1 year					
x_S	0.520	0.614	0.652	0.673	0.741
x_N	5.260	5.708	5.893	5.995	6.417
$\frac{x_S}{x_N}$	0.099	0.108	0.111	0.112	0.115
10 year					
x_S	0.535	0.630	0.669	0.690	0.759
x_N	0.575	0.629	0.651	0.664	0.712
$\frac{x_S}{x_N}$	0.931	1.002	1.027	1.040	1.066
	(b) For different initial habit level h_0				
Bond maturity	200	300	400	500	600
1 year					
x_S	0.632	0.576	0.520	0.464	0.407
x_N	6.014	5.637	5.260	4.883	4.505
$\frac{x_S}{x_N}$	0.105	0.102	0.099	0.095	0.090
10 year					
x_S	0.650	0.593	0.535	0.478	0.421
x_N	0.661	0.618	0.575	0.532	0.488
$\frac{x_S}{x_N}$	0.983	0.959	0.931	0.900	0.862
	(c) For different investment horizon T				
Bond maturity	1 year	5 years	10 years	20 years	30 years
1 year					
x_S	0.686	0.613	0.563	0.527	0.520
x_N	3.655	3.879	4.241	4.846	5.260
$\frac{x_S}{x_N}$	0.188	0.158	0.133	0.109	0.099
10 year					
x_S	0.693	0.622	0.574	0.541	0.535
x_N	0.433	0.448	0.478	0.535	0.575
$\frac{x_S}{x_N}$	1.602	1.389	1.200	1.011	0.931

The table shows the optimal portfolio strategy with one nominal bond under real habit formation. x_S/x_N is the stock-to-bond ratio. We consider 1-year nominal bond and 10-year nominal bond, respectively. x_S (x_N) is the demand for the stock (the nominal bond). The parameter values are as follows: $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 1.

the optimal portfolio lean towards the bond because the stock can be used neither for hedging purpose nor ensuring future minimum consumption. Panel (c) illustrates the horizon effect. It turns out that while the stock demand decreases with investment horizon, the bond demand increases, since longer horizon generates larger value of the habit bond and higher volatility of future investment opportunities. Comparison between Table 2 and Table 3 shows that for given parameter values, the optimal portfolio share in the stock is higher in the incomplete market than it is in the complete market and the difference is larger for the case with long-term bond. The higher demand for the stock stems from the fact that $\mathrm{d}z_S$ is calibrated to be negatively correlated with both $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$. As the bonds have negative loadings on $\mathrm{d}z_r$ and $\mathrm{d}z_\pi$, the correlation between the nominal returns between the stock and the bonds is positive, which dampens the stock investment in the myopic portfolio. Unreported results show that in the case of one bond, the optimal portfolio takes lower interest risk exposure but higher inflation risk exposure than does the optimal portfolio in the complete market. Because the correlation between $\mathrm{d}z_S$ and $\mathrm{d}z_r$ is much higher than that between $\mathrm{d}z_S$ and $\mathrm{d}z_\pi$, the decreased correlation effect associated with lower interest rate exposure outweighs the increased correlation effect associated with higher inflation exposure. Moreover, since the interest risk exposure decreases with the bond maturity in the incomplete market case, the correlation effect diminishes accordingly.

Now we turn to the optimal portfolio strategy under nominal habit formation. Table 4 shows the results for different habit parameters and investment horizon. Some interesting changes emerge as compared to the optimal portfolio strategy under real habit formation shown in Table 2. First, as shown in panel (a), the presence of habit persistence induces larger inflation risk exposure and this effect intensifies with habit strength, which is in sharp contrast to the decreasing inflation risk exposure in the real habit case. Moreover, for any given habit strength, the optimal portfolio under nominal habit formation has much larger loadings on the inflation risk than it does under real habit formation. These distinctions are consequences of different risk profiles of the hedge portfolio and the subsistence portfolio under different types of habit formation: while these two portfolios under real habit formation are only subject to the interest rate risk, those under nominal habit formation carry the expected inflation risk through the habit bond $\hat{f}$, because $\hat{f}$ is comprised of nominal zero-coupon bonds rather than inflation-indexed zero-coupon bonds. As a result, the impact of the inflation risk on the optimal portfolio is substantially amplified. Second, although the equity exposure

Table 4: Optimal portfolio strategy in complete market under nominal habit formation

(a) For different habit strength ($\beta - \alpha$)

$\beta - \alpha$	L_r	L_π	x_S	x_N^1	x_N^{10}	x_I^1
0.1	-9.701	-4.466	0.549	23.046	-2.085	0.782
0.2	-9.911	-3.617	0.602	25.929	-2.506	0.857
0.3	-10.021	-3.257	0.634	27.198	-2.688	0.893
0.4	-10.103	-3.078	0.641	27.939	-2.793	0.919
No habit	-10.454	-2.857	0.703	29.673	-3.015	1.000

(b) For different initial habit level h_0

h_0	L_r	L_π	x_S	x_N^1	x_N^{10}	x_I^1
200	-10.512	-4.318	0.628	26.220	-2.448	0.893
300	-10.112	-4.391	0.591	24.632	-2.271	0.838
400	-9.701	-4.466	0.549	23.046	-2.085	0.782
500	-9.304	-4.538	0.509	21.458	-1.902	0.728
600	-8.911	-4.614	0.465	19.873	-1.706	0.668

(c) For different investment horizon T

T	L_r	L_π	x_S	x_N^1	x_N^{10}	x_I^1
1	-4.536	-2.798	0.684	6.560	-0.421	0.961
5	-5.513	-3.039	0.612	10.057	-0.791	0.868
10	-6.701	-3.548	0.569	13.538	-1.082	0.813
20	-8.512	-4.218	0.501	18.968	-1.648	0.794
30	-9.711	-4.466	0.549	23.035	-2.085	0.782

The table shows the optimal portfolio strategy in complete market under nominal habit formation. L_r (L_π) is the sensitivity of the optimal portfolio to innovations in r (π). x_S, x_N^1 x_N^{10} and x_I^1 are the fractions of wealth invested in the stock, the 1-year nominal bond, the 10-year nominal bond and the 1-year inflation-indexed bond respectively. The parameter values are as follows: $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 1.

and interest risk exposure remains decreasing in habit strength, they get higher as compared to the real habit case due to the stronger leverage effect: inflation erodes the habit bond price, thereby leaving more free wealth. Third, the optimal demand for the inflation-index bond becomes dependent on the habit parameters and investment horizon, because the subsistence portfolio is left uninsured against unexpected inflation. As the habit bond price increases, which is associated with stronger habit persistence and longer horizon, the optimal inflation-index bond holding declines. Fourth, panel (c) shows that the horizon effect on the inflation risk sensitivity is reversed. This is also due to the bigger impact of the inflation risk on the optimal portfolio strategy.

Table 5: Optimal portfolio strategy with one nominal bond under nominal habit formation

(a) For different habit strength ($\beta - \alpha$)

Bond maturity	0.1	0.2	0.3	0.4	No habit
1 year					
x_S	0.577	0.635	0.663	0.680	0.741
x_N	7.012	6.618	6.470	6.402	6.417
$\frac{x_S}{x_N}$	0.082	0.096	0.102	0.106	0.115
10 year					
x_S	0.593	0.652	0.680	0.697	0.761
x_N	0.811	0.751	0.728	0.716	0.712
$\frac{x_S}{x_N}$	0.732	0.868	0.935	0.973	1.069

(b) For different initial habit level h_0

Bond maturity	200	300	400	500	600
1 year					
x_S	0.658	0.617	0.577	0.536	0.495
x_N	7.298	7.155	7.012	6.869	6.725
$\frac{x_S}{x_N}$	0.090	0.086	0.082	0.078	0.074
10 year					
x_S	0.676	0.635	0.593	0.552	0.511
x_N	0.836	0.824	0.811	0.798	0.786
$\frac{x_S}{x_N}$	0.809	0.771	0.732	0.691	0.650

(c) For different investment horizon T

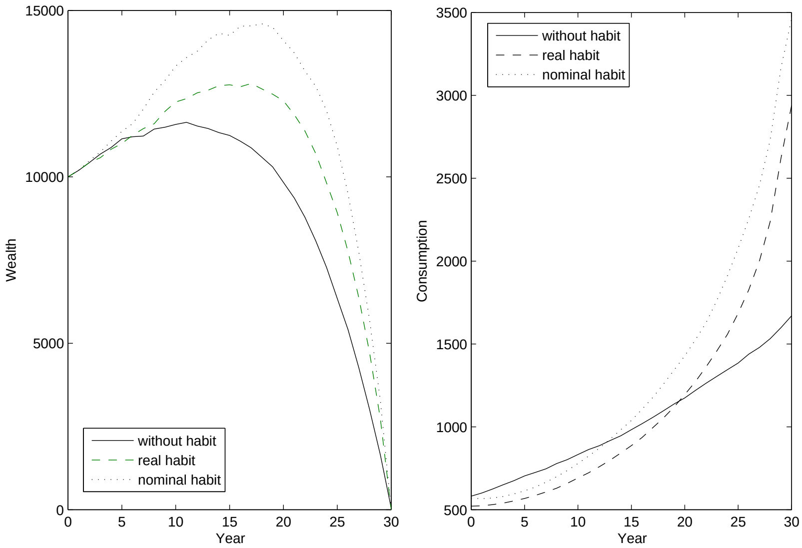
Bond maturity	1 year	5 years	10 years	20 years	30 years
1 year					
x_S	0.686	0.622	0.588	0.575	0.577
x_N	3.684	4.265	5.108	6.323	7.012
$\frac{x_S}{x_N}$	0.186	0.146	0.115	0.091	0.082
10 year					
x_S	0.694	0.632	0.600	0.589	0.593
x_N	0.437	0.501	0.598	0.736	0.811
$\frac{x_S}{x_N}$	1.588	1.261	1.003	0.801	0.732

The table shows the optimal portfolio strategy with one nominal bond under nominal habit formation. We consider 1-year nominal bond and 10-year nominal bond, respectively. x_S (x_N) is the demand for the stock (the nominal bond). x_S/x_N is the stock-to-bond ratio. The parameter values are as follows: $\alpha = 0.3$, $\beta = 0.4$ (varying in panel (a)), $h_0 = 400$ (varying in panel (b)) and $T = 30$ (varying in panel (c)). The current interest rate and current expected inflation are set at the unconditional means $\bar{r}$ and $\bar{\pi}$, respectively. Other parameters are shown in Table 1.

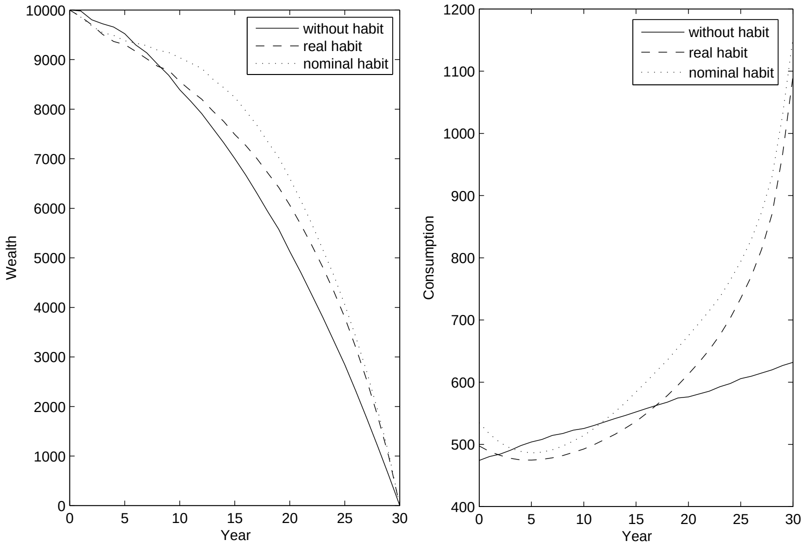
Table 5 reports the optimal portfolio strategy with one nominal bond under nominal habit formation. Comparison between Table 3 and Table 5 reveals that the demand for both risky assets grows because of the increase in free wealth. The stock-to-bond ratio is lower under nominal habit formation than it is under real habit formation, implying that the composition of the portfolio leans more towards the bond in the former setting. This tilt stems from the higher hedge demand and subsistence demand induced by the inflation risk.

Finally, we investigate the expected wealth and expected consumption under different types of habit formations, which are illustrated in Figure 1. Panel (a) shows that all three types of investors accumulate wealth in the early periods and decumulate wealth in the late periods. The accumulation is slowest for the non-habit investor, modest for the real habit investor and fastest for the nominal habit investor. Compared with the habit investors, the non-habit investor does not have to reserve a fraction of wealth for ensuring future subsistence consumption and enjoy higher consumption in the early periods, which is clearly displayed in the right graph. In the late periods, however, the consumption of the habit investors exceeds that of the non-habit investor because of the higher saving rate generated by habit formation. The nominal habit investor has higher wealth and consumption than the real habit investor over the whole life-cycle, both because the nominal habit investor has more free wealth to invest in stocks and benefit more from equity risk premium and because she has a higher marginal propensity to consume on average than the real habit investor.

The wealth accumulation phase arises from the equity risk premium implied by the estimates in Brennan and Xia (2002), which seems unrealistically high in the current market circumstances. Therefore, it is of interest to study the case with lower equity risk premium, which is shown in panel (b). When equity risk premium is set at a lower level, the wealth of three types of investors decumulates over the whole life-cycle. Interestingly, in face of worse market conditions, the habit investors begin with higher consumption than their counterpart, but reduce spending for some periods, because they have to drive down the habit level and increase saving to ensure that future habit consumption can be sustained.



(a) $\lambda_S = 0.343$



(b) $\lambda_S = 0.200$

Figure 1: Expected wealth and consumption under different types of habit formation. Panel (a) and (b) show the expected wealth and expected consumption for high equity risk premium ($\lambda_S = 0.358$) and low equity risk premia ($\lambda_S = 0.200$), respectively. In each panel, the left graph plots the expected wealth and the right graph plots the expected consumption. The solid line is for the case without habit formation, the dashed line is for the case with real habit formation and the dotted line is for the case with nominal habit formation.

5 Conclusion

In this paper, we have derived the optimal portfolio and consumption policies for an investor with habit formation in preferences and subject to inflation risk. Specifically, we considered two types of habit formation: one is based on real past consumption, while the other on nominal past consumption, which is motivated by money illusion. We also studied the case in which there is only one nominal bond available. The optimal strategy was expressed explicitly in terms of the solution to a linear partial differential equation.

The optimal portfolio is a combination of three portfolios: a myopic portfolio, a hedge portfolio and a subsistence portfolio. The effects of inflation on the optimal strategy turn out to depend on the the type of habit formation. Under real habit formation, the importance of inflation risk is little because of the absence of interaction between habit persistence and expected inflation. On the contrary, inflation risk plays a much bigger role in the case of nominal habit formation, because it modifies the risk profile of both the hedge demand and speculative demand and raises the inflation risk exposure of the overall optimal portfolio. Moreover, while the optimal portfolio takes full hedge against unexpected inflation risk under real habit formation, it leaves the subsistence portfolio uninsured under nominal habit formation. The dependence on the type of habit formation is robust to the incompleteness of the financial market. Another interesting observation in the case of one bond is that the optimal portfolio is tilted more towards bonds under nominal habit formation than under real habit formation.

There are several avenues for future research. First, the dependence of optimal strategy on the type of habit formation raises a need for empirically testing whether and to what extent households have money illusion in forming their habit. Although there are a bunch of papers providing evidence in support of habit formation, to the best of our knowledge none of them takes into account households' attitude towards inflation. Second, as human capital is a large component of households' wealth, it is interesting to add labor income to this model and study the optimal strategy for habit-households in the wealth accumulation phase, which contrasts with the wealth decumulation setting assumed in this paper.

A Proof of Theorem 1

We first derive the solution to the dual problem without habit formation formulated in Schroder and Skiadas (2002),

$$\max_{C_t} \quad \mathrm{E}\left[\int_0^T \mathrm{e}^{-\delta t}\frac{(\frac{\hat{C}_t}{\Pi_t})^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] \tag{56}$$

$$\text{s.t.} \quad \mathrm{E}\left[\int_0^T \frac{\hat{m}_t}{\hat{m}_0}\frac{\hat{C}_t}{\Pi_t}\mathrm{d}t\right] \leq \frac{\hat{W}_0}{\Pi_0}. \tag{57}$$

The dual problem is closely related to the one solved by Brennan and Xia (2002). The only difference is the introduction of an inflation-index bond, which serves to complete the market. Following Brennan and Xia (2002), one can solve the dual problem using the martingale approach. The indirect utility function is,

$$\hat{J}_t = \frac{\hat{Q}_t^{\gamma}\hat{w}_t^{1-\gamma}}{1-\gamma}, \tag{58}$$

the optimal dual consumption strategy is

$$\hat{c}_t = \frac{\hat{w}_t}{\hat{Q}_t}, \tag{59}$$

and the optimal dual portfolio strategy is

$$\hat{x}_t^* = (\sigma')^{-1}\left(\frac{-\hat{\phi}}{\gamma}\right) + (\sigma')^{-1}(\sigma_{\hat{Q}t} + \xi), \tag{60}$$

where $\sigma_{\hat{Q}t}$ is the percentage volatility vector of the process $\hat{Q}$ defined by

$$\hat{Q}_t = \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\frac{\delta(s-t)}{\gamma}}\left(\frac{\hat{m}_s}{\hat{m}_t}\right)^{1-\frac{1}{\gamma}}\mathrm{d}s\right]. \tag{61}$$

It follows from Schroder and Skiadas (2002) that there is an isomorphism between the primal problem with linear habit formation and the dual primal problem without habit formation:

$$\hat{c}_t = c_t - h_t, \tag{62}$$

$$\hat{w}_t = \frac{w_t - h_t f_t}{1 + \alpha f_t}, \tag{63}$$

$$\hat{m}_t = m_t (1 + \alpha f_t), \tag{64}$$

$$\hat{\phi} = \phi + \frac{\alpha f_t}{1 + \alpha f_t} \sigma_{ft}. \tag{65}$$

It is important to note that $\hat{m}$, which corresponds to the habit-adjusted pricing kernel, depends on habit formation via f. Substituting (64) into (61) yields

$$\hat{Q}_t = g_t (1 + \alpha f_t)^{\frac{1-\gamma}{\gamma}}, \tag{66}$$

where g_t has to be solved numerically. The PDE for g_t in (35) follows from Equation (16) in Munk (2008).

Using (58), (62) and (63), the indirect utility function in the primal problem can be obtained

$$\begin{aligned} J_t &= \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-\delta(s-t)} \frac{(c_s^* - h_s)^{1-\gamma}}{1-\gamma} \mathrm{d}s \right] \\ &= \mathrm{E}_t \left[\int_t^T \mathrm{e}^{-\delta(s-t)} \frac{(\hat{c}_s^*)^{1-\gamma}}{1-\gamma} \mathrm{d}s \right] \\ &= \hat{J}_t \\ &= \frac{g_t^\gamma (w_t - h_t f_t)^{1-\gamma}}{1-\gamma} \end{aligned} \tag{67}$$

Proposition 1 in Schroder and Skiadas (2002) shows that if $\hat{c}_t^*$ is the optimal primal consumption strategy, the dual optimal consumption strategy $\hat{c}_t$ satisfies,

$$c_t^* = h_t + \frac{\hat{c}_t^*}{\hat{w}_t} \frac{w_t - h_t f_t}{1 + \alpha f_t}. \tag{68}$$

Combining (59), (66) and (68) yields the optimal primal consumption strategy c^* in (36).

Following Proposition 8 in Schroder and Skiadas (2002), one can derive the relationship between the optimal primal and dual portfolio strategy in the presence of inflation under real habit formation. Proposition 1 and Proposition 3 in Schroder and Skiadas (2002) imply

$$m_t \left(w_t + \frac{h_t}{\alpha} \right) = \hat{m}_t \left(\hat{w}_t + \frac{h_t}{\alpha} \right) \tag{69}$$

Using integration by parts, we obtain

$$m_t \mathrm{d}w_t + \left(w_t + \frac{h_t}{\alpha} \right) \mathrm{d}m_t = \hat{m}_t \mathrm{d}\hat{w}_t + \left(\hat{w}_t + \frac{h_t}{\alpha} \right) \mathrm{d}\hat{m}_t + \mathrm{d}(BV)_t. \tag{70}$$

Dividing both sides of (70) and using the fact (from Proposition 3) that $(1+\alpha\hat{f}_t)(\hat{w}_t/w_t) = 1 - (h_t\hat{f}_t)$, we obtain

$$\frac{\mathrm{d}w_t}{w_t} = \left(1 - \frac{h_t\hat{f}_t}{w_t} \right) \frac{\mathrm{d}\hat{w}_t}{\hat{w}_t} + \left(1 + \frac{h_t}{\alpha w_t} \right) (\hat{\phi}' - \phi')\mathrm{d}z_t + \mathrm{d}(BV)_t \tag{71}$$

On the other hand, the budget equations in the primal and dual markets imply

$$\frac{\mathrm{d}w_t}{w_t} = \mathrm{d}(BV)_t + (x_t'\sigma - \xi')\mathrm{d}z_t \tag{72}$$

$$\frac{\mathrm{d}\hat{w}_t}{\hat{w}_t} = \mathrm{d}(BV)_t + (\hat{x}_t'\sigma - \xi')\mathrm{d}z_t. \tag{73}$$

Combining the last three equations, and matching the martingale parts yields,

$$x_t^* = \left(1 - \frac{h_t f_t}{w_t} \right) \hat{x}_t^* + \left(1 + \frac{h_t}{\alpha w_t} \right) (\sigma')^{-1} \frac{\alpha f_t}{1 + \alpha f_t} \sigma_{ft} + \frac{h_t f_t}{w_t} (\sigma')^{-1} \xi. \tag{74}$$

It is straightforward to verify that

$$\sigma_{\hat{Q}t} = \sigma_{gt} + \left(\frac{1}{\gamma} - 1 \right) \frac{\alpha f_t}{1 + \alpha f_t} \sigma_{ft}. \tag{75}$$

Substituting this into $\hat{x}_t$ and the resulting expression into (74), we obtain (37).

B Proof of Theorem 2

We follow De Jong (2008) in deriving the optimal portfolio strategy in the incomplete market. As derived by Brennan and Xia (2002), the optimal real wealth in the dual problem evolves as,

$$\mathrm{d}\ln w_t = \mathrm{d}(BV)_t + \left(-\frac{\hat{\phi}'}{\gamma} + \sigma'_{\hat{Q}t}\right)\mathrm{d}z_t \tag{76}$$

where BV is short for some bounded variation process and can be different in each occurrence of abbreviation. On the other hand, if $\hat{x}$ is vector of the portfolio weights to the stock and the bond in the dual problem, the real wealth process is given by

$$\mathrm{d}\ln w_t = \mathrm{d}(BV)_t + (\hat{x}'_t\sigma_I - \xi')\,\mathrm{d}z_t \tag{77}$$

Equating (76) and (77) yields the optimal portfolio strategy in the complete market. However, the optimal wealth process can not be achieved by dynamic trading in the available assets. Instead, we derive the optimal dual portfolio by minimizing the norm of the difference between the optimal and feasible wealth dynamics

$$\min \quad ||\left(\hat{x}'_t\sigma_I - \xi' + \frac{\hat{\phi}'}{\gamma} - \sigma'_{\hat{Q}t}\right)\mathrm{d}z_t|| \tag{78}$$

with $||a'\mathrm{d}z|| = a'\rho a$. The first order condition is

$$\sigma_I\rho\left(\sigma'_I\hat{x}_t - \xi + \frac{\hat{\phi}}{\gamma} - \sigma_{\hat{Q}t}\right) = 0, \tag{79}$$

and therefore the optimal dual portfolio strategy is given by

$$\hat{x}_t = (\sigma_I\rho\sigma'_I)^{-1}\left(-\sigma_I\rho\frac{\hat{\phi}}{\gamma} + \sigma_I\rho(\sigma_{\hat{Q}t} + \xi)\right). \tag{80}$$

To obtain the optimal primal portfolio strategy, we minimize the norm of the difference between the optimal strategy in two economies[10],

$$\min \quad ||\left(x'_t\sigma_I - \left(1 - \frac{h_tf_t}{w_t}\right)\hat{x}'_t\sigma_I - \left(1 + \frac{h_t}{\alpha w_t}\right)\frac{\alpha f_t}{1+\alpha f_t}\sigma'_{ft} - \frac{h_tf_t}{w_t}\xi'\right)\mathrm{d}z_t||. \tag{81}$$

[10]The relationship between the optimal strategy in two economies is shown in Proposition 8 of Schroder and Skiadas (2002).

The first order condition is

$$\sigma_I\rho\left(\sigma_I'x_t - \left(1-\frac{h_tf_t}{w_t}\right)\sigma_I'\hat{x}_t - \left(1+\frac{h_t}{\alpha w_t}\right)\frac{\alpha f_t}{1+\alpha f_t}\sigma_{ft} - \frac{h_tf_t}{w_t}\xi\right) = 0, \tag{82}$$

and therefore the optimal primal portfolio strategy in the incomplete market is given by

$$x_t = \frac{w_t - h_tf_t}{w_t}\hat{x}_t + \left(1+\frac{h_t}{\alpha w_t}\right)\frac{\alpha f_t}{1+\alpha f_t}(\sigma_I\rho\sigma_I')^{-1}\sigma_I\rho\sigma_{ft} + \frac{h_tf_t}{w_t}(\sigma_I\rho\sigma_I')^{-1}\sigma_I\rho\xi. \tag{83}$$

Plugging (75) into (80) and the resulting expression into (83) yields (38).

C Proof of Theorem 3

The proof is similar to that of Theorem 1, except with the isomorphism under real habit formation replaced by the isomorphism under nominal habit formation[11], which is shown as follows,

$$\hat{c}_t = c_t - h_t, \tag{84}$$

$$\hat{w}_t = \frac{w_t - h_t\hat{f}_t}{1+\alpha\hat{f}_t}, \tag{85}$$

$$\hat{m}_t = m_t(1+\alpha\hat{f}_t), \tag{86}$$

$$\hat{\phi} = \phi + \frac{\alpha\hat{f}_t}{1+\alpha\hat{f}_t}\sigma_{\hat{f}t}. \tag{87}$$

It should be noted that the primal problem with real habit formation and that with nominal habit formation are associated with the same dual problem. Based on the solution to the dual problem derived in the proof of Theorem 1, we have

$$\hat{Q}_t = \hat{g}_t(1+\alpha\hat{f}_t)^{\frac{1-\gamma}{\gamma}}. \tag{88}$$

Again, the PDE for $\hat{g}_t$ in (48) follows from Equation (16) in Munk and Sørensen (2004). It is straightforward to verify that the indirect utility function and the optimal consumption

[11]The proof of the isomorphism under nominal habit formation in the presence of inflation risk is available upon request.

strategy in the primal problem are given by

$$J_t = \frac{\hat{g}_t^{\gamma}(w_t - h_t \hat{f}_t)^{1-\gamma}}{1-\gamma}, \tag{89}$$

$$c_t^* = h_t + (1+\alpha \hat{f}_t)^{-\frac{1}{\gamma}} \frac{w_t^* - h_t \hat{f}_t}{\hat{g}_t}. \tag{90}$$

Similar to the real habit case, one can show under nominal habit formation, the relationship between the optimal dual portfolio strategy $\hat{x}_t^*$ and the optimal primal portfolio strategy x_t^* is given by

$$x_t^* = \left(1 - \frac{h_t \hat{f}_t}{w_t}\right) \hat{x}_t^* + \left(1 + \frac{h_t}{\alpha w_t}\right) (\sigma')^{-1} \frac{\alpha \hat{f}_t}{1+\alpha \hat{f}_t} \sigma_{\hat{f}t}. \tag{91}$$

It is straightforward to verify that

$$\sigma_{\hat{Q}t} = \sigma_{\hat{g}t} + \left(\frac{1}{\gamma} - 1\right) \frac{\alpha \hat{f}_t}{1+\alpha \hat{f}_t} \sigma_{\hat{f}t}. \tag{92}$$

Substituting this into $\hat{x}_t$ and the resulting expression into (91), we obtain (50).

D Proof of Theorem 4

The proof is similar to that of Theorem 2, except with the relationship between the optimal strategy in two economies under real habit formation replaced with that under nominal habit formation as shown in (91).

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