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Comparison of UFR Implementations in Europe

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Comparison of UFR implementations in Europe

by

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Abstract

Several ultimate forward rate methods have been proposed to extrapolate the risk-free yield curve, which aims to value long-term liabilities. Different UFR methods may lead to different values for the long-term liabilities. Also different UFR methods may have different impact on the solvency risk of the insurer. In this thesis, I compare five UFR methods in terms of their valuation impact and solvency risk. It turns out that some UFR methods tend to offer a higher price than the others. And the difference of UFR liabiltiy values can be up to 2.32% in this thesis. Besides, if the UFR value of the Swedish and EIOPA methods is set to be around the market UFR, then there is no material difference between the solvency risk corresponding to the UFR method.What's more, given the UFR value of the Swedish and EIOPA methods higher than the market UFR, the SCR of Swedish UFR is higher than that of EIOPA UFR, which is about 2%. This number decreases after changing the convergence period from 40 years to 10 years.

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Contents

Chapter 1

Introduction

According to EIOPA(2013), the time to maturity of pension obligation of Life insurance company and pension fund may be longer than the assets available in the market to construct the reliable risk-free yield curve for the valuation of the obligation. In order to obtain the fair obligation value, many methods are provided to extraoplate the risk-free yield curve.

EIOPA proposes the Ultimate Forward Rate (UFR) extrapolation method in the QIS5 (2010), based on the Smith-Wilson technique by Smith and Wilson (2001), which is a very important part of the Solvency II package.The EIOPA UFR has been under discussion ever since its born. Rebel (2012) suggested to make the combination of EIOPA UFR and maket data after the last liquid point, which is adopted by the Dutch regulator. Recently, the Dutch regulator (2013) formulated a new UFR method using six criteria. The Swedish regulator also believes that the EIOPA UFR is unworkable and has introduced the linear UFR extrapolation method (2013). The French extrapolation method is mainly based on the Vasicek-Fong method (2001).The German regulator publishes the discount curve up to the maturity of 50 years (2009), and does not provide the official extrapolation method for those beyond 50 years. Therefore, I would assume that the German life insurance company makes the last liquid forward data-50 years one-year forward rate- as the UFR value, which is assumed to be constant for the maturity beyond 50 years. The impact of UFR methods above except the old Dutch UFR will be compared in this thesis.

For the impact of UFR extrapolation method, lots of literatures are related to the EIOPA UFR. Kocken et.al.(2012) mainly focuses on the risk impact, who finds that the interest rate risk concentrates on the LLP and proposes a revised version of UFR method according to that. The EIOPA (2013) also conducts the anlaysis on its UFR method and aknowledges that the EIOPA UFR may delay the regulation. According to the CRO(2010), the choice of extrapolation method will influence the market value of the liability and the amount of the solvency capital requirement. In that paper, two simplified extrapolation methods are compared to illustrate the valuation impact. In this thesis, I would like to compare different UFR implementation in Europe in terms of the valuation impact and the solvency risk.

Therefore, my research qestions are:

1. What's the difference with respect to the valuation of the five methods?

This question is answered by the difference in the pension liablity valued by different UFR method in terms of three aspects: pension liability value, market consistency of pension liability and the characteristic of the UFR value.The comparison is taken under three types of shape of market yield curve.

In regard to the pension liability value, it can be expected that three events may occur with a high probability given that the EIOPA and Swedish UFR is higher than the market UFR.The first event is that the Swedish pension liability is smaller than the French liability. The second event is that the Dutch pension liability is larger than the EIOPA liability. The last event is that the EIOPA liability is smaller than the Swedish liablity. The difference of liability value can be up to 2.32%.

In regard to the market consistency comparison, the market consistency level of Dutch and French liabilities are higher than those of Swedish and EIOPA. The absolute deviation of the UFR liability and market liability ranges from 0% to 4.47% depends on different UFR methods and different average duration of pension fund.

In regard to the characteristic of the UFR value, the French UFR value varies a lot, which is quite unstable. On the contrary, other UFR methods remain nearly unchanged for different shapes of yield curves.

Furthermore, it is found that the differences of the UFR valuation impact increase with the increase of the average duration.

2. What are the consequences for solvency risk of the five methods?

The solvency risk is measured by the Solvency Capital Requirement (SCR) in this thesis. It turns out that the SCR calculated by Monte Carlo Simulation method is different from that calculated by the standard formula.Due to the limit of time, the thesis focuses on the results of the simulation method, which are:

When the UFR value of EIOPA and Swedish (These two methods share the same UFR value) is set to be higher than the market UFR, then the solvency risk of pension fund employing EIOPA UFR method and Swedish UFR method is lower than that of French method and Dutch method, given that the UFR value of EIOPA and Swedish is chosen to be higher than market UFR.Besides, the SCR of Swedish UFR is higher than that of EIOPA UFR, which is about 2%.

Difference in solvency risk increases with the increase of the average duration.

These conclusions are robust according to the sensitivity analysis except for the French UFR method, which turns out to be quite sensitivity with respect to the FSP. What's more, it is found that: when the EIOPA and Swedish UFR value is set to be around the market UFR, the SCR for different UFR methods can be regarded as no difference. When the UFR is set to be relatively higher, than the SCRs of Swedish and EIOPA are lower than the SCRs of Dutch and French. Besides, the Swedish SCR is higher than the EIOPA SCR. When the UFR is set to be relatively lower, than the SCRs of Swedish and EIOPA are higher than the SCRs of Dutch and French. The Swedish SCR is lower than the EIOPA SCR in this case. What's more, the difference between the Swedish SCR and EIOPA SCR narrows down after changing the convergence period from 40 years to 10 years.

Notice that the conclusion does not include anything about the German UFR method. This is due to the yield curve obtained by employing the German UFR method deviates too much from the market yield curve and the other UFR curves, which you may observe in Table 4.3 . Considering the time is limited and reliability of the conclusion, I would just present the results related to the German UFR method without analysis; therefore no conclusion here is related to the German UFR methods.

The thesis is constructed as follow:

In Chapter 2, five UFR methods are illustrated.

In Chapter 3, the term structure of interest rate model is introduced, which will be used to model the market yield curve.Besides, the impact of past interest rate risk on the comparison of valuation impact is also analyzed in this chapter.

In Chapter 4, under different shapes of current market yield curve, the valuation impact of different UFR methods are compared based on the pension liabilities discounted by the corresponding UFR yield curve.

In Chapter 5, the solvency capital requirements corresponding to different UFR methods are compared. The sensitivity analysis of the comparison for the solvency capital requirement with respect to extrapolation parameters, very long-term forward rate implied by the market yield curve (market UFR), the interest rate hedging percentage and Vasicek model parameter.

In Chapter 6, conclusion.

Chapter 2

Technical Specification

UFR method is used to extrapolate the market yield curve. Different UFR method has different impact on the valuation of pension liabilities and on the solvency risk. In order to compare extrapolation effects, it would be necessary to get to know how to construct the corresponding UFR yield curve. In this chapter, five UFR methods-Dutch UFR, Swedish UFR, German UFR , French UFR and EIOPA UFR-are illustrated, which will be used to extrapolate the current market yield curve and future market yield curve in the following chapters.These notation will be used:

$P(t, t+l)$: the price of a zero-coupon default-free bond with maturity l at time t

$z(t, t+l)$: the continously compounded zero coupon rate at time t with maturity$l$, which is obtained by $z(t, t+l) = -\frac{1}{l} log(p(t, t+l))$

$f(t, u, u+l)$: the u-year default-free annually compounded forward rate with maturity l valued at time t. ($f_c(t, u, u+l)$ is continously compounded)

u_j: cash payment dates for the observed coupon paying bonds, $j = 1, 2, ..., J$, where J is the total number of the cash payment dates

c_{ij}: the cash flow that coupon paying bond i with the maturity of $u_{l(i)}$ needs to pay at time u_j, for $i = 1, 2, ..., N$; $j = 1, 2, ..., J$

$r_{t,t+l}$: the one-year swap rate with maturity l at time t

UFR: the ultimate forward rate, annually compounded, one-year forward rate in the very long term. ($UFR_c$is continuously compounded).

LLP: last liquid point, no market data utilized after the LLP.

FSP: first smoothing point, some market data utilized after the FSP. [In this thesis, FSP is used to represent the point that enters the extrapolation for simplicity.]

α: determines the speed of convergence to the UFR after the FSP

2.1 Dutch UFR

A Dutch UFR method was introduced by the Dutch regulator in 2013.

2.1.1 Method to construct the non-extrapolated part of the yield curve

Before the FSP, the method to obtain the yield curve is the same as that in the Financial Assessment Framework:

The Euro swap rates for maturities of 1-10 years, 12 years, 15 years, 20 years are used to construct the yield curve before the FSP. For $j < u_{l(i)}$,$c_{ij} = r_{t,t+u_{l(i)}}$; $j = u_{l(i)}$,$c_{ij} = r_{t,t+u_{l(i)}} + 1$; $j > u_{l(i)}$,$c_{ij} = 0$.

The zero-coupon rate with maturity of 1 year at time$t$ $z(t, t + 1)$can be derived from the one-year swap rate by:

$$(1 + r_{t,t+1})e^{-z(t,t+1)} = 1 \tag{2.1}$$

The zero-coupon rate with maturity of 2 year at time t $z(t, t+2)$can then be obtained from the $z(t, t + 1)$and the two-year swap rate by:

$$r_{t,t+2}e^{-z(t,t+1)} + (1 + r_{t,t+2})e^{-2z(t,t+2)} = 1 \tag{2.2}$$

$$\Rightarrow z(t, t + 2) = -\frac{1}{2}log(\frac{1 - r_{t,t+2}e^{-z(t,t+1)}}{1 + r_{t,t+2}}) \tag{2.3}$$

All the zero coupon rate with maturity $u_{l(i)}(u_{l(i)} \leq 10)$can be obtained in the same way as above, namely based on the formula :

$$\sum_{j=1}^{l(i)} c_{ij}e^{-u_j z(t,t+u_j)} = 1 \tag{2.4}$$

because no market data is omitted for a given maturity before 10 years.

However, after 10 years,some swap rates does not exist for a given maturity, for instance the maturity of 11 years; hence interpolation is needed. This is based on the assumption that the intermediate one-year forward rate is constant. For instance, the value of $f(t, 10, 12)$ can be expressed by:

$$r_{t,t+12}e^{-z(t,t+1)} + r_{t,t+12}e^{-2z(t,t+2)} + ... + r_{t,t+12}e^{-11z(t,t+11)} + (1 + r_{t,t+12})e^{-12z(t,t+12)}$$

$$= r_{t,t+12}[\sum_{l=1}^{10} e^{-lz(t,t+1)} + e^{-10z(t,t+10)} \sum_{l=1}^{2} e^{-lf_c(t,10,12)}] + e^{-10z(t,t+10)+2f_c(t,10,12)} = 1. \tag{2.5}$$

And then the zero coupon rate with the maturities of 11 years and 12 years can be expressed by:

$$z(t, t+11) = \frac{10z(t, t+10) + f_c(t, 10, 12)}{11} \tag{2.6}$$

$$z(t, t+12) = \frac{10z(t, t+10) + 2f_c(t, 10, 12}{12}) \tag{2.7}$$

Other zero coupon rates with maturities larger than 10 years and less than 20 years can be obtained in the same way as the zero coupon rates with maturities of 11 years and 12 years.

2.1.2 Method to construct the extrapolated part of the yield curve

After the FSP, a new method based on the affine term strucutre model and Kalman filter is employed.

a) UFR

The UFR is determined by the moving average of the 20-year forward rate with the 1 year maturity over the previous 10 year, which is recalculated once a month by:

$$UFR(t) = \frac{1}{120} \sum_{m \in M(t)} f(m, 20, 21) \tag{2.8}$$

where $M(t)$ stands for the set of the 120 month-ends immediately prior to time t.

$$UFR_c(t) = In(1 + UFR(t)) \tag{2.9}$$

b) Affine term structure model

The affine term structure assumes:

$$z(t, l) = A(l) + B(l)x(t) \tag{2.10}$$

where

$$B(l) = \frac{1 - e^{-\alpha l}}{\alpha l} \tag{2.11}$$

The definition of $B(l)$ is based on the Vasicek (1977) model to ensure that the zero coupon rate converges to a stable level; αdetermines the convergence rate, $\alpha = 0.10$.

Hence, the corresponding forward rate can be derived from (2.10), that is:

$$f_c(t,u,u+l) = \frac{A(u+l)(u+l) - A(u)u}{l} + \frac{B(u+l)(u+l) - B(u)u}{l} x(t) \quad (2.12)$$

plug the formula (2.11) and assumes that the forward rate will converge to the UFR_C , the expression (2.12) becomes:

$$f_c(t,u,u+l) = UFR_C(t) + B(l)e^{-\alpha u}x(t) \quad (2.13)$$

Take the limit $l \to 0$, we get

$$F(t,u) = UFR_C(t) + e^{-\alpha u}x(t) \quad (2.14)$$

where $F(t,u)$is the the u-year continously compounded instantaneous forward rate valued at time t

Therefore,we have:

$$e^{-\alpha u}x(t) = F(t,u) - UFR_C \quad (2.15)$$

Plug (2.15) in (2.13),we have:

$$f_c(t,u,u+l) = UFR_C(t) + (F(t,u) - UFR_C)B(l) \quad (2.16)$$

c) Last liquid forward rate

The instantaneous forward rate $F(t,u)$ in the (2.16) cannot be observed from the market. The UFR regulator approximates it based on the Kalman filter theory, because the market data after the first smoothing point is not liquid enough. Data from different sources are used in order to better approximate the instantaneous forward rate$F(t{,}20)$.According to Langejan (2013), the last liqurd forward rate is estimated by using the zero coupon rates after the first smoothing point for maturities of 25,30,40 and 50 years. The first smoothing point that is recommended is 20 years, and the weights provided for 25,30,40 and 50 years are 8/15, 4/15, 2/15, 1/15 respectively.

$$f_c^*(t) = \beta f_c^*(t-1) + (1-\beta)w(f_c(t,20,25) + \frac{1}{2}f_c(t,20,30) + \frac{1}{4}f_c(t,20,40) + \frac{1}{8}f_c(t,20,50)) \quad (2.17)$$

where $f_c^*(t)$ is the estimation of the instantaneous forward rate $F(t,20)$, and is called last liquid forward rate. β is the weight of the previous day LLFR,w provides the unbiased weighted average of the post-FSP forward rate . The UFR committe sets $\beta = \frac{1}{2}$, $w = \frac{8}{15}$, $f_c^*(0) = UFR_C(t)$.

If the first smoothing point is changed from 20 years to 30 years, then there are only the zero coupon rates with maturities of 40 and 50 years for the estimation of the Future term structure sensitivity analysis. Since the weight are constantly halved with the decrease of liquidity, I set the weight of the 40 years to be twice of that of 50 years. Therefore, the instantaneous forward rate $F(t, 30)$ can be expressed as:

$$f_c^*(t) = \beta f_c^*(t-1) + (1-\beta)(2f_c(t, 20, 40) + f_c(t, 20, 50)) \tag{2.18}$$

where β is the weight of the previous day LLFR, which is still set to be equal $\frac{1}{2}$.

d) Post-FSP forward rate

Replace the instantaneous forward rate $F(t, 20)$ by the LLFR $f_c^*(t)$, according to the (2.16), the post-FSP forward rate $f_c(t, 20, 20+l)$ can be extrapolated by:

$$f_c(t, 20, 20+l) = UFR_C(t) + (f_c^*(t) - UFR_C(t))B(l) \tag{2.19}$$

e) Post-FSP zero coupon rate

The post-FSP zero coupon rate can be expressed as:

$$z(t, t+20+l) = \frac{20z(t, t+20) + lf_c(t, 20, 20+l)}{20+l} \tag{2.20}$$

2.2 Swedish UFR

The Euro swap rates with maturities of 1-10 years, 12 years, 15 years, 20 years will be used. The market forward rate $f(t, l-1, l)$ is calculated in the same way as in the Financial Assessment Framework, which is stated in the 'before the FSP ' of the CIE UFR section. The linear extrapolation method provides the forward rate for the discount curve:

$$f^*(t, l-1, l) = (1-w(l)).f(t, l-1, l) + w(l)UFR \tag{2.21}$$

where

$$w(l) = \begin{cases} 0, & l < FSP \\ \frac{l-FSP}{T_2-FSP+1}, & FSP < l \leq T_2 \\ 1, & l > T_2 \end{cases} \tag{2.22}$$

$UFR = 4.2$ denotes the time when the forward rate for the discount curve achieves the convergence level UFR.

Therefore, the zero coupon rate before the FSP constructed in the same way as the CIE UFR, while after the FSP, an linear extrapolation is employed. The discrete zero coupon rate$z_d(t, l)$ can be obtained by:

$$z_d(t, l) = ((1 + f(t, l-1, l).(1 + z_d(t, l-1))^{l-1})^{\frac{1}{l}} - 1 \tag{2.23}$$

For comparison purpose, the annually compounded zero coupon rate obtained will be transformed into the continuously compounded zero coupon rate:

$$z(t, l) = In(1 + z_d(t, l)) \tag{2.24}$$

2.3 German UFR

The Euro swap rates with maturities of 1-10 years, 12 years, 15 years, 20 years,25 years, 30 years, 40 years, 50 years will be used. The German discount curve is constructed by the sum of the average zero coupon rates of the last seven years and a surcharge.

2.3.1 Average zero coupon rate

The way to derive the zero coupon rate is the same as that of the Financial Assessment framework. The forward rate with maturity of 1 year after 50 years is also assumed to be equal to the 40-year forward rate with the maturity of 10 years. Hence, the average zero coupon rate can be expressed as:

$$\overline{z}(t, l) = \frac{1}{84} \sum_{k=t-83}^{t} z(k, l) \tag{2.25}$$

2.3.2 Surcharge

The surcharge is the same for all maturity, which is calculated as follows:

Let $t_k$denote the average maturity of the bonds of the corporate index, $\bar{t}_{aver}$ denote the average of $t_k$over the past seven years:

$$\bar{t}_{aver} = \frac{1}{84} \sum_{k=t-83}^{t} t_k \tag{2.26}$$

Let $U_k$denote the yield corporate bond index, then the average yield on the corporate bond index $\overline{U_k}$over the past 7 years is :

$$\overline{U}_{aver} = \frac{1}{84} \sum_{k=t-83}^{t} U_k \tag{2.27}$$

Let $z(t, \bar{t}_{aver})$ denote the zero coupon rate with the maturity of $\bar{t}_{aver}$ at time t. Hence, the surcharge for all maturities is :

$$A_{aver} = \overline{U}_{aver} - z(t, \bar{t}_{aver}) \tag{2.28}$$

In this thesis, the surcharge of the German UFR curve is approximated through a regression model. To capture the relation between the surcharge and the short rate, the regression model is assumed to be:

$$A_t = a + br_t + \epsilon_t, \tag{2.29}$$

which is assumed to satisfy the ordinary least squares conditions.The surcharge A_t is calculated according to (2.28).The Markit iBoxx EUR Corporates AA (ISIN: DE0006600083), both Redemption yields and average duration are taken from January 1999 to May 2014. After calculating the 84 months arithmetic mean of Redemption yields and average duration for each month, the zero-coupon rate with the same maturity as the 84 months arithmetic mean of the average duration can be taken from the website of Deutsche Bundesbank. The Euribor 3 months is taken to represnet the short rate r_t, which lasts from January 2005 to May 2014. After running the OLS regression in Matlab, I obtain:

$$a = 0.027, b = -0.6064.$$

The 95% confidence interval for a is [0.0258 0.0283], for b is [-0.6547, -0.5581]. p-value for a is 7.599*e-67, for b is 1.163e-44.

Hence, the coefficients are all significant. The surcharge of German curve will be approximated by:

$$\widehat{A}_t = \widehat{a} + \widehat{b}r_t \tag{2.30}$$

2.3.3 German Yield curve

The average of the zero coupon rate with the maturity of l over the past 7 years can be expressed as:

$$\overline{z}(t, l) = \frac{1}{84} \sum_{k=t-83}^{t} z(k, l) \tag{2.31}$$

The German yield curve is constructed by the sum of a surcharge and the average zero coupon rates over the past 7 years. In order words, the yield curve can be expressed by:

$$A_{aver} + \overline{z}(t, l) \tag{2.32}$$

where A_{aver}may be calculated according to (2.30) given the corresponding short rate.

2.4 French UFR

2.4.1 Method to construct the French yield curve

The French discount curve is constructed by the Vasicek-Fong method according to Institut des Actuaires (2001).

In order to fit the discount function by polynomial splines and keep the linear relationship between the price of the product and the discount function at the same time, the Vasicek and Fong suggested to tranform the argument of the discount function instead of the discount function itself. That is to say: Let

$$X = 1 - e^{-al}, for 0 \leq X < 1 \tag{2.33}$$

Then the A(X) can be defined by:

$$P(t,l) = P(-\frac{log(1-X)}{a}) = A(X) = (1-X)^{\frac{z(t,l)}{a}}, for 0 \leq X < 1 \tag{2.34}$$

The function A(X) turns out to be a power function. Hence, it can be fitted by the piecewise polynomial splines. The knot is :

$$X_c = 1 - e^{-al_c}, for 0 < X_c < 1 \tag{2.35}$$

We have:

$$A(X) = \begin{cases} P_1(X) = a_0 + a_1 X + a_2 X^2 + a_3 X^3, & for X \leq X_c \\ P_2(X) = P_1(X) + (X - X_C)^2 (b_0 + b_1 X + b_2 X^2 + b_3 X^3), & for X \geq X_c \end{cases} \tag{2.36}$$

Constraints that the discount function should meet:

When the maturity approaches infinity, the discount function should approaches 0.Therefore, we have:

$$P(\infty) = A(1) = P_2(1) = 0 \tag{2.37}$$

Besides, when the maturity is zero, the discount function should be equal to 1. Therefore, we have:

$$P(0) = A(0) = a_0 = 1 \tag{2.38}$$

The coupon paying bonds based on swap rates with the maturities of 1-10 years, 12 years, 15 years, 20 years,25 years, 30 years, 40 years, 50 years will be used. The market price of the coupon paying bond i can be expressed as the follows, which is equal to 1:

$$m_i = \sum_{j=1}^{t(i)} c_{ij} A(X(u_j)) \tag{2.39}$$

Constraints by the equation (2.4.4) and (2.4.5), the coefficients $(a_{i,}b_j)$can be solved by OLS to achieve the minimization of the:

$$\sum_{i=1}^{N}(m_i^{'} - m_i)^2. \tag{2.40}$$

So far, $A(X)$, namely the discount function $P(t,l)$, has been obtained. Accordingly, the zero coupon rate can be derived by:

$$z_c(t,l) = -\frac{1}{l}log(A(X)) \tag{2.41}$$

2.4.2 Extra assumption to the French UFR method

Since there is no illustration about how to determine the parameter a for the transformation function (2.33)in the Institut des Actuaires (2001), I follow the guidance from the original paper of Vasick and Fong (1982), namely parameter a is chosen to minimize the sum of square error of the regression (2.40). What's more, according to Vasick and Fong (1982), the parameter a is the limiting value of the long-term instantaneous forward rate.

Furthermore, there is no illustration about how to choose the knot X_c, which is equivalent to choose $l_c$in (2.35). Since the knot divide the discount curve into two part, I would assume that the knot is the point that enters to the extrapolation of the yield curve. Hence $l_c$is chosen to be equal to the FSP.

2.5 EIOPA UFR

2.5.1 Method to construct the non-extrapolated part of the yield curve

The non-extrapolated part of the EIOPA discount curve is provided by CRO Forum/ CFO Forum, based on the smoothed regression spline technique by Barrie and Hibbert. The aim is to better fit the market curve.

2.5.2 Method to construct the extrapolated part of the yield curve

The Smith-Wilson method is employed to extrapolate the term structure, which has the discount function of the following form:

$$P(t,t+l) = e^{-UFR_c.l} + \sum_{i=1}^{N} \zeta_i (\sum_{j=1}^{J} c_{ij} W(l,u_j)), l \geq 0 \tag{2.42}$$

with the symmetric wilson function:

$$W(l,u_j) = e^{-UFR_c(l+u_j)}.\{\alpha.min(l,u_j) - 0.5.e^{-\alpha.max(l,u_n)}.(e^{\alpha.min(l,u_n)} - e^{-\alpha.min(l,u_j)})\} \tag{2.43}$$

We have:

$$m_i = \sum_{j=1}^{J} c_{ij} P(u_j), i = 1,2,3,...N \tag{2.44}$$

Use the vector space notation, we have:

$m = (m_{1,}m_2, m_{3,}...m_N)^T$, m is a vector of one, which represents the price of the observed coupon paying bonds.

$p = (p(t,t+u_1), p(t,t+u_2), ...p(t,t+u_J))^T.p$ is a vector of the price of the zero-coupon default free bonds.

$\zeta = (\zeta_{1,}\zeta_{2,}..., \zeta_N)^T.\zeta$ is a vector of parameters to fit the known yield curve.

$\mu = (e^{-UFR_c u_1}, e^{-UFR_C.u_2}, ..., e^{-UFR_C.u_J})^T$.

$C$is the $N*J$cash flow matrix; W is the $J*J$ wilson function matrix.

From (2.44), we have:

$$m = CP \tag{2.45}$$

According to in (2.42), we may rewrite (2.45) as:

$$m = CP = C\mu + (CWC^T)\zeta \tag{2.46}$$

Hence:

$$\zeta = (CWC^T)^{-1}(m - C\mu) \tag{2.47}$$

So far, the discount function $p(l)$ has already obtained. Therefore, the zero coupon rate can be derived by:

$$z_c(t,l) = -\frac{1}{l} log(p(l)) \tag{2.48}$$

Notice that although the French UFR method does not explicitly illustrate that from which point it starts to extrapolate, neither does the German UFR method, I would compare the UFR methods based on the yield curve beyond 20 years. This is due to three other UFR methods-Dutch UFR, EIOPA UFR and Swedish UFR starts to extrapolate from 20 years. I would assume that the main difference between the UFR curves lies beyond the 20 years. Therefore, the UFR curves before 20 years are all constructed to be the same as the market yield curve for simplicity purpose in the following analysis. Besides, the point that starts to extrapolate the market yield curve will be called as the Frist smoothing point (FSP).

Chapter 3

Term structure model

In this chapter, the Vasicek term structure model is first illustrated, the parameters of which are then provided to approximately describe the Europe market. And then the shapes of the yield curve implied by the Vasicek model are explored with the proof offered in the Appendix 7.1 . We have already seen that Dutch and German UFR method depends on the past interest rate, hence it would be interesting to learn the past interest rate impact on the comparison of UFR implementation, which is analyzed at the end of this part.

The Vasicek model by Vasicek (1977) is chosen to describe the dynamic evolution of the short rate, from which the current market yield curve can be constructed and the future yield curve would be simulated. The first reason why I would like to use the term structure model instead of the realistic yield curve fitted by market data is because that the fitted market yield curve could not provide the zero coupon rate with the maturity beyond the liquid market data, without which I would not be able to compare the market consistency of the UFR curves.The second reason is that the term structure model makes the simulation of the future short rates become possible, which would be needed to answer the reasearch question about the comparison between the solvency risk. Besides, the Vasicek Model is chosen due to the tradeoff between simplicity and realistic description of the market yield curve.It is easy to simulate and also captures three common shapes of the market yield curve, hence makes it possible to compare the UFR implementation under different shapes of the yield curve and draw the general conclusion.

The Vasicek Model can be expressed by:

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t, \alpha > 0, \tag{3.1}$$

where W_t is Brownian motion under the real world measure.

Parameters	Value
a	24.75%
β	3.25%
σ	0.64%
λ	-15%

Table 3.1 – Values of Vasicek Parameters

$\lambda$is the market price of risk, which is assumed to be constant. This parameter expresses the compensation that the investors require for each unit of exposure to the interest rate risk according to Schumacher (2013) .

Vasicek (1977) gives the form of the term structure of interest rate in terms of yield:

$$z(t, t+T) = z(\infty) + [r(t) - z(\infty)]\frac{1}{\alpha T}(1 - e^{-\alpha T}) + \frac{\sigma^2}{4\alpha^3 T}(1 - e^{-aT})^2, T \geq 0 \quad (3.2)$$

where:

$$\lim_{T\to\infty} z(t, t+T) = z(\infty) = \beta - \frac{\lambda\sigma}{\alpha} - \frac{\sigma^2}{2\alpha^2} \quad (3.3)$$

3.1 Values for parameters of the Vasicek model

Since this thesis focuses on European solvency regulation. I take the value for the parameters of the Vasicek model from the Herrala (2009) , which calibrates the parameters of the vasicek model by using the one week Euro Interbank Offered Rate (Euribor) data from January 18,1999 to Nov. 18, 2008. As for the value of the market price of risk, I set it to be -15%, which is in line with the report by committee Don (2009).

3.2 Shape of the yield curve

In order to compare the UFR implementation under the general circumstances, the Vasicek Model is chosen to capture three shapes of the market yield curve-Increasing shape, humped shape and Decreasing shape. The intuition of the shape of yield curve and the proof of the shape of the Vasicek yield curve are offered, which offer the basis for the following construction of the different shapes of market yield curve.

The idea of proof of the shape of the yield curve is based on the derivative of yield function $z(t, t+T)$ with respect to the time to maturity T, denoted by $z'(t, t+T)$.

For the increasing yield curve , according to the Vasicek (1997), if the level of the current short rate $r(t)$ in the Vasicek model satisfies

$$r(t) \leq R(\infty) - \frac{1}{4}\frac{\sigma^2}{\alpha^2} \tag{3.4}$$

then the yield curve is monotonically increasing. Hence,$z'(t, t+T)$ will be shown to be positive for all $T \in [0, +\infty)$ if current short rate satisfies (3.4). This type of shape may be an indication of economic growth. The investors expect that the future interest rate will go up, therefore they require higher return for holding the long term interest rate product.

For the humped shape yield curve, namely the yield curve first slope upward and then downward, according to the Vasicek (1997), if the level of the current short rate $r(t)$ in the Vasicek model satisfies

$$R(\infty) - \frac{1}{4}\frac{\sigma^2}{\alpha^2} < r(t) < R(\infty) + \frac{1}{2}\frac{\sigma^2}{\alpha^2} \tag{3.5}$$

then the yield curve is a humped shape. Hence,$z'(t, t+T)$ will be shown to be first positive and then negative for $T \in [0, +\infty)$ if current short rate satisfies (3.5). According to the explantion by Fisher (2001) , this type of shape may be due to the risk-premium dominates at the beginning and then the expectation effect (together with the convexity effect) dominate.

For the decreasing yield curve, according to the Vasicek (1997), if the level of the current short rate $r(t)$ in the Vasicek model satisfies

$$r(t) \geq R(\infty) + \frac{1}{2}\frac{\sigma^2}{\alpha^2,} \tag{3.6}$$

then the yield curve is monotonically decreasing.Hence, $z'(t, t+T)$ will be shown to be negative for all $T \in [0, +\infty)$ if current short rate satisfies (3.6) .This type of shape may suggest that the future interest rate is expected to decline, which is often preceded with economic recession.

For the detail of the proof, please refer to the Appendix 7.2 .

3.3 Past interest rate impact analysis

As can be seen from the technical specification in Chapter 2, the Dutch UFR method and the German UFR method is related to the past interest rate. Therefore, it would be interesting to see if the past interest rate information can infuence the comparison of the valuation impact of different UFR methods. Here I assume three past short rate scenarios with an horizon of 10 years: increasing short rate scenario, constant short rate scenario and decreasing short rate scenario.According to the Schumacher (2013), in practice, the overnight rate is affected by various factors that term structure models usually do not aim to cover, so that the three-month rate is often considered to provide a better proxy for the short rate than the overnight rate. Therefore, in order to determine the starting value for the decreasing and increasing scenarios, I take the Euribor 3 months from January,01,1999 to June, 27, 2014 into account, which is presented in Figure 3.1 on page 19[1] The lowest short rate (about 0.1%) is chosen as the starting short rate for the increasing scenario, and the highest short rate (about 5.4%) is chosen as the starting short rate for the decreasing scenario. The evolution of the past short rate is assumed to be a linear line passing the starting short rate in ten years ago and the short rate at the current time.Providing the past short rate scenarios, the current UFR yield curve is then obtained by extrapolating the market yield curve according to the corresponding UFR method. Afterwards, the pension liabilities valued by different UFR yield curves are compared under three past short rate scenarios to learn the influence of the past interest rate on the comparison of the valuation impact of different UFR methods.

3.3.1 Construction of the current UFR yield curves based on the three past short rate scenarios

The humped shape yield curve is taken do the analysis. According to Section 3.1 and Section 3.2, the current short rate is chosen to be 3.62% to produce the humped shape market yield curve. Five UFR methods are applied to extrapolate the market yield curve from 20 years onwards, where three past interest scnarios are considered. Afterwards, the projected cash flows of the pension funds are discounted by the UFR curves.

When the past short rate is decreasing, the following UFR yield curves are obtained:

When the past short rate is constant, the following UFR yield curves are obtained:

When the past short rate is increasing, the following UFR yield curves are obtained:

[1] Source: www.homefinance.nl

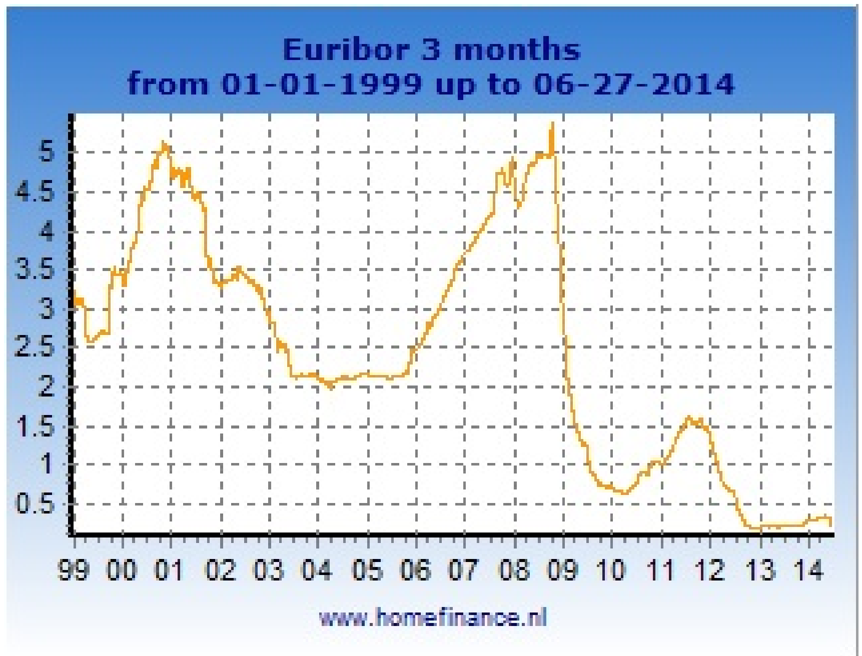

Figure 3.1 – Euribor 3 months

3.3.2 Past interest rate impact on the comparison of the valuation of the UFR methods

To check the impact of the past interest rate on the comparison of the UFR valuation, projected cash flows of pension liability are assumed. The projected cash flows of pension liabilities for the three pension funds-young pension fund, middle pension fund and old pension fund- are assumed to share the same amount of undiscounted cash flows 10 billion euros but different average durations. Young pension fund has the duration of 36. Middle pension fund has the duration of 22. Old pension fund has the duration of 13.You may find the cash flow patterns in the Appendix. The value of pension liability is obtained by discounting the projected cash flows of the pension funds by the UFR curves and market yield curve. Let L stands for liability. For comparison purpose, the elements in the table below are calculated by

$$\frac{L_{UFR} - L_{market}}{L_{market}} \tag{3.7}$$

As we may see from the table, there is nothing change for the pension liability calculated by the Swedish, EIOPA and French methods with the change of the past interest rate. This is because these extrapolation methods don't rely on the past

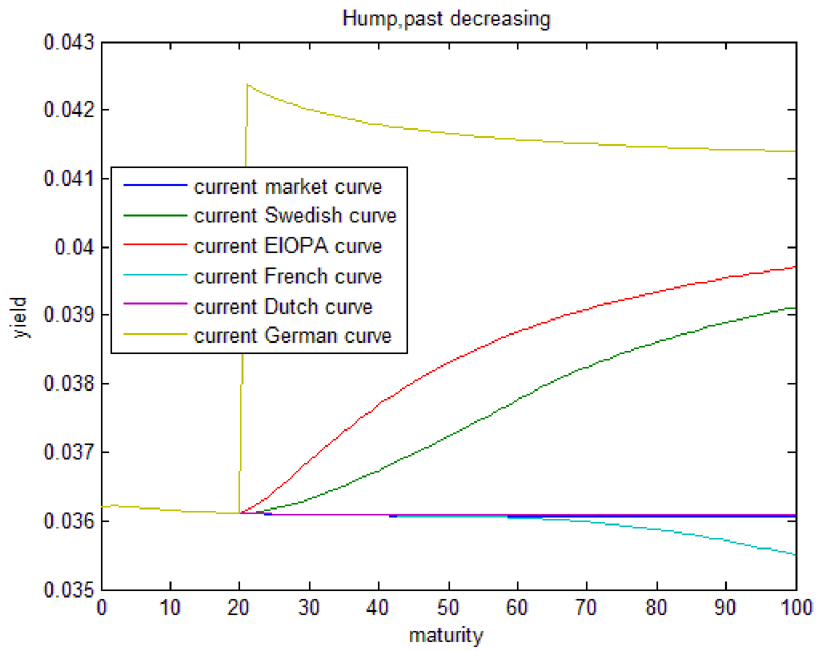

Figure 3.2 – Humped shape yield curve, past decreasing

past decreasing	Young fund	Middle fund	Old fund	UFR
Swedish	-2.16%	-0.52%	-0.15%	4.2%
EIOPA	-4.46%	-1.28%	-0.38%	4.2%
French	0.01%	0.00%	0.00%	2.4%
Dutch	-0.05%	-0.01%	0.00%	3.6%
German	-15.49%	-6.91%	-2.34%	3.6%

Table 3.2 – Past short rate is decreasing. Pension liability comparison under humped shape yield curve

past constant	Young fund	Middle fund	Old fund	UFR
Swedish	-2.16%	-0.52%	-0.15%	4.2%
EIOPA	-4.46%	-1.28%	-0.38%	4.2%
French	0.01%	0.00%	0.00%	2.4%
Dutch	0.00%	0.00%	0.00%	3.6%
German	-13.86%	-6.04%	-2.03%	3.6%

Table 3.3 – Past short rate is consant. Pension liability comparison under humped shape yield curve

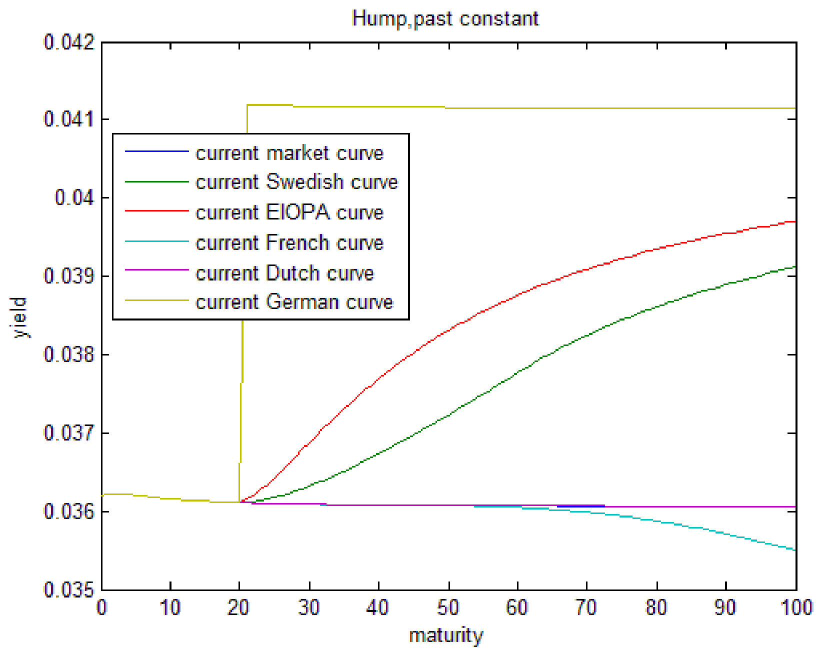

Figure 3.3 – Humped shape yield curve, past constant

past increasing	Young fund	Middle fund	Old fund	UFR
Swedish	-2.16%	-0.52%	-0.15%	4.2%
EIOPA	-4.46%	-1.28%	-0.38%	4.2%
French	0.01%	0.00%	0.00%	2.4%
Dutch	0.02%	0.02%	0.02%	3.6%
German	-4.25%	-4.25%	-4.25%	3.6%

Table 3.4 – Past short rate is increasing. Pension liability comparison under humped shape yield curve

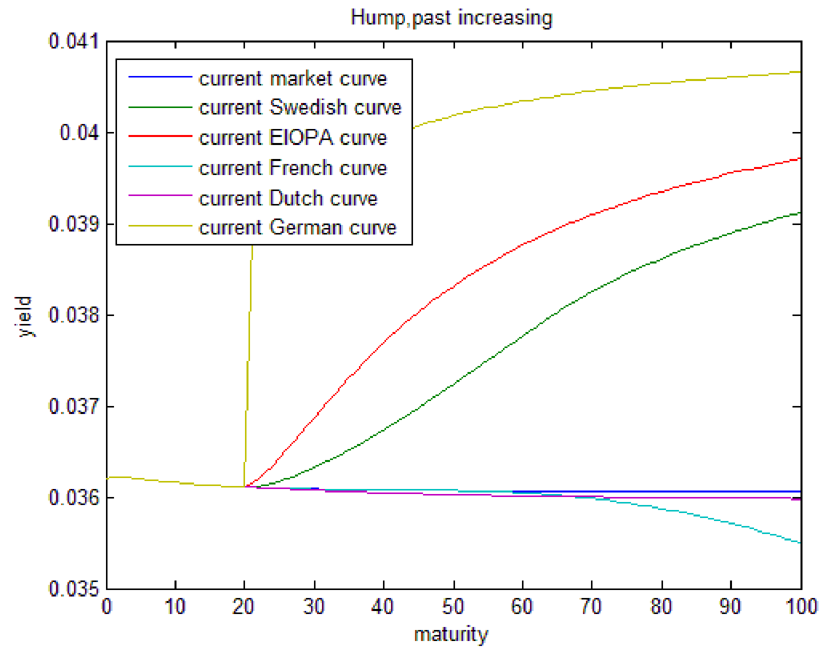

Figure 3.4 – Humped yield curve, past increasing

information. According to (2.8) and (2.17) in Chapter 2, the past interest rate may influence the valuation of liability through the determinations of the instantaneous forward rate and the ultimate forward rate.

According to the Table 3.2,3.3 and 3.4, the past interest rate does not have much influence on the Dutch UFR, which seems quite stable. What's more, the largest change for the Dutch pension liability is only 7 basis points(0.02%-(-0.05%)). Even though mathematically, the relation between French liability and Dutch liability change with the change of past interest rate. This change is only caused by the change of Dutch liability,which is so small (up to 0.07%) that we may say that the impact of the past interest rate on the comparison is ignorable when the market yield curve is humped shape.

In regard to the normal yield curve and inverse yield curve, the past doesn't change the order of the UFR liabilities valued by the corresponding UFR methods.Hence, they are not analyzed here.

As a result, we may draw the conclusion that the impact of the past interest rate on the comparison of valuation impact is ignorable. Therefore, I only assume the case where the past short rate remains constant.

Chapter 4

UFR method comparison in terms of the valuation impact

The current UFR yield curves are obtained by extrapolating the market yield curve from 20 years onwards, according to the UFR method technical specification in Chapter 2. To obtain the general conclusion, the comparison is taken under three types of shapes of market yield curves. Therefore, the UFR curves under three types of market yield curves are constructed firstly. Afterwards, the pension liabilities valued by different UFR yield curves are compared.

4.1 Construction of the current UFR yield curves for three shapes of market yield curve

According to Section 3.1and Section 3.2, the current short rate is chosen to be 4.5% to produce the inverted market yield curve. Five UFR methods are applied to extrapolate the market yield curve from 20 years onwards.

According to Section 3.1and Section 3.2, the current short rate is chosen to be 3.62% to produce the humped shape market yield curve. Five UFR methods are applied to extrapolate the market yield curve from 20 years onwards.

According to Section 3.1and Section 3.2, the current short rate is chosen to be 1% to produce the normal market yield curve. Five UFR methods are applied to extrapolate the market yield curve from 20 years onwards.

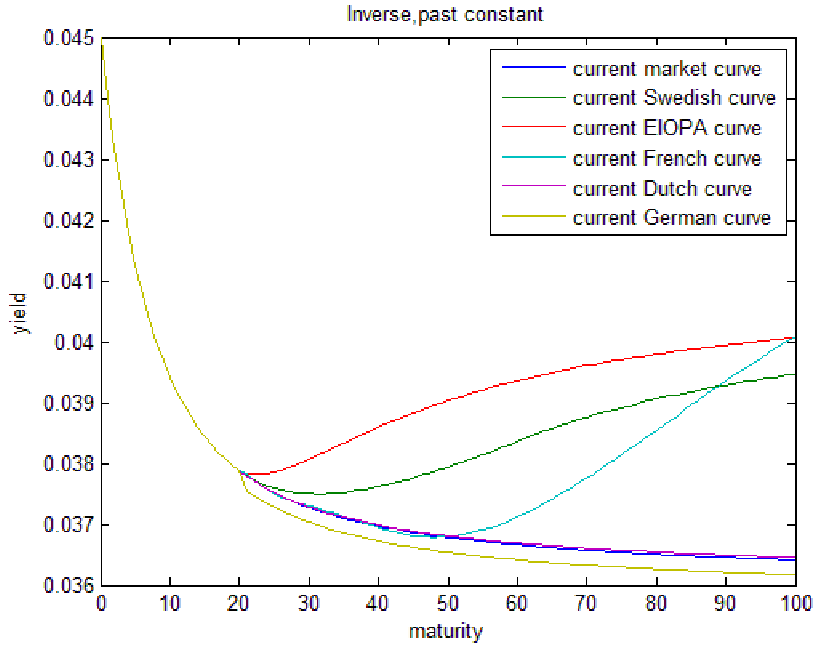

Figure 4.1 – Inverted yield curve

4.2 Valuation impact comparison in terms of pension liability

The value of pension liability is obtained by discounting the cash flows of the pension funds modeled above by the UFR curves. Let L stands for liability. For comparison purpose, the elements in the table below are calculated by:

$$\frac{L_{UFR} - L_{market}}{L_{market}} \tag{4.1}$$

Firstly,under all three shapes and all pension funds,the Swedish liability is smaller than the French liability.The market forward curve shows that the one-year forward rates beyond 20 years are lower than the 4.2%.The French method produces the same yield rates at the data points,interpolates and extrapolates according to that, while the Swedish method takes the linear combination of the market forward rate and the UFR (4.2%). Therefore, the Swedish curve could be higher than the French curve. Hence, the Swedish liability is lower than the French liability.

Secondly,under all three shapes and all pension funds, the Dutch liability is larger than EIOPA liability because Dutch UFR committee employs the moving average

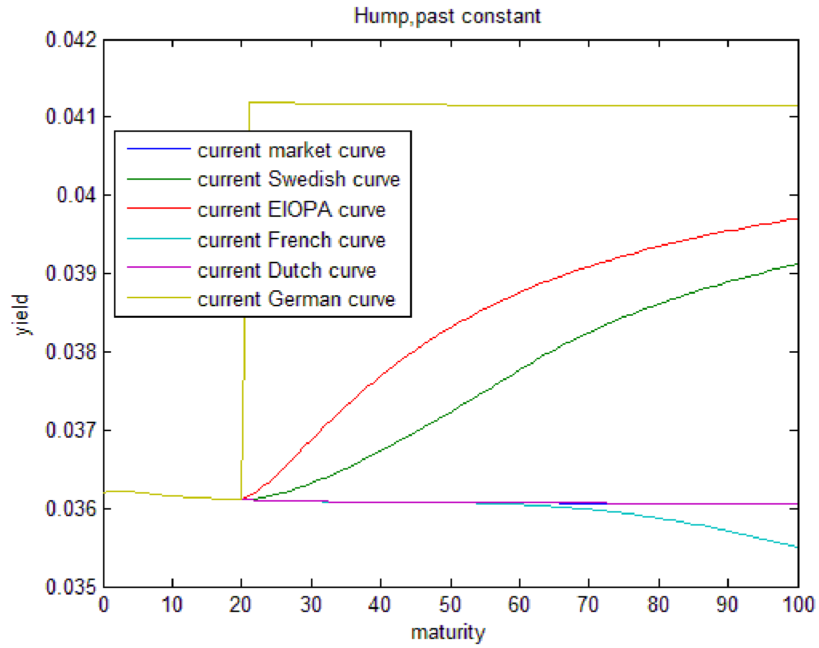

Figure 4.2 – Humped shape yield curve

r0=4.5%	Young fund	Middle fund	Old fund	UFR
Swedish	-2.15%	-0.52%	-0.15%	4.2%
EIOPA	-4.47%	-1.28%	-0.38%	4.2%
French	-0.22%	-0.03%	0.00%	4.9%
Dutch	-0.05%	-0.01%	0.00%	3.6%
German	0.74%	0.32%	0.11%	3.6%

Table 4.1 – Inverted yield curve pension liability comparison

r0=3.62%	Young fund	Middle fund	Old fund	UFR
Swedish	-2.16%	-0.52%	-0.15%	4.2%
EIOPA	-4.46%	-1.28%	-0.38%	4.2%
French	0.01%	0.00%	0.00%	2.4%
Dutch	0.00%	0.00%	0.00%	3.6%
German	-13.86%	-6.04%	-2.03%	3.6%

Table 4.2 – Humped yield curve pension liability comparison

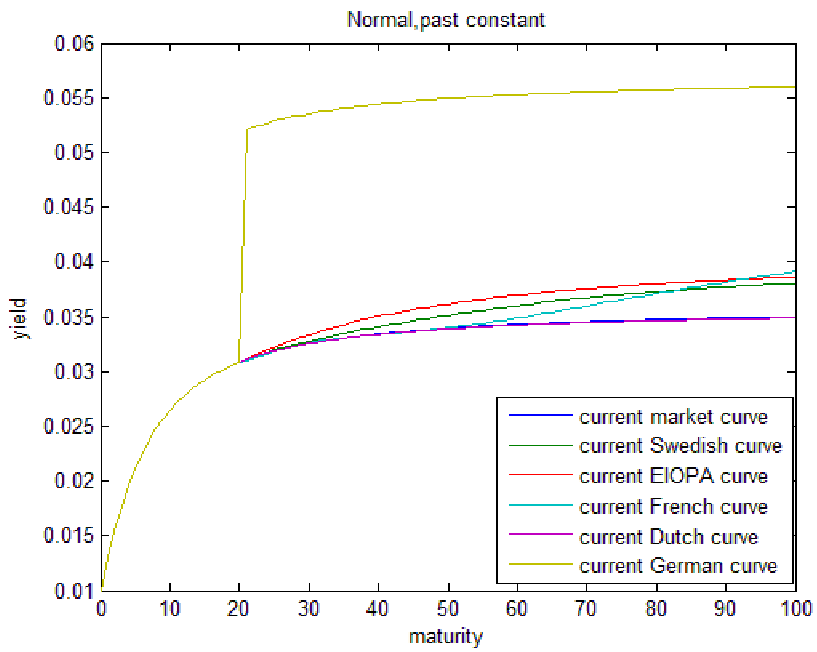

Figure 4.3 – Normal yield curve

r0=1%	Young fund	Middle fund	Old fund	UFR
Swedish	-2.17%	-0.53%	-0.15%	4.2%
EIOPA	-4.40%	-1.27%	-0.38%	4.2%
French	-0.24%	0.00%	0.01%	4.9%
Dutch	0.13%	0.03%	0.01%	3.6%
German	-42.89%	-19.74%	-6.82%	3.6%

Table 4.3 – Normal yield curve pension liability comparison

of the 20-year forward rates to approximates the Ultimate forward rate, which is indeed quite stable and a closer approximation of the very long term forward rate implied by the Vasicek Model: 3.6%. As for the EIOPA UFR method, 4.2% is set to be higher than the long-term forward rate implied by the market yield curve. The relatively higher UFR value for EIOPA method may lead to a higher yield curve than the Dutch one, which explains the reason why Dutch liability is always higher than EIOPA liability. Besides, the lack of market data beyond the first smoothing point (20years) might lead to over dependence on the UFR for the EIOPA method.

Thirdly, under all three shapes and all pension funds, the EIOPA liability is smaller than Swedish liability. This may be due to the fact that the Swedish method still employs the market data after the FSP, while the EIOPA method doesn't. Therefore, the influence of the UFR is larger for the EIOPA. Their UFR is higher than the market UFR; therefore, the larger influence the UFR, the higher the UFR yield curve could be.

For the comparison between Swedish liability and EIOPA liabillity, one finding may worth mentioning. For the Swedish curve, the extrapolation part is the linear combination of the swap curve forward rate and the UFR, while for the EIOPA UFR, its extrapolation part is the linear combination of the FSP forward rate and the UFR according to Kocken.et.al. (2012). If we look at the one-year forward rate curve for the normal market yield curve, we may observe that the one-year forward rate of the swap curve beyond the First smoothing point (20 years in this case) increases. Since both curves share the same UFR (4.2%), the EIOPA curve will be lower than the Swedish curve due to the lower forward rate, if we assumes that the weight of EIOPA assigned to the UFR is the same as that of Swedish assigned to the UFR. That is to say that the EIOPA liability will be higher than the Swedish liability under the normal market yield curve. However, we may observe that for the EIOPA liability is smaller than the Swedish liability under all three shapes and all pension funds. it may not be a good idea to assume the Smith-Wilson one-year forward rate as the linear combination of the one-year forward rate at the FSP and the UFR when comparing with the Swedish one-year forward rate.

$$Forward_{t-1,t}^{SW} = (1-w)Forward_{FSP-1,FSP} + wUFR \tag{4.2}$$

$$Forward_{t-1,t}^{Swedish} = (1-w)Forward_{t-1,t} + wUFR \tag{4.3}$$

where$t > FSP$.

The relationship mentioned above exists for three different shapes and three pension funds, therefore it is expected that this relationship will remained unchanged with a high probability in general situation.

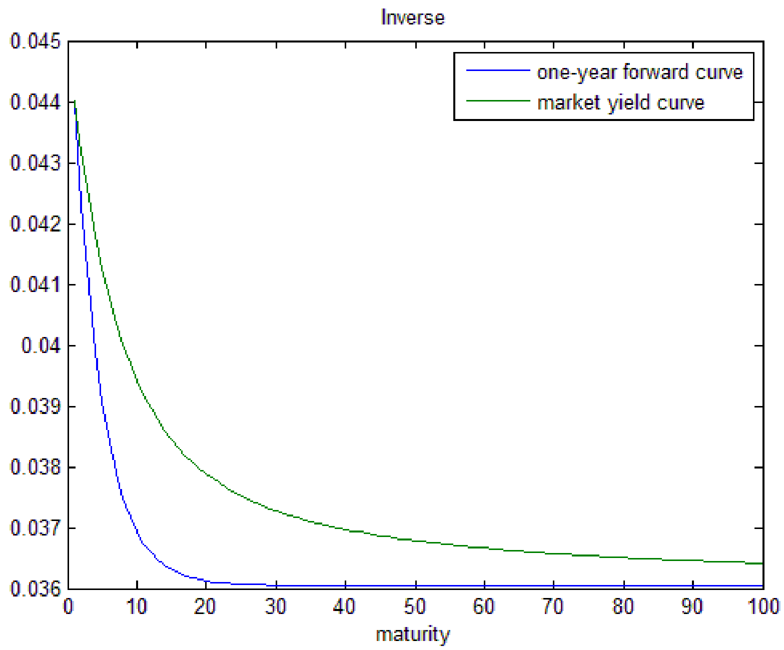

Figure 4.4 – One-year swap forward curve for the inverted yield curve

There are no material difference between the Dutch liability and French liability. Both of them are quite close to the market liability. This could be due to the regression method that the French UFR method takes to fit the market curve is better than those whose UFR are based on the expert opinion. The UFR of the Dutch method is determined by the moving average of the 20-year forward rate over the past ten years. Using the 120 months long-term forward rate (20 years) to approximate Ultimate forward rate looks more reliable and transparent.

Besides, compared with other UFR methods, the French UFR value varies a lot, which is quite unstable. On the contrary, other UFR methods remain nearly unchanged for different shapes of yield curves.

Finally, the change of the liability for the young pension fund is the largest, followed by the middle pension fund and then the young pension fund. This can be explained by the different average duration between these three pension funds. For instance, the young pension fund is influenced most by the extrapolation methods, because it has the longest duration.

Summary

In regard to the pension liability value, it can be expected that three events may occur with a high probability given that the EIOPA and Swedish UFR is higher than

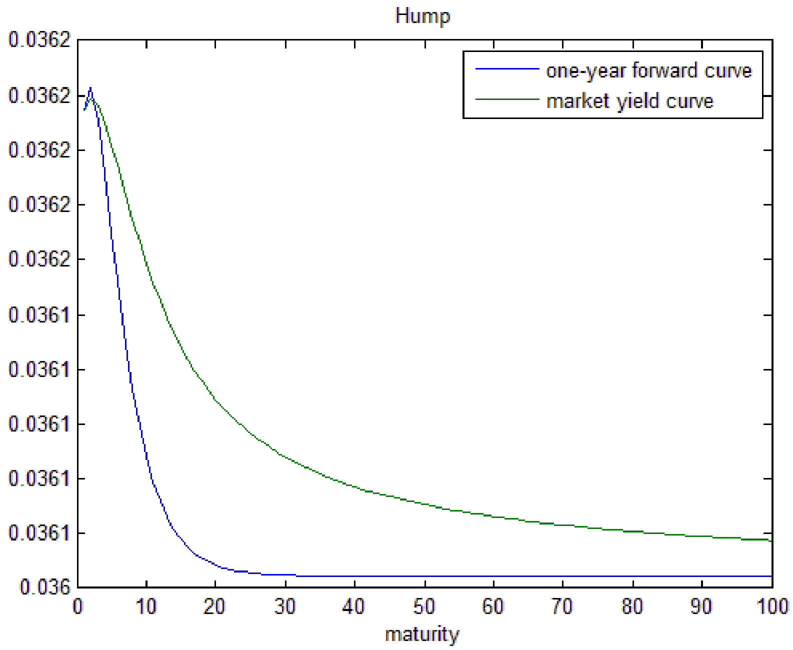

Figure 4.5 – One-year swap forward curve for the humped yield curve

the market UFR.The first event is that the Swedish pension liability is smaller than the French liability. The second event is that the Dutch pension liability is larger than the EIOPA liability. The last event is that the EIOPA liability is smaller than the Swedish liablity.Difference in the UFR liability values may be measured by the difference between the corresponding elements in table. For instance, the difference of liability value can be up to 2.32%, which is obained by the -2.15%-(-4.47%) from the table.

In regard to the market consistency comparison, the market consistency level of Dutch and French liabilities are higher than those of Swedish and EIOPA.According to the equation, the elements in the table reflects the market consistency of the corresponding UFR liability, which show that the absolute deviation of the UFR liability and market liability ranges from 0% to 4.47% (Table -) depends on different UFR methods and different average duration of pension fund.

In regard to the characteristic of the UFR value, the French UFR value varies a lot, which is quite unstable. On the contrary, other UFR methods remain nearly unchanged for different shapes of yield curves. This can be seen from the last column labeled by UFR in Table.

Furthermore, it is found that the differences of the UFR valuation impact increase

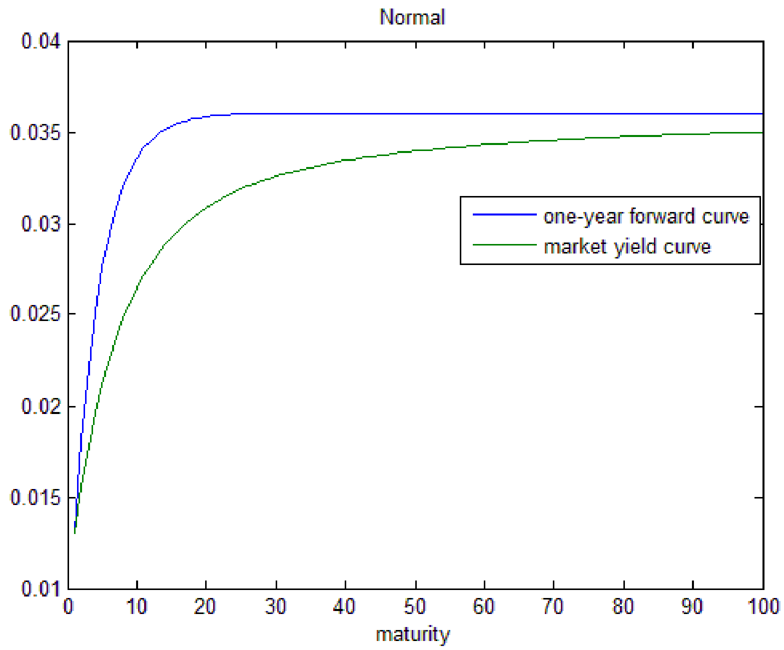

Figure 4.6 – One-year swap forward curve for the normal yield curve

with the increase of the average duration.

Chapter 5

UFR method comparison in terms of the solvency risk

Different extrapolation method may lead to different solvency risk, which would be measured by the solvency capital requirement (SCR) in this thesis. SCR is the amount of fund that the insurers and reinsurers need to hold to make sure they can meet their obligations in the coming year with the confidence level of 99.5%. In this thesis, only interest rate is considered; therefore only the interest rate risk module of SCR is calculated. Two methods to calculate the solvency capital requirement (SCR) for the interest rate risk are illustrated- the simulation method and the standard formula in Solvency II.And then the UFR methods are compared in terms of the SCR. Finally, in order to obtain robust conclusions, sensitvity analysis with respect to the input parameters of the UFR methods , the Vasicek Model parameters and interest rate risk hedging level is conducted.

5.1 Calculation of solvency capital requirement (SCR)

According to Solvency II, the solvency capital requirement corresponds to the value at risk of the basic own funds of insurance or reinsurance companies with the confidence of 99.5% over a one-year period. The basic own fund is equal to asset minus liability.In this thesis, only the interest rate risk is considered; therefore only interest rate sensitive assets and liabiliities are included in the solvency risk analysis.The cash flows of the interest rate sensitivity asset are assumed to be exactly the same as the cash flows of the pension liability for each pension fund to realize that the interest rate risk is fully hedged.

Assuming the solvency capital requirement (SCR) is invested in the one-year risk-free asset (P.Devolder, 2011), according to the definition of Solvency II, SCR can then be solved from:

$$PA_{t+1} + SCR.e^{z(t,t+1)} < L_{t+1} = 1 - 99.5\% \tag{5.1}$$

Equivalently

$$PL_{t+1} - A_{t+1} \leq SCR.e^{z(t,t+1)} = 99.5\% \tag{5.2}$$

Where$t$ is the current time, L_{t+1} and $A_{t+1}$is the liability and asset in the next year. From 5.2 , $SCR.e^{z(t,t+1)}$is the 99.5% quantile of the loss distribution $F(L_{t+1}-A_{t+1})$,let $q_{99.5\%}(F) = SCR.e^{z(t,t+1)}$.

The SCR can be calculated through Monte-Carlo simulation. 10,000 scenarios of short rate are simulated using the Vasicek (1977) Model. Since the typical term structure model is monotonically increasing, the initial value of the short rate is set to be 0.01 with the Vasicek parameter values given in Section 3.1. For each scenario, the market yield curve may be obtained by 3.2 . Afterwards, UFR yield curves are derived by applying the corresponding UFR method to extrapolate the market yield curve. The projected liability cash flow of each pension fund may be valued by the UFR yield curves, from which we may get the corresponding UFR liability value. Since the asset is valued based on the mark-to-market value, the market yield curve is applied to value the projected cash flow of the asset for each pension fund. So far, the UFR liability value L_{t+1} and asset value $A_{t+1}$for each scenario have been obtained. Then the 99.5% quantile of the loss distribution $L_{t+1} - A_{t+1}$can be estimated by ranking the $L_{t+1} - A_{t+1}$values for 10,000 scenarios in ascending order and choose the 99.5th observation as the estimation for the corresponding SCR.

The SCR can also be calculated according to the standard formula given by the EIOPA(2012b). It is based on two stress scenarios: the upward interest rate shock and the downward interest rate shock. The value of SCR is determined by the type of shock that requires the higher the highest capital requirement including the loss absorbing capacity of technical provisions. The loss absorbing capacity means that the the insurer can change the future bonus rates in response to the shock CEIOPS(2012b). In this thesis, we assume that the insurer does not have this capacity. Therefore, we are calculating the gross SCR .

Specifically, the current market yield curve is constructed according to 3.2 by setting the current short rate to be 0.01 as in the Monte Carlo simulation method. Afterwards, the current UFR curve is obtained by employing the corresponding UFR method. And then the altered term structures for each yield curve including the current market yield curve are derived by multiplying the current yield curve by $(1 + s^{up})$ and $(1 + s^{down})$,where $s^{up}$and $s^{down}$are the value of upward shock and that of downward shock provided by the EIOPA (2012b). Note the absolute change in the yield is required to be at least one percent, hence for those unstressed yield lower than 0.01, the doward stressed yield is assumed to be 0. For each UFR method, the asset is valued by the market yield curve while the liability is valued by the corresponding UFR curve. For instance, to calculate the SCR of the Dutch UFR method, the following steps are taken:

1. Calculate the net asset before shock BOF_b: asset valued by the current market yield curve minus the liability valued by the current Dutch UFR yield curve.
2. Calculate the net asset after downward shock BOF_a^{down}: asset valued by the downward market yield curve minus liability valued by the downward Dutch UFR yield curve.
3. Calculate the net asset after upward shock BOF_a^{up}: asset valued by the upward market yield curve minus liability valued by the upward Dutch UFR yield curve.
4. Determine the SCR: $SCR = max(BOF_b - BOF_a^{down}, BOF_b - BOF_a^{up})$.

5.2 Comparison of SCR calculated by simulation method for different UFR method

The impact of different UFR implementation is compared with respect to the solvency capital requirement, which is calculated according to the simulation method illustrated in the Section 5.1. The reason why I would like to use the simulation method to calculate the SCR is due to the fact that Vasicek model makes the simulation of future market yield curve become possible, hence it looks more transparent and understandable to calculate the SCR than simply using the standard formula provided by the EIOPA. Using the simulation method to calculate the SCR for different UFR method is based on the 5.2. Specifically, for each UFR method, the corresponding SCR is calculated according to:

$$PL_{t+1}^{UFR} - A_{t+1}^{market} \leq SCR^{UFR}.e^{z(t,t+1)} = 99.5\% \tag{5.3}$$

which is the amount of money that the insurers and reinsurers need to hold to make sure that they can meet their pension obligation in the next year with the confidence level of 99.5%.

The result of the SCR for different UFR method is presented in the figure below.The value of SCR are percentages in terms of the liability valued by the current market yield curve, which is for the convenience of comparison. Besides, you may observe that some SCRs shown below are negative. Although it is required by the EIOPA that when you get a negative value for SCR, you need to make it equal to

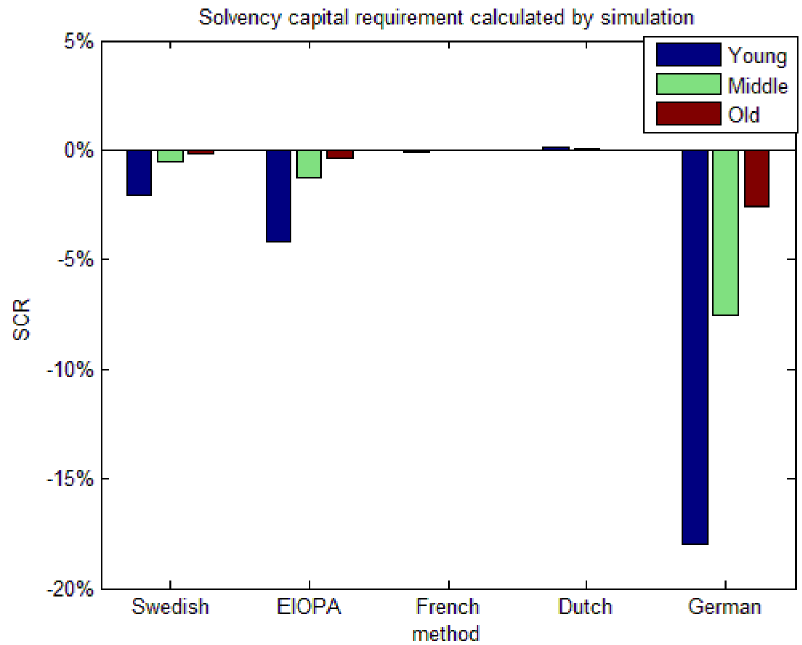

Figure 5.1 – Solvency capital requirement calculated from the simulation method

zero, I would still keep the negative value for SCR. Because even though the different UFR methods may lead to different SCR, replacing the negative SCR value to zero will cover this difference, which may be important.

The existence of negative SCR is due to my assumption that the cash flows of liabilities are perfectly matched by those of assets. According to the simulation calculation method, when the UFR yield curve is higher than the market yield curve, then the liability is lower than the asset. As a result, the SCR becomes negative. For the EIOPA method and Swedish method, their UFR value is set to be 4.2%, which is higher than the market UFR 3.6%. This leads to the higher EIOPA yield curve and Swedish yield curve, and hence the negative SCR value.

Analysis:

As one can be seen from the figure, the EIOPA requires the lowest capital requirement, therefore it provides the lowest policy protection among all the UFR methods concerned. Because the lower the solvency capital requirement, the later the supervisory intervention. This may be due to the relatively high ultimate forward rate and no market data after the 20 years. In our model, the market yield curves are all shared the same long-term yield rate and long-term forward rate, which is about 3.6%. Hence,on one hand, setting the ultimate forward rate to be 4.2% is relatively higher than the market. On the other hand, no market data after the first smoothing point (20 years) makes the EIOPA curve has a higher probability to deviate further away from the market curve. Therefore, this sort of optimism may be dangerous.

The solvency capital requirement for the Swedish is higher than the that for EIOPA. The same overestimate of the UFR (4.2%) may be corrected by the market data beyond the FSP, which drives the Swedish curve dowards.

Besides, the value of market SCR is zero according to my fully cash flows matching assumption;therefore, the closer to the horizontal line in the figure, the closer to the market SCR. It is observable that both the French SCR and Dutch SCR are quite close to the market SCR, which means that they have a high level of market consistency.This could be due to the regression method that the French UFR method takes, which aims to fit the market curve using the market data only. As for the Dutch method, its UFR value is determined by the moving average of the 20-year forward rate over the past ten years, which is also another approach to extrapolate using the market data only.

The French and Dutch SCR underestimate the market SCR, which may be explained by the too high UFR value they used, as illustrated above. Since the French and Dutch SCRs has a high level of market consistency and the EIOPA and Swedish SCRs is lower than the market SCR, we may get the conclusion that the EIOPA and Swedish SCRs are lower than the Dutch and French SCR.

Finally, we can observe that the variance of the SCR for the young pension fund is the highest, then followed by the middle pension fund. As for the old pension fund, it is hard to observe the difference. Besides,the solvency capital requirement of the old pension fund changes the least by changing the market yield curve to the UFR curves for all UFR methods, then followed by the middle pension fund. The young pension fund has the largest change.This is again due to the different average duration of the pension fund. The longer duration, the larger influence by the extrapolation methods.

Conclusion:

1. The solvency risk of pension fund employing EIOPA UFR method and Swedish UFR method is lower than that of French method and Dutch method, given that the UFR value of EIOPA and Swedish is chosen to be higher than market UFR.Besides, the SCR of Swedish UFR is higher than that of EIOPA UFR, which is about 2%.

2. The solvency risk of old pension fund employing UFR methods is the closest to market solvency risk, and it varies the least among different UFR methods. In regard to the young pension fund, its solvency risk employing UFR methods deviates the most from the market solvency risk, and it varies the most among different methods. The the solvency risk of middle penison fund lies between the young pension fund

and the old pension fund.

5.3 Comparison of SCR calculated by standard formula

The standard formula of calculating the SCR is provided by the EIOPA in the solvency II.In this section, the impact of different UFR implementation is compared with respect to the solvency capital requirement calculated by the standard formula. The reason why I include the standard formula calculation is to compare the result with that obtained by the simulation part. The calculation of SCR of the UFR method based on standard formula is illustrated in section 5.1, more specifically:

$$SCR = max(A_t^{market} - L_t^{UFR} - (A_t^{downmarket} - L_t^{downUFR}), A_t^{market} - L_t^{UFR} - (A_t^{upmarket} - L_t^{upUFR}))$$

where $A_t^{downmarket}$ represents the asset valued by the downward market yield curve, $L_t^{downUFR}$represents the liability vlaued by the downward UFR yield curve. Other variables may be explained in the same way.

The result of the SCR for different UFR method is presented in the figure below.

Compared with the SCR calculated by the simulation method, it is observable that the Swedish SCR and EIOPA SCR are larger than the French SCR and Dutch SCR. This is due to the standard formula also takes into the current liability into account. As is already analyzed in the Chapter 5, the Swedish and EIOPA current liabilities are lower than the French and Dutch current liabilities given a UFR higher than the the very long-term forward rate implied by the market curve. Since the assets of all UFR curve measured by the same market yield curve, the current net assets of EIOPA and Swedish are higher than those of French and Dutch method. Even though after the shock, the net assets of EIOPA and Swedish valued are higher than those of French and Dutch method, it is possible that the Swedish and EIOPA UFR insurers are required to hold more capital than French and Dutch UFR insurers. So reasoning may be applied to the relationship between Swedish SCR and EIOPA SCR.

As for the market consistency, we may observe that the French UFR method and the Dutch UFR method are still quite market consistency, and they are more closer to the market SCR than the Swedish and EIOPA method.

The variance of the young pension fund SCR is still higher than that of the middle penison fund SCR. The variance of the old pension fund SCR is the smallest.

Summary: The comparison of solvency risk is measured by the solvency capital requirement. It turns out that the result of comparison may be different for different

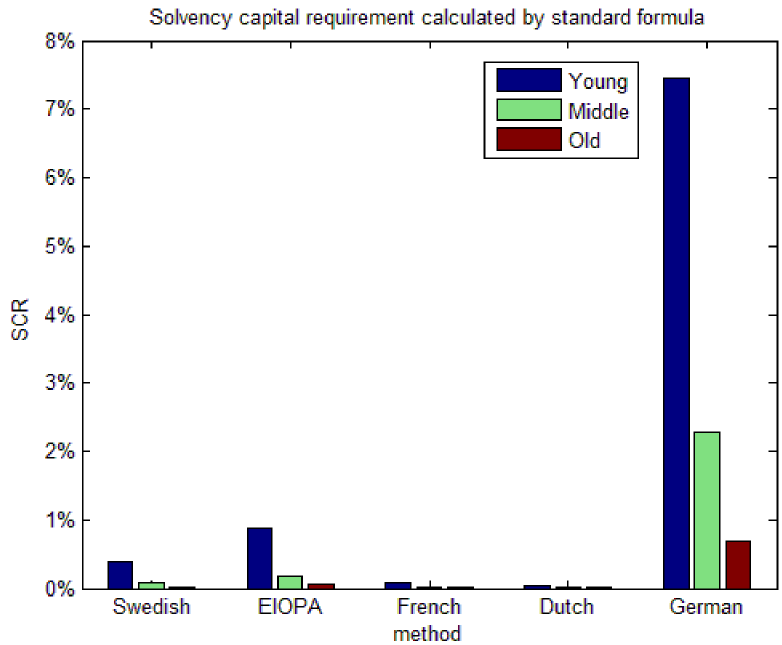

Figure 5.2 – Solvency capital requirement calculated from the standard formula

way to calculate the SCR. In regard to the market consistent level of SCR and the variance of different pension funds, both methods lead to the same result.

5.4 Sensitivity analysis

The extrapolation methods are mainly influenced by three parameters: the first smoothing point, the convergence period, the value of the ultimate forward rate.Other factors may affect the comparison result as well. Therefore, the sensitivity analysis will be conducted not only with respect to the three extrapolation parameters but also with respect to the very long-term forward rate implied by the market yield curve, the interest rate partial hedging strategy and the Vasicek model parameter.

5.4.1 Level of the ultimate forward rate

The EIOPA admits the fact that setting the ultimate forward rate to be 4.2% is a simplifying assumption; hence it would be interesting to take the sensitivity analysis with respect to the level of the ultimate forward rate. Among five UFR methods, two methods-Swedish method and EIOPA method- employ the 4.2% ultimate forward rate assumption. Therefore, changing the value of UFR 4.2% would cannot change the solvency capital requirement of other methods.The market UFR is defined as the limiting value of the one-year forward rate implied by the market yield curve.The sensitivity analysis of UFR aims to find out the impact of the choice of ultimate forward rate on the comparison of different solvency capital requirement corresponding to different UFR methods.

The sensitivity analysis is taken by changing the UFR value from 3.4% to 4.4% with 10 basis points as one step. The SCRs of UFR methods are obtained correspondingly. They are compared according to the following equation. The results are shown in the figure 5.3, 5.4 and 5.5.

$$\frac{SCR_{Swedish} - SCR_{French}}{L_{market}} \tag{5.4}$$

$$\frac{SCR_{Swedish} - SCR_{Dutch}}{L_{market}} \tag{5.5}$$

$$\frac{SCR_{EIOPA} - SCR_{French}}{L_{market}} \tag{5.6}$$

$$\frac{SCR_{EIOPA} - SCR_{Dutch}}{L_{market}} \tag{5.7}$$

$$\frac{SCR_{Swedish} - SCR_{EIOPA}}{L_{market}} \tag{5.8}$$

As can be seen from the figures, the absolute value of slopes in the young pension fund are all larger than those of their counterparts in the middle pension fund. The absolute value of slopes in the old pension is the smallest. Therefore, the sensitivity with respect to the UFR value of the difference in SCR decreases with the decline of pension fund average duration. This is because the longer the average duration, the larger the influence by the extrapolation method.

It is observable that when the UFR value is chosen to be around the market UFR, the SCR for different UFR methods are nearly the same.Actually, there is no material difference within the range of market UFR plus and minus 10 basis points. The reason why the UFR value is chosen to be around the market UFR, the closer the SCR of different UFR curves is because that the Dutch and French SCR are relatively closer to the market SCR.The closer UFR is to the market UFR, the closer SCR of Swedish and French UFR to the market SCR, and hence to the Dutch and French UFR. The existence of this range is due to the fact that the value of UFR is not the only difference between all those UFR methods.

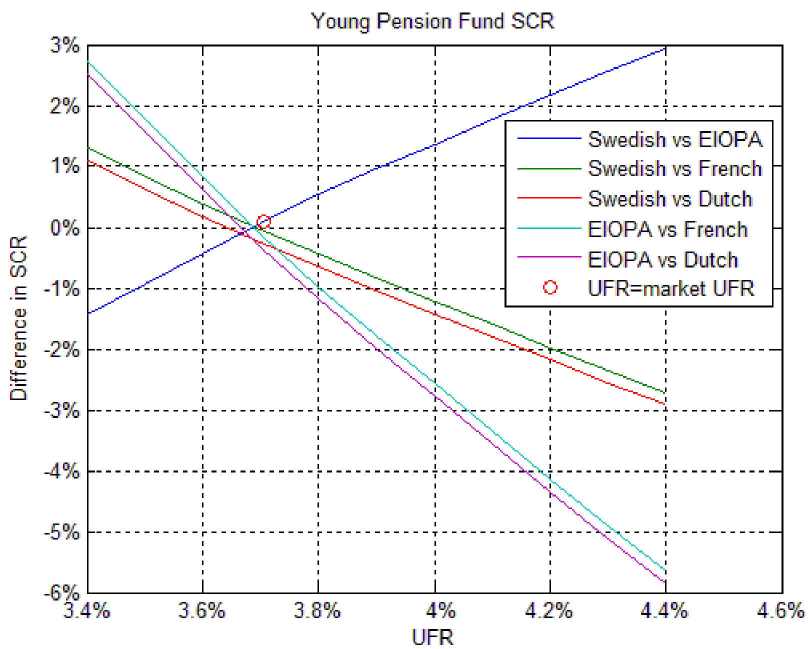

Figure 5.3 – Sensitivity analysis with respect to the value of Swedish and EIOPA UFR of the difference in Solvency Capital Requirement for young pension fund

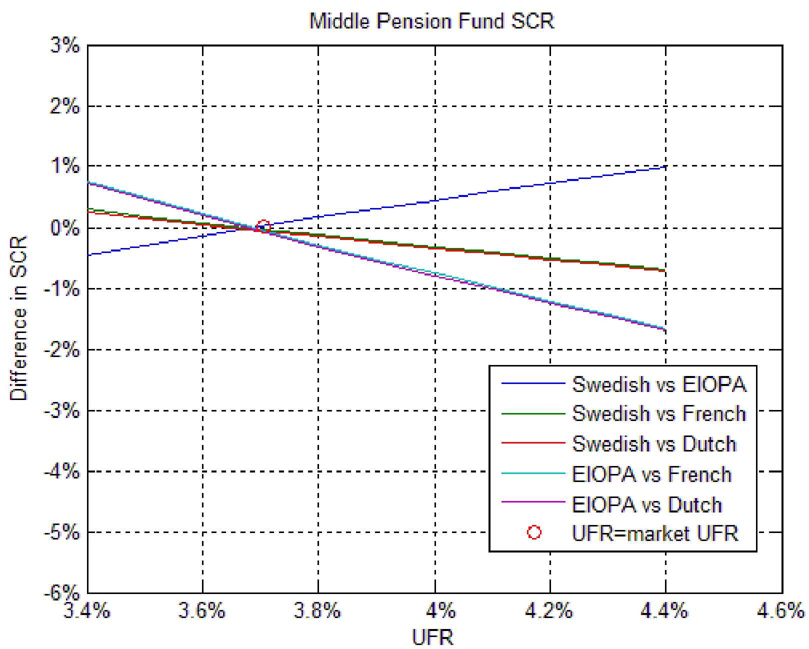

Figure 5.4 – Sensitivity analysis with respect to the value of Swedish and EIOPA UFR of the difference in Solvency Capital Requirement for middle pension fund

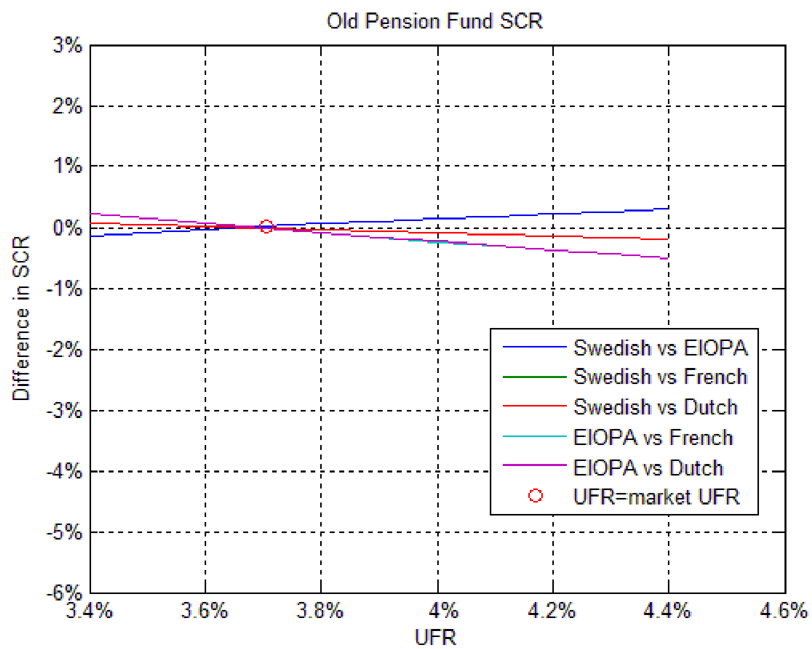

Figure 5.5 – Sensitivity analysis with respect to the value of Swedish and EIOPA UFR of the difference in Solvency Capital Requirement for old pension fund

When the UFR value is chosen to be at least 10 basis points higher than the market UFR, the Swedish and EIOPA SCRs are all lower than the French and Dutch SCRs. In order to reach the relatively higher UFR value, the EIOPA yield curves and the Swedish yield curves have to be relatively higher than the French yield curve and Dutch yield curve, which decrease the Swedish and EIOPA pension liabilities.Since the interest rate risk hedge are the same for all UFR methods, the lower the pension liabilitity in the next year could be , the lower the solvency capital requirement.

Furthermore, the SCR of Swedish is higher than the SCR of EIOPA, which is consistent with the conclusion that is drawn in the 5.2. This is because the Swedish UFR method still uses the market data beyond the first smoothing point, which reduce the error caused by the too high UFR setting, and hence drives the Swedish yield curve relatively lower than the EIOPA yield curve. As a result, the Swedish liability is higher than the EIOPA liabiltiy, which explains the relativley higher SCR of Swedish.

When the UFR value is chosen to be at least 10 basis points lower than the market UFR, the Swedish and EIOPA SCRs are all higher than the French and Dutch SCRs. Same reasoning can be applied here as in the case for the higher UFR value.

Besides,the SCR of Swedish is lower than the SCR of EIOPA. This is because the market data beyond the first smoothing point reduce the error casued by the too low UFR setting, which makes the Swedish yield curve relatively higher than the EIOPA yield curve. Following the same reasoning, the Swedish SCR is lower than the EIOPA SCR. Therefore, the relation between SCR and EIOPA also depends on the value of UFR.

Conclusion:

1. When the ultimate forward rate value is set to be around the market UFR, that is within the market UFR plus or minus 10 basis points, the SCR for different UFR methods can be regarded as no difference. When the UFR is set to be relatively higher, than the SCRs of Swedish and EIOPA are lower than the SCRs of Dutch and French. Besides, the Swedish SCR is higher than the EIOPA SCR. When the UFR is set to be relatively lower, than the SCRs of Swedish and EIOPA are higher than the SCRs of Dutch and French. Besides, the Swedish SCR is lower than the EIOPA SCR.

2. The sensitivity with respect to the UFR value of the difference in SCR decreases with the decline of pension fund average duration.Also, for each UFR value, we may observe that the differences of SCR for different UFR method increase with the increase of the average duration.As one can see, the differences in the young pension fund are the largest. This is consistent with what is found in Section 5.2.

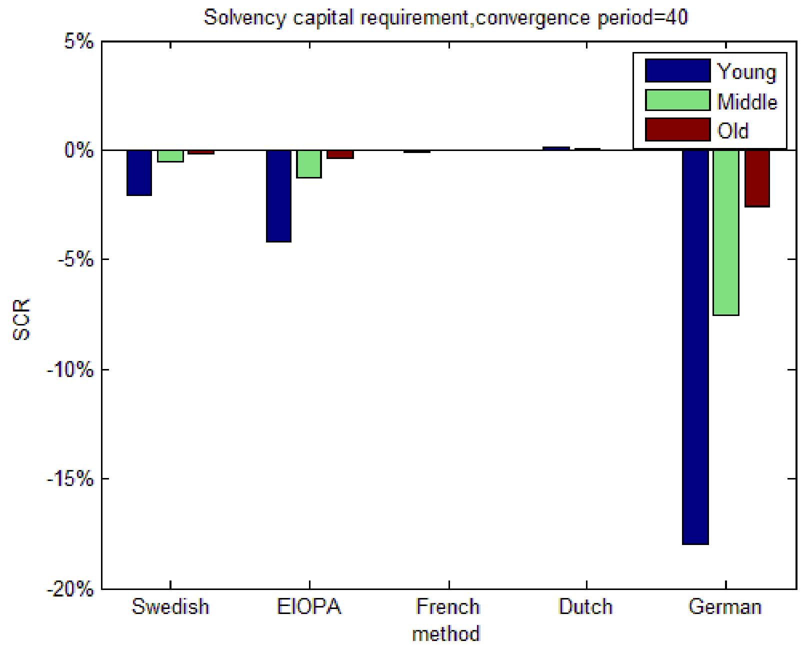

Figure 5.6 – For convergence period =40, SCR comparison for different UFR methods

5.4.2 Convergence period

Although EIOPA proposes a convergence period of 40 years, the insurers and European parliament prefers a convergence period of 10 years.The risk that the convergence period will be changed to 10 years is quite high according to Van.wessel(2014). Therefore, I would conduct the sensitivity analysis with respect to the converegence period here. The convergence period for the EIOPA,Swedish is 40 years currently. But the convergence period for the Dutch method and French method is infinity.In this thesis,the sensitivity analysis with respect to the convergence period will be conducted with respect to the convergence period of 40 years to 10 years.

Changing the convergence period from 40 years to 10 years, the figures 5.6 and 5.7 show that the Swedish SCR and EIOPA SCR are all decreasing.Since the current interest rate is set to be quite low, which is only 1%, it is expected that the one-year forward rate in the FSP is lower than the Swedish and EIOPA UFR (4.2%) even after the upward shock. If that is the case, shorter convergence period leads to higher yield curves for both Swedish method and EIOPA method. What's more, it is easy to see that the Swedish SCR decreases more than the EIOPA SCR,which indicates that Swedish SCR is more sensitive to the convergence period than the EIOPA does.

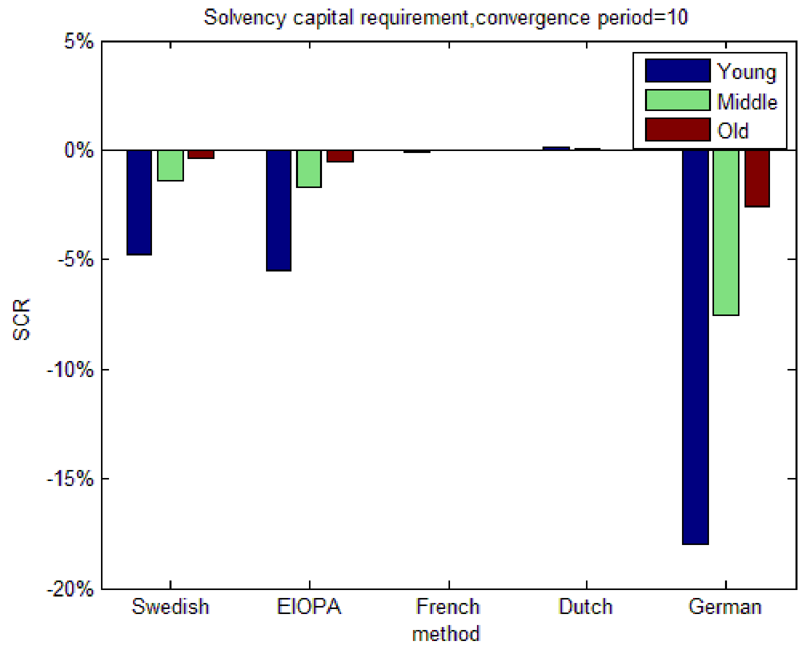

Figure 5.7 – For convergence period =10, SCR comparison for different UFR methods

The reason is that there is one more effect for reducing the convergence period for the Swedish method, which is reduction of the market data. In this case, reducing the convergence period from 40 years to 10 years also cancels the market data for the maturity mroe than 30 years for the Swedish extrapolation, which increase the influence of UFR value.Even though the influence of UFR value on the Swedish curve is increased, the influence of the UFR value on the EIOPA is still larger, which explains why the EIOPA SCR is still lower than the Swedish SCR after changing to 10 years convergence period.

Conclusion: the difference between the Swedish SCR and EIOPA SCR narrows down after changing the convergence period from 40 years to 10 years, because Swedish SCR is more sensitive to the convergence period than EIOPA SCR.

5.4.3 First smoothing point

According to Kocken.et. al (2012), the liquidity of the bond and swap market is still liquid up to 30 years. It is possible that the European solvency regulator may change the FSP from 20 years to 30 years. Therefore, it would be necessary to check if the conclusion is robust or not for this change.

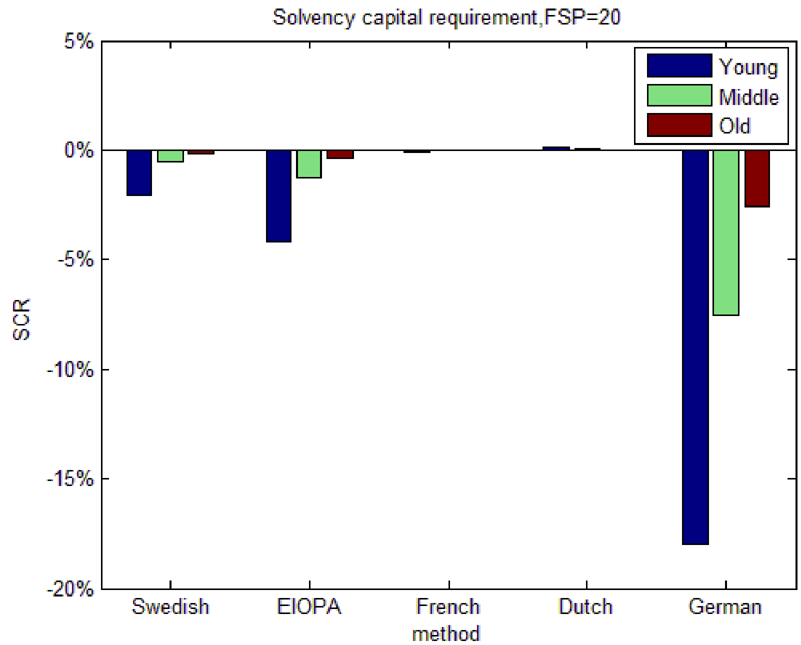

Figure 5.8 – For FSP=20, SCR comparison for different UFR methods

When changing the first smoothing point from 20 years to 30 years, we may observe that the SCR of French method changes a lot, which indicates that its SCR is quite sensitive to the choice of the first smoothing point.Since there is no illustration about how to choose the first smoothing point in the official document about the French discount curve, I simply assume that the knot of the transformation of the price function be the first smoothing point. Therefore,when changing the first smoothing point from 20 years to 30 years, the knot changes accordingly, which may lead to the large change of the solvency capital requirement. However, I am not sure how the first smoothing point is determined for the French yield curve and whether it is related to the knot in this way or not. I couldn't provide too much comments on this phenomenon here.

Without considering the impact on the French SCR, the SCR of other methods are all closer to the SCR of the market yield curve.The SCR of different UFR methods becomes more similar when the FSP changes from 20 years to 30 years. This is because the longer the first smoothing point, the larger the influence of the market data, the lesser the difference between the market curves and the UFR curves.Besides, the Dutch SCR is not sensitive to the choice of FSP.

Conclusion:

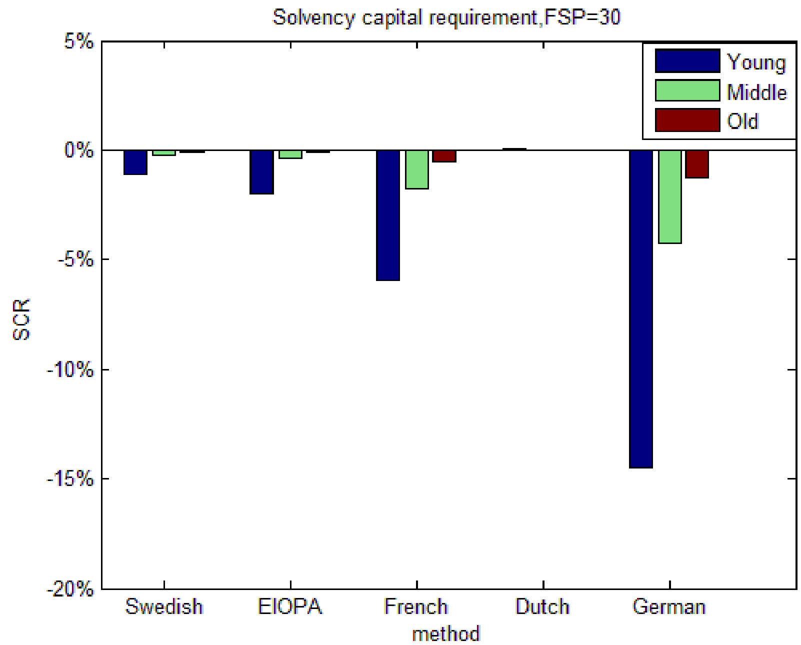

Figure 5.9 – For FSP=30, SCR comparison for different UFR methods

French SCR is quite sensitive to the choice of FSP, which leads to the result that French SCR becomes lower than the other SCRs. But this is not necessarily the case for the real French SCR, because I am not sure about whether my assumption for the French UFR method is really the one that is adopted by the French regulator or not. If it is really the case, then we can say that the French SCR is very sensitive to the choice of FSP. Swedish and EIOPA SCRs are still lower than the Dutch SCR, and the Swedish SCR is higher than the EIOPA SCR. which is consistent with the conclusion in Section 5.2.

5.4.4 Very long-term one-year forward rate implied by the market yield curve

Since conclusion drawn in section 5.2 is related to the comparison between the UFR value and the long-term one-year forward rate implied by the market yield curve (market UFR), it would be necessary to change the market UFR to check if the conclusion will change. I change the Vasicek Parameters to make the market UFR to be about 4.7%. The SCR for UFR method are as follows:

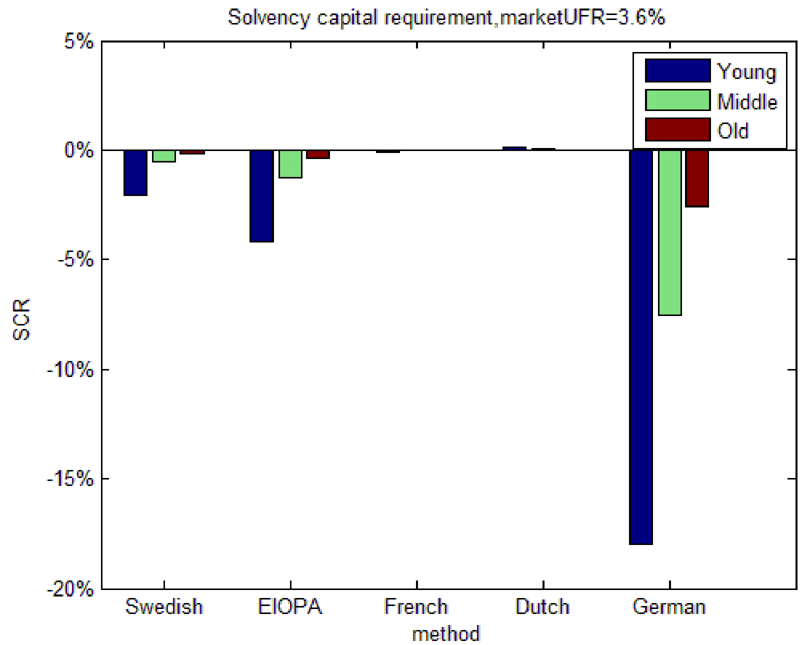

Figure 5.10 – For Market UFR=3.6%, SCR comparison for different UFR methods

It is quite obvious that the SCRs of Swedish and EIOPA do become larger than those of French and Dutch when the market UFR is larger than the UFR value (4.2%) for the Swedish and EIOPA UFR method. This result is consistent with what is found in the UFR value sensitivity.

5.4.5 60% interest rate risk hedging

According to Wyman (2013), about 60% of the assets are government bonds and corporate bonds.Hence, I would like to check if the conclusion would still exists for the 60% interest rate hedging case. Now the total asset for the insurer also include the stock, which is not related to the interest rate risk. The cash flows of the interest rate related asset are 60% of the cash flows of the liability. The stock value is assumed to be equal for all UFR method, which does not influence the comparison of SCR. Hence, the the SCR below does not include the stock value in its calculation.

It is quite obvious that the SCRs of Swedish and EIOPA are still larger than those of French and Dutch when changing the interest rate risk from 100% to 60%. Besides, we may notice that the Swedish SCR is higher than that of EIOPA. What's more, the differences of SCR for different UFR method increase with the increase of

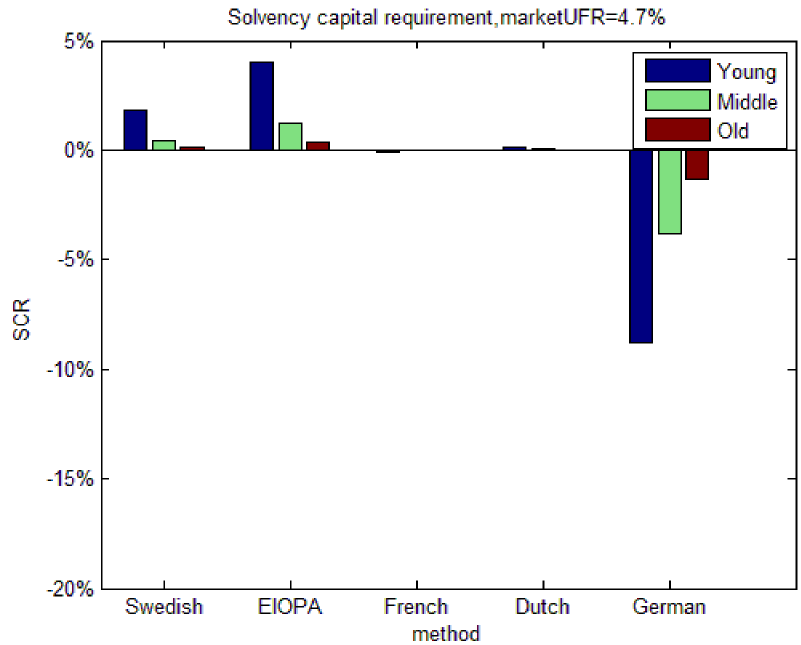

Figure 5.11 – For Market UFR=4.7%, SCR comparison for different UFR methods

the average duration. Hence, the conclusion in Section 6.2 exists even in the partial interest rate hedging case.

5.4.6 Model parameter robustness analysis

According to many empirically studies for instance Ball and Torous (1996), the drift parameters $\alpha$and β are hard to estimate. Therefore, to check the robustness of the conclusion, I would calculate the SCR for the UFR methods for the β plus and minus 10 basis points. The results are as followed:

As can be seen from the graph, so long as the error of estimating the β is not high enough to make the market UFR higher than the EIOPA and Swedish UFR, the conclusion drawn in the Section 5.2 remains the same. After carrying out the sensitivity with respect to the value of β by plus and minus 20 basis points, the conclusion turns out to be robust.

Summar of the sensitivity analysis:

For all the sensitivity analysis conducted above, the conclusion drawn in the Section 5.2 is robust except for the French UFR method, which turns out to be quite sensitivity with respect to the FSP.

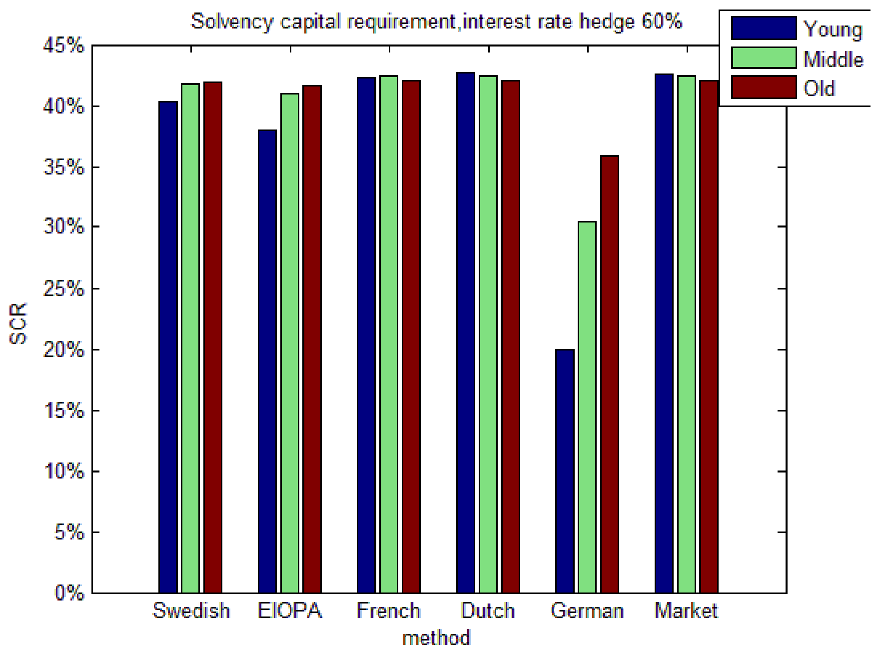

Figure 5.12 – For interest rate risk hedge=60% ,SCR comparison for different UFR methods

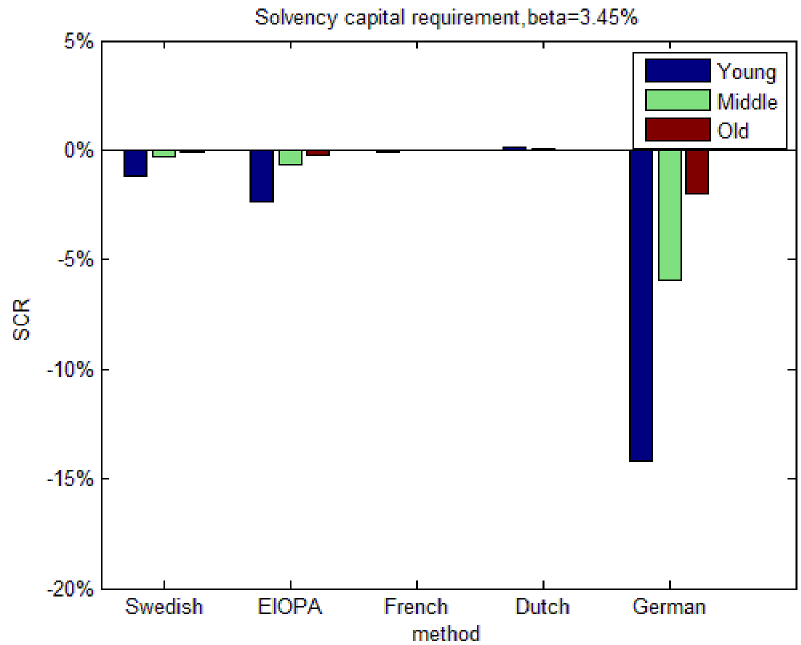

Figure 5.13 – For $\beta = 3.45\%$, SCR comparison for different UFR methods

What's more, when the EIOPA and Swedish UFR is set to be around the market UFR, there is no material difference of SCR. When it is set to be higher than the market UFR, the EIOPA and Swedish SCRs are lower than the French and Dutch SCRs. Besides, the Swedish SCR is higher than the EIOPA SCR. When it is set to be lower than the market UFR, the result is just the opposite. For different UFR method.the sensitivity with respect to the UFR value of the difference in SCR decreases with the decline of pension fund average duration.

The decrease in convergence period from 40 years to 10 years narrows down the difference between the Swedish SCR and EIOPA SCR.

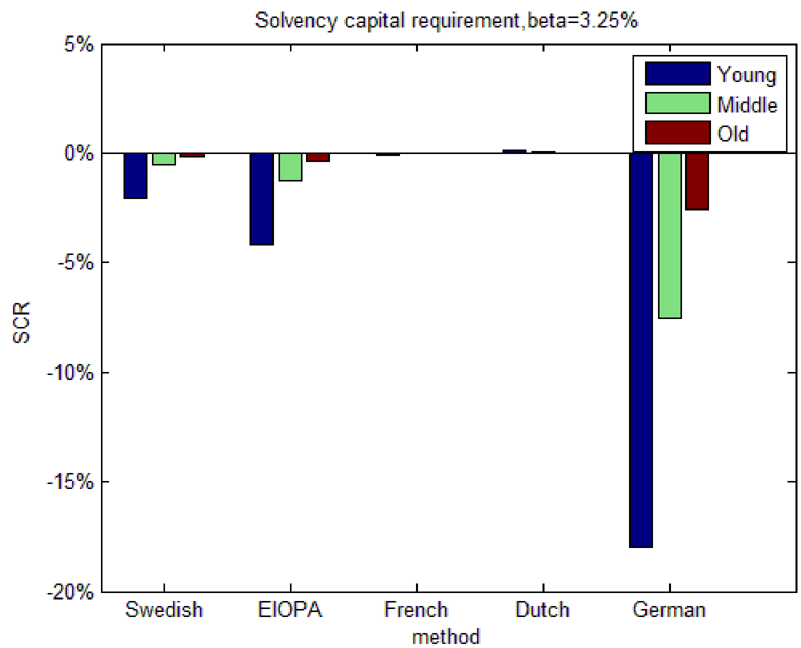

Figure 5.14 – For $\beta = 3.25\%$, SCR comparison for different UFR methods

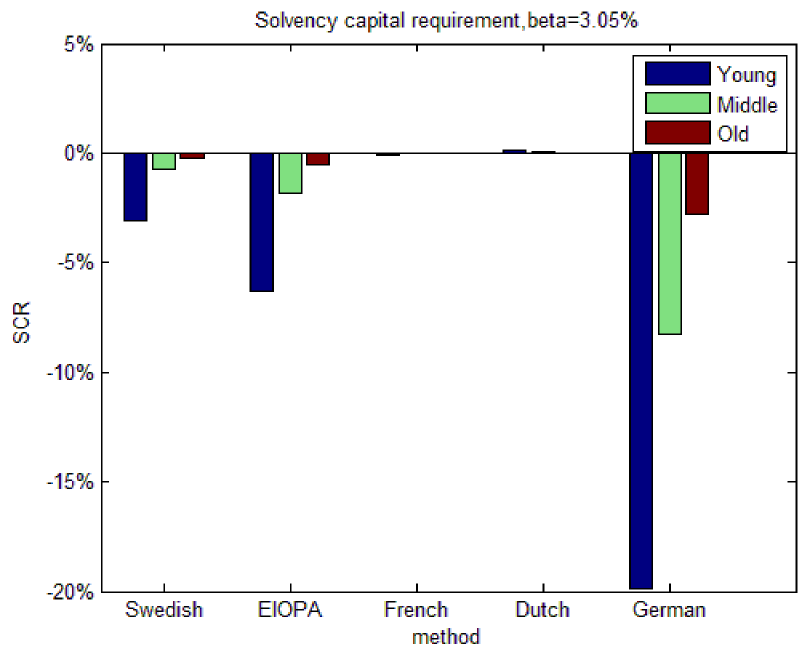

Figure 5.15 – For $\beta = 3.05\%$, SCR comparison for different UFR methods

Chapter 6

Conclusion

After checking the methodology of modelling the German UFR method with the Deutsche Bundesbank, the yield curve obtained by employing the German UFR method still deviates too much from the market yield curve and the other UFR curves. Considering the time is limited and reliability of the conclusion, I would just present the results related to the German UFR method without analysis; therefore no conclusion here is related to the German UFR methods.

Though the past interest rate information partly determines the Dutch UFR curve, its impact on the comparison between different UFR methods considered in this thesis is not material; therefore the past short rates are assumed to be constant and equal to the current short rate in the comparison of different UFR implementation consequences.

In regard to the comparison of the UFR valuation impact,it can be expected that three events may occur with a high probability given that the EIOPA and Swedish UFR is higher than the market UFR.The first event is that the Swedish pension liability is smaller than the French liability. The second event is that the Dutch pension liability is larger than the EIOPA liability. The last event is that the EIOPA liability is smaller than the Swedish liablity. The difference of liability value can be up to 2.32%. What's more, the market consistency level of Dutch and French liabilities are higher than those of Swedish and EIOPA. The absolute deviation of the UFR liability and market liability ranges from 0% to 4.47% depends on different UFR methods and different average duration of pension fund. Besides, the French UFR value varies a lot, which is quite unstable. On the contrary, other UFR methods remain nearly unchanged for different shapes of yield curves. Finally, the differences of the UFR valuation impact increase with the increase of the average duration.

One finding that may be worth mentioning is that it may not be a good idea to assume the Smith-Wilson one-year forward rate as the linear combination of the one-year forward rate at the FSP and the UFR when comparing with the Swedish one-year forward rate. Because if you assume that their weights are equal, then you are likely to draw the wrong conclusion about the comparison result based on your linear combination assumption about the SW one-year forward rate.

In regard to the comparison of the UFR solvency risk, when the EIOPA and Swedish UFR value is set to be around the market UFR, the SCR for different UFR methods can be regarded as no difference. When the UFR is set to be relatively higher, than the SCRs of Swedish and EIOPA are lower than the SCRs of Dutch and French. Besides, the Swedish SCR is higher than the EIOPA SCR. When the UFR is set to be relatively lower, than the SCRs of Swedish and EIOPA are higher than the SCRs of Dutch and French. The Swedish SCR is lower than the EIOPA SCR in this case.

The difference between the Swedish SCR and EIOPA SCR narrows down after changing the convergence period from 40 years to 10 years. The French UFR method is quite sensitivity to the choice of FSP.

Chapter 7

Appendix

7.1 Proof of the shape of yield curve

The proof of the shape of yield curve implied by the Vasicek Model is as follow:

The function for the yield curve is:

$$z(t, t+T) = z(\infty) + [r_t - z(\infty)]\frac{1}{\alpha T}(1 - e^{-\alpha T}) + \frac{\sigma^2}{4\alpha^3 T}(1 - e^{-\alpha T})^2, T \geq 0 \quad (7.1)$$

Define

$$b(T) = \frac{1 - e^{-\alpha T}}{\alpha} \rightarrow b'(T) = e^{-\alpha T} \quad (7.2)$$

$$h(T) = \frac{b(T)}{T} \quad (7.3)$$

$$r(t) - z(\infty) = \triangle \quad (7.4)$$

Then the yield curve becomes:

$$z(t, t+T) = z(\infty) + \frac{b(T)}{T}\triangle + \frac{\sigma^2}{4\alpha}\frac{1}{T}b^2(T) = z(\infty) + \frac{b(T)}{T}[\triangle + \frac{\sigma^2}{4\alpha}b(T)] = z(\infty) + h(T)[\triangle + \frac{\sigma^2}{4\alpha}b(T)] \quad (7.5)$$

So

$$z'(t, t+T) = h'(T)[\triangle + \frac{\sigma^2}{4\alpha}b(T)] + h(T)\frac{\sigma^2}{4\alpha}b'(T) = h'(T)\{\triangle + \frac{\sigma^2}{4\alpha}[b(T) + \frac{h(T)}{h'(T)}b'(T)]\} \quad (7.6)$$

Define

$$g(T) = b(T) + \frac{h(T)}{h'(T)} e^{-\alpha T} \tag{7.7}$$

So

$$z'(t, t+T) = h'(T)[\triangle + \frac{\sigma^2}{4\alpha} g(T)] \tag{7.8}$$

First I will show $h'(T) < 0$ for all T.

$$h(T) = \frac{1 - e^{-\alpha T}}{\alpha T} \tag{7.9}$$

$$h'(T) = \frac{\alpha T e^{-\alpha T} - 1 + e^{-\alpha T}}{\alpha T^2} = \frac{\alpha T + 1 - e^{\alpha T}}{\alpha T^2 e^{\alpha T}} \tag{7.10}$$

According to taylor expansion, $\alpha T + 1 < e^{\alpha T}$, therefore, the numerator is negative. It is obvious that the denominator is positive. Hence, $h'(T) < 0$.

Now show that the $g(T)$ is monotonically increasing function with

$$\lim_{T \to 0} g(T) = -\frac{2}{\alpha} \tag{7.11}$$

$$\lim_{T \to \infty} g(T) = \frac{1}{\alpha} \tag{7.12}$$

From the definition, we may learn that

$$g(T) = \frac{1 - e^{-\alpha T}}{\alpha} + \frac{T(1 - e^{-\alpha T})e^{-\alpha T}}{\alpha T e^{-\alpha T} + e^{-\alpha T} - 1} \tag{7.13}$$

When $T \to 0$, the first term $\frac{1-e^{-\alpha T}}{\alpha} \to 0$.The second term $\frac{T(1-e^{-\alpha T})e^{-\alpha T}}{\alpha T e^{-\alpha T} + e^{-\alpha T} - 1} \to \frac{0}{0}$.So the second term needs to use the L'Hopital Rule.

$$\lim_{T \to 0} g(T) = \lim_{T \to 0} \frac{T(1 - e^{-\alpha T})e^{-\alpha T}}{\alpha T e^{-\alpha T} + e^{-\alpha T} - 1} = \lim_{T \to 0} \frac{e^{-\alpha T} - \alpha T e^{-\alpha T} - e^{-2\alpha T} + 2\alpha T e^{-2\alpha T}}{-\alpha^2 e^{-\alpha T} T}$$

$$= \lim_{T \to 0} \frac{-\alpha e^{-\alpha T} - \alpha[e^{-\alpha T} + T(-\alpha e^{-\alpha T})] + 2\alpha e^{-2\alpha T} + 2\alpha[e^{-2\alpha T} + T(-2\alpha e^{-2\alpha T})]}{-\alpha^2(-\alpha e^{-\alpha T} T + e^{-\alpha T})}$$

$$= \frac{2\alpha}{-\alpha^2} = -\frac{2}{\alpha} \tag{7.14}$$

$$\lim_{T \to \infty} g(T) = \lim_{T \to \infty} (\frac{1 - e^{-\alpha T}}{\alpha} + \frac{T(1 - e^{-\alpha T})e^{-\alpha T}}{\alpha T e^{-\alpha T} + e^{-\alpha T} - 1}) \tag{7.15}$$

Since

$$\lim_{T\to\infty} e^{-\alpha T} = 0 \tag{7.16}$$

The first term

$$\frac{1-e^{-\alpha T}}{\alpha} \to \frac{1}{\alpha} \tag{7.17}$$

The second term can be rearranged as

$$\frac{\frac{1-e^{-\alpha T}}{\alpha}\frac{\alpha T}{e^{\alpha T}}}{\frac{\alpha T}{e^{\alpha T}} + e^{-\alpha T} - 1}$$

According to taylor expansion, $e^{\alpha T}$always grow faster than αT,hence $\frac{\alpha T}{e^{\alpha T}} \to 0$ when $T \to \infty$.Consequently, the second term approaches to zero. As a result

$$\lim_{T\to\infty} g(T) = \frac{1}{\alpha}$$

According to the book by Claus Munk (1993), $g(T)$ can be proved to be monotonically increasing by tedious manipulations.Therefore:

$$g'(T) > 0$$

Hence, the maximum value of g(T) is

$$g_{max}(T) = \lim_{T\to\infty} g(T) = \frac{1}{\alpha}$$

The minimum value of g(T) is

$$g_{min}(T) = \lim_{T\to 0} g(T) = -\frac{2}{\alpha}$$

Back to equation ( ),$h'(T) < 0$

$$z'(t, t+T) = h'(T)[\triangle + \frac{\sigma^2}{4\alpha}g(T)]$$

If $\triangle + \frac{\sigma^2}{4\alpha}g_{max}(T) = r(t) - R(\infty) + \frac{\sigma^2}{4\alpha}\frac{1}{\alpha} = r(t) - R(\infty) + \frac{\sigma^2}{4\alpha^2} \le 0$, then for all $T \in [0, +\infty)$,$\triangle + \frac{\sigma^2}{4\alpha}g(T) < 0$. $z'(t, t+T) > 0$,the yield curve is monotonically increasing.

If $\triangle + \frac{\sigma^2}{4\alpha}g_{min}(T) = r(t) - R(\infty) - \frac{\sigma^2}{4\alpha}\frac{2}{\alpha} = r(t) - R(\infty) - \frac{\sigma^2}{2\alpha^2} \ge 0$, then for all $T \in [0, +\infty)$,$\triangle + \frac{\sigma^2}{4\alpha}g(T) > 0$. $z'(t, t+T) < 0$,the yield curve is monotonically decreasing.

If $R(\infty) - \frac{1}{4}\frac{\sigma^2}{\alpha^2} < r(t) < R(\infty) + \frac{1}{2}\frac{\sigma^2}{\alpha^2}$, then the yield curve will first increase and then decrease, namely the yield curve is a humped shape.

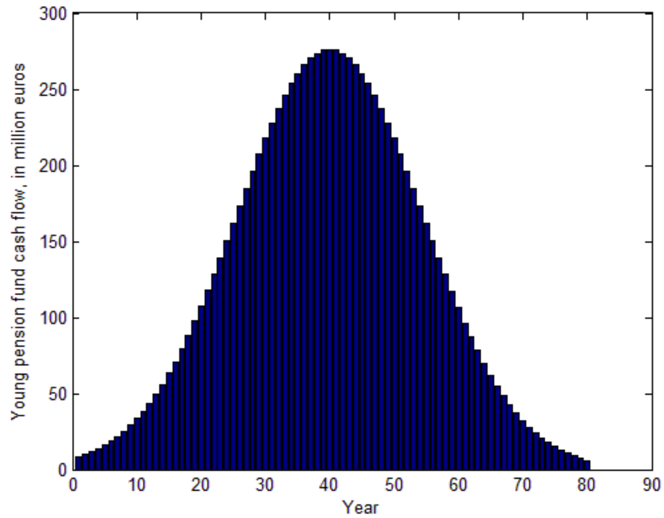

Figure 7.1 – Cash flow of the young pension fund

7.2 Three pension funds projected cash flows

The projected cash flows of the three pension funds are as follows:

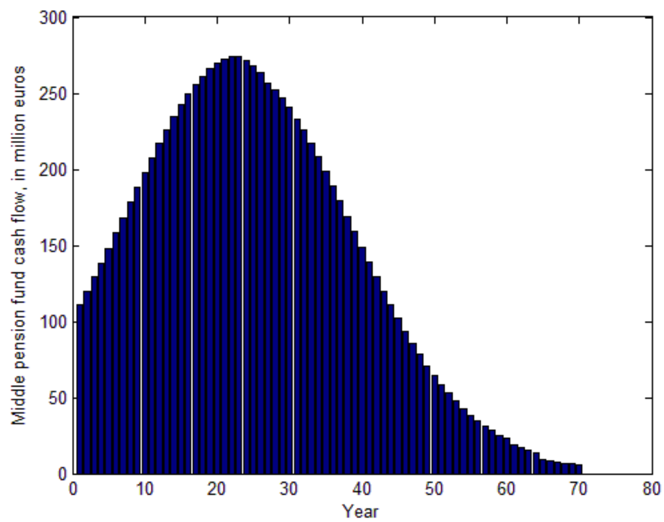

Figure 7.2 – Cash flow of the middle pension fund

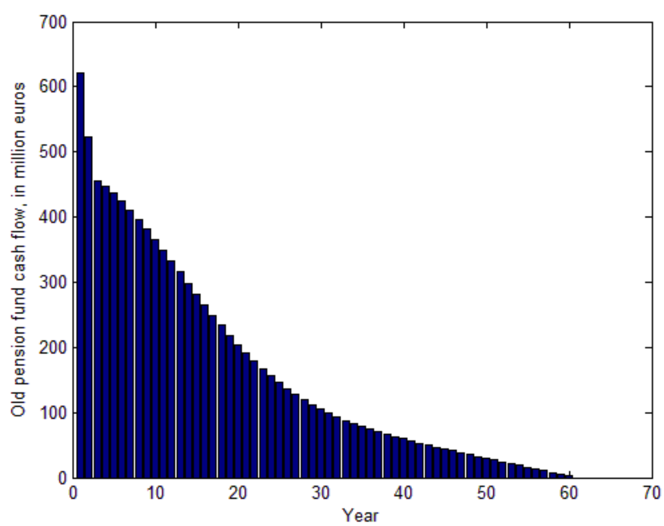

Figure 7.3 – Cash flow of the old pension fund

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