Benefit Provision in a Cyclic Economy

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Abstract

We discuss business cycle effects on the management of a trust fund set up to provide regular income on a continuing basis. Fund managers must find a balance between short-term and long-term variability of income. In our model the managers know that the expected return is mean-reverting, but they have limited capability of learning the true current state of the cycle. We consider policies that are optimal under constant relative risk aversion, and we contrast these with a parametric class of policies in which the asset mix is fixed and the estimated business cycle variable is only used in the consumption decision. In a calibration exercise, we find that optimal decisions are based on an exponentially weighted history of past asset returns with a half-time of about seven years. The tradeoff between short-term and long-term variability of income is illustrated by simulation results.

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1 Introduction

An important concern in fund management is the stability of benefit provision. Random shocks to investment returns affect the fund value along the business cycle, and create a tension between short-term and long-term variability of income. For instance, when the value of the assets goes down due to adverse returns, reducing income in a less than proportional way is attractive from the short-term point of view, but it does bring a risk that the fund will be drained and future income levels will be impaired.

In this paper we analyze the problem of benefit provision in a framework that takes into account cyclical effects as well as uncertainty surrounding these effects. A specific application that we have in mind relates to long-lived trust funds, such as the ones that are used for charitable causes supported from an initial donation. The setting of the paper can however also be thought of as representing a simplified form of problems faced by pension funds. In the paper we use the terms “income”, “benefits”, and “consumption” as synonyms, which correspond to various applications of which one may think.

We use as a benchmark a standard dynamic portfolio selection model in which the investor has a utility function with constant relative risk aversion. We choose a CRRA utility index for simplicity and the availability of closed-form solutions, but in our numerical applications we also explore recursive (Epstein-Zin) preferences. We find that this standard portfolio selection model does well in representing the risk-return tradeoff, but is less apt to deal with the tension between short-run and long-run variability of income that was mentioned above. To capture this aspect, we bring in a set of ad-hoc policies in which the investment mix is fixed. These policies serve to express a procyclical/countercyclical dimension in consumption, which differs from the usual aggressive/defensive dimension in investment.

Our work relates to the literature on life cycle investment as well as to consumer theory. Optimal investment and consumption in a setting that allows for business cycles have been investigated for instance by Merton (1971), Kim and Omberg (1996), Wachter (2002), and Campbell et al. (2004), under various assumptions concerning the correlation between the innovations to returns and the innovations to expected future returns. In most of this literature, attention is paid in particular to the asset allocation decision, which in the context of equilibrium analysis has implications for asset pricing, rather than to the consumption decision. The theory of consumption, as it has been reformulated by Deaton (1991) and Carroll (1992), tends to emphasize labor income risk rather than investment risk in its pursuit of explanations of the spending and saving behavior of individuals. Labor income risk has been included in studies of life cycle investment behavior for instance by Cocco et al. (2005) and Benzoni et al. (2007). While these latter papers focus on the impact of labor income risk on the investment decision, in the present paper our main interest is in the impact of returns risk on the consumption decision. As opposed to the literature on hyperbolic discounting (cf. Laibson (1997)) which analyzes the conflict between the short-term and the long-term level of consumption, one of our aims is to describe the tension between the short-term and the long-term volatility of consumption.

The remainder of the article is organized as follows. The model is presented in Section 2. Section 3 shows optimization results and presents formulas relating to the performance of fixed-mix strategies. In section 4, we work out the consequences of the theoretical development in a calibrated model, and Section 5 concludes. Various technical issues are relegated to Appendices.
2 Model formulation

2.1 Modeling the business cycle

Perhaps the simplest model that incorporates a business cycle effect is obtained from the standard Black-Scholes model by making the drift term for the risky asset subject to an Ornstein-Uhlenbeck process. A model of this form was already introduced by Merton (1971) who used the term “De Leeuw hypothesis” to refer to it. The model that we shall use is described by the following set of stochastic differential equations:

\begin{align}
    dB_t &= rB_t \, dt \\ 
    \frac{dS_t}{S_t} &= \mu_t \, dt + \sigma_1 \, dZ_{S,t} \\ 
    d\mu_t &= \theta(\bar{\mu} - \mu_t) \, dt + \sigma_2 \, dZ_{\mu,t}.
\end{align}

Here, \( B_t \) denotes the price per unit of a riskless asset (bond), and \( S_t \) is the price per unit of a risky asset (stock). In our calibrations later on, we work with inflation-indexed quantities, so that the parameter \( r \) denotes the real (as opposed to nominal) interest rate. The variable \( \mu_t \) defines the instantaneous (inflation-corrected) drift of the stock price; it is itself subject to an OU process with mean reversion level \( \bar{\mu} \) and mean reversion coefficient \( \theta > 0 \). Furthermore, \( \sigma_1 \) and \( \sigma_2 \) are constant positive volatility parameters. The processes \( Z_{S,t} \) and \( Z_{\mu,t} \) are jointly standard Brownian motions which we assume, as in Kim and Omberg (1996) and Campbell et al. (2004), to be imperfectly correlated; the correlation coefficient is denoted by \( \rho \). Below it is always assumed that the long-term equity premium is positive, i.e. \( \bar{\mu} > r \).

We shall take the drift parameter process \( \{\mu_t\} \) as not directly observed. Here we differ from Kim and Omberg (1996) and from most of the literature in which various business cycle indicators have been considered. In effect, we assume that the history of stock prices is the only source of dynamic information available to the investor. This stance may be radical, but it does allow us in a relatively simple manner to model a situation in which the investor is confronted with limits to learning. On the other hand we do assume that the investor does know all constant parameters in the model. This is a strong although not unusual assumption, which we adopt in order to focus on uncertainty relating to the business cycle. In this setting we can combine the methods of Kim and Omberg (1996) and Wachter (2002) with the innovations representation, to be discussed below, in order to arrive at an explicit solution.

The model (1) would reduce to the framework used by Brennan (1998) if we would take \( \theta = 0 \) and \( \sigma_2 = 0 \). In this setting, the equity premium is unknown but constant; by observing equity returns, investors are able to remove asymptotically all uncertainty concerning the true value of this parameter. In this paper however, it is a standing assumption that both the mean reversion coefficient \( \theta \) and the volatility \( \sigma_2 \) of expected equity return are positive. The approaches of Brennan (1998) and this paper could be combined to incorporate the effects of both parameter uncertainty and uncertainty concerning dynamic variables, but in the context of the present paper we do not carry out this extension.

The model (1) becomes affine when it is rewritten in terms of logarithmic price variables, and its dynamical properties are established by standard calculations which are briefly summarized in Appendices A.1 and A.2 for the convenience of the reader. While in the standard Black-Scholes model the variance of the log stock price at time \( t \) as seen from time 0 is a linear function of \( t \), the long-horizon variability of \( S_t \) in the model (1) can be proportionally more or less than the short-horizon
variability, depending on the parameter values. Specifically, we have
\[
\frac{d}{dt} \text{Var } \ln S_t \bigg|_{t=0} = \sigma_1^2
\]  
(2)
and
\[
\lim_{t \to \infty} \frac{d}{dt} \text{Var } \ln S_t = \sigma_1^2 + 2\frac{\sigma^2}{\theta} \sigma_1 \rho + \sigma_1^2
\]  
(3)
so that the rate of increase of the variance of the log stock price is less for large \(t\) than for small \(t\) if and only if
\[
\rho < -\frac{\sigma^2}{2\theta \sigma_1}.
\]  
(4)
The function \(\frac{d}{dt} \text{Var } \ln S_t\) is decreasing for all \(t\) if and only if the stronger condition
\[
\rho < -\frac{\sigma^2}{\theta \sigma_1}
\]  
(5)
holds. On the other hand, a reduction of the rate of increase of the variance of the log stock price from the initial value \(\sigma_1^2\) takes place on some interval of the form \((0, t_0)\) with \(t_0 > 0\) as soon as the correlation coefficient \(\rho\) is negative. Derivations of these results can be found in Appendix A.2.

2.2 Asset mix and benefit policy

Consider now an investor who lives in a world described by (1) and who wants to derive income from returns on investments. Let the investor’s wealth at time \(t\) be denoted by \(W_t\). The investor has to decide on the asset mix as well as on the amount of funds withdrawn for consumption. Both decision variables will be modeled as continuous-time variables. The portfolio weight of risky assets at time \(t\) will be denoted by \(x_t\); in other words, at time \(t\) the amount invested in risky assets is \(x_t W_t\). The rate of extraction of funds at time \(t\), expressed as a proportion of wealth, is denoted by \(c_t\). The amount of funds withdrawn during the time interval from \(t\) to \(t + \Delta t\), where \(\Delta t\) is small, is therefore approximately equal to \(c_t W_t \Delta t\). The rate of consumption in absolute terms is denoted by \(C_t := c_t W_t\). The investor’s wealth process is given by
\[
\frac{dW_t}{W_t} = x_t \frac{dS_t}{S_t} + (1 - x_t) \frac{dB_t}{B_t} - c_t \, dt
\]  
\[
= (r + x_t (\mu_t - r) - c_t) \, dt + x_t \sigma_1 \, dZ_{s,t}.
\]  
(6)
A simple approach that could be followed in this context, and one that serves as a benchmark in practice, is to let both the asset mix and the consumption-wealth ratio be constant in time. A typical recommendation would be that the percentage withdrawn annually should be equal to the long-term average annual return on assets. If \(x\) denotes the percentage of wealth held in risky assets, this recommendation would come down, in the context of the model given by (1), to taking \(c_t = c\) with
\[
c = x \bar{\mu} + (1 - x) r.
\]  
(7)
In the standard Black-Scholes model (\(\mu_t = \bar{\mu}\) for all \(t\)), this choice ensures that the wealth process is a martingale, so that in particular the expected value at time 0 of the wealth at time \(t\) is, for any \(t\), equal to the wealth at time 0.

The formula (7) is subject to correction in the context of the business cycle model (1), when both the correlation coefficient \(\rho\) and the volatility \(\sigma_2\) of the instantaneous expected stock return are nonzero. A calculation in Appendix A.1 shows that in
the model (1) the asymptotic exponential growth rate of $EW_t$ under the constant-
proportion policy with constant asset mix is zero if and only if
\[ c = x\bar{\mu} + (1 - x)r + \frac{1}{2}x^2\left(\frac{\sigma_2^2}{\theta^2} + 2\frac{\sigma_2^2}{\theta}\sigma_1\rho\right). \] (8)
The correction term with respect to (7) that appears here is negative if and only
if the condition (4) holds, that is, if and only if the underlying model (1) shows
reduced long-term risk of stocks. In other words, if the correlation between shocks
to returns and shocks to expected returns has a sufficiently negative value so that
the long-term risk of stocks is reduced, then the implementation of the simple rule
(7) will cause the expected value of wealth to be exponentially decreasing as a
function of time.

One consequence of this is the following. Consider an investor who is looking for
the constant asset mix that will allow the highest possible constant consumption-
wealth ratio, subject only to the constraint that the expected value of wealth should
be constant. Under the assumptions of the standard Black-Scholes model, this is
not a well-posed problem; the higher the proportion of wealth in risky assets, the
higher consumption levels can be realized on average. In the model (1) under the
condition (4) however, we find
\[ x^* = \frac{(\bar{\mu} - r)\theta^2}{-2\theta\sigma_1\sigma_2\rho - \sigma_2^2}. \] (9)
Therefore even on the basis of an analysis in terms of expected values, without
regard to risk, the portfolio weight of risky assets is limited.

Below we will carry out optimization exercises on the basis of criteria that do
take risk into account, allowing both the asset mix and the consumption-wealth
ratio to be adjusted dynamically to available information at a given moment in
time. The policy in which both are constant is still a convenient benchmark against
which the dynamic policies can be compared. It is then useful to note that the
benchmark for consumption, given the asset mix, is given by (8) rather than by (7).

2.3 The innovations representation
The analysis of the model (1) can be simplified by making use of the so called
innovations representation. This representation was originally developed for appli-
cations in filtering theory (see for instance Kailath (1970)) but it can be used also
for optimization problems, in particular when, as in the problem discussed in this
paper, there is no influence of the control action on the flow of information (Fleming
Björk et al. (2010)). The innovations approach is based on replacing unobserved
state variables by variables that represent the conditional distribution of the state
vector given information from observed quantities up to the current time. The ap-
proach is especially effective when the conditional distribution can be summarized
by a few parameters. In the case of the model (1), the conditional distribution of $\mu_t$
is Gaussian for all $t \geq 0$ if the distribution of $\mu_0$ is Gaussian, so that we can work
just with expectation and variance.

Let the mean and variance of the conditional distribution of $\mu_t$, given the obser-
vations $(S_u)_{0 \leq u \leq t}$, be denoted by $m_t$ and $V_t$ respectively. Assume that, at the
beginning of the observation interval, the distribution of $\mu_0$ is Gaussian with mean
$m_0$ and variance $V_0$. Then it follows (cf. Liptser and Shiryaev (1978)) that
\[ \begin{align*}
    dm_t &= \theta(\bar{\mu} - m_t) \, dt + \frac{\sigma_1\sigma_2\rho + V_t}{\sigma_1^2} \left( \frac{dS_t}{S_t} - m_t \, dt \right) \\
    dV_t &= \left[ -2\theta V_t + \sigma_2^2 - \sigma_1^{-2}(\sigma_1\sigma_2\rho + V_t)^2 \right] dt,
\end{align*} \] (10) (11)
where $\rho$ is the correlation coefficient between the two Brownian motions $Z_{S,t}$ and $Z_{\mu,t}$. Equation (10) summarizes the investor’s update of the investment opportunity over time. Intuitively, the update combines two considerations: it takes into account that there is a tendency towards the long-term average $\bar{\mu}$, but is also includes a correction term that is derived from the difference between the most recently observed asset return that is actually observed and its expected value (the “innovation”). The magnitude and the sign of the correction are determined by the quantity $\kappa_t$ defined by

$$
\kappa_t = \frac{\sigma_1 \sigma_2 \rho + V_t}{\sigma_1^2}.
$$

(12)

It is seen from (11) that the variance $V_t$ follows a deterministic differential equation, so that $V_t$ is a deterministic function of time and consequently the same holds for $\kappa_t$.

The cumulative process that is driven by the normalized innovations is called the innovations process. We write it as $Z'_t$, so that

$$
dZ'_t = \sigma_1^{-1} \left( \frac{dS_t}{S_t} - m_t \, dt \right), \quad Z'_0 = 0.
$$

(13)

A fundamental result is that the innovations process is a standard Brownian motion whose natural filtration coincides with the natural filtration of the process $\{S_t\}$ (Liptser and Shiryaev (1978), Gennotte (1986)). Rewriting the stochastic differential equation for the stock price $S_t$ in terms of the innovations process, we arrive at the innovations representation of the process $\{S_t\}$:

$$
dS_t = m_t \, dt + \sigma_1 \, dZ'_t.
$$

(14a)

$$
dm_t = \theta (\bar{\mu} - m_t) \, dt + \kappa_t \sigma_1 \, dZ'_t.
$$

(14b)

This model is equivalent to the original one in terms of the joint statistics of the processes $\{S_t\}$ and $m_t$. An importance difference is that the innovations model is driven by a single Brownian motion, whereas there are two Brownian motions in the model (1). In particular, while the market described by the original model is incomplete, the one described by the innovations representation is complete. There is no contradiction, because the latter market is strictly smaller. The original model allows for instance to define payoffs of the form $C_T = F(S_T, \mu_T)$, whereas in the innovations model we can only allow payoffs that depend on the stock price trajectories. However, the market as described by the innovations representation is sufficient for the purposes of this paper.

The conditional expected return at time $t$ is given in terms of the observed returns from time $0$ to time $t$ by

$$
m_t = (1 - \exp (-\theta t - \int_0^t \kappa(u) \, du)) \bar{\mu} + \left( \exp (-\theta t - \int_0^t \kappa(u) \, du) \right) m_0
+ \int_0^t \exp \left( -\theta(t-s) - \int_s^t \kappa(u) \, du \right) \kappa(s) \left( \frac{dS_s}{S_s} - \bar{\mu} \, ds \right).
$$

(15)

This shows that the conditional expected return is built up from a convex combination of an initial estimate and the long-term return, together with a weighted average of deviations of realized returns from the long-term expected return.

2.4 The stable phase

In the presence of state uncertainty, the investor’s assessment of the investment opportunity set is generally imperfect in the sense that the estimation error, as
described by the variance $V_t$ of the conditional distribution, cannot be ignored even if realized returns are observed continuously over a very long horizon. As the length of the observation interval tends to infinity, the variance of the estimator converges to a constant given by

$$V = \sigma_t^2 \left[ \sqrt{\theta^2 + 2\theta \frac{\sigma_2^2}{\sigma_1^2} + \frac{\sigma_2^2}{\sigma_1^2}} - \left( \theta + \frac{\sigma_2^2}{\sigma_1^2} \right) \right]$$

(cf. Appendix A.5). We refer to the situation in which the variance of the estimator can be assumed to be equal to this constant as the **stable phase**. In this phase, there is a balance between on the one hand the reduction of uncertainty that is due to the observation of new asset returns, and other hand the increase of uncertainty that is due to the shocks to the expected return that cannot be observed directly. The effect of continuous updating is just to keep the uncertainty about the true value of the unobserved state variable at the same level. This describes a situation in which there are limits to learning.

In the stable phase, the innovations model turns into a model with constant coefficients:

$$\frac{dS_t}{S_t} = m_t \, dt + \sigma_1 \, dZ'_t$$

(17a)

$$dm_t = \theta (\bar{\mu} - m_t) \, dt + \kappa \sigma_1 \, dZ'_t$$

(17b)

where

$$\kappa = \frac{\sigma_1 \sigma_2 \rho + \bar{V}}{\sigma_1^2} = \sqrt{\theta^2 + 2\theta \frac{\sigma_2^2}{\sigma_1^2} + \frac{\sigma_2^2}{\sigma_1^2}} - \theta.$$  

(18)

From the expression given for $\kappa$, it is seen that the sign of $\kappa$ is the same as the sign of the quantity $\sigma_2/\sigma_1 + 2\theta \rho$, which in turn is the same as the sign of the difference between the long-term and the short-term rate of increase of the variance of the stock price as a function of time. In other words, the situations in which there is reduction of the riskiness of stocks in the long run are exactly the ones in which the value of the parameter $\kappa$ is negative. This situation is the one that will be mostly considered in this paper.

Equation (18) also shows that the parameter $\kappa$ is bounded from above by $\sigma_2/\sigma_1$, with equality if and only if the correlation $\rho$ is equal to 1, and that $\kappa$ is bounded from below by $-\sigma_2/\sigma_1$, with equality if and only if $\rho = -1$ and $\theta - \sigma_2/\sigma_1 \geq 0$. As a consequence, the inequality $|\kappa_{\sigma_1}| \geq \sigma_2$ holds; in other words, when the correlation between the Brownian motions driving the returns and the expected returns is imperfect, the volatility of the conditional expectation of the level of expected returns is less than the volatility of the level itself, as should be expected.

The evolution of the conditional expected return $m_t$ can in the stable phase be expressed in terms of the deviations of observed asset returns from their long term average, as follows:

$$dm_t = (\theta + \kappa)(\bar{\mu} - m_t) \, dt + \kappa \left( \frac{dS_t}{S_t} - \bar{\mu} \, dt \right).$$

(19)

The expression (15) is therefore reduced to

$$m_t = (1 - e^{-(\theta + \kappa)t}) \bar{\mu} + e^{-(\theta + \kappa)t} m_0 + \kappa \int_0^t e^{-(\theta + \kappa)(t-s)} \left( \frac{dS_s}{S_s} - \bar{\mu} \, ds \right).$$

(20)

It is seen from (18) that the quantity $\theta + \kappa$ is positive under our standing assumption that the mean reversion coefficient $\theta$ is positive and the additional assumption that
the correlation between the two Brownian motions in (1) is imperfect. Therefore the factor in (20) that multiplies the initial estimate \( m_0 \) tends to zero as \( t \) increases, whereas the coefficient multiplying the long-term average \( \bar{\mu} \) tends to 1. The integral term in (20) represents an exponentially weighted moving average of deviations of realized returns from the long-term average. The strength of the decay of the weights of past observations is determined by the quantity \( \theta + \kappa \).

It follows from (1c) and (10) that, in the stable phase, the difference between the estimated expected return \( m_t \) and the actual expected return \( \mu_t \) satisfies the following SDE:

\[
d(m_t - \mu_t) = -((\theta + \kappa)(m_t - \mu_t) + \kappa \sigma_1 \, d\mathbb{Z}_{S,t} - \sigma_2 \, d\mathbb{Z}_{\mu,t}).
\]  

(21)

The estimation error \( e_t := m_t - \mu_t \) therefore follows a mean-reverting process with zero long-term mean. The variance of the corresponding stationary distribution is given, on the basis of (21), by

\[ \text{var}(e_t) = \frac{\kappa^2 \sigma_1^2 - 2 \rho \kappa \sigma_1 \sigma_2 + \sigma_2^2}{2(\theta + \kappa)}. \]  

(22)

It can be verified by direct calculation that this quantity is equal to the limit value \( \bar{V} \) of the variance of the conditional distribution as given in (16).

3 Optimization

We consider an investor who seeks to optimize benefit policy as well as asset allocation policy in the context of the model (1), with given values of the fixed parameters. However the investor does not have direct access to the current value of the parameter \( \mu_t \) that represents the current state of the business cycle; information concerning this variable is only available indirectly, through observation of the realized asset returns. Decisions concerning benefits and asset allocation at any given time have to be taken on the basis of the stock price history up to that time.

The stochastic differential equation that describes the evolution of the investor’s wealth is given in (6). By using the innovations representation of the process \( \{S_t\} \), we can also describe the wealth process by the SDE

\[
dW_t = [r + x_t(m_t - r)]W_t \, dt - C_t \, dt + x_t \sigma_1 W_t \, d\mathbb{Z}'_t.
\]  

(23)

where, as before, \( W_t \) denotes wealth at time \( t \), \( x_t \) is the fraction of wealth held in stocks at time \( t \), and \( C_t \) is the rate of consumption at time \( t \). The SDE for the process \( \{m_t\} \) is given by (14b), and more specifically by (17b) in the stable phase. It may be noted that the model that we arrive at in this way is formally identical to the one used by Wachter (2002). However, the interpretation of the variables is different. Wachter (2002) takes the variable denoted by \( m_t \) above as the actual expected return at time \( t \), whereas in our paper it is only the conditional expectation that is formed on the basis of observed asset returns. In particular, we are able to incorporate a value of the correlation coefficient \( \rho \) that is different from \(-1\) whereas Wachter (2002) forces the correlation to be perfectly negative. The difference in interpretation of the variables gives rise to a different interpretation of the parameters, hence to different values of the parameters when calibration is done, and hence to different policies.
3.1 The classical Merton problem

We start by briefly reviewing CRRA optimization under constant expected returns. This problem was solved by Merton (1971). The optimization problem is

$$\sup_{C_t \geq 0, x_t \in \mathbb{R}} E \left[ \int_0^T e^{-\eta t} \left( \frac{C_t^{1-\gamma}}{1-\gamma} + \psi e^{-\eta T} \frac{W_T^{1-\gamma}}{1-\gamma} \right) dt \right]$$

subject to $dW_t = \left[ (r + x_t(\bar{\mu} - r))W_t - C_t \right] dt + x_t \sigma W_t dZ_t$

$W_T \geq 0$

where $\gamma > 0$ is the constant rate of relative risk aversion, $\eta > 0$ denotes the subjective discount rate, and $\psi > 0$ is a parameter that indicates the importance of bequest. The control variables are $x_t$, the fraction of wealth invested in risky assets, and $C_t$, the consumption rate. The optimal investment in risky assets in the Merton problem is independent of time and of current wealth, and is given by

$$x_t^* = \frac{\bar{\mu} - r}{\gamma \sigma^2}. \quad (25)$$

The optimal ratio of consumption to wealth $c_t = C_t/W_t$ is given by

$$c_t^* = \frac{A}{1 + (\psi^{1/\gamma} A - 1) e^{-A(T-t)}} \quad (26)$$

where

$$A = \frac{\eta + r(\gamma - 1)}{\gamma} + \frac{\gamma - 1}{2 \gamma^2} \left( \frac{\bar{\mu} - r}{\sigma} \right)^2. \quad (27)$$

When the difference $T - t$ is large, the influence of the parameter $\psi$ is small and the consumption/wealth ratio is nearly constant in time and close to the value $A$. This value is maintained across the full planning interval when $\psi$ is determined by the rule

$$\psi = A^{-\gamma}. \quad (28)$$

In long-horizon problems, it therefore seems reasonable to take this as a benchmark value for $\psi$. The optimal policy then takes the form of a policy that has already been described as a benchmark above, namely one in which investments are given by a fixed-mix portfolio, and consumption is a fixed percentage of wealth.

We define the benchmark value of the consumption/wealth ratio as the one that causes expected wealth to be constant when the fraction of wealth invested in risky assets is given by (25). This value is given by

$$b := x_t^* \mu + (1 - x_t^*) r = r + \frac{1}{\gamma} \left( \frac{\mu - r}{\sigma} \right)^2. \quad (29)$$

Comparing this to the level given by (27), we see that the optimal consumption/wealth ratio in the Merton problem can be written as

$$A = \frac{\gamma - 1}{\gamma} b + r + \frac{1}{\gamma} \eta. \quad (30)$$

In other words, the optimal ratio is a convex combination of the benchmark value, the riskless interest rate, and the subjective discount rate. In this combination, the benchmark and the riskless rate always have equal weights, whereas the weight of the subjective discount rate is one if the coefficient of relative risk aversion $\gamma$ is equal to 1 and tends to zero when $\gamma$ tends to infinity. For a fixed value of $\gamma$, the optimal ratio can be higher or lower than the benchmark value (i.e. wealth may increase or
decrease in expectation), depending on the value of the subjective discount rate $\eta$. Equality holds when
$$\eta = r + \frac{\gamma + 1}{2\gamma} \left( \frac{\mu - r}{\sigma} \right)^2.$$ (31)
In situations in which stability of capital is a natural desideratum, this equation can be thought of as defining a canonical value of $\eta$ given a value for $\gamma$; or conversely, given a value for the subjective discount rate, the equation can be used to define a canonical value for the coefficient of relative risk aversion.

### 3.2 Variable expected return

In a situation in which expected return is variable according to the business cycle model discussed in Section 2, the Merton model is formulated as follows, where we make use of the innovations representation (14):

$$\sup_{C_t \geq 0, x_t \in \mathbb{R}} E \left[ \int_0^T e^{-\eta t} \frac{C_t^{1-\gamma}}{1-\gamma} dt + \psi e^{-\eta T} W_T^{1-\gamma} \right]$$ (32a)

subject to
$$dW_t = [(r + x_t(m_t - r))W_t - C_t] dt + x_t \sigma_1 W_t dZ_t$$ (32b)
$$dm_t = \theta(\bar{\mu} - m_t) dt + \kappa_t \sigma_1 dZ_t$$ (32c)
$$W_T \geq 0.$$ (32d)

Assuming there exists a solution of the dynamic optimization problem (32), we can characterize the solution in terms of a PDE. The following theorem summarizes the solution that is thus provided for the investor’s decision problem with unobservable expected return.

**Theorem 3.1** For the investor facing the consumption and investment problem in the presence of state uncertainty, the optimal consumption/wealth ratio, $c^*_t$, and the optimal portfolio plan, $x^*_t$, are given by

$$c^*_t = \frac{1}{F(m_t, t)}$$ (33)
$$x^*_t = \frac{1}{\gamma} \frac{m_t - r}{\sigma_t^2} + \frac{\sigma_1 \sigma_2 \rho + V_t}{\sigma_1^2} \frac{1}{F(m_t, t)} \frac{\partial F}{\partial m_t}(m_t, t)$$ (34)

where $F(m_t, t)$ solves the following partial differential equation:

$$\frac{\partial F}{\partial t}(m_t, t) = 1 - \left[ \frac{\eta}{\gamma} - 1 - \frac{1}{2\gamma^2} \left( \frac{m_t - r}{\sigma_1} \right)^2 \right] F(m_t, t)$$
$$+ \left[ \theta(\bar{\mu} - m_t) + \frac{1 - \gamma}{\gamma} \frac{m_t - r}{\sigma_1} \sigma_1 \sigma_2 \rho + V_t \right] \frac{\partial F}{\partial m_t}(m_t, t)$$
$$+ \frac{1}{2} \left( \frac{\sigma_1 \sigma_2 \rho + V_t}{\sigma_1} \right)^2 \frac{\partial^2 F}{\partial m_t^2}(m_t, t)$$ (35)

with the boundary condition
$$F(m_T, T) = \psi^{1/\gamma}.$$

A proof of this theorem is provided in Appendix A.4. The first term in (34) is of the same form as the solution to the standard Merton problem, with the conditional expected return taking the place of the constant expected return in the Merton
The optimal consumption and investment strategies depend on the conditional expected return $m_t$, and hence, as discussed in Section 2, they can be looked as functions of an exponentially weighted average of excess returns.

The process followed by optimal consumption can be derived from (33) together with (14b). In particular it is of interest to consider the volatility of consumption. Define $\mu_C(m_t, t)$ and $\sigma_C(m_t, t)$ by

$$\frac{dC_t}{C_t} = \mu_C(m_t, t) \, dt + \sigma_C(m_t, t) \, dZ'_t. \tag{36}$$

An application of Itô’s formula to (33) using (17b) shows that

$$\sigma_C(m_t) = \left[ x^* - \frac{\kappa_t}{F(m_t, t)} \frac{\partial F}{\partial m_t}(m_t, t) \right] \sigma_1. \tag{37}$$

In view of (34), this implies that

$$\sigma_C(m_t, t) = \frac{1}{\gamma} \frac{m_t - r}{\sigma_1}. \tag{38}$$

The instantaneous volatility of consumption is therefore the same as what it would be in the classical Merton model, when the expected return is taken equal to its current estimate. In other words, there is no horizon effect in the volatility of consumption, and the level of the volatility varies in tandem with the current estimate of the expected return.

### 3.3 Solution in the stable phase

An investigation of the investor’s decision in the stable phase is of interest, since it leads to insights on how an investor with “sufficient experience” will consume and invest. In the stable phase, the PDE in Theorem 3.1 reduces to

$$- \frac{\partial F}{\partial t}(m_t, t) = 1 - \left[ \frac{\eta}{\gamma} - 1 - \frac{\gamma}{\gamma} \left( \frac{m_t - r}{\sigma_1} \right)^2 \right] F(m_t, t) + \left[ \theta(\bar{\mu} - m_t) + \frac{1 - \gamma}{\gamma} \kappa(m_t - r) \right] \frac{\partial F}{\partial m_t}(m_t, t) + \frac{1}{2} \kappa^2 \sigma_1^2 \frac{\partial^2 F}{\partial m_t^2}(m_t, t). \tag{39}$$

This equation can be solved in closed form (cf. Kim and Omberg (1996), Wachter (2002)). To prevent situations in which the solution reaches infinity in a finite time, we assume that the coefficient of relative risk aversion exceeds 1 (risk aversion stronger than log utility). The assumption that $\gamma > 1$ is empirically relevant, since it is supported by many empirical studies of risk aversion (see e.g. Friend and Blume (1975), Pindyck (1988), and Szpiro (1986)) as well as by the literature on the equity premium puzzle. The solutions of the PDEs in Kim and Omberg (1996) and Wachter (2002) suggest the following guessed form of the solution of (39):

$$F(m_t, t) = \int_t^T e^{H(m_t, s-t)} \, ds + \psi^{1/\gamma} e^{H(m_t, T-t)} \tag{40}$$

where

$$H(m_t, \tau) = \frac{1 - \gamma}{\gamma} \left[ \frac{1}{2} A_1(\tau) \left( \frac{m_t - r}{\sigma_1} \right)^2 + A_2(\tau) \left( \frac{m_t - r}{\sigma_1} \right) + A_3(\tau) \right]. \tag{41}$$

The three functions, $A_1$, $A_2$, and $A_3$, can be solved by substituting (40) back into (39), and then working out the resulting three differential equations in $A_1$, $A_2$, and
The equations are nonlinear; however, it is shown in Appendix A.6 that the equations can be solved in terms of a system of linear ODEs with constant coefficients. The following solutions are obtained:

\[ A_1(\tau) = \frac{b_0 e^{\gamma \tau} - e^{-\gamma \tau}}{(q + c_0)e^{\theta} + (q - c_0)e^{-\gamma \tau}} \]  

(42a)

\[ A_2(\tau) = \frac{b_0 c_1}{q} \frac{e^{\gamma \tau} + e^{-\gamma \tau} - 2}{(q + c_0)e^{\theta} + (q - c_0)e^{-\gamma \tau}} \]  

(42b)

\[ A_3(\tau) = r \tau + \int_0^\tau \left[ c_1 A_2(x) + \frac{1}{2} \kappa^2 A_1(x) - \frac{1}{2} b_2 A_2^2(x) \right] dx \]  

(42c)

where

\[ b_0 = \frac{1}{\gamma}, \quad b_2 = \frac{\gamma - 1}{\gamma}, \quad c_0 = \frac{\gamma - 1}{\gamma}, \quad \kappa + \theta, \quad c_1 = \theta \frac{\mu - r}{\sigma_1} \]  

(43)

and

\[ q = \sqrt{c_0^2 + b_0 b_2} = \sqrt{\frac{1}{\gamma} (\kappa + \theta)^2 + \frac{1}{\gamma} \theta^2}. \]  

(44)

Therefore, the optimal strategy of the investor is solved as shown in the following lemma.

**Lemma 3.2** In the stable phase, and under the assumption that the coefficient of relative risk aversion \( \gamma \) is larger than 1, the optimal consumption strategy is given, as a function of \( m_t \) and \( t \), by

\[ C_t = \left[ \int_t^T e^{H(m_t, s-t)} ds + \psi^{1/\gamma} e^{H(m_t, T-t)} \right]^{-1}. \]  

(45)

The optimal portfolio strategy is given, as a function of \( m_t \) and \( t \), by

\[ x_t^* = \frac{1}{\gamma} \frac{m_t - r}{\sigma_t^2} + \frac{1 - \gamma}{\gamma} \frac{\kappa}{\sigma_1} \int_t^T [A_1(s-t) \frac{m_t - r}{\sigma_1} + A_2(s-t) e^{H(m_t, s-t)}] ds \]  

\[ + \frac{1 - \gamma}{\gamma} \frac{\psi^{1/\gamma}}{\sigma_1} \int_t^T e^{H(m_t, s-t)} ds + \psi^{1/\gamma} e^{H(m_t, T-t)} + \frac{1 - \gamma}{\gamma} \frac{\kappa}{\sigma_1} \int_t^T e^{H(m_t, s-t)} ds + \psi^{1/\gamma} e^{H(m_t, T-t)} \]  

(46)

The first term in (46) does not depend on the investment horizon; it is the optimal portfolio strategy that would follow by assuming that the current estimate \( m_t \) of the expected return will remain constant in the future. The second and third terms express corrections that take the variability of conditional expected returns into account, so that they can be referred to as allocations for intertemporal hedging purposes. The third term arises from bequest motives, for this term disappears if the investor does not derive utility from bequest (\( \psi = 0 \)).

While (45) and (46) may first appear complicated, they can used to provide insights into the decision of the experienced investor in the presence of state uncertainty. In particular we can make statements concerning the dependence of decision variables on various parameters.

**Proposition 3.3** Assume that the investor is more risk averse than log utility (\( \gamma > 1 \)), and that the long-term mean of the equity risk premium is positive (\( \bar{\mu} - r > 0 \)). Then as long as the conditional risk premium, \( m_t - r \), is positive, the optimal consumption/wealth ratio is increasing in the conditional expected return \( m_t \).

The proof of this proposition follows from Appendix A.6, which shows that \( A_1(\tau) \) and \( A_2(\tau) \) are both positive when \( \gamma > 1 \) and \( \bar{\mu} - r > 0 \). The condition \( m_t - r > 0 \) then
guarantees that the partial derivative of $H(m_t, \tau)$ with respect to $m_t$ is negative. In other words, the function $H(m_t, \tau)$ is decreasing in $m_t$. It then follows immediately that the optimal consumption/wealth ratio as given by (45) is increasing in $m_t$.

It is natural that the consumption/wealth ratio should increase with $m_t$ when $m_t - r > 0$, since higher values of $m_t$ correspond to better economic prospects. When $m_t$ is equal to $r$, the sign of $\partial H / \partial m_t(m_t, \tau)$ is still positive, but when $m_t$ is decreased further the sign of the partial derivative will become negative at a point that depends on $\tau$. When $m_t$ is given successively smaller values, the sign of the partial derivative of the optimal consumption/wealth ratio with respect to the estimated expected return $m_t$ will turn zero and then become negative. This means that, as $m_t$ is decreased, the corresponding optimal consumption/wealth ratio reaches a minimum and then starts to increase as $m_t$ is decreased further.

It may seem strange that the consumption/wealth ratio should be increased when economic prospects turn from bleak to even bleaker. However, it should be noted that in our problem setting there is no constraint on short selling. Given that short positions are allowed, negative expected excess returns offer business opportunities of equal attractiveness as positive ones do.

When the mean reversion level $\bar{\mu}$ is substantially above the riskfree rate, as would be the case under typical choices of the parameters, the occurrence of values of the estimated expected return well below the riskless rate would be rare. The typical situation then is the one in which the consumption/wealth ratio is increasing with the estimated expected return. It is also a usual assumption that the correlation between the observed excess return on investments and the estimated expected return is negative. As a consequence, CRRA optimal policies will typically lead to a reduction of the consumption/wealth ratio following favorable returns, and an increase of the ratio following adverse returns on assets. Such policies can be described as countercyclical.

**Proposition 3.4** Assume that the investor’s preference and the financial market dynamics satisfy the following conditions:

1. the investor is more risk averse than log utility ($\gamma > 1$);
2. the long-term mean of equity risk premium is positive ($\bar{\mu} - r > 0$);
3. $\kappa < 0$, or equivalently $\sigma_2 / \sigma_1 + 2\theta \rho < 0$.

Then as long as the conditional risk premium, $m_t - r$, is positive, the intertemporal hedging demand is positive, namely, the optimal allocation to the risky asset is higher than the corresponding myopic allocation. Furthermore, when there is no utility from bequest ($\psi = 0$), the optimal allocation to risky assets is increasing in the investment horizon. If the same inequalities hold except that the sign of $\kappa > 0$ is reversed, then all conclusions are reversed as well. In the special case in which $\kappa = 0$, the optimal investment plan does not depend on the investment horizon.

This proposition is proved in Appendix A.7. The assumption of no utility of bequest is made for simplicity.

Finally we consider sensitivities with respect to the strength of the bequest motive. The sensitivity of the consumption-wealth ratio with respect to the strength of this motive is immediate from (45).

**Proposition 3.5** The optimal consumption-wealth ratio is decreasing in the strength of bequest motives, $\psi$. This result does not depend on sign assumptions and is indeed intuitive: the more should be left as bequests, the less should be consumed. Concerning the dependence of the portfolio strategy on the bequest motives, we can state the following.
Proposition 3.6 Assume that (i) $\gamma > 1$; (ii) $\bar{\mu} - r > 0$; and (iii) $\kappa < 0$. Then as long as the conditional risk premium, $m_t - r$, is positive, the optimal allocation to the risky asset increases with the strength of bequest motives, $\psi$. Conversely, if the same inequalities hold except that the sign of $\kappa$ is reversed, then the opposite is true.

The proof of this proposition is given in Appendix A.8.

3.4 Fixed-mix investment policy

In the calibration study that is carried out below, it will be seen that the optimal policies that have been derived above may call for fairly aggressive market timing. It is a conceivable that a fund board might not be comfortable with the implementation of such policies, given the uncertainties that surround the assumptions on which the optimization model is based. As an alternative, the board might consider using a simple fixed-mix policy for investments, while still adapting consumption to the estimated phase of the business cycle. The use of a well-defined long-term asset mix is a standard recommendation for pension funds (OECD (2006)), and in many countries funds are subject to legal portfolio limits that are expressed in terms of percentages of total asset value (OECD (2011)).

When the fraction of wealth invested in risky assets is fixed, the only remaining decision variable concerns consumption. Under constant relative risk aversion, the optimal consumption/wealth ratio does not depend on wealth due to the homothetic property of the optimization problem; given a long-horizon perspective, the optimal policy should also not depend on calendar time. In other words, we are looking for strategies under which consumption at time $t$ is given by a rule of the form

$$C_t = \phi(m_t)W_t \tag{47}$$

where $\phi(\cdot)$ is a smooth policy function.

The effect of a policy of the general form (47) on the instantaneous volatility of consumption can be described as follows. By a calculation similar to one in subsection 3.2 (cf. (37)) we find

$$\sigma_C = \left( x + \frac{\phi'(m_t)}{\phi(m_t)} \kappa \right) \sigma_1. \tag{48}$$

The fraction $x$ of wealth invested in risky assets is normally positive, and $\phi(m_t)$ must be positive as well. Under our usual assumptions, the parameter $\kappa$ is negative. To reduce the instantaneous volatility of consumption, the derivative $\phi'(m_t)$ should therefore be positive and not too large in absolute value.

An analytical solution of the optimization problem with fixed-mix investment under constant relative risk aversion does not appear to be available. The problem may be solved numerically, for instance by the method briefly outlined in Appendix A.9. Results from calculations in specific cases (cf. subsection 4.3) suggest that the dependence of the optimal consumption/wealth ratio on the estimated expected return is close to being linear. This is in marked contrast with the results obtained from optimization with variable asset mix; in calculations in Section 4 on the basis of the analytical results of this section, the optimal asset mix is found to depend approximately linearly on the estimated expected return, whereas optimal consumption shows an approximately quadratic dependence. As noted in the discussion following Prop. 3.3, the curvature upwards of the consumption/wealth ratio for very low values of the estimated expected return can be related to the absence of a short-selling constraint. When the investment mix is fixed as a long position, it becomes plausible that the consumption/wealth ratio should depend approximately linearly on the estimated expected return.
Consumption policies that are linear in the estimated expected return, in combination with fixed-mix investment policies, are of interest for their tractability. If the consumption/wealth ratio is given by
\[ c_t = \alpha + \beta (m_t - \bar{\mu}) \] (49)
and the asset mix is given by a constant \( x \), then the model (23) becomes
\[ dW_t = \left[ r + x(m_t - r) - (\alpha + \beta (m_t - \bar{\mu})) \right] W_t \, dt + x \sigma_1 W_t \, dZ'_t. \] (50)
Introducing the new variable \( w_t = \log W_t \), we can write the joint model for \( w_t \) and \( m_t \) as
\[ dw_t = \left( (1 - x) r - \alpha + \beta \bar{\mu} + (x - \beta) m_t - \frac{1}{2} x^2 \sigma_1^2 \right) dt + x \sigma_1 \, dZ'_t \] (51a)
\[ dm_t = \theta (\bar{\mu} - m_t) \, dt + \kappa \sigma_1 \, dZ'_t. \] (51b)
This model is linear, so that in particular the joint distribution of \( w_t \) and \( m_t \), for any given value of \( t \), is a bivariate normal distribution. As a result it is possible to write explicit expressions for the expectation and variance of consumption
\[ C_t = (\alpha + \beta m_t) \exp(w_t) \] (52)
at time \( t \); see Appendix A.10 for details. The long-term behavior of wealth is described by
\[ Ew_t = \left( (1 - x) r + x \bar{\mu} - \frac{1}{2} x^2 \sigma_1^2 - \alpha \right) t + O(1) \] (53)
\[ \text{var}(w_t) = \frac{\sigma_1^2}{\theta^2} (x(\kappa + \theta) - \beta \kappa)^2 t + O(1) \] (54)
where in both cases \( O(1) \) refers to a term that remains bounded as \( t \to \infty \). To reach a stationary distribution (other than zero) of wealth and consumption as \( t \) tends to infinity, we therefore need to define the parameters \( \alpha \) and \( \beta \) by
\[ \alpha = (1 - x) r + x \bar{\mu} - \frac{1}{2} x^2 \sigma_1^2 \] (55a)
\[ \beta = \frac{\kappa + \theta}{\kappa x}. \] (55b)
The condition (55a) has a straightforward interpretation; it is the condition for the expectation of log wealth to remain constant in a model with constant expected return \( \bar{\mu} \) and constant investment mix \( x \). The parameter \( \kappa + \theta \) is normally positive while \( \kappa \) is negative under our usual assumptions as noted before. If we also assume a long position in risky assets \( (x > 0) \), then the rule (55b) leads to a negative value of the sensitivity coefficient \( \beta \).

Given that our usual assumptions imply a negative correlation between asset returns and the estimated expected return, the effect of a negative value of \( \beta \) is that the consumption-wealth ratio is increased after favorable returns and decreased after disappointing returns. In particular, when returns are negative so that wealth has gone down and consequently consumption would already be reduced when the consumption/wealth ratio is kept constant, a negative-\( \beta \) policy calls for an even further reduction of consumption. Such a policy increases the short-term volatility of consumption but decreases the volatility of wealth, which in turn reduces the variability of consumption in the long run. The policy that is defined by (55), for a given value of the proportion of wealth held in risky assets, will be referred to as the procyclical benchmark. Policies that follow from this benchmark are parametrized by the asset mix \( x \). A singular case arises when \( x = 0 \) (no investments in risky assets); the benchmark given by (55) then reduces to the riskless policy of keeping all assets...
in a fixed-interest account and using only the interest payments for consumption, which is a policy that should be characterized as acyclical rather than procyclical.

The short-term volatility of consumption is given by (48). This expression is dependent on $m_t$ in the case of a linear policy, but in order to minimize average volatility it would be reasonable to choose $\alpha$ and $\beta$ such that the quantity in (48) vanishes when $m_t = \bar{\mu}$. This suggests the rule

$$\beta = -\frac{\alpha}{\kappa} x$$

which may be contrasted with (55b). The value for $\beta$ that is found from (56) is positive under the usual assumptions that $\kappa$ is negative and $x$ is positive. The policy that is defined, for a given asset mix $x$, by (55a) and (56) will be referred to as the \textit{countercyclical benchmark}. Just as in the case of the procyclical benchmark, the related policies are parametrized by the chosen asset mix $x$, and reduction to the riskless policy occurs when $x = 0$.

4 Calibration and discussion

4.1 Numerical values

In this section, we present some representative numerical illustrations and discussions. The parameter values underlying the calibration exercises are drawn from Barberis (2000) and Wachter (2002), and summarized in Table 1. Recall that we work with inflation-indexed quantities, so that the riskless interest rate reflects the real interest rate rather than the nominal rate. The number of decimals provided is not meant to be indicative of the level of accuracy that can actually be achieved.

<table>
<thead>
<tr>
<th>Parameter Description</th>
<th>Notation</th>
<th>Parameter Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Riskless rate</td>
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</tr>
<tr>
<td>Volatility of stock price</td>
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</tr>
<tr>
<td>Volatility of expected return</td>
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</tr>
<tr>
<td>Unconditional mean of expected return</td>
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<tr>
<td>Mean-reverting parameter of expected return</td>
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</tr>
<tr>
<td>Correlation coefficient</td>
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</tr>
<tr>
<td>Subjective discount rate</td>
<td>$\eta$</td>
<td>0.0624</td>
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</table>

Table 1: \textbf{Parameter values used in the numerical analysis} The parameter values are calculated based on Barberis (2000) and Wachter (2002). All parameters are in annual units.

Under the parameter values as given in Table 1, the asymptotic variance of the difference between the conditional expected return and the actual expected return is $\bar{V} = 9.3237 \cdot 10^{-4}$; the corresponding value of the standard deviation is 3.05 percentage points. Relative to the value 5.76% of the unconditional mean $\bar{\mu}$, this is substantial. In other words, within the calibrated model there are clearly noticeable limits to learning.

The convergence of the solution to the quadratic differential equation (11) to the stable equilibrium at $\bar{V}$ is superexponential; in fact, for any given value of $\varepsilon > 0$ there is an upper bound to the amount of time needed for the solution to reach the value $\bar{V}$, starting from an arbitrary initial value $V_0 > \bar{V} + \varepsilon$ (cf. Appendix A.5). In the present numerical setting, it turns out that the time required for the variance of the conditional distribution to be reduced below 1.01 times the asymptotic value is maximally 31 years, regardless of the initial variance. Given that this is well within
the time span covered by financial market data, we feel motivated to assume that convergence has taken place, so that we limit discussion to the stable phase.

The parameter $\kappa$, which determines the strength in the stable phase of the correction of the conditional expected return on the basis of the observed actual return, takes the value $-0.1715$. The fact that $\kappa$ is negative implies that the expected return is adjusted downwards in response to asset returns that are higher than expected, and vice versa. Such behavior is caused by the assumed negative correlation between asset price shocks and shocks in expected asset returns.

The negative sign of $\kappa$ also means that the rate of increase of the variance of the price of the risky asset is less for longer horizons; indeed, what might be called the effective long-term volatility (cf. (3)) is given by

$$\sqrt{\frac{\sigma_2^2}{\theta^2} + 2\frac{\sigma_2}{\theta} \sigma_1 \rho + \sigma_1^2} = 0.0555$$

which is considerably less than the short-term volatility $\sigma_1 = 0.1510$. The correlation coefficient $\rho$ is sufficiently negative so that the inequality (5) is satisfied; indeed, we have $\rho = -0.9351$ whereas the right hand side of (5) evaluates to $-0.8376$. This implies that the variance of the log stock price is concave as a function of time.

The volatility of the estimated expected return is $|\kappa \sigma_1| = 0.0259$; not surprisingly, this is less than the volatility of the expected return itself, namely $\sigma_2 = 0.0343$.

The standard deviation of the stationary distribution of the estimated expected return is

$$\bar{\sigma}_m := \frac{|\kappa \sigma_1|}{\sqrt{2\theta}} = 0.0352.$$

Correspondingly, the long-term probability of expected return being negative in our current model is about 11%, whereas the probability of the estimated expected return to be negative is about 5%.

The relation between the current estimate of expected return and previous excess returns is given by (20). In the calibrated model, we have $\theta + \kappa = 0.0997$. Under this parameter value, if we take the difference between time 0 in (20) and current time $t$ to be 50 years, the coefficient that multiplies the unconditional return is 0.9932, so that the coefficient multiplying the initial estimate is 0.0068. It therefore seems reasonable to describe the conditional expected return as the unconditional expected return plus an exponentially weighted moving average of excess returns (positive or negative). The associated half time is equal to $(\log 2)/(\theta + \kappa) = 6.95$; this means that close to one half of the weight in the moving average is carried by observations that have been made more than seven years ago.\(^1\) The calibrated model therefore implies a fairly long time scale for memory.

The relationship (31) that was formulated at the end of subsection 3.1 can be used to define a “canonical” value of the coefficient of risk aversion $\gamma$, when the subjective discount rate $\eta$ is given. This is the value of $\gamma$ which in the classical Merton model would cause the optimal consumption/wealth ratio to be such that the expected wealth level remains constant in time. Given the parameter values in Table 1, the corresponding canonical value for $\gamma$ is found to be $\gamma = 4.013$.

At the end of subsection 2.2 it has been noted that there is an impact of the variability of expected returns on the performance of policies that use a constant asset mix and a constant consumption/wealth ratio. While in the standard Black-Scholes model there is no limit to the consumption/wealth ratios that can be realized while still keeping the expected level of wealth (and therefore also the expected absolute level of consumption) constant, in a business cycle there is an upper limit that is obtained by inserting the value for the asset mix (9) into the expression (8)

\(^1\)Note that, for $a > 0$, we have $\int_0^\infty ae^{-ax} \, dx = 1$ and $\int_0^\infty \frac{1}{x} \log a e^{-ax} \, dx = \frac{1}{2}$. 

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for the corresponding benchmark value of the consumption/wealth ratio. Under
the parameter values as given in Table 1, it turns out that the maximal constant
consumption/wealth ratio that can be maintained under a fixed-mix investment
policy, without draining the fund in expectation, is 5.90%. To realize this, the
fraction of wealth invested in risky assets that must be chosen is 206.9% (i.e. a
short position in bonds needs to be taken). Given that there is no upper bound
under the Black-Scholes assumptions, it appears that the impact of business cycle
effects is quite substantial.

4.2 Optimal CRRA investment and consumption policy
Under the CRRA criterion (32a), the optimal consumption and investment policies
are given by (45) and (46) respectively. Figure 1 shows the optimal consumption

![Optimal consumption and investment strategies](image)

Figure 1: **Optimal consumption and investment strategies** The figure shows the dependence on the estimated expected return of consumption-wealth ratio (Panel A) and allocation to the risky asset (Panel B) in strategies that are optimal with respect to power utility, for different values of the coefficient of relative risk aversion ($\gamma \in \{2, 5, 10, 50\}$). The horizon $T$ is 100 years, and $\psi$ has the value given by (28). Other parameter values are as shown in Table 1.

and investment plans for a range of estimated values of expected return. The
value range shown roughly corresponds to the 95% confidence interval with respect
to the asymptotic distribution of $m_t$. Data in numerical form are given in Table
2. Observe first that the more risk tolerant the investor is (lower $\gamma$), the more
The table shows optimal consumption/wealth ratios and stock weights for various values of the risk aversion parameter \( \gamma \) and for three different values of the conditional expected return. The long-term average of the conditional expected return is \( \bar{\mu} = 0.0576 \), its long-term standard deviation is \( \bar{\sigma} = 0.0352 \).

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( m_t = \mu )</th>
<th>( \mu + \sigma_m )</th>
<th>( \mu - \sigma_m )</th>
<th>( m_t = \mu )</th>
<th>( \mu + \sigma_m )</th>
<th>( \mu - \sigma_m )</th>
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<td>6.78%</td>
<td>123.6%</td>
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<tr>
<td>5</td>
<td>4.83%</td>
<td>4.56%</td>
<td>5.25%</td>
<td>70.38%</td>
<td>27.14%</td>
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<tr>
<td>10</td>
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<td>3.58%</td>
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<td>16.98%</td>
</tr>
</tbody>
</table>

Table 2: CRRA optimal strategies

Responsive consumption and investment are to the variation of the estimated return. Secondly, consistent with Prop. 3.3, the consumption/wealth ratio is increasing with the estimated expected return. Likewise, a higher estimated expected return leads the investor to allocate more to the stock.

The optimal portfolio strategy is represented in (46) as a sum of three terms. The first term does not depend on the horizon length and is usually referred to as the “myopic” component. The other two terms are jointly known as the “dynamic” component, which provides intertemporal hedging. The numerical importance of the three terms is demonstrated in Fig. 2 for the case in which the estimated expected return \( m_t \) is equal to its long-term average \( \mu \), and for a long horizon (\( T = 100 \)). As predicted by Prop. 3.4, given that the parameter \( \kappa \) in the calibrated model is negative, the sign of the dynamic component is positive. The results of calculations for different values of \( m_t \) (not shown in the figure) indicate that the dynamic component is fairly insensitive to the current value of the estimated expected return, whereas, as is also obvious from the expression in (46), the myopic part depends strongly on \( m_t \). At high levels of risk aversion, the dynamic component contributes substantially to the allocation to risky assets, leading to an allocation to risky assets that is more than twice as large as the allocation that would be chosen on the basis of the myopic component only. The same conclusion remains true at horizons longer than 10 years. It is also seen from the figure that the third term in (46), when the bequest weight is determined by the benchmark value (28), is numerically unimportant at the 100-year horizon. At shorter horizons however, this term contributes more and can even dominate the second term.

The behavior of consumption over time is illustrated in Fig. 3 for three different values of the coefficient of risk aversion \( \gamma \). Initial wealth is taken to be 100 units. The policy that is obtained from taking \( \gamma = 50 \) is, as expected, quite conservative, but still it generates an expected level of consumption that is substantially above the level 1.68 which would result from investing everything into riskless assets and withdrawing only the riskless return. The strategy obtained from taking \( \gamma = 5 \) achieves a high level of benefits which on average is even increasing in time. The downward trend of its 5% quantile is limited by the upward trend of the mean; the quantile remains above the level 3 even at longer horizons (not shown in the figure). The 5% quantile of the strategy associated to \( \gamma = 10 \) becomes approximately constant after some time, like the one that corresponds to \( \gamma = 5 \), but at a slightly lower level.

It was shown in subsection 3.2 that the response of consumption to shocks in the economy is completely determined by the myopic component. It follows from (38) that, when the estimated expected return is at its long-term average and the risk aversion coefficient \( \gamma \) is equal to 5, the instantaneous volatility of consumption is 5.40%. The volatility depends on the current value of the estimated expected return. In our calibration the long-term average of the expected return, \( \bar{\mu} \), is equal

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Figure 2: **Components of stock weight**  The figure shows the components of the allocation to risky assets when the current estimate of its expectation is equal to the unconditional mean. The investment horizon is 100 years, and the bequest weight $\psi$ is for each value of $\gamma$ set equal to the value given by (28). Other relevant parameter values are as shown in Table 1.

Figure 3: **Consumption under variable-mix strategies**  The figure shows the average behavior in time of consumption, as well as 5% and 95% quantiles, for strategies that are optimal under power utility with various values of the risk aversion parameter $\gamma$. The plot is based on 10,000 scenarios that have been generated under the model (1)–(10) with $V_t = \bar{V}$. The initial condition for $m_t$ is $\bar{\mu}$; the initial condition for $\mu_t$ is random, following a normal distribution with expectation $\bar{\mu}$ and variance $\bar{V}$. The initial wealth is 100 units.

to 5.76%; its standard deviation is $|\kappa\sigma_1|/\sqrt{2\theta}$, which equals 3.52 percentage points.
Table 3: **Approximately optimal rules** The table shows linear consumption rules that approximate optimal policies under fixed-mix investment policies and under different optimization criteria. The symbol \( \gamma \) denotes the coefficient of relative risk aversion, \( \zeta \) is the inverse of the intertemporal elasticity of substitution, \( x \) is the fraction of wealth held in risky assets, \( \alpha \) is the consumption/wealth ratio when the estimated expected return is equal to its long-term average \( \bar{\mu} = 0.0576 \), \( \beta \) is the coefficient of sensitivity of the consumption/wealth ratio to the estimated expected return, and \( \bar{\sigma}_m \) denotes the long-term standard deviation of the estimated expected return. The numerical value of \( \bar{\sigma}_m \) is 0.0352. The final column shows the instantaneous volatility of consumption when the current value of the estimated expected return is equal to its long-term average, relative to the volatility of consumption when the consumption/wealth ratio would be kept constant.

At one standard deviation above its long-term average, the estimated expected return causes the volatility of consumption to be 10.06% (still assuming \( \gamma = 5 \)); this occurs because of the larger allocation to risky assets in this case. As is apparent from (38), these figures for the volatility of consumption are divided by 2 and by 10 for the policies defined by \( \gamma = 10 \) and by \( \gamma = 50 \) respectively.

### 4.3 Fixed-mix policies

Numerical experiments were carried out using the method described in Appendix A.9 for a number of different preference specifications, including specifications of the Epstein-Zin type. In all cases it was found that the optimal consumption policy is close to a linear function (with nonzero intercept) of the estimated expected return. Results of linear policies that approximate the optimal policies are shown in Table 3. The results show small positive values of the sensitivity coefficient \( \beta \). As a result, the instantaneous volatility is in all cases slightly less than what it would be under a policy of constant consumption/wealth ratio. All consumption policies in the table can therefore be described as mildly countercyclical.

Under a 50/50 asset mix, the benchmark value as determined by (55a) of the parameter \( \alpha \) which determines the average consumption/wealth ratio is 3.43%; under the more conservative 20/80 mix, this figure is reduced to 2.45%. For the sensitivity coefficient \( \beta \), there are two benchmarks, which have been referred to above as procyclical and countercyclical respectively. The procyclical benchmark given by (55b) minimizes the long-term variance of consumption; in our calibration, the value of this benchmark is \(-0.29\) in the case of the 50/50 mix, and \(-0.12\) in the case of the 20/80 mix. On the other hand, the values of the sensitivity coefficient \( \beta \) suggested by the rule (56), which aims to minimize the instantaneous volatility of consumption, are given by 0.10 and 0.03 in the 50/50 and 20/80 case respectively, assuming that \( \alpha \) is determined by (55a). Comparing this to the values for the sensitivity coefficient shown in Table 3, it is seen that these values are about one-tenth of the countercyclical benchmark, quite independently of the choice of the risk aversion parameter and the coefficient of elasticity of intertemporal substitution.

Simulation results are shown in Fig.4 for the two benchmark policies, for two
Figure 4: Consumption under fixed-mix strategies  The figure shows the average behavior in time of consumption, as well as 5% and 95% quantiles, for various fixed-mix strategies with linear consumption rules. The terms “first benchmark” and “second benchmark” refer to the rule (55) and the combination (55a–56) respectively. The benchmark for the average consumption/wealth ratio $\alpha$ is given by (55a). The plot is based on 10 000 scenarios that have been generated under the model (1–10) with $V_t = \bar{V}$. The initial condition for $m_t$ is $\bar{\mu}$; the initial condition for $\mu_t$ is random, following a normal distribution with expectation $\bar{\mu}$ and variance $\bar{V}$.

The initial wealth is 100 units.

policies that are derived from optimization, and for the policy of having a fixed consumption/wealth ratio ($\beta = 0$). The results as shown apply to the case of a 50/50 mix; under a 20/80 mix similar results are obtained, albeit at lower levels of consumption. All policies that use the benchmark value of the average consumption/wealth ratio $\alpha$ are seen to do well in keeping the expected value of consumption constant through time. The same holds true for the policy that is obtained from optimization with respect to the CRRA criterion with $\gamma = 5$. The policy that is obtained from Epstein-Zin utility with $\gamma = 5$ and $\zeta$ (the inverse of the coefficient elasticity of intertemporal substitution) equal to 50 is conservative; since it uses a relatively low consumption/wealth ratio, wealth is increasing on average, and therefore the expected absolute level of consumption is increasing in time. This policy is the best of all considered policies in terms of expected level of consumption after 50 years. Of course, that result does come at the cost of relatively low consumption in the first thirty years.

The quantiles of the procyclical benchmark shown in Fig. 4 are indicative of asymptotic stationarity, as predicted. The stationary distribution shows a rather wide spread compared to the distributions of consumption produced by the other policies, even after 50 years. The countercyclical benchmark is quite effective in suppressing the short-term variability of income. At longer horizons, the variability of wealth becomes a more important factor in determining the variance of consumption under this benchmark. After 50 years, the spread of the distribution that it generates, as judged by the 5% and 95% quantiles, is larger than that of the simple policy which keeps the consumption/wealth ratio always fixed at the benchmark
value. This policy in turn will be improved upon in terms of quantiles by the procyclical benchmark at even longer horizons (not shown in the figure); for the upper quantile, this happens after about 120 years, while the lower quantiles cross after approximately 350 years. The highest level of the 5% quantile at all horizons beyond 30 years is provided by the conservative policy, which generates an upward trend of expected consumption.

Comparing Fig. 4 to Fig. 3, it is seen that substantial improvements can be obtained from market timing, even though we assume in this paper that the true current status of the business cycle can at any time only be known up to a substantial estimation error. It should be emphasized that we have also assumed that investors are perfectly informed about the dynamics of the economy.

Under the 50/50 asset mix, we found a rather wide spread of the stationary distribution generated by the procyclical benchmark. The spread can be reduced by making the asset mix more conservative. The mean-variance tradeoff for the procyclical benchmark is shown in Figure 5.

Figure 5: Mean-variance tradeoff The figure shows mean vs. standard deviation for the stationary distribution generated by the rule (55). The circles correspond to various values of the percentage of wealth invested in risky assets; the lowest value is zero and the step size is two percentage points. The 50/50 mix is indicated by a cross.

5 Conclusions

We have analyzed the problem of long-term income provision from an initial endowment. We have done this within the context of a business cycle model which exhibits limits to learning, in the sense that the true state of the cycle can only be known up to a certain random error. While the setting of the paper is derived most directly from the situation of a trust fund, the framework can also be viewed as a simplified representation of problems faced by pension funds, and in this sense the conclusions of the paper are relevant to the pensions industry as well.
We have approached the problem partly using optimization and partly by an investigation of a particular class of strategies. In the optimization part, we have built on the work of Kim and Omberg (1996) and Wachter (2002) to derive a closed-form solution under power utility. The solution leads to a class of strategies, parameterized by the coefficient of risk aversion $\gamma$, which prescribe both the consumption decision and the investment decision at each moment in time as a function of the estimated state of the business cycle. Motivated in part by robustness concerns, we have also investigated a class of strategies in which the investment mix is constant and only the income paid depends on the estimated business cycle variable. This class exhibits more clearly the conflict between short-term and long-term variability of income. We have identified a “procyclical benchmark” which minimizes the long-term variability within the chosen class, and a “countercyclical benchmark” which minimizes the short-term variability. These strategies respond in opposite ways to asset returns.

In a calibration exercise based on parameter values drawn from the literature, we have found that the uncertainty in determining the status of the business cycle from observed asset returns is substantial; the standard deviation of the error in estimating the value of the current expected return is more than 3 percentage points. Both consumption and investment decisions are optimally based on an index computed on the basis of exponential weighting from past asset returns. The weight of past observations declines slowly with their age; almost one half of the total weight is carried by observations that are more than 7 years old. In spite of the uncertainty in establishing the true state of the business cycle, optimal policies apply aggressive market timing for usual values of the coefficient of relative risk aversion. Simulation results show that in this way it is possible, with an initial endowment of 100 units, to keep up a level of benefits that averages around 5 units per year across scenarios and across time for the first 50 years, while the 5% quantile of the distribution at long horizons lies at a level of about 3.4 units per year. These figures should be compared to the calibrated values of 5.76% for the average expected return on risky assets, and 1.68% for the riskless rate (all quantities are inflation-corrected).

For a wide range of preference specifications both under CRRA and under recursive utility, numerical optimization carried out under the constraint of a fixed investment mix produces consumption policies that respond to asset returns in the same direction as the countercyclical benchmark does, but with a sensitivity of only about one-tenth of that of the benchmark. The procyclical benchmark keeps the variance of consumption as seen from time 0 within bounds for an indefinite period; however the associated mean-variance tradeoff does not appear very attractive. When the results under the fixed-mix constraint are compared to those that were obtained without that constraint, it is clear that there is substantial value in market timing even in the presence of state uncertainty, while model uncertainty is still assumed absent.

A typical trust fund objective is to provide a stable income over an indefinite period. The CRRA criterion is somewhat related to this objective, but cannot be said to express it fully. In such a situation, optimization acts as a method of limiting the search for acceptable strategies from the vast space of all possible dynamic policies to a subset that can be described in terms of just a few parameters, such as the coefficient of relative risk aversion $\gamma$ and the subjective discount rate $\eta$. Alternatively, one can work with ad-hoc parametrizations, as we did in the context of fixed-mix policies. In the situation of this paper, the ad-hoc parametrization in fact appears to provide a more extensive range of possible ways to balance short-term and long-term variability of consumption than the CRRA parametrization does. The CRRA objective focuses on how much risk an agent wants to take, but this is not quite the same as finding a balance between short-term and long-term variability. It may be surmised that criteria of the Epstein-Zin type would provide
more scope to express this issue, but the outcomes of calculations in this paper do not provide much support for that conjecture.

We have used a relatively simple model, chosen with an eye towards tractability and towards enabling a focus on the effects of the business cycle. The real interest rate was taken to be constant. Given results as for instance in Sangvinatsos and Wachter (2005), one may expect that the CRRA optimization problem for the business cycle model is still analytically solvable when the model is extended with an affine term structure model. Such an extension would make the model more realistic, and it would allow a study of the role of interest rate movements in consumption and investment decisions. Even when the model is extended, parameter uncertainty and other forms of model risk still need to be taken into account, and robust policies should be developed whose performance does not critically depend on particular model assumptions. An extension of the present study to include model uncertainty may for instance be carried out along the lines of Wachter and Warusawitharana (2009).

In terms of the tradeoff between short-term and long-term variability of income, none of the strategies discussed in this paper is completely satisfactory. The variable-mix policies that are best in terms of expected consumption level generate short-term volatilities of consumption that can reach 10% or more. The countercyclical benchmark with fixed-mix investment suppresses short-term volatility quite effectively, but performs less well in the long run. The procyclical benchmark is capable of reaching low levels of long-horizon variability, but this requires that investments in risky assets are kept low, so that the corresponding levels of consumption are fairly low as well. The CRRA and Epstein-Zin policies for the fixed-mix case do not appear to produce solutions that are essentially different from the ones that are obtained from the simple parametrization in terms of linear policies. Alternative strategies, perhaps making use of collar options (selling upward potential to buy downward protection), may be capable of achieving a better compromise between short-term and long-term variability.

A Appendices

A.1 Means and (co)variances in a two-dimensional linear SDE

Several models in this paper take the form

\[
\begin{bmatrix} w_t \\ \mu_t \end{bmatrix} = \left( \begin{bmatrix} 0 & a \\ 0 & -\theta \end{bmatrix} \right) \begin{bmatrix} w_t \\ \mu_t \end{bmatrix} + \begin{bmatrix} b \\ \theta \mu_0 \end{bmatrix} dt + \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix} d\begin{bmatrix} Z_{w,t} \\ Z_{\mu,t} \end{bmatrix}
\]

(57)

where \( w_t \) represents log stock price or log wealth; \( Z_w \) and \( Z_\mu \) are Brownian motion with correlation coefficient \( \rho \), and initial conditions \( w_0 \) and \( \mu_0 \) are given. For any given \( t \), the joint distribution of \( w_t \) and \( \mu_t \) is bivariate normal. We show here how the means and the variance-covariance matrix can be obtained from a method based on left eigenvectors. First of all, it is immediate from (57) that

\[
E \mu_t = e^{-\theta t} \mu_0 + (1 - e^{-\theta t}) \bar{\mu}.
\]

(58)

The generator matrix that appears in (57) has a left eigenvector \([ \theta \ a ]\) associated to the eigenvalue at 0. Using this eigenvector, we find

\[
\frac{d}{dt}(\theta E w_t + a E \mu_t) = \theta (b + a \bar{\mu}) dt
\]

so that

\[
\theta E w_t + a E \mu_t = \theta w_0 + a \mu_0 + \theta (b + a \bar{\mu}) t.
\]

(59)
An explicit expression for \( Ew_t \) can be derived easily from this formula together with (58).

Turning now to the variances and covariance, we can apply Itô’s formula to \((w_t - Ew_t)^2\), \((w_t - Ew_t)(\mu_t - E\mu_t)\), and \((\mu_t - E\mu_t)^2\), and then take expectations to arrive at the following system of ODEs for the variances \( V_w \) and \( V_\mu \) and the covariance \( V_{w\mu} \):

\[
\frac{d}{dt} \begin{bmatrix} V_\mu \\ V_{w\mu} \\ V_w \end{bmatrix} (t) = \begin{bmatrix} -2\theta & 0 & 0 \\ a & -\theta & 0 \\ 0 & 2a & 0 \end{bmatrix} \begin{bmatrix} V_\mu \\ V_{w\mu} \\ V_w \end{bmatrix} (t) + \begin{bmatrix} \sigma_1^2 \\ \sigma_1 \sigma_2 \rho \\ \sigma_2^2 \end{bmatrix}
\]

(60)

with initial conditions \( V_\mu(0) = V_{w\mu}(0) = V_w(0) = 0 \). The generator matrix in this equation has eigenvalues \( 0, -\theta, \) and \( -2\theta \), with left eigenvectors given by \([a^2 \ 2a \ \theta \ \theta^2] \), \([a \ \theta \ 0] \), and \([1 \ 0 \ 0] \). Using these eigenvectors in the same way as above, we find

\[
a^2 V_\mu(t) + 2a\theta V_{w\mu}(t) + \theta^2 V_w(t) = (a^2 \sigma_2^2 + 2a\theta\sigma_1\sigma_2\rho + \theta^2 \sigma_1^2)t
\]

(61a)
\[
a V_\mu(t) + \theta V_{w\mu}(t) = (a\sigma_2^2 + \theta\sigma_1\sigma_2\rho) \frac{1 - e^{-\theta t}}{\theta}
\]

(61b)
\[
V_\mu(t) = \frac{\sigma_2^2}{2} \frac{1 - e^{-2\theta t}}{2\theta}.
\]

(61c)

Explicit expressions for the variances and the covariance are immediate from these formulas.

A.2 Short-term and long-term variance of the stock price

To analyze the behavior of the variance of the stock price in the model (1), we can use the results of subsection A.1 taking \( w_t = \ln S_t \), \( a = 1 \), and \( b = -\frac{1}{2} \sigma_1^2 \). It follows from (61) that the variance of the log stock price \( V_w(t) \) satisfies

\[
\lim_{t \to \infty} \frac{dV_w}{dt}(t) = \frac{\sigma_2^2}{\theta^2} + \frac{2\sigma_2}{\theta} \sigma_1 \rho + \sigma_1^2.
\]

(62)

From (60), it is immediate that

\[
\frac{dV_w}{dt}(0) = \sigma_1^2.
\]

(63)

Therefore, the long-term rate of increase of the variance of the stock price per unit of time is less than the short-term rate of increase in case

\[
\frac{\sigma_2^2}{\theta^2} + \frac{2\sigma_2}{\theta} \sigma_1 \rho < 0
\]

(64)

which is equivalent to condition (4) in the text. From (61) one can also compute that

\[
\frac{d^2V_w}{dt^2}(t) = \frac{2e^{-\theta t}}{\theta} (\sigma_2^2 + \theta\sigma_1\sigma_2\rho - \sigma_2^2 e^{-\theta t}).
\]

This expression is negative for all \( t > 0 \) if and only if \( \sigma_2^2 + \theta\sigma_1\sigma_2\rho < 0 \). From the above it is furthermore noted that \( (d^2V_w/dt^2)(0) = 2\sigma_1\sigma_2\rho \), so that \( dV_w/dt \) is decreasing for values of \( t \) close to 0 if \( \rho < 0 \).

A.3 Expected wealth under constant asset mix and constant consumption/wealth ratio

The combination of (1c) and (6) with \( x_t = x \) and \( c_t = c \) for all \( t \) constitutes a model of the form (57) with

\[
w_t = \ln W_t, \quad a = x, \quad b = (1 - x)r - c - \frac{1}{2}x^2 \sigma_1^2.
\]
It is reasonable to guess that the indirect utility function is of the form

\[ E_w = (b + a\bar{\mu})t + O(1) \quad (65) \]

\[ V_w(t) = x^2 \left( \frac{\sigma_1^2}{\theta^2} + 2\frac{\sigma_2}{\theta}\sigma_1\rho + \sigma_1^2 \right) t + O(1) \quad (66) \]

where \( O(1) \) refers to a term that remains bounded as \( t \) tends to infinity. Consequently, we have

\[
\lim_{t \to -\infty} \ln \frac{E W_t}{t} = b + a\bar{\mu} + \frac{1}{2}x^2 \left( \frac{\sigma_1^2}{\theta^2} + 2\frac{\sigma_2}{\theta}\sigma_1\rho + \sigma_1^2 \right)
\]

\[
= x\bar{\mu} + (1 - x)r - c + \frac{1}{2}x^2 \left( \frac{\sigma_1^2}{\theta^2} + 2\frac{\sigma_2}{\theta}\sigma_1\rho \right). \quad (67)
\]

The right hand side vanishes when \( c \) takes the value given by (8) in the text.

### A.4 Proof of Thm. 3.1

We opt to solve the problem through dynamic programming. The indirect utility function in the intermediate consumption case is

\[
J(W_t, m_t, t) = \sup_{(C_{t}, x_{t}) \in [r, T]} E_{W_t, m_t, t} \left[ \int_t^T e^{-\eta(s-t)} \frac{C_t^{1-\gamma}}{1-\gamma} ds + \psi e^{-\eta(T-t)} W_t^{1-\gamma} \right]
\]

The Hamilton-Jacobi-Bellman (HJB) equation associated with this dynamic programming problem is

\[
\eta J(W_t, m_t, t) = \sup_{C_t \geq 0, x_t \in \mathbb{R}} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + J_t(W_t, m_t, t) \right. \\
+ J_W(W_t, m_t, t) \left[ W_t \left( r + x_t (m_t - r) \right) - C_t \right] \\
+ \frac{1}{2} J_{WW}(W_t, m_t, t) W_t^2 \sigma_1^2 x_t^2 + J_m(W_t, m_t, t) (\bar{\mu} - \theta m_t) \\
+ \frac{1}{2} J_{mm}(W_t, m_t, t) \left( \frac{\sigma_1\sigma_2\rho + V_t}{\sigma_1} \right)^2 \\
+ J_{Wm}(W_t, m_t, t) W_t x_t \sigma_1 \left( \frac{\sigma_1\sigma_2\rho + V_t}{\sigma_1} \right) \} \quad (69)
\]

with the terminal condition \( J(W_T, m_T, T) = \psi \frac{W_T^{1-\gamma}}{1-\gamma} \), where the subscripts of \( J \) denote the obvious partial derivatives. The first-order condition with respect to \( C_t \) is

\[
C_t^{\star-\gamma} = J_W(W_t, m_t, t).
\]

Then the optimal consumption strategy is

\[
C_t^{\star} = J_W(W_t, m_t, t)^{-\frac{1}{\gamma}}.
\]

Similarly, it follows from the first-order condition with respect to \( x_t \) that the optimal portfolio strategy is given by

\[
x_t^{\star} = - \frac{J_W(W_t, m_t, t)}{W_t J_{WW}(W_t, m_t, t) \sigma_1^2} (m_t - r) - \frac{J_{Wm}(W_t, m_t, t)}{W_t J_{WW}(W_t, m_t, t) \sigma_1^2} \left( \sigma_1\sigma_2\rho + V_t \right).
\]

It is reasonable to guess that the indirect utility function is of the form

\[
J(W, m_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} F(m_t, t)^\gamma \quad (70)
\]
where
\[ F(m_t, t)^\gamma = (1 - \gamma)J(1, m_t, t). \]

From the terminal condition \( J(W_T, m_T, T) = \psi_{W_T}^{1-\gamma} \), it follows that
\[ F(m_T, T) = \psi^{1/\gamma}. \]

From this guessed form of the indirect utility function, \( J \), and its relevant derivatives, it follows that the optimal consumption-wealth ratio and the optimal portfolio plan are given by (33) and (34). Finally, inserting the relevant derivatives and the candidate optimal strategy into the HJB equation (69) yields the PDE (35).

A.5 The explicit solution of \( V_t \) and its value in the stable phase

The ODE (11) characterizing \( V_t \) is of Riccati type. Such equations can be written in the form
\[ dx(t) = -a(x(t) - x_1)(x(t) - x_2) \, dt \]  
with \( a > 0 \) and \( x_2 > x_1 \). Given any initial value \( x(0) > x_1 \), the solution converges to the stable equilibrium at \( x_2 \). In the case of the equation (11), the two roots are
\[ x_{1,2} = \sigma_1^2 \left[ \pm \sqrt{\theta^2 + 2\rho\sigma_2 \sigma_1^2 + \frac{\sigma_2^2}{\sigma_1^2} - (\theta + \rho\sigma_2 \sigma_1^2)} \right]. \]  
This shows that the variance \( V_t \) will converge to the equilibrium value \( \bar{V} = x_2 \) whenever at least one of the conditions \( V_0 > 0 \) and \( |\rho| < 1 \) is satisfied. When \( x(0) > x_2 \) in (71), then \( x(t) < x_2 \) for all \( t \), and since the equation (71) can be written as
\[ a(x_2 - x_1) \, dt = \frac{dx}{x - x_1} - \frac{dx}{x - x_2}, \]
it follows that for all \( t > 0 \)
\[ t = \frac{1}{a(x_2 - x_1)} \left( \ln \frac{x(t) - x_1}{x(t) - x_2} - \ln \frac{x(0) - x_1}{x(0) - x_2} \right). \]
The solution can be written down explicitly from this. It is also noted from the above that the time \( t(x_0; \varepsilon) \) at which the solution reaches the value \( x_2 + \varepsilon \) from the initial value \( x(0) = x_0 > x_2 + \varepsilon \) is given by
\[ t(x_0; \varepsilon) = \frac{1}{a(x_2 - x_1)} \left( \ln \frac{x_2 + \varepsilon - x_1}{\varepsilon} - \ln \frac{x_0 - x_1}{x_0 - x_2} \right). \]
Consequently, we have
\[ \lim_{x_0 \to -\infty} t(x_0; \varepsilon) = \frac{1}{a(x_2 - x_1)} \left( \ln (x_2 + \varepsilon - x_1) + \ln \frac{1}{\varepsilon} \right). \]
The numerical value that is cited in subsection 4.1 can be derived from this formula together with (72).

A.6 Solving the equations for \( A_1(\tau) \) and \( A_2(\tau) \)

In this appendix we show how the nonlinear differential equations that are obtained by substituting (40) into (39) can be solved by relating them to a system of linear
ordinary differential equations with constant coefficients. The equations resulting from the substitution are

\[
A_1'(t) = b_0 - b_1 A_1(t) - b_2 A_2^2(t) \tag{73a}
\]

\[
A_2'(t) = -c_0 A_2(t) + c_1 A_1(t) - b_2 A_1(t) A_2(t) \tag{73b}
\]

\[
A_3'(t) = r + c_1 A_2(t) + \frac{1}{2} \kappa^2 A_1(t) - \frac{1}{2} b_2 A_2^2(t) \tag{73c}
\]

where the parameters \(b_0\), \(b_2\), \(c_0\), and \(c_1\) are as defined in (43), \(b_1 = 2c_0\), and \(q\) is as defined in (44). Using a classical technique (cf. for instance Ince (1956), p. 24), we can introduce a new function of time by the differential equation \(B_0'(t) = b_2 B_0 A_1(t)\) with the initial condition \(B_0(0) = 1\). Also introduce \(B_1(t) := B_0(t) A_1(t)\) and \(B_2(t) := B_0(t) A_2(t)\). Then it is readily verified that the three functions \(B_0(t)\), \(B_1(t)\), and \(B_2(t)\) satisfy the following constant-coefficient system of linear differential equations:

\[
\frac{d}{dt} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} (t) = \begin{bmatrix} 0 & b_2 & 0 \\ b_0 & -b_1 & 0 \\ 0 & c_1 & -c_0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} (t), \quad \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} (0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{74}
\]

Functions \(A_1(t)\) and \(A_2(t)\) that satisfy (73) can be obtained by solving the above system and setting

\[
A_i(t) = B_i(t) B_0^{-1} (t) \quad (i = 1, 2). \tag{75}
\]

The solution becomes infinite (i.e. there is a finite escape time) when \(B_0(t)\) reaches 0. However, under the assumption that \(\gamma > 1\), the coefficients \(b_0\) and \(b_2\) are both positive, which implies that the first quadrant is an invariant set for the pair \((B_0(t), B_1(t))\); indeed we have \(B_1' > 0\) when \(B_1 > 0\), \(B_0 > 0\), and \(B_0' > 0\) when \(B_0 = 0\), \(B_1 > 0\). Therefore, under this assumption, \(B_0(0)\) is positive for all \(t \geq 0\), and both \(A_1(t)\) and \(A_2(t)\) are defined for all \(t\). The solutions \(A_1(t)\) and \(A_2(t)\) obtained from (75) remain the same when the functions \(B_i(t)\) are multiplied by the same factor, and therefore we are free to modify the matrix appearing in (74) by adding a constant times the unit matrix. Making use of this freedom, and noting the relation \(b_1 = 2c_0\), we can define the solutions of (73) by (75) together with the equations

\[
\frac{d}{dt} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} (t) = \begin{bmatrix} c_0 & b_2 & 0 \\ b_0 & -c_0 & 0 \\ 0 & c_1 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} (t), \quad \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} (0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{76}
\]

The eigenvalues of the matrix that appears here are 0 and \(\pm q\) where

\[
q = \sqrt{c_0^2 + 4b_0b_2} = \sqrt{2^{-1}(\kappa + \theta)^2 + \frac{1}{\gamma} \theta^2}.
\]

Therefore the solutions consist of a linear combination of the exponential functions \(e^{q t}\) and \(e^{-q t}\) plus a constant. The coefficients are found by making use of the initial conditions \((B_0(0) = 1, B_0'(0) = c_0 B_0(0) + b_2 B_1(0) = c_0,\) and so on), and we obtain

\[
B_0(t) = \frac{q + c_0}{q} e^{qt} + \frac{q - c_0}{q} e^{-qt} \tag{77a}
\]

\[
B_1(t) = \frac{b_0}{2q} (e^{qt} - e^{-qt}) \tag{77b}
\]

\[
B_2(t) = \frac{b_0 c_1}{2q^2} (e^{qt} + e^{-qt} - 2). \tag{77c}
\]

The formulas for \(A_1(\tau)\) and \(A_2(\tau)\) in (42) follow from the above via (75), whereas the expression for \(A_3(\tau)\) is a matter of straightforward integration.
We note some basic properties of the functions \( A_i(\tau) \). The derivatives of \( A_1(\tau) \) and \( A_2(\tau) \) are given by

\[
A_1'(\tau) = \frac{4b_0q^2}{((q + c_0)e^{\tau} + (q - c_0)e^{-\tau})^2}
\]

and

\[
A_2'(\tau) = 2b_0c_1q(e^{\tau} - e^{-\tau}) + c_0(e^{\tau} + e^{-\tau} - 2)
\]

respectively. If \( \gamma > 1 \) then \( q > |c_0| \), so that \( A_1'(\tau) > 0 \) for all \( \tau \geq 0 \); under the additional assumption that \( \bar{\mu} > r \) which implies \( c_1 > 0 \), we also have that \( A_2'(\tau) > 0 \) for all \( \tau \geq 0 \). Given that \( A_1(0) = 0 \) and \( A_2(0) = 0 \), it follows under the stated assumptions that \( A_1(\tau) > 0 \) and \( A_2(\tau) > 0 \) for all \( \tau > 0 \). Moreover we have

\[
\begin{align*}
\lim_{\tau \to \infty} A_1(\tau) &= \frac{b_0}{q + c_0} \quad (78) \\
\lim_{\tau \to \infty} A_2(\tau) &= \frac{b_0c_1}{q(q + c_0)} \quad (79) \\
\lim_{\tau \to \infty} \frac{A_2(\tau)}{\tau} &= r + \frac{1}{2} \left( \frac{b_0c_1^2}{q^2} + \frac{\gamma}{\gamma - 1} (q - c_0) \right) \quad (80)
\end{align*}
\]

### A.7 Proof of Prop. 3.4

The intertemporal hedging demand is by definition given by the last two terms on the right in the expression (46) for the optimal investment decision. Under the assumed inequalities, the positivity of this term follows from Appendix A.6. It is also immediate from (46) that the signs of the dynamic component and its derivative with respect to the investment horizon are reversed when the same inequalities hold except that the sign of \( \kappa \) is opposite, and that the dynamic component vanishes when \( \kappa = 0 \).

To prove the statement concerning the dependence on the investment horizon, let \( t \) and \( m_1 \) be fixed. To alleviate the notation somewhat, write

\[ h(s) = e^{H(m_1, s)}, \quad \tau = T - t. \]

The optimal portfolio strategy (46) in case \( \psi = 0 \) can then be written in the form

\[
x_t^* = \frac{1}{\gamma} \frac{m_t - r}{\sigma^2_t} + \frac{1 - \gamma \kappa}{\gamma \sigma_1} \left[ \frac{m_t - r}{\sigma_1} \int_0^\tau h(s)A_1(s) \, ds + \int_0^\tau h(s)A_2(s) \, ds \right] + \frac{\int_0^\tau h(s)A_2(s) \, ds}{\int_0^\tau h(s) \, ds}. \quad (81)
\]

Define

\[
g(\tau) = \frac{\int_0^\tau h(s)A(s) \, ds}{\int_0^\tau h(s) \, ds}
\]

where \( A(\tau) \) may refer to \( A_1(\tau) \) or to \( A_2(\tau) \). We have

\[
g'(\tau) = h(\tau) \left[ \frac{A(\tau) \int_0^\tau h(s) \, ds - \int_0^\tau A(s)h(s) \, ds}{(\int_0^\tau h(s) \, ds)^2} \right].
\]

The numerator in the fraction above is positive both when \( A(\tau) = A_1(\tau) \) and when \( A(\tau) = A_2(\tau) \) because both functions are increasing. Since \( h(\tau) \) is positive as well, it follows that \( g'(\tau) \) is positive. On the basis of the assumptions \( m_t - r > 0, \gamma > 1, \) and \( \kappa < 0 \), the statement in the proposition follows.
A.8 Proof of Prop. 3.5

Writing the optimal portfolio policy in a form similar to (81), one sees that the proposition is proved if one shows that, under the stated assumptions, the function $f(\bar{\psi})$ defined by

$$f(\bar{\psi}) = \int_0^\tau h(s)A(s)ds + \bar{\psi}h(\tau)A(\tau)$$

is increasing both when $A(\tau) = A_1(\tau)$ and when $A(\tau) = A_2(\tau)$. This follows by writing

$$f(\bar{\psi}) = A(\tau) + \frac{A(\tau)\int_0^\tau h(s)ds - \int_0^\tau A(s)h(s)ds}{\int_0^\tau h(s)ds + \bar{\psi}h(\tau)}$$

and noting that the numerator in the fraction on the right is positive by the argument given in Appendix A.7.

A.9 Numerical optimization method

We describe here a simple approximation scheme to find optimal policies under CRRA preferences or, more generally, Epstein-Zin preferences. The method is described here under the assumption that the fraction $x$ of wealth invested in risky assets is constant. First of all, the continuous-time specification is replaced by a discrete-time specification as follows:

$$V_T = \psi C_T$$

$$V_t = \left(\eta \Delta t \frac{C_t^{1-\gamma}}{1-\gamma} + (1 - \eta \Delta t) \left(E_t[V_{t+\Delta t}^{1-\gamma}]\right)^{1-\gamma}\right)^{1-\gamma}$$

where $\psi$ is a constant that expresses strength of the bequest motive, $\eta$ is the subjective discount factor, $C_t$ is the rate of consumption at time $t$, $\zeta = 1/\tau$ where $\tau$ is the coefficient of elasticity of intertemporal substitution, and $\gamma$ is the coefficient of constant relative risk aversion. The CRRA case is obtained when $\zeta$ is equal to $\gamma$.

Write $V_t$ as a function of the state variables and time

$$V_t = V(t, m_t, W_t)$$

and solve recursively:

$$V(T, m_T, W_T) = \psi W_T$$

$$V(t, m_t, W_t) = \max_{C_t} \left(\eta \Delta t \frac{C_t^{1-\gamma}}{1-\gamma} + (1 - \eta \Delta t) Z_t^{1-\gamma}\right)^{1-\gamma}$$

with

$$Z_t = \left(E_t[(V(t + \Delta t, m_t + \Delta t, W_t + \Delta t))^{1-\gamma}]\right)^{1-\gamma}$$

where expectation is taken on the basis of (17b) and (23). It follows from the homotheticity property of the problem formulation that the value function is of the form

$$V(t, m_t, W_t) = k(t, m_t) W_t$$

where the function $k(t, m_t)$ can be solved from the recursion

$$k(T, m_T) = \psi$$

$$k(t, m_t) = \max_{C_t} \left(\eta \Delta t \frac{C_t^{1-\gamma}}{1-\gamma} + (1 - \eta \Delta t) \hat{Z_t}^{1-\gamma}\right)^{1-\gamma}$$

with

$$\hat{Z_t} = \left(E_t[(k(t + \Delta t, m_t + \Delta t)W_t + \Delta t/W_t))^{1-\gamma}]\right)^{1-\gamma}$$
where $c_t$ is as usual the consumption/wealth ratio. To evaluate the expectation in (83c) numerically for given values of $c_t$, one can use the approximations

$$m_{t+\Delta t} \approx \theta(\bar{\mu} - m_t) \Delta t + \kappa \sigma_1 \Delta Z'$$

$$W_{t+\Delta t}/W_t \approx \exp \left((r + x(m_t - r) - c_t - \frac{1}{2} r^2 \sigma_t^2) \Delta t + x \sigma_1 \Delta Z' \right)$$

with $\Delta Z' \sim N(0, \Delta t)$. The optimization problem in (83b) can be solved by means of a derivative-free optimizer. Once this has been done at a given point in the time grid for a range of values of the estimated expected return $m_t$, a convenient approximation to the relative value function $k(t, m_t)$ as a function of $m_t$ can be obtained by regression on a set of polynomial basis functions. Iteration may be continued until a given convergence criterion is satisfied, in order to find a solution that corresponds to a long time horizon.

### A.10 Fixed-mix investments and linear consumption rule

The parameters that describe the joint distribution of $w_t$ and $m_t$ can be obtained as in Appendix A.1. The mean and the variance of consumption as given by (52) are subsequently found by using the following fact: if $X$ and $Y$ follow a bivariate normal distribution, then (with the usual meanings of the symbols)

$$E(Xe^Y) = (\mu_X + \sigma_X \sigma_Y \rho_{XY}) e^{\mu_Y + \frac{1}{2} \sigma_Y^2}$$

and

$$\text{var}(Xe^Y) = [(\mu_X + 2\sigma_X \sigma_Y \rho_{XY})^2 + \sigma_X^2] e^{2\mu_Y + \sigma_Y^2} - (\mu_X + \sigma_X \sigma_Y \rho_{XY})^2 e^{2\mu_Y + \sigma_Y^2}.$$ 

(87)

These formulas can be obtained by direct integration.

### References


