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Labor Income and the Demand for Long-Term Bonds

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Abstract

The riskless nature, in real terms, of inflation-linked bonds has led to the conclusion that inflation-linked bonds should constitute a substantial part of the optimal investment portfolio for long-term investors. This conclusion is reached in models where investors do not receive labor income during the investment period. Since such an income stream is often indexed with inflation, labor income in itself constitutes an implicit holding of real bonds. As such, the optimal investment in inflation-linked bonds is substantially reduced. By extending recently developed simulation-based techniques, we are able to determine the optimal portfolio choice among inflation-linked bonds, nominal bonds, and stocks for investors endowed with an indexed stream of income. We find that the fraction invested in inflation-linked bonds is much smaller than reported in the literature, the duration of the optimal nominal bond portfolio is lengthened, and the utility gains of having access to inflation-linked bonds are substantially reduced. We investigate as well the robustness of our results to time-variation in bond risk premia, the riskiness of labor income, and correlation between labor income risk and financial risks.

Keywords: Inflation linked bonds, optimal lifetime investment, simulation-based portfolio choice

JEL codes: C15, C63, E43, G11

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1 Introduction

The market for inflation-linked securities grows rapidly as many governments have decided to issue inflation-linked bonds on either local or global inflation indices, see, for instance, Deacon, Derry, and Mirfendereski (2004) for a recent overview. Given that inflation-linked bonds can be viewed upon as a riskless real investment, it has been argued by both academics and practitioners that inflation-linked bonds should constitute a substantial part of the investment portfolio of long-term investors. This argument has been formalized in Campbell and Viceira (2001) and Campbell, Chan, and Viceira (2003). These papers present models in which inflation-linked bonds are prominently present in a long-term investor's optimal portfolio. Moreover, Campbell and Viceira (2001) find utility gains of having access to inflation-linked bonds in the order of magnitude of 1-19% of the optimal consumption to wealth ratio for various investors.

On the other hand, the life-cycle literature, consider, for instance, the contributions by Cocco, Gomes, and Maenhout (2005), Gomes and Michaelides (2005), Bodie, Merton, and Samuelson (1992), Heaton and Lucas (1997), Viceira (2001), Campbell and Cocco (2003), Gomes and Michaelides (2004), and Munk and Sorensen (2005), has shown the importance of accounting for both income received by the investor over the investment period and portfolio constraints. Especially when labor income cannot be capitalized via financial markets due to omnipresent borrowing constraints, labor income forms a non-traded asset that influences portfolio choice. The effects can be separated into two parts. First, Gollier and Pratt (1996) and Elmendorf and Kimball (2000) have shown that increasing idiosyncratic risk triggers a reduction in the financial risk an investor is willing to bear. As a consequence, the presence of idiosyncratic labor income risk induces an effective increase in the risk aversion of the individual. Secondly, the correlation between labor income risk and financial risks is important for two reasons. On the one hand, traded assets can be used to hedge part of labor income risk. As a consequence, the optimal portfolio contains a hedging demand to offset unfavorable changes in labor income. On the other hand, as shown by, for instance, Munk and Sorensen (2005), the value of human capital becomes investor specific, if labor income is non-tradable. This implicit value is, among other factors, determined as well by the correlations between labor income risk and financial risks that are priced. Hence, different correlations between labor income risk and financial risks induce different hedging demands and implicit values of human capital. This can have serious implications for the composition of the optimal portfolio.

This paper integrates both literatures. We are initially concerned with optimal long-term bond demand of an investor who is entitled to indexed labor income that is non-tradable. In the absence of labor income, Campbell and Viceira (2001) have shown that the optimal investment portfolio contains large fractions invested in inflation-linked bonds and document sizeable utility gains from having access to inflation-linked bonds. Our baseline specification for labor income postulates that real labor income risk is uncorrelated with financial risks, like in the benchmark models of Cocco et al. (2005). The non-tradable position in labor income is then a mixture of a fixed position in inflation-linked bonds and an idiosyncratic risk component. The former component reduces the demand for inflation-linked bonds as the investor is interested in the optimal allocation of total wealth, which is the sum of financial wealth and human capital. The second component induces an effective increase in the risk aversion of the individual, which increases the demand for inflation-linked bonds, as these instruments constitute

a riskless real investment. We consider the impact of both effects and, in case the demand for inflation-linked bonds is indeed reduced in the optimal portfolio, we infer the reduction in utility gains from having access to inflation-linked bonds. Moreover, we infer the sensitivity of these results to correlation between labor income risk and financial risks.

Secondly, in several countries inflation-linked bonds are not available and the investor's asset menu is restricted to nominal bonds, leaving the possibility of bonds linked to a different country's inflation for future research. In this particular case, we investigate the impact of labor income on the demand for nominal bonds. Long-term nominal bonds are characterized by having a modest exposure to the real interest rate and a large exposure to expected inflation. Medium-term bonds, on the other hand, have a larger real interest rate sensitivity and a smaller expected inflation exposure. A similar trade-off as before is present. As labor income imposes a large real interest rate exposure on the investor, this induces a shift in the optimal portfolio towards long-term bonds, which lengthens the duration of the optimal nominal bond portfolio. However, the idiosyncratic risk component results in an effective increase in the investor's risk aversion, which leads in the investment problem without labor income to a shorter duration of the optimal nominal bond portfolio, see Campbell and Viceira (2001). The ultimate effect of incorporating labor income in an investment problem with only nominal bonds is thus ambiguous.

Finally, recent contributions by Brennan and Xia (2000, 2002), Campbell and Viceira (2001), and Sangvinatsos and Wachter (2005) have shown that realistic models of financial markets generate optimal portfolios for which the ratio of long-term nominal bonds to equities is increasing in the investor's risk aversion. This rationalizes common investment advises as summarized in Canner, Mankiw, and Weil (1997). We show that this conclusion is robust to the introduction of labor income into the investment problem.

Our model of the financial market accommodates time-variation in bond risk premia. After all, there is abundant empirical evidence that indicates that bond risk premia are not constant over time, see, among others, Campbell and Shiller (1991), Dai and Singleton (2002), and Cochrane and Piazzesi (2005). Sangvinatsos and Wachter (2005) show that abstracting from time-variation in bond risk premia in investment problems leads to substantial utility losses. Similarly, Aït-Sahalia and Brandt (2001) and Campbell et al. (2003) have shown the importance to account for information on term structure variables to predict bond risk premia and to construct optimal portfolios. We extend the models of Brennan and Xia (2002) and Campbell and Viceira (2001) by modeling the price of real interest rate risk and expected inflation affine in the real interest rate and expected inflation, respectively. We impose this structure to enhance the interpretation of the implications of time-variation in bond risk premia. The model is estimated on the basis of US data as of 1959 up to 2002.

We find that the inflation risk premium, i.e., the difference in expected returns on nominal and inflation-linked bonds with a particular maturity, is increasing in the level of expected inflation and decreasing in the level of real interest rates. The real term premium is increasing in the level of real interest rate, whereas the nominal term premium is increasing in both the real interest rate and expected inflation.

The contribution of this paper is threefold. First of all, we determine the role of nominal and inflation-linked long-term bonds for long-term investors who receive labor income during the period of

investing. We infer the implications for the optimal composition of the optimal portfolio, as well as for the value added of inflation-linked bonds. As we consider a finite horizon problem, we are able to identify horizon effects that are triggered by time-variation in both interest rates and bond risk premia. In absence of labor income, this complements the analysis of Campbell and Viceira (2001).

We find that the role of inflation-linked bonds is substantially reduced once we account for labor income. The optimal portfolio is tilted towards long-term nominal bonds or equities, depending on the assets available for investing. When the investor's asset menu is confined to nominal bonds, it turns out to be optimal to lengthen the duration of the optimal nominal bond portfolio as we account for labor income. The utility gains of having access to inflation-linked bonds are reduced considerably, especially when the investor's initial wealth is modest. In terms of horizon effects, we find that the ratio of long-term nominal bonds to inflation-linked bonds is increasing in the investment horizon. This results from the fact that horizon effects in expected inflation tend to dominate the horizon effects in real interest rates. Concerning the ratio of nominal bonds to stocks, we do find that this ratio is increasing in the risk aversion of the investor. As a consequence, it is possible, at least qualitatively, to rationalize the recommendations provided by popular investment advisors.

The second main contribution is to assess the relevance of time-variation in bond risk premia, the amount of idiosyncratic labor income risk, and correlations between labor income risk and financial risks for the investment problem with labor income. Time-variation in bond risk premia turns out to have a dramatic impact on both the optimal bond portfolios and the value added of inflation-linked bonds. However, the decline in utility gains from having access to inflation-linked bonds due to incorporation of labor income into the investment problem is preserved. Next, the impact of changing the amount of idiosyncratic labor income risk is relatively small. Finally, we find that correlation between labor income risk and real interest rate risk has limited effects on the optimal portfolio composition. On the other hand, the correlation between labor income risk and either expected inflation risk or equity risk has strong implications for the optimal portfolio composition. Changing the correlation between labor income risk and expected inflation risk erodes or strengthens the role of labor income as a hedge against inflation. We find, quite expectedly, that different correlations induce large changes in the optimal portfolio. It turns out that the value added of enriching the investor's asset menu with inflation-linked bonds is decreasing in the correlation between labor income risk and expected inflation risk.

The third contribution is methodologically. We extend the simulation-based approach to portfolio choice of Brandt, Goyal, Santa-Clara, and Stroud (2005) to include labor income and accommodate portfolio constraints. Apart from these extensions, we modify an important approximation that has recently been criticized by DeTemple, Garcia, and Rindisbacher (2003, 2005). The modified approximation overcomes the shortcomings mentioned in DeTemple et al. (2003, 2005), results in a simple optimization problem that can be solved fast, and improves the accuracy of the approximation proposed in Brandt et al. (2005). Apart from determining the optimal portfolio, we show how the method can be used to decompose the total optimal portfolio into a myopic demand and several hedging demands, even in the presence of labor income. This facilitates a better understanding of the composition of the total optimal portfolio.

The plan of this paper is as follows. Section 2 explains the financial market in which the investor operates. We outline the preference structure of the investor, as well as the labor income process

to which the investor is entitled. In addition, we provide the optimal continuous time solution and summarize the numerical procedure that we use to solve the constrained problem in the presence of labor income. Section 3 presents the estimation results of our financial market in Section 2. Section 4 considers the investment problem in absence of labor income. In Section 5, we incorporate labor income into the investment problem. Next, Section 6 investigates the robustness of the results to time-variation in bond risk premia, idiosyncratic labor income uncertainty, and different correlations between labor income risk and financial risks. Section 7 concludes and four appendices contain proofs and technical issues.

2 The economy and investor's preferences

2.1 The financial market

Our financial market accommodates time-variation in bond risk premia. The model we propose is closely related to Brennan and Xia (2002), Campbell and Viceira (2001), and Sangvinatsos and Wachter (2005). Brennan and Xia (2002) and Campbell and Viceira (2001) propose two factor models of the term structure, where the factors are identified as the real interest rate and expected inflation. Both models assume that bond risk premia are constant. Sangvinatsos and Wachter (2005) use a three factor term structure model with latent factors and accommodate time-variation in bond risk premia, in line with Duffee (2002). We consider a model like Brennan and Xia (2002) and Campbell and Viceira (2001), but do accommodate time-variation in bond risk premia.

The securities that are possibly present in the asset menu of the investor are a stock (index), nominal bonds, inflation-linked bonds, and a nominal money market account. We start with a model for the instantaneous real interest rate, r, which is assumed to be driven by a single factor, X_1,

$$r_t = \delta_r + X_{1t},\ \delta_r > 0. \tag{1}$$

In order to accommodate the first-order autocorrelation in the real interest rate, we model X_1 to be mean-reverting around zero, i.e.,

$$dX_{1t} = -\kappa_1 X_{1t} dt + \sigma_1^\intercal dZ_t,\ \sigma_1 \in \mathbb{R}^4,\ \kappa_1 > 0, \tag{2}$$

where $Z \in \mathbb{R}^{4\times 1}$ is a vector of independent Brownian motions driving the uncertainty in the financial market.

In order to link the real and nominal side of the economy, we postulate a process for the (commodity) price index Π

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \sigma_\Pi^\intercal dZ_t,\ \sigma_\Pi \in \mathbb{R}^4, \tag{3}$$

where π_t denotes the instantaneous expected inflation. Instantaneous expected inflation is assumed to be affine in a second factor, X_2,

$$\pi_t = \delta_\pi + X_{2t},\ \delta_\pi > 0, \tag{4}$$

where the second term structure factor exhibits the mean-reverting dynamics

$$dX_{2t} = -\kappa_2 X_{2t} dt + \sigma_2^\intercal dZ_t,\ \sigma_2 \in \mathbb{R}^4,\ \kappa_2 > 0. \tag{5}$$

Concerning the stock (index), S, we postulate

$$\frac{dS_t}{S_t} = (R_t + \eta_S)\, dt + \sigma_S^{\intercal} dZ_t,\ \sigma_S \in \mathbb{R}^4, \tag{6}$$

where R_t is the nominal instantaneous interest rate to be derived later (see (11)) and η_S a constant equity risk premium. We are mainly interested in the optimal bond demand when the investor is entitled to a particular income stream. This provides the main motivation to abstract from stock return predictability, despite increasing empirical evidence that there is a certain degree of predictability, see, for instance, Ang and Bekaert (2003), Campbell and Yogo (2005), and Brennan and Xia (2005).

To complete our model, we specify an affine term structure for the term structure of interest rates by assuming that the prices of risk are affine in the term structure factors. More precisely, in the nominal state price density, $\phi^{\$}$,

$$\frac{d\phi_t^{\$}}{\phi_t^{\$}} = -R_t dt - \Lambda_t^{\top} dZ_t, \tag{7}$$

we assume that the time-varying prices of risk, i.e. Λ_t, are affine in the term structure factors X_1 and X_2,

$$\Lambda_t = \Lambda_0 + \Lambda_1 X_t, \tag{8}$$

with $X_t = (X_{1t}, X_{2t})$. I.e., we adopt the essentially affine model as proposed by Duffee (2002). In the nomenclature of Dai and Singleton (2000), the model proposed can be classified as $\mathbb{A}_0(2)$.

This specification accommodates time-variation in bond risk premia as advocated by, for instance, Dai and Singleton (2002) and Cochrane and Piazzesi (2005). As we assume the equity risk premium to be constant, we have

$$\sigma_S^{\intercal} \Lambda_t = \eta_S, \tag{9}$$

which restricts Λ_1. We further restrict the risk premia such that the price of real interest rate risk is determined only by the level of the real interest rate, whereas the price of (pure) expected inflation risk depends solely on expected inflation. We impose this separation of the nominal and real world to enhance the interpretation of time-variation in bond risk premia and its implications for portfolio choice. In terms of the model parameters, this implies that the first two rows of Λ_1 form a diagonal matrix. Next, the price of unexpected inflation risk cannot be identified on the basis of data on the nominal side of the economy alone. We impose that the part of the price of unexpected inflation risk that cannot be identified using nominal bond data equals zero. Since inflation-linked bonds have been launched in the US only as of 1997, the data available is insufficient to estimate this price of risk accurately. This restriction is in line with the recent literature, see for instance Ang and Bekaert (2004) and Campbell and Viceira (2001).

Interestingly, Kothari and Shanken (2004), Roll (2004), and Hunter and Simon (2005) use data on TIPS to infer the properties of real bonds. Hunter and Simon (2005) find that real bonds do not significantly extend the investment opportunity set using both conditional and unconditional Mean-Variance spanning tests. As Hunter and Simon (2005) mention, this conclusion is based on an analysis of a period in which inflation has been low.

Given the nominal state price density in (10), we find for the real state price density, ϕ,

$$\begin{aligned} \frac{d\phi_t}{\phi_t} &= -(R_t - \pi_t + \sigma_\Pi^\intercal \Lambda_t)dt - (\Lambda_t^\top - \sigma_\Pi^\intercal)dZ_t \\ &= -r_t dt - (\Lambda_t^\top - \sigma_\Pi^\intercal)dZ_t. \end{aligned} \tag{10}$$

As a consequence, we obtain for the instantaneous nominal interest rate

$$\begin{aligned} R_t &= r_t + \pi_t - \sigma_\Pi^\intercal \Lambda_t \\ &= \delta_R + (\iota_2^\intercal - \sigma_\Pi^\intercal \Lambda_1)\, X_t, \end{aligned} \tag{11}$$

where $\delta_R = \delta_r + \delta_\pi - \sigma_\Pi^\intercal \Lambda_0$. The conditions specified in Duffie and Kan (1996) to ensure that both nominal and real bond prices are exponentially affine in the state variables are satisfied. Hence, we find for the prices of a nominal bond at time t with a maturity $t + \tau$,

$$P(t, t+\tau) = \exp(A(\tau) + B(\tau)^\top X_t), \tag{12}$$

and for a inflation-linked bond

$$P^R(t, t+\tau) = \exp(A^R(\tau) + B^R(\tau)^\top X_t), \tag{13}$$

where $A(\tau)$, $B(\tau)$, $A^R(\tau)$, $B^R(\tau)$, and the corresponding derivations are provided in Appendix A. Note that the nominal price process of a real bond is scaled by changes in the price index, i.e. the nominal price process of a real bond evolves as

$$\frac{d\left(\Pi_t P^R(t, t+\tau)\right)}{\Pi_t P^R(t, t+\tau)} = \left(R_t + B^R(\tau)^\top \Sigma_X \Lambda_t + \sigma_\Pi^\top \Lambda_t\right) dt + \left(B^R(\tau)^\top \Sigma_X + \sigma_\Pi^\intercal\right) dZ_t. \tag{14}$$

Campbell and Viceira (2001) and Campbell et al. (2003) infer the role of inflation-linked bonds in the optimal asset allocation. The former paper has constant risk premia for both stocks and bonds, which is more restrictive than the model considered here. Campbell et al. (2003) capture the dynamics of the financial market with a VAR-model. In modeling inflation-linked bonds, Campbell et al. (2003) make the assumption that the (log) expectations hypothesis holds for the real term structure. Finally, Campbell et al. (2003) also abstract from portfolio constraints. More importantly, both papers do not account for labor income during the investment period. Munk and Sorensen (2005) do account for labor income, but restrict their analysis to the real side of the economy and the bond considered is therefore a real bond. Due to the fact that Munk and Sorensen (2005) abstract from inflation, the different role of nominal and inflation-linked bonds cannot be identified.

We summarize the financial market for future reference. Denote the state vector containing both term structure factors by X_t and define

$$X_t = \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix},\ \Sigma_X = \begin{bmatrix} \sigma_1^\intercal \\ \sigma_2^\intercal \end{bmatrix},\ K_X = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}. \tag{15}$$

We have the following dynamics of the state variables

$$dX = -KX dt + \Sigma_X dZ. \tag{16}$$

Assume that asset menu consists of the stock (index), n nominal bonds, and m real bonds. Denote the vector containing the prices at time t of n nominal bonds by P_t and the vector containing the m real bonds P_t^R. The dynamics are then given by

$$d\begin{bmatrix} P_t \\ P_t^R \\ S_t \end{bmatrix} = diag\left(\begin{bmatrix} P \\ P^R \\ S \end{bmatrix}\right)\left(\left(\begin{bmatrix} R_t\iota_{m+n+1} + \begin{bmatrix} B\Sigma_X \\ B^R\Sigma_X \\ \sigma_S^\intercal \end{bmatrix}\Lambda_t \end{bmatrix} dt + \begin{bmatrix} B\Sigma_X \\ B^R\Sigma_X \\ \sigma_S^\intercal \end{bmatrix} dZ_t\right),$$

where $B \in \mathbb{R}^{n\times 2}$ and $B^R \in \mathbb{R}^{m\times 2}$ containing the factor exposures of nominal and real bonds. We denote the volatility matrix of the traded assets in the sequel by Σ.

2.2 The investor's labor income and preferences

The investor operating in the financial market of Section 2.1 is endowed with an income stream, which will be denoted by Y_t in real terms. We postulate for the dynamics of labor income, in line with Viceira (2001),

$$\frac{dY_t}{Y_t} = \left(g - \frac{\sigma_Y^2}{2}\right)dt + \sigma_Y dZ_t^Y. \tag{17}$$

Moreover, in line with the baseline case of Cocco et al. (2005), we assume initially that Z^Y is independent of Z. In Section 6.3, we infer the impact of correlations between labor income innovations and either the real interest rate, expected inflation, or equity risk. More general models have been proposed to model labor income by, for instance, Campbell and Cocco (2003), Cocco et al. (2005), Gomes and Michaelides (2005), Woolley (2004), and Munk and Sorensen (2005). The main extensions encompass an age-dependent or interest-rate dependent income growth, transient shocks to labor income, substantial income drops in case of being laid off. Since we do not aim to model life-cycle behavior of individuals in this paper, we abstract from these refinements.

We assess the impact of the income process on the optimal portfolio of an investor that is active in the before-mentioned financial market. The effect of being entitled to a certain income stream is well-understood in simple financial markets, see for instance Viceira (2001), Cocco et al. (2005), and Gomes and Michaelides (2005). In general, there are two important determinants of the effect of labor income on the optimal asset allocation. First of all, the correlations between labor income risk and financial risks are important and have two effects. Once the investor is able to hedge labor income uncertainty with financial assets, the optimal portfolio will contain an additional hedging demand, see for instance Viceira (2001) and Munk and Sorensen (2005), to hedge labor income uncertainty. For instance, if stock returns are positively correlated with labor income uncertainty, a negative investment in stocks can be used to hedge labor income uncertainty. After all, a negative shock to labor income is then generally accompanied with a positive return on the investment portfolio. The second effect of introducing correlations is that the (implicit) value of human capital is affected, whenever labor income is correlated with financial risks that are priced.

The second relevant component is the amount of idiosyncratic labor income uncertainty. The literature on background risk, see for instance Gollier and Pratt (1996) and Elmendorf and Kimball (2000), has illustrated that background risk substitutes for financial risk. Hence, an increase in the idiosyncratic labor income risk induces an effective increase in the risk aversion of the individual.

Our main contribution to this literature is our more flexible model of the financial market in which we allow for both stochastic interest rates and inflation rates, as well as time-variation in bond risk premia. Therefore, we can assess the possibly different impact of labor income on the optimal portfolio when the investor can choose between either nominal bonds or both nominal and inflation-linked bonds. In addition, we can assess the effect of labor income on the value added, as measured in utility terms, of having access to inflation-linked bonds.

The individual obtains utility from real terminal wealth that has been accumulated during the investment period $[t,T]$. Consequently, the problem faced by the individual can be formalized as

$$\begin{aligned} V_1\left(t,T,X_t,w_t\right) &= \max_{x_s\in\mathcal{K},s\in[t,T]} \mathbb{E}_t\left(\frac{1}{1-\gamma}w_T^{1-\gamma}\right) \\ &= w_t^{1-\gamma}V_1\left(t,T,X_t,1\right), \end{aligned} \tag{18}$$

where $w_t = W_t/\Pi_t$ denotes real wealth. The maximization is subject to the dynamic budget constraint

$$\frac{dW_t}{W_t} = \left(R_t + x_t^\intercal \Sigma\Lambda_t\right)dt + x_t^\intercal \Sigma dZ_t, \tag{19}$$

where $x_t \in \mathbb{R}^{n+m+1}$ denotes the fraction of nominal wealth invested in the different assets.

The CRRA utility index summarizes the preferences of the individual and $\mathcal{K}$ is the set to which the portfolio fractions of financial wealth invested, x_t, are constrained, which is in this case $\mathcal{K} = \mathbb{R}^{n+m+1}$. W_t denotes the nominal (financial) wealth of the investor at time t. In this paper we focus on the problem where utility is obtained only from terminal wealth and abstract from intermediate consumption. The application we have in mind is an individual saving for retirement. In an individual context, one may argue that people saving for retirement often contribute a fixed fraction of labor income, say 20%, on the basis of some form of mental accounting or precommitment. In this case, the income process is proportional to the labor income process to which the individual is entitled. Gomes, Michaelides, and Polkovnichenko (2004) show in a life-cycle context that the utility costs of fixing the savings rate exogenously are surprisingly small, given that the savings rate is set properly.

2.3 Optimal portfolio choice without labor income

We consider first the investment problem for an investor without labor income and without trading constraints, i.e. $\mathcal{K} = \mathbb{R}^{n+m+1}$. In this case, the maximization in (18) can be solved analytically in continuous time.

The situation without labor income and unconstrained continuous time investing is a special case of the model of Sangvinatsos and Wachter (2005). Their results regarding the optimal portfolio in the continuous time setting are discussed in Appendix B, where we show in addition that the optimal investment in the risky assets can be decomposed in a convenient way, namely

$$\begin{aligned} x_t^* = \underbrace{\frac{1}{\gamma}\left(\Sigma\Sigma^\intercal\right)^{-1}\Sigma\Lambda_t + \left(1-\frac{1}{\gamma}\right)\left(\Sigma\Sigma^\intercal\right)^{-1}\Sigma\sigma_\Pi}_{\text{Myopic demand}} \\ \underbrace{+\frac{1}{\gamma}\left(\Sigma\Sigma^\intercal\right)^{-1}\Sigma\sigma_1\left(\frac{V_{1r}}{V_1}\right)}_{\text{Hedging demand real rate}} \quad \underbrace{+\frac{1}{\gamma}\left(\Sigma\Sigma^\intercal\right)^{-1}\Sigma\sigma_2\left(\frac{V_{1\pi}}{V_1}\right)}_{\text{Hedging demand expected inflation}}, \end{aligned} \tag{20}$$

and Appendix B shows that the function V_1 is exponentially quadratic in both term structure factors. V_{1X} denotes the partial derivative of V_1 with respect to X. Equation (20) shows that the optimal portfolio has three components. The first part of the first component,

$$\frac{1}{\gamma}\left(\Sigma\Sigma^{\intercal}\right)^{-1}\Sigma\Lambda_t, \tag{21}$$

is the standard myopic demand that maximizes the continuous time Sharpe ratio $\sqrt{\Lambda_t^{\intercal}\Lambda_t}$. The second part of the first component,

$$\left(1-\frac{1}{\gamma}\right)\left(\Sigma\Sigma^{\intercal}\right)^{-1}\Sigma\sigma_{\Pi}, \tag{22}$$

replicates unexpected inflation as far as possible. Both components (21) and (22) constitute the myopic demand. The other two components are hedging demands. As shown by Sangvinatsos and Wachter (2005), these hedging demands are induced by time-variation in the investment opportunity set to the extent that real interest rates and prices of risks are affected. In order to assess the relevance of both hedging portfolios, Sangvinatsos and Wachter (2005) propose to set certain parameters to zero in V_{1X}/V_1. A disadvantage of this approach is that the different components no longer sum to the total hedging portfolio. In our model of the financial market, the factors have a clear financial interpretation which facilitates an alternative decomposition of the total hedging portfolio. As such,

$$\frac{1}{\gamma}\left(\Sigma\Sigma^{\intercal}\right)^{-1}\Sigma\sigma_1\left(\frac{V_{1r}}{V_1}\right), \tag{23}$$

provides the hedging demand that arises due to time-variation in the real interest rate and

$$\frac{1}{\gamma}\left(\Sigma\Sigma^{\intercal}\right)^{-1}\Sigma\sigma_2\left(\frac{V_{1\pi}}{V_1}\right) \tag{24}$$

constitutes the hedging portfolio that comes from time-variation in expected inflation rates. Obviously, additivity of the two hedging components to the total hedging demand is preserved, which is a desirable property. An additional advantage is that this construction of hedging demands allows a natural extension to the discrete time setting, possibly with labor income. It should be noted that time-variation in the real rate triggers two hedging demands. First of all, if prices of risk are constant, time-variation in the real interest rate introduces a hedging demand, as shown by Brennan and Xia (2002) and Campbell and Viceira (2001). Secondly, prices of risk co-vary with the real interest rate, which leads to an additional hedging demand. Time-variation in expected inflation induces a hedging demand only due to the fact that prices of risk co-vary with expected inflation.

Secondly, we consider an investor who can trade annually and is prohibited to take short positions, i.e.[1] $\mathcal{K} = \{x \mid x \geq 0, x^{\top}\iota \leq 1\}$. The reduced trading frequency and portfolio constraints can be motivated, for instance, by the presence of transaction costs. For this constrained discrete time investment problem, we maximize (18) subject to the budget constraint in real terms,

$$w_{t+1} = w_t\left(x_t^{\top}r_{t+1}^{e} + r_{t+1}^{f}\right), \tag{25}$$

[1] It has been argued by, among others, Davis, Kubler, and Willen (2003) and Cocco et al. (2005) that borrowing to invest is often possible to a limited extent in practice, albeit that the borrowing rate exceeds the lending rate, i.e. so-called endogenous borrowing constraints. We confine ourselves to exogenous borrowing constraints throughout this paper.

where $r^e_{t+1} = R^e_{t+1}\Pi_t/\Pi_{t+1}$ denotes the real excess return, $r^f_{t+1} = P(t,t+1)^{-1}\Pi_t/\Pi_{t+1}$ represents the real return on the nominal money market account, and recall that $w_t = W_t/\Pi_t$ denotes real wealth. The corresponding value function in (18) is denoted by V_2. Using Bellman's principle of optimality, we find

$$V_2(t,T,X_t,w_t) = \max_{x_t\in\mathcal{K}} \mathbb{E}_t\left(V_2(t+1,T,X_{t+1},w_{t+1})\right) \tag{26}$$

Exploiting the homogeneity of the power utility index, provides

$$V_2(t,T,X_t,w_t) = w_t^{1-\gamma} V_2(t,T,X_t,1). \tag{27}$$

Substituting these results in (26), we find

$$\begin{aligned} V_2(t,T,X_t,1) &= \max_{x_t\in\mathcal{K}} \mathbb{E}_t\left(V_2(t+1,T,X_{t+1},1)\left(\frac{w_{t+1}}{w_t}\right)^{1-\gamma}\right) \\ &= \max_{x_t\in\mathcal{K}} \mathbb{E}_t\left(V_2(t+1,T,X_{t+1},1)\left(x_t^\top r^e_{t+1} + r^f_{t+1}\right)^{1-\gamma}\right). \end{aligned} \tag{28}$$

These first two investment problems consider investors who are not entitled to any form of income during the period of investing. Sangvinatsos and Wachter (2005) solve the unconstrained continuous time problem in absence of labor income and portfolio constraints[2]. Campbell and Viceira (2001) consider the optimal portfolio choice between long-term bonds in a discrete time setting, but do not account for labor income or time-variation in bond risk premia. Campbell et al. (2003) do allow for time-variation in risk premia, but they cannot easily accommodate portfolio constraints. Moreover, the results presented in Campbell and Viceira (2001) and Campbell et al. (2003) are derived in infinite horizon models. Therefore, an analysis of horizon effects is possible only implicitly and we extend their results therefore as well by addressing the investor's investment horizon.

2.4 Optimal portfolio choice with labor income

We now consider an investor who receives a certain stream of labor income during the period of investing. We consider the investment problem in which the investor can trade annually and investing is subject to borrowing and short-sale constraints, i.e. $\mathcal{K} = \{x \mid x \geq 0, x^\top\iota \leq 1\}$. In the presence of labor income, it has been shown by, for instance, Bodie et al. (1992) that it is optimal to borrow excessively against future labor income, especially in early stages of the life-cycle. For reasons of moral hazard, this is generally infeasible in practice.

For convenience, we assume that the investor receives labor income annually. The discrete time counterpart of (17) is given by

$$Y_{t+1} = Y_t \exp(g+\xi_{t+1}),\ \xi_{t+1} \overset{i.i.d.}{\sim} N\left(0,\sigma^2_\xi\right), \tag{29}$$

where the innovations in the labor income process are initially uncorrelated with innovations in the financial market, as in the baseline case of Cocco et al. (2005). In Section 6.3 we discuss the effects of labor income shocks that are correlated with market shocks.

[2] Sangvinatsos and Wachter (2005) consider the essentially affine 3-factor model that has been advocated by Duffee (2002) and enrich this model with an inflation and stock (index) process. We consider a more parsimonious model in which the factors have a clear financial interpretation in order to enhance the interpretation of the results.

For the investment problem where the investor receives labor income during the period of investing, we maximize (18) subject to the budget constraint

$$w_{t+1} = w_t \left(x_t^\top r_{t+1}^e + r_{t+1}^f \right) + Y_{t+1}. \tag{30}$$

The corresponding value function in (18) is denoted by V_3. The homogeneity of the utility index can be used in this case to show that the optimal portfolio is independent of the permanent component of labor income,

$$V_3(t, T, X_t, w_t, Y_t) = Y_t^{1-\gamma} V_3(t, T, X_t, \bar{w}_t, 1), \tag{31}$$

where $\bar{w}_t = w_t / Y_t$, i.e. normalized real wealth. If we substitute these results in (26), we find

$$\begin{aligned} V_3(t, T, X_t, \bar{w}_t, 1) &= \max_{x_t \in \mathcal{K}} \mathbb{E}_t \left(V_3(t+1, T, X_{t+1}, \bar{w}_{t+1}, 1) \left(\frac{Y_{t+1}}{Y_t} \right)^{1-\gamma} \right) \\ &= \max_{x_t \in \mathcal{K}} \mathbb{E}_t \left(V_3(t+1, T, X_{t+1}, \bar{w}_{t+1}, 1) \exp(g + \xi_{t+1})^{1-\gamma} \right). \end{aligned} \tag{32}$$

2.5 Summary of the numerical approach

As portfolio problems in a discrete time setting generally do not allow for analytical solutions, numerical techniques have been used to solve such asset allocation problems. In life-cycle models, numerical dynamic programming is the leading solution technique. The numerical issues given our number of state variables and sources of uncertainty should not be underestimated. The present section discusses the main ideas of our solution technique and details can be found in Appendix E.

Brandt et al. (2005) have shown how to extend the simulation-based valuation methods of Longstaff and Schwartz (2001) and Tsitsiklis and vanRoy (2002) to the area of portfolio selection. Both in valuation of American options and in dynamic asset allocation problems, the solution can be derived easily once certain conditional expectations are known. In pricing American options, this is the continuation value of the option, whereas in case of portfolio selection, the conditional expectation of future utility is of particular interest. However, in both problems, the conditional expectations cannot be calculated analytically in most cases.

The idea is to approximate conditional expectations by a projection on a set of basis functions in the state variables. In order to estimate the projection coefficients, we simulate first of all M paths of both state variables and asset returns. The projection coefficients are subsequently estimated via a cross-sectional regression across all simulated paths. Once the conditional expectations have been approximated, the principle of dynamic programming is used to solve for the optimal portfolio.

We adjust the approach of Brandt et al. (2005) considerably in order to be able to handle portfolio constraints and labor income. Apart from these extensions, we modify an approximation proposed in Brandt et al. (2005). In general, it is very time-consuming to optimize over the optimal portfolio in every branch, as every function evaluation requires a cross-sectional regression over the M paths, where M is typically large. Brandt et al. (2005) suggest to determine a fourth order Taylor expansion of the utility index. In solving for the optimal portfolio on the basis the resulting polynomial, Brandt et al. (2005) suggest an iterative procedure to optimize efficiently over all paths at a certain time point simultaneously. Yet, as noted by DeTemple et al. (2003, 2005), this iterative procedure is not guaranteed to converge.

We circumvent this approximation by exploiting the fact that the projection coefficients that follow from the cross-sectional regression across all paths are smooth functions of the portfolio weights. Therefore, we determine a polynomial expansion of the projection coefficients in the portfolio weights. This results in a quadratic optimization problem that can be solved fast. This alternative approximation turns out to be highly accurate and avoids the iterative procedure. Appendix E compares both approximations for an example proposed in Brandt et al. (2005). The results indicate that our approach enhances the approximation of Brandt et al. (2005), especially when the trading frequency is relatively low, which is the prime case where the approximation of Brandt et al. (2005) tends to be inaccurate.

Secondly, due to the presence of the income stream, the portfolio becomes dependent on the level of normalized real wealth. We solve this problem by combining the conventional approach of discretizing the state space with the simulation-based approach. We specify a grid for wealth at each point in time and combine these grid points with the simulated values of the state variables. We refer to Appendix E for a rigorous discussion of the simulation-based approach to portfolio choice in the presence of portfolio constraints and an income stream.

The before-mentioned numerical procedure can be extended to decompose the optimal portfolio into myopic and hedging demands that are induced by either time-variation in the real interest rate or expected inflation. In absence of labor income, the myopic demand has been defined as the optimal portfolio allocation that solves a single period investment problem. Therefore, the total hedging demand that arises in a multi-period problem is obtained by subtracting the myopic demand from the total demand. As shown by Samuelson (1969), the optimal portfolio in a multi-period problem coincides with the optimal solution in a single period problem, if interest rates are constant and asset returns are i.i.d. Therefore, the myopic demand can be calculated as well by solving the multi-period problem in which we reset the state variables to their initial values at every time step in the simulation procedure. In Appendix C, we show that this approach can be specialized to the case where the investor only hedges time-variation in the real rate or expected inflation. However, as discussed in detail in Appendix C, such decompositions do not necessarily sum to the total demand. In our empirical application, the differences are negligible and the decomposition significantly enhances the understanding of the composition of the total demand.

In this investment problem with labor income, the conventional definition of hedging demands, namely the optimal strategy that solves the single period problem, cannot be used. After all, an investor with multiple periods ahead has a different entitlement to labor income than an individual that faces a single period investment problem. However, in absence of labor income, resetting the state variables results in the same optimal portfolio, independent of the investment horizon, as the single period (myopic) portfolio. Therefore, we extend the concept to the case with labor income and reset the state variables after a single period simulation in order to determine the myopic demand in the presence of labor income. This enables a decomposition of the optimal portfolio choice into the myopic and total hedging demand. As before, the total hedging demand can be decomposed into a hedging demand induced by time-variation in the real rate and time-variation in expected inflation rates. We refer to Appendix C for further details.

3 Data and estimation

In this section, we estimate our specification of the financial market in Section 2.1. Section 3.1 describes the data that we use in estimation, while in Section 3.2, we provide the estimation results and illustrate the fit of the model.

3.1 Data

In order to estimate our specification of the financial market, we use monthly data as of January 1959 up to May 2002. The monthly US government yield data are the same as in Duffee (2002) and Sangvinatsos and Wachter (2005) up to December 1998. These data are taken from McCulloch and Kwon up to February 1991 and extended using the data in Bliss (1997) up to December 1998. We extend these time series up to May 2002 using the data that have been obtained via Rob Bliss[3]. We use six yields with maturities 3 months, 6 months, 1, 2, 5, and 10 years.

Data on the price index has been obtained from the website of the Bureau of Labor Statistics. We use the CPI-U index to represent the relevant price index for the investor. The CPI-U index represents the buying habits of the residents of urban and metropolitan areas in the US[4]. We assume, in line with the literature, that the price index used to index inflation-linked bonds coincides with the price index that is relevant for the investor. Finally, we use returns on the CRSP value-weighted NYSE/Amex/Nasdaq index for stock returns.

3.2 Estimation

The Kalman filter with unobserved state variables X_{1t} and X_{2t} is used to estimate the model by maximum likelihood. Following De Jong (2000), Brennan and Xia (2002), and Campbell and Viceira (2001), we assume that all yields have been measured with error. Details on the estimation procedure are in Appendix D.

The relevant processes in estimation are $K_t = (X_t, \log \Pi_t, \log S_t)$ for which the joint diffusion can be written as

$$d\begin{bmatrix} X_t \\ \log \Pi_t \\ \log S_t \end{bmatrix} = \left(\begin{bmatrix} 0_{2\times 1} \\ \delta_\pi - \frac{1}{2}\sigma_\Pi^\intercal \sigma_\Pi \\ \delta_R + \eta_S - \frac{1}{2}\sigma_S^\intercal \sigma_S \end{bmatrix} + \begin{bmatrix} -K_X & 0_{2\times 2} \\ e_2^\top & 0_{1\times 2} \\ \left(\iota_2^\top - \sigma_\Pi^\intercal \Lambda_1\right) & 0_{1\times 2} \end{bmatrix} \begin{bmatrix} X_t \\ \log \Pi_t \\ \log S_t \end{bmatrix} \right) dt + \Sigma_K dZ_t, \quad (33)$$

with

$$\Sigma_K = \begin{bmatrix} \Sigma_X \\ \sigma_\Pi^\intercal \\ \sigma_S^\intercal \end{bmatrix}. \quad (34)$$

An unrestricted volatility matrix, Σ_K, would be statistically unidentified and, therefore, we impose

[3] We are grateful to Rob Bliss for providing the yield data.

[4] See `www.bls.gov` for further details.

that the volatility matrix is lower triangular, in line with Sangvinatsos and Wachter (2005), i.e.

$$\Sigma_K = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ \sigma_{12} & \sigma_2 & 0 & 0 \\ \sigma_{\Pi(1)} & \sigma_{\Pi(2)} & \sigma_{\Pi(3)} & 0 \\ \sigma_{S(1)} & \sigma_{S(2)} & \sigma_{S(3)} & \sigma_{S(4)} \end{bmatrix}. \tag{35}$$

Given this normalization, we can easily accommodate the parametric restrictions on the prices of risk. First of all, we assume that the price of real interest rate risk and the price of expected inflation risk are driven only by respectively the real interest rate and expected inflation. Hence, we find

$$\begin{aligned} \Lambda_t &= \Lambda_0 + \Lambda_1 X_t \\ &= \begin{bmatrix} \Lambda_{0(1)} \\ \Lambda_{0(2)} \\ 0 \\ * \end{bmatrix} + \begin{bmatrix} \Lambda_{1(1,1)} & 0 \\ 0 & \Lambda_{1(2,2)} \\ 0 & 0 \\ * & * \end{bmatrix} X_t, \end{aligned} \tag{36}$$

where the $*$ in the last row indicate that these parameters are chosen to satisfy the restriction that the equity risk premium is constant ($\sigma_S^\intercal \Lambda_0 = \eta_S$ and $\sigma_S^\intercal \Lambda_1 = 0$).

The estimation results are presented in Table 1, together with the standard errors on the basis of the outer product gradient estimator. The parameters σ_{u_1},..., σ_{u_6} are the volatilities of the measurement errors of the bond yields at the six maturities that have been used in estimation.

Insert Table 1

We find first of all that the level parameters, i.e. δ_R and δ_π, are close to the estimates reported by Sangvinatsos and Wachter (2005) (respectively 5.6% and 4.0%)[5]. It is well-known that the means of the short rate and expected inflation resulting from these models are usually low relative to the sample counterparts. However, we confirm the finding of Sangvinatsos and Wachter (2005) that when we adopt the essentially affine model for the term structure and we incorporate inflation data into the estimation, the means are estimated properly.

In line with Brennan and Xia (2002) and Campbell and Viceira (2001), we find that expected inflation is a much more persistent process than the real interest rate (i.e. $\kappa_1 > \kappa_2$). The instantaneous correlation between expected inflation and the real interest rate is slightly negative (-15%), in line with Brennan and Xia (2002) and Ang and Bekaert (2004). Hence, the Mundell-Tobin effect is supported by our estimates. We find that innovations in stock and bond returns are negatively correlated with inflation innovations, in line with Sangvinatsos and Wachter (2005).

Concerning the prices of risk, we find that the unconditional price of real interest rate risk is higher than the unconditional price of expected inflation risk, i.e. $\left|\Lambda_{0,(1)}\right| > \left|\Lambda_{0,(2)}\right|$, which is in line with Campbell and Viceira (2001) and Brennan and Xia (2002). This implies that the Sharpe ratio of nominal bonds will be lower than the Sharpe ratio of real bonds, simply due to the fact that real bonds

[5] Sangvinatsos and Wachter (2005) use data from 1952 up to 1998 to estimate their model for the financial market.

have no exposure to the expected inflation factor, which is awarded a lower compensation for risk. The equity risk premium is estimated to be 5.4%, reflecting the historical equity risk premium.

Apart from the unconditional term premia, we are interested in the impact of the enriched structure by allowing time-variation in bond risk premia. As pointed out by Dai and Singleton (2002), time-varying risk premia can be caused by time-varying volatilities and time-varying prices of risk. We confine ourselves to time-variation in the prices of risk. We find that the price of real interest rate risk significantly decreases with the level of the real interest rate and the price of expected inflation is significantly decreasing in the level of expected inflation. These estimates imply that nominal bonds become more attractive relative to real bonds when expected inflation is high. After all, in these cases, the price of expected inflation risk is low. As nominal bond returns are negatively correlated with expected inflation, the inflation risk premium is high when expected inflation is high, where we define the inflation risk premium as the difference between the risk premium on a nominal bond and a real bond with the same maturity, in line with Campbell and Viceira (2001). Next, we find that the price of real interest rate risk is negatively correlated with the real interest rate. This implies that the real term premium arising from real interest rate risk is high if the real interest rate is high. Interestingly, due to the Mundell-Tobin effect, real bonds of a particular maturity have a larger real interest rate exposure than nominal bonds with the same maturity. As a consequence, a high real interest rate implies a low price of real interest rate risk and, therefore, the risk premium on real bonds increases more than on nominal bonds. Thus, a high real interest rate tends to dampen the inflation risk premium, whereas high expected inflation rates tend to amplify the inflation risk premium. The same effects regarding the interplay between the real rate, expected inflation, and the inflation risk premium result as well from the general equilibrium model of Buraschi and Jiltsov (2005).

We summarize several implications of the estimates that are reported in Table 1. First of all, we consider the risk premia on both nominal and real bonds, as well as their volatilities and the inflation risk premium. Table 2 provides the results when the factors equal their unconditional expectation.

Insert Table 2

First of all, the nominal bond risk premia are in line with the estimates reported in Campbell and Viceira (2001). Secondly, real bonds tend to be much safer than nominal bonds, which is caused by the fact that real bonds do not have exposure to the expected inflation factor. Thirdly, the unconditional inflation risk premium for a 3M bond equals 22bp and 101bp for a 10Y bond. Buraschi and Jiltsov (2005) estimate the short-term inflation risk premium to be 25bp and the long-term at 70bp in a general equilibrium setting. Ang and Bekaert (2004) report an estimate of 97bp and Campbell and Viceira (2001) 110bp for long-term bonds. Hence, our estimates of the inflation risk premium are in line with the literature.

Table 3 provides the correlations between returns on nominal and real bonds, as well as between bond and stock returns. In addition, Table 3 reports the correlation between the risk premia on both nominal and inflation-linked bonds that we consider and the returns on the (possibly) traded assets.

Insert Table 3

We find that stock returns and nominal bond returns are positively correlated, in line with Sangvinatsos and Wachter (2005). Sangvinatsos and Wachter (2005) report a correlation between stock and nominal bond returns with different maturities that varies between $19.1 - 21.2\%$, which is close to our estimates. The correlation between nominal bond returns resembles their estimates as well. We find that the correlation among inflation-linked bonds is high, which is caused by the fact that we postulate a single factor term structure model for the real term structure. In addition, the correlation between long-term real bonds and nominal bonds is modest. This results from the fact that long-term nominal bonds have mainly exposure to expected inflation, whereas long-term inflation-linked bonds have only exposure to the real interest rate, see Figure 1. Since expected inflation and the real interest rate are negatively correlated, we find a small negative correlation between long-term nominal and inflation-linked bond returns.

Insert Figure 1

Panel B of Table 3 provides the implied correlations between the risk premia of both nominal and inflation-linked bonds that we consider and the returns on the traded assets. First of all, inflation-linked bond returns are negatively correlated with the real interest rate. Hence, a negative price of real interest rate risk translates into a positive risk premium for inflation-linked bonds. Since the price of real interest rate risk is negatively correlated with the real interest rate, the real bond risk premium is positively correlated with the real interest rate. Therefore, real bond returns are negatively correlated with the risk premium on real bonds.

Nominal bonds, on the other hand, are exposed to both the real rate and expected inflation factor. Moreover, both exposures are negative and the exposure to expected inflation exceeds the real interest rate exposure substantially, see Figure 1. The price of expected inflation risk is positively correlated with the real rate, but negatively correlated with expected inflation. As a negative price of real interest rate and expected inflation risk translates into positive bond risk premia, nominal bond returns are negatively correlated with nominal bond risk premia. Medium-term nominal bonds have a substantial real interest rate exposure, which induces a negative correlation between inflation-linked bond returns and medium-term nominal bond risk premia. Long-term nominal bonds, on the other hand, have hardly any exposure to the real rate, but mainly to the expected inflation factor. Due to the fact that the price of expected inflation risk is positively correlated with the real rate, we find that long-term nominal bond risk premia are positively correlated with inflation-linked bond returns.

In Table 4, we compare several summary statistics that follow from the discretized model to their raw sample counterparts. This provides additional insights in the fit of the model. Table 4 provides an overview of several important statistics for stock returns, inflation, 3M, 1Y, 5Y, and 10Y nominal bonds, on a monthly basis.

Insert Table 4

We find that the processes for stock returns and inflation are fitted properly. It is well-known that the means of the yields are often underestimated in models with constant risk premia and which do not exploit the information in inflation data to estimate the parameters. We therefore adopt the essentially affine term structure model as proposed by Duffee (2002) which largely solves the problem, once we incorporate inflation data into the estimation. Alternatively, Campbell and Viceira (2001) model bond risk premia to be constant and set the means equal to their sample counterparts.

Concerning the parameters of the income process, we rely on Viceira (2001). Hence, we choose the expected log growth in real income equal to three percent and the volatility of the log income growth is equal to ten percent per year. Initially, we assume that the correlation between the innovations of the labor income process and the financial assets is equal to zero, in line with the baseline specification Cocco et al. (2005). We infer in Section 6.2 the effects of the amount of idiosyncratic labor income risk. In Section 6.3, we assess the impact of correlations between labor income innovations and financial risks.

We consider an investor saving for retirement and allocating a fraction θ of its labor income to the retirement account. Hence, we exogenously fix the savings rate. Gomes et al. (2004) illustrate in a life-cycle context that exogenously fixing the savings rate results in small utility costs, provided that the savings rate is set properly. Moreover, it is noteworthy that choosing a different level θ only matters for the initial wealth considered. After all, the homogeneity property of the power utility index enables a factorization of the value function of the form $V(\bar{w}_0, \theta) = \theta^{1-\gamma} V(\bar{w}_0/\theta, 1)$. Hence, the results remain valid if one wants to consider a different savings rate, albeit that the interpretation of initial wealth has to be adjusted accordingly.

Denote the labor income at time t by L_t and assume

$$L_{t+1} = L_t \exp(g + \xi_{t+1}),$$

and $Y_{t+1} = \theta L_{t+1}$, implying

$$Y_{t+1} = Y_t \exp(g + \xi_{t+1}).$$

We set the savings rate equal to $\theta = 20\%$. In many countries, mandatory pension schemes enforce a substantial fraction of income to be invested for retirement purposes. In addition, employers often contribute a significant amount to pension funds. In an optimized life-cycle context, this fraction will obviously vary over time, implying that youngsters will save less than older people, see for instance Cocco et al. (2005) and Gomes and Michaelides (2005). We consider investment horizons up to 30 years, which means that we abstract from the early part of the life-cycle in which accumulation of funds is generally modest. For instance, Cocco et al. (2005) report savings in the order of magnitude of 6 months of labor income during the first decade of the life-cycle.

As mentioned earlier, the optimal portfolio is no longer independent of normalized wealth when we account for the income stream of the investor. Therefore, we have to determine the optimal portfolio for different initial values of normalized wealth. We consider two cases, namely one where the individual has accumulated only twenty percent of an annual salary, i.e. $\bar{w}_0 = 1$, and secondly, the case where the individual has saved 5 annual incomes, i.e. $\bar{w}_0 = 25$. By doing this, we consider rather extreme cases and the main effects are identifiable. It is important to note though that when the investment horizon increases, the investor has relatively more periods in which labor income is received. Hence, the value of human capital is larger in these cases. This implies, together with the fact that the role of labor income will be smaller if initial wealth is larger, we expect the results for $\bar{w}_0 = 25$ and short investment horizons to resemble closely the results without labor income.

4 Optimal portfolio choice without labor income

In this section, we examine the optimal portfolio choice when the investor does not receive labor income during the investment period. First of all, we determine the optimal portfolio choice for an investor who can trade continuously and is not subject to borrowing or short-sale constraints. Subsequently, we consider the discrete time problem, in which the optimal portfolio has to satisfy borrowing and short-sale constraints. To infer the optimal bond demand, we restrict the menu of assets to contain solely bonds in Section 4.1. In Section 4.2, we consider the optimal allocation between stocks and bonds, where we confine the bond part of the asset menu to be either nominal or real. These menus of assets have also been considered by Campbell and Viceira (2001) and enable us to assess the impact of labor income on the long-term bond demand as well as the stock-bond mix in Section 5. We extend the literature by drawing a comparison between the optimal (unconstrained) continuous time portfolio choice and the (constrained) discrete time portfolio choice in a financial market which accommodates time-variation in interest rates and inflation rates, as well as bond risk premia. In addition, by considering a finite horizon investment problem, we are able to address the effect of the investment horizon. Throughout, we consider the investment horizons $T = 1, 5, 10, 20$, and 30. We characterize three types of investors on the basis of their preference to bear risk, namely aggressive ($\gamma = 3$), moderate ($\gamma = 5$), and conservative ($\gamma = 7$).

4.1 Optimal bond demand

Table 5 provides the optimal portfolio for an investor who can trade continuously and is not subject to portfolio constraints. The asset menu contains 3Y nominal bonds, 10Y nominal bonds, and cash. Table 6 considers the optimal portfolio when the menu of assets is enriched with a long-term inflation-linked bond. Since the financial market in Section 2.1 postulates a two factor term structure model, two nominal bonds with different maturities suffice to create any exposure to the real interest rate and expected inflation factor. When a single inflation-linked bond is added to this asset menu, the financial market is completed and unexpected inflation risk can be hedged as well. Hence, any additional bond is redundant in this problem. Using the decomposition of the optimal portfolio in (20), we can identify the hedging demands induced by time-variation in either the real interest rate or expected inflation. In addition, we report the exposures to the real interest rate and expected inflation implied by the optimal portfolio.

Table 5 shows that the myopic demand contains a long position in 3Y nominal bonds that is financed by a short position in 10Y nominal bonds and borrowing cash. These positions are induced by the investor's desire to have a substantial exposure to the real interest rate. This is mainly caused by the large price of real interest rate risk in comparison to the price of expected inflation risk for aggressive investors, whereas conservative investors aim at synthesizing an inflation-linked bond. As follows from our estimation results in Table 1, we find expected inflation to be more persistent than the real interest rate. As a consequence, $B_2(\tau)$ will be larger than $B_1(\tau)$ in absolute values for every τ in absolute terms[6] and $B_1(\tau)/B_2(\tau)$ is decreasing in τ, see Figure 1. Therefore, to establish a large exposure to the real

[6] We will denote the exposure of a nominal bond to the real interest rate by B_1 and the exposure to expected inflation by B_2.

interest rate and thereby keeping a modest exposure to expected inflation, the investor needs to hold long the 3Y nominal bond and short the 10Y nominal bond.

Concerning the hedging demands, Table 5 provides the decomposition of the hedging portfolio as proposed in (20). We find substantial horizon effects, but the allocations implied by the myopic demand are extremely large and tend to dominate the optimal portfolio. The last two columns of Table 5 provide the exposures to the real interest rate and expected inflation as implied by the optimal portfolio. Comparing these numbers to Figure 1, we find that especially the exposure to the real interest rate is exceptionally large. There does not exist a single nominal bond with a particular maturity that can be used to create such a real interest rate exposure. The expected inflation exposure, on the other hand, tends to be more reasonable in the sense that these exposures are attainable with a single nominal bond. Such extreme positions in the different nominal bonds have been reported as well by Brennan and Xia (2002), Campbell and Viceira (2001), and Sangvinatsos and Wachter (2005) [7].

Insert Table 5

Insert Table 6

Table 6 assesses the role of inflation-linked bonds in the optimal portfolio. We enrich the asset menu with an inflation-linked bond with a maturity of 10 years. As two nominal bonds with different maturities suffice to create any exposure to the real interest rate and expected inflation factor, the role of real bonds is confined to the hedging demand for unexpected inflation. This improved ability to hedge unexpected inflation implies that the exposures to the real interest rate and expected inflation change marginally.

Insert Table 7

Apart from optimal portfolios, we can determine the utility gains[8] from enriching the investor's asset menu with inflation-linked bonds. The results are presented in the first column of Table 7. These utility gains have been determined as the fraction of initial wealth the individual must have additionally in case of the nominal asset menu to be indifferent to that asset menu and the one that does contain inflation-linked bonds. In the presence of labor income, we determine the fraction of total wealth an

[7] Hence, the extreme positions tend to arise due to the investor's desire to have an relatively extreme exposure to the real interest rate. Similar results have been found in Brennan and Xia (2002) and Van Hemert, De Jong, and Driessen (2005). We remark that these results suggest an interesting role for (real) interest rate derivatives, like interest rate and inflation swaps. After all, it is often remarked that the extreme positions arising from these kind of models are impossible in practice due to all kind of market imperfections. However, swaps are frequently used by institutional investors. These contracts can be seen as a very particular long-short position in bonds and hence can play a key role in constructing optimal portfolios to attain the desired exposures.

[8] Formally, we solve for ϕ in

$$\mathbb{E}_t\left[\frac{((1+\phi)W_{1T})^{1-\gamma}}{1-\gamma}\right] = \mathbb{E}_t\left[\frac{(W_{2T})^{1-\gamma}}{1-\gamma}\right],$$

i.e.

$$\phi = \left(\frac{\mathbb{E}_t\left[(W_{2T})^{1-\gamma}\right]}{\mathbb{E}_t\left[(W_{1T})^{1-\gamma}\right]}\right)^{\frac{1}{1-\gamma}} - 1,$$

where W_{1T} denotes optimal terminal wealth when the investor has no access to inflation-linked bonds and W_{2T} indicates optimal terminal wealth when the asset menu does contain inflation-linked bonds.

investor requires additionally in case of the nominal asset menu to be indifferent to having access to an asset menu that is enriched with inflation-linked bonds.

The value added of inflation-linked bonds is limited in this case to the ability to hedge unexpected inflation. In line with Brennan and Xia (2002), we find the value added of inflation-linked bonds to be rather limited to at most 1% for conservative long-term investors.

Next, we consider an investor who can trade on an annual basis and is subject to borrowing and short-sale constraints. Especially the inability of the investor to short particular assets or to borrow cash to invest in risky assets may increase the value added of real bonds substantially, see Campbell and Viceira (2001). Hence, we compare the same asset menus as before, in which the investor can select either between 3Y and 10Y nominal bonds, or this menu is enriched with a 10Y inflation-linked bond. In both cases, a nominal money market account is available to the investor as well. However, as can be deduced from Figure 1, the exposures attainable with the latter asset menu are exactly the same as with an asset menu that contains only 10Y nominal and inflation-linked bonds, when short-selling and borrowing cash is prohibited. We remark that this is an empirical finding and may not be the case for different values of the model parameters. Therefore, we confine henceforth the asset menu in the presence of inflation-linked bonds to contain only 10Y nominal and 10Y inflation-linked bonds, without loss of generality.

Table 8 reports the optimal portfolio allocations when investors are subject to borrowing and short-sale constraints and can trade only annually for both asset menus.

Insert Table 8

At least three aspects of Table 8 are noteworthy. First of all, as the risk aversion of the individual increases, the investor tilts the optimal portfolio towards 3Y nominal bonds. As 10Y nominal bonds have a large expected inflation-exposure, these securities are considered to be risky for conservative investors. As 3Y nominal bonds have a larger real interest rate exposure than 10Y nominal bonds, the investor shortens the duration of the optimal nominal bond portfolio as the risk aversion increases. This is in line with Campbell and Viceira (2001). If inflation-linked bonds are available, this effect is more prominent. Inflation-linked bonds allow the investor to build a larger real interest rate exposure without incurring a larger expected inflation exposure. We find indeed that as the risk aversion of the investor increases, the optimal portfolio shifts gradually towards inflation-linked bonds.

Secondly, when we consider the impact of the investment horizon, we find substantial horizon effects. For instance, when the investor can select between 3Y and 10Y nominal bonds, we find that an aggressive investor with a horizon of five years allocates 25% to long-term nominal bonds, whereas an investor with an investment horizon of thirty years allocates 58% of its wealth to 10Y nominal bonds. Hence, we find that the duration of the optimal nominal bond portfolio is increasing in the investment horizon, in line with Brennan and Xia (2002). This is apparent as well from the allocation between 10Y inflation-linked and 10Y nominal bonds. After all, lengthening the duration of a nominal bond portfolio is tantamount to a reduction in the real interest rate exposure and an increase in the expected inflation exposure. We indeed find that increasing the investor's investment horizon triggers a shift from long-term inflation-linked to long-term nominal bonds.

Thirdly, the decomposition of the optimal portfolio into a myopic demand and two hedging demands for either variation in the real interest rate and expected inflation provides a further understanding of the total bond demand. Interestingly, we find that the investment in 3Y nominal and 10Y inflation-linked bonds largely channels through the myopic demand. The demand for 10Y nominal bonds emerges due to the investor's desire to hedge time-variation in expected inflation. This hedging demand is generally long in 10Y nominal bonds and short in either 3Y nominal or 10Y inflation-linked bonds. This results from the fact that time-variation in expected inflation affects the investment opportunity set via the price of expected inflation risk, which is the most significant determinant of the risk premium on long-term nominal bonds. Table 3 shows that 10Y nominal bonds are negatively correlated with its risk premium and hence a positive position in 10Y nominal bonds can be used to hedge time-variation in expected inflation. Interestingly, as mentioned in Appendix C, it is not necessarily the case that the myopic and hedging demands sum to the total portfolio. However, we find empirically that the difference is marginal.

Finally, we remark that the standard errors resulting from the simulation-based approach are relatively small. The standard errors are at most 45bp. We find that the standard errors are increasing with the investment horizon, the riskiness of the asset mix, and there is a mixed effect of increasing the risk-aversion of the investor. Brandt et al. (2005) remark that when the relative risk aversion of the individual increases, standard errors decrease due to a more prudent investment strategy. On the other hand, the increased curvature of the value function tends to amplify the dispersion. We experience the same trade-off as an increase in the relative risk averseness has ambiguous effects for the accurateness of the optimal portfolio[9].

The second column of Table 7 determines the utility gains when the investor's asset menu contains real bonds as compared to the case where the investor can only select between nominal bonds. We find that the utility gains are sizeable, in line with Campbell and Viceira (2001). In all cases, the investor benefits from having access to real bonds. The larger the investment horizon and the more conservative the investor, the larger the utility gains. For instance, a conservative long-term investor gains almost 8%, whereas a moderately risk averse long-term investor gains somewhat more than 6%.

Our findings are largely in line with Campbell and Viceira (2001). However, as we consider a finite horizon investment problem, we are able to identify the horizon effects. Interestingly, we find that constrained investors allocate increasingly less wealth to inflation-linked bonds as the investment horizon increases. Hence, the horizon effects triggered by time-variation in expected inflation tend to dominate the horizon effects caused by real interest rates. This becomes especially apparent using the decomposition as outlined in Appendix C.

4.2 Optimal portfolio choice between stocks and bonds

In Tables 9 and 10 we consider the optimal portfolio choice between equities and long-term bonds, where the long-term bond is either nominal or real. In both cases, the investor has available a nominal cash account as well. In Table 9, we consider initially an investor who can trade continuously and who

[9] Detailed results on the accuracy of the simulation-based approach are available upon request.

is not subject to borrowing and short-sale constraints. We decompose the optimal portfolio according to the decomposition proposed in (20).

Insert Table 9

When the investor can choose between stocks and long-term nominal bonds, we find that stocks constitute a prominent part of the optimal portfolio which is mainly driven by the high equity risk premium. As the investor becomes more conservative, both stocks and long-term nominal bonds are relatively risky. Therefore, the investor chooses to invest a substantial part in cash, in line with Campbell and Viceira (2001) and Sangvinatsos and Wachter (2005). Interestingly, our decomposition shows that long-term nominal bonds are of little help to hedge real interest rate risk. As Figure 1 indicates, long-term nominal bonds have mainly exposure to expected inflation and only a modest exposure to the real interest rate.

We identify strong horizon effects in the demand for nominal bonds. As follows from Table 3, nominal bond returns are negatively correlated with nominal bond risk premia. As a consequence, a long position in nominal bonds can be used to hedge adverse changes in nominal bond risk premia. The horizon effects induced by time-variation in the real interest rate are very weak. Remark that the hedging demands are not necessarily monotone in the relative risk aversion parameter. After all, when the investor becomes more conservative, hedging becomes more important, which induces an increase in the hedging portfolio. On the other hand, the speculative demand reduces and therefore hedging time-variation in risk premia becomes less relevant, which leads to a reduction in the hedging portfolio.

Next, we consider the optimal portfolio when the investor allocates wealth between stocks, 10Y inflation-linked bonds, and cash. In this case, the myopic demand is characterized by enormous investments in inflation-linked bonds. Moreover, since inflation-linked bonds are negatively correlated with both the real interest rate and the risk premium on long-term inflation-linked bonds, the hedging demand for long-term real bonds is positive. As a consequence, the optimal portfolio contains large investments in inflation-linked bonds, regardless of the investor's investment horizon or risk preferences.

Table 10 considers the case where the investor can trade annually and is short-sale and borrowing constrained.

Insert Table 10

When the investor's asset menu contains stocks, long-term nominal bonds, and cash, we find the constraints only to be binding for aggressive investors. For the moderately risk averse as well as for the conservative investor, the fraction invested in equities is close to the exact continuous time solution. This can be regarded as well as a robustness check of the simulation-based approach.

Next, we do find that long-term conservative investors should have a higher ratio of long-term nominal bonds to stocks in the optimal portfolio, which is qualitatively in line with common investment advises as summarized in Canner et al. (1997). Nevertheless, quantitatively, the ratio can be very different for different investment horizons or risk preferences. We find that the fraction invested in long-term nominal bonds is reduced as the investor becomes more conservative. This is caused by the fact that long-term nominal bonds are risky due to their large expected inflation exposure. This effect is

amplified when the investor can only trade in discrete time, which explains why the fraction invested in the discrete time setting is somewhat lower than in continuous time. Regarding the composition of the optimal portfolio, we find that conservative investors invest mainly in 10Y nominal bonds for hedging motives. The investment in stocks turns out to be largely for speculative purposes. As before, we find that time-variation in the real rate hardly generates hedging demands.

Consider next the optimal asset allocation when the investor can invest in stocks, 10Y inflation-linked bonds, and cash. Since real bonds do not have the risky expected inflation exposure of nominal bonds, more conservative investors invest large fractions in real bonds, reducing the investment in equities and cash as compared to the asset menu with nominal bonds. For the reported precision, we are not able to detect any horizon effects. The reason is that investor do not have a strong incentive to hedge real interest rate risk in the constrained, discrete time setting and none of the assets can be used to hedge time-variation in expected inflation. This is reflected by the absence of hedging demands. The investor behaves myopically, provided this asset menu. In line with Munk and Sorensen (2005), we find that the ratio of inflation-linked bonds to stocks is increasing in the risk aversion of the investor.

Concerning the accuracy of the simulation-based portfolio choice, we find that the standard errors remain modest, even though the risky stock investment results in a somewhat larger dispersion of the estimated optimal portfolios. Nevertheless, the largest standard error we experience is 1.03% for the case where the investor is conservative and the investment horizon equals 30 years.

We conclude this section by remarking that the findings reported resemble the results of Campbell and Viceira (2001) for a fixed investment horizon. We show that their results largely carry over to the case where the investor obtains utility of terminal wealth and bond risk premia are time-varying. We complement their analysis by identifying the horizon effects, allowing for time-variation in bond risk premia, and providing a convenient decomposition of the total optimal portfolio that turns out to provide interesting insights in the constrained, discrete time problem.

5 Optimal portfolio choice in the presence of labor income

We consider the investment problem of an investor who receives a certain income stream during the investment period. We distinguish in this section two cases, namely one where initial wealth is low relative to the initial income, i.e. $\bar{w}_0 = 1$, which corresponds to having accumulated twenty percent of an annual salary in our set-up. Secondly, we consider the case where initial wealth is relatively high, i.e. $\bar{w}_0 = 25$, which corresponds to an accumulated wealth of five annual incomes. As the role of labor income is likely to be smaller as initial wealth is relatively larger and since the value of human capital is lower for shorter investment horizons, we expect the latter case to resemble more closely the results without labor income, especially for short investment horizons.

We consider the same asset menus as in the previous section. Recall that in the benchmark case we consider, it has been assumed that real labor income risk is idiosyncratic and the factors equal their unconditional expectation. In Section 6, we infer the robustness of these results to time-variation in bond risk premia, the amount of labor income uncertainty, and finally correlations between labor income uncertainty and financial risks.

5.1 Optimal bond demand

In the presence of labor income, Tables 11 and 12 provides the optimal asset allocation between either 3Y and 10Y nominal bonds or between 10Y inflation-linked and 10Y nominal bonds. In both cases, a nominal money market account is available to the investor as well.

Insert Tables 11 and 12

In order to understand the results, it is useful to recall the two main additional effects when the investor is endowed with labor income. First of all, the non-tradable position in labor income is a mixture of a fixed position in inflation-linked bonds and an idiosyncratic risk component. The former component increases the real interest rate exposure of the investor, which tends to decrease the desire for assets with a large real interest rate exposure, like inflation-linked bonds and 3Y nominal bonds. The second component induces an effective increase in the risk aversion of the investor, which tilts the optimal portfolio to either 3Y nominal or inflation-linked bonds, as we have seen in Section 4.1. Hence, the resulting effect remains an empirical question. The second important effect arises when labor income risk is correlated with returns on traded assets. First of all, these financial assets may be used to hedge unfavorable income changes, see, for instance, Viceira (2001) and Munk and Sorensen (2005). Secondly, when labor income risk is correlated with financial risks that are priced, the (implicit) value of human capital will be affected.

In our baseline specification, we assume that labor income uncertainty is idiosyncratic. In Section 6.2, we assess the effect of the amount of idiosyncratic labor income risk, whereas Section 6.3 infers the effects of correlations between labor income innovations and either the real interest rate, expected inflation or equity risk.

Tables 11 and 12 show that the impact of labor income on the optimal nominal bond portfolio is substantial. When initial wealth is low, see Table 11, which is especially the case for young individuals, we find that the optimal portfolio is heavily tilted towards long-term nominal bonds. For instance, a conservative investor with an investment horizon of twenty years invests 70% in 3Y nominal bonds and 30% in 10Y nominal bonds. In the presence of labor income and low initial wealth, these figures change to 2% in 3Y nominal bonds and 98% in 10Y nominal bonds. Hence, the implied real interest rate exposure is the dominant factor here. This triggers a demand for financial risks, other than the real interest rate. As 10Y nominal bonds are characterized by a larger expected inflation exposure and a smaller real interest rate exposure, we find that the duration of the optimal nominal bond portfolio is lengthened due to the presence of labor income. Interestingly, the decomposition of the optimal portfolio in Table 11 illustrates that for aggressive investors, the shift in the optimal portfolio channels largely via the myopic demand. For moderately risk averse and conservative investors, for which hedging time-variation in expected inflation is important, it turns out that the shift in the optimal portfolio is driven to a large extent by hedging motives as well. In addition, remark that the additivity of the portfolio components to the total portfolio is (almost) preserved in the presence of labor income.

When initial wealth is relatively high, see Table 12, which is mainly the case for older investors, we find that the results are close to the case without labor income, especially for short investment horizons.

Next, we enrich the asset menu with 10Y inflation-linked bonds. We then find a dramatic reduction

in the optimal fraction invested in inflation-linked bonds when initial wealth is relatively low, see Table 11. The optimal portfolio allocates large fractions to long-term nominal bonds, contrasting the case without labor income. For instance, a moderately risk averse investor with an investment horizon of 20 years invests half of its wealth in inflation-linked bonds. Once we account for labor income, all wealth is invested in long-term nominal bonds. In line with the portfolio problem without labor income, we find that when the investment horizon increases, the optimal portfolio shifts from long-term inflation-linked bonds to long-term nominal bonds. When initial wealth is high, see Table 12, the effects are mainly visible for an investor with a long investment horizon. This is a result of the fact that the human capital of the investor is larger in this case. Again, we find that for aggressive investors, the optimal portfolio changes channel via the myopic demand. For moderately risk averse and conservative investors, hedging motives turn out to be important as well.

To assess the impact of labor income on the value added as measured in utility terms, Table 7 reports the utility gains of having access to inflation-linked bonds. We find that the value added of having access to real bonds is reduced by approximately 30-40% in comparison to the case where the investor is not entitled to a labor income stream. For instance, the utility gains of a risk averse investor with an investment horizon of 30 years are reduced from 7.68% to 4.34%. When we consider the case where initial wealth is relatively high, we find that the reduction in utility gains is generally below 10% in order of magnitude. Hence, albeit that the introduction of labor income implies a substantial reduction in the utility gains of having access to inflation-linked bonds, the utility gains remain sizeable.

In sum, we find that properly accounting for labor income reduces the fraction invested in inflation-linked bonds and reduces the utility gains that the investor experiences from having access. This effect is most prominent for long-term conservative investors with relatively low initial wealth. When the investor's asset menu is confined to nominal bonds, we find that the duration of the optimal nominal bond portfolio is lengthened as labor income is incorporated into the investment problem.

5.2 Optimal portfolio choice between stocks and bonds

We consider the asset menu that contains either equities and 10Y nominal bonds or equities and 10Y inflation-linked bonds. In both cases, a nominal money market account is available to the investor as well. Tables 13 and 14 provides the optimal portfolios corresponding to these asset menus.

Insert Table 13 and 14

We find that the optimal portfolio is shifted almost fully towards equities when initial wealth is low, see Table 13. This result has been established for equities and inflation-linked bond by Munk and Sorensen (2005). We extend their results to the case where the investor's asset menu contains stocks and long-term nominal bonds. In case of inflation-linked bonds and equities, the investor is hardly able to build up any exposure to the expected inflation factor and we find the fraction invested in stocks to be even larger than when the investor can select among equities and nominal bonds. When we consider the decomposition of the optimal portfolio into the myopic demand and hedging demands for time-variation in the real interest rate and expected inflation, we find that the incorporation of labor income implies that the myopic portfolio constitutes the largest part of the optimal portfolio.

This contrasts the investment problem without labor income, in which the total portfolio contains a substantial hedging portfolio to offset unfavorable changes in expected inflation. The introduction of labor income tilts the myopic demand towards equities, thereby reducing the investor's desire to hedge time-variation in nominal bond risk premia. Interestingly, where the optimal portfolio contains a large investment in cash when the investor has only access to equities and nominal bonds in absence of labor income, once we account for labor income in the investment problem, this is no longer the case. Labor income provides a proper hedge against inflation and real interest rate risk and hence the desire to hedge inflation risk is reduced, which translates into an increased demand for equities.

When initial wealth is relatively high, see Table 14, we find again that the optimal portfolios in presence of labor income closely match the optimal portfolios in the absence of labor income.

Next, it is important to note that the earlier established result in absence of labor income, i.e. that the ratio of long-term bonds to stocks is increasing in the investor's risk aversion, remains valid, regardless whether the long-term bonds are nominal or real. This implies that the popular recommendations by investment advisors as summarized by Canner et al. (1997) are qualitatively robust to the introduction of labor income. Quantitatively, the results can depend strongly on the amount of initial wealth and labor income, as well as on the state of the economy, as illustrated in the next section. Hence, it is hard, not to say impossible, to provide proper investment advice without additional knowledge of the financial situation of the individual, like the amount of initial wealth, outside investments, and the investment horizon.

The results can be summarized as follows. The inclusion of labor income in the asset allocation problem tilts the optimal portfolio towards equities, reducing the amount invested in long-term bonds and cash. Hence, the role of long-term bonds is substantially mitigated in the presence of labor income. When initial wealth is high and labor income is a less prominent source of funding, the results are in line with the results in absence of labor income.

6 Financial robustness

6.1 Time-variation in bond risk premia

Our model of the financial markets as presented in Section 2.1 differs from the models of Campbell and Viceira (2001) and Brennan and Xia (2002) by allowing for time-variation in bond risk premia. Our model is dissimilar to the model of the financial market of Sangvinatsos and Wachter (2005) to the extent that we identify the factors to be the real rate and expected inflation, whereas Sangvinatsos and Wachter (2005) use three latent factors as driving forces of the term structure of interest rates. In modeling the prices of risk, we carry the separation between the nominal and the real side of the economy through by allowing the price real interest rate risk to depend only on the real interest rate, whereas the price of (pure) expected inflation risk depends only on expected inflation. Hence, it is natural to ask what the impact is of time-variation in the prices of risk on the optimal demand for long-term bonds and the impact on the value added of inflation-linked bonds.

We consider two different cases. First of all, we impose that the current real interest rate is two (unconditional) standard deviations above its unconditional expectation. Next, we consider the case where

the current level of expected inflation is two (unconditional) standard deviations above its unconditional expectation. When the real interest rate and expected inflation equal their unconditional expectations, the risk premia on 3Y nominal, 10Y nominal, and 10Y real bonds equal 1.25%, 1.98%, and 0.97%, see Table 2. When the real interest rate is two standard deviations above its unconditional expectation, these risk premia change to 2.53%, 1.99%, and 3.01%. Since $\Lambda_{1(1,1)}$ is estimated to be negative, the price of real interest rate risk is decreasing in the real interest rate, which translates into high bond risk premia in periods of high real rates. At the same time, since $\sigma_{12} < 0$, the inflation risk premium is decreasing in the real interest rate, which explains why the inflation risk premium on 10Y nominal bonds is low in this case. When expected inflation is two standard deviations above its unconditional expectation, the risk premia on these long-term bonds equal 3.10%, 7.86%, and 0.89%, respectively. Increases in expected inflation hardly change real bond risk premia, but do increase nominal bond risk premia and hence the inflation risk premium, since $\Lambda_{1(2,2)}$ is estimated to be negative.

Table 15 considers the portfolio implications of currently high real interest rates or expected inflation rates, both for an investor who is not endowed with labor income and for an investor who does receive labor income and has low initial wealth. We confine ourselves to the case where initial wealth is low, since the previous section has pointed out that the results for high levels of initial wealth closely resemble the results without labor income. The asset menu considered contains 3Y nominal and 10Y nominal bonds.

Insert Table 15

The optimal portfolio is tilted towards 3Y nominal bonds during periods of relatively high real interest rates. This holds true for both the case with and without labor income and is most pronounced for aggressive investors. This results from the fact that the real interest rate factor carries a lower price of risk, whereas in this case the expected inflation factor receives a higher price of risk. As 10Y nominal bonds have a larger exposure to expected inflation and a smaller exposure to the real interest rate than 3Y nominal bonds, the investor shortens the duration of the optimal nominal bonds portfolio. As inflation-linked bonds are perfectly suited to build up real interest rate exposure without incurring expected inflation exposure, the utility gains of having access to inflation-linked bonds increase as compared to the benchmark case, which is depicted in Table 7. The effects are most pronounced for aggressive investors with relatively short investment horizons.

Next, we consider the situation of a financial market that is characterized by high expected inflation rates. This results in a rise of the inflation risk premium, which makes 3Y nominal and 10Y inflation-linked bonds less attractive for myopic investors relative to long-term nominal bonds. The reason is that both 10Y inflation-linked bonds and 3Y nominal bonds have a larger real interest rate and a smaller expected inflation exposure than 10Y nominal bonds. This makes 10Y nominal bonds relatively more attractive in this state of the financial market.

The implications for the value added of inflation-linked bonds are depicted in Table 7. In line with the results on the optimal portfolios, we find the value added on inflation-linked bonds is lower in this case. Interestingly, the effects of the high real interest rate are most prominently visible for investors with relatively short investment horizons. When the current expected inflation is high, this has a dramatic impact on the utility gains for all investment horizons, and especially for aggressive investors.

The difference in horizon effects follows from the fact that expected inflation is far more persistent than the real interest rate. Hence, a shock to expected inflation has more important consequences for all investment horizons than a shock to the real interest rate. Consider for instance a conservative investor with an investment horizon of 20 years. The utility gains in the benchmark case when the investor is entitled to labor income equal 3.03%. An increase in the real interest rate increases the utility gain to 3.12%. On the other hand, an increase in expected inflation causes a drop in the utility gains to 1.89%.

However, when we compare an investor who is endowed with a stream of labor income to an investor who does not receive labor income during the investment period for the different values of the state variables, the reduction in utility gains for a long-term investor remain substantial. For instance, a conservative investor with an investment horizon of 20 years experiences a reduction in the utility gains of having access to inflation-linked bonds from 5.12% to 3.12% in case of high real interest rates and from 2.64% to 1.89% in periods with high expected inflation rates.

6.2 The impact of labor income uncertainty

The benchmark specification of the labor income that has been used postulates that labor income risk is idiosyncratic, in line with Cocco et al. (2005). In this section, we assess the impact of a reduction in the idiosyncratic real labor income uncertainty. It has been shown by, for instance, Letendre and Smith (2001) that the impact of different degrees of idiosyncratic labor income risk tends to be minor. Table 15 report the optimal portfolio allocations when we set labor income risk to zero, i.e. $\sigma_\xi = 0$.

As Viceira (2001) illustrates, an increase in idiosyncratic labor income risk induces an effective increase in the risk aversion of the investor. Hence, when we consider the case where real labor income risk is absent, the investor tends to select a riskier portfolio. After all, labor income constitutes in this case a portfolio of inflation-linked bonds and the investor has thus a fixed position in an asset that is riskless in real terms. Table 15 shows that the optimal portfolio indeed gradually shifts from 3Y nominal to 10Y nominal bonds. However, the effects are in almost all cases negligible, except for conservative long-term investors. However, in line with previous research, we find that these effects are rather weak.

This is confirmed as well by the reduction in utility gains when the investor's income is safer in real terms, see Table 7. Only long-term conservative investors experience a further reduction in utility gains in the order of magnitude of ten percent in comparison with an investor who is not endowed with labor income.

6.3 The impact of correlation between labor income risk and financial risks

Up to this point, we have assumed that labor income is always fully indexed with inflation and that innovations in labor income are uncorrelated with real interest rates, expected inflation, or equity risk. As such, labor income constitutes a excellent hedge against inflation. In this section, we are interested in the effects of correlations of labor income innovations with either the real interest rate, expected inflation or equity risk. As shown by Viceira (2001) for the case with correlation between equity returns and labor income innovations and by Munk and Sorensen (2005) for correlations between equity returns, real interest rates, and labor income innovations, such correlations trigger an additional hedging demand. For instance, if labor income innovations are positively correlated with real interest

rates, a long position in inflation-linked bonds can serve as a hedge against labor income uncertainty. After all, a negative shock to the real interest rate is then accompanied by a decrease in labor income, but the hedging portfolio provides a positive return on inflation-linked bonds.

A second effect that arises when there are correlations between labor income innovations and financial risks is that the value of human capital changes. Albeit that the value of non-tradable labor income is investor specific, as shown by, for instance, Munk and Sorensen (2005), the implicit value is still dependent on the correlation of labor income innovations with financial risks that are priced. Munk and Sorensen (2005) show numerically that the implicit value of human capital is decreasing in the correlation between labor income risk and equity risk. In our case, an increasing correlation between labor income innovations and either real interest rate or expected inflation is likely to increase the implicit value of human capital if the factors equal their unconditional expectations. After all, the corresponding prices of risk, $\Lambda_{0(1)}$ and $\Lambda_{0(2)}$, are estimated to be negative, see Table 1.

Hence, the results in this section will be driven by the interplay between the hedging potential of the different assets and the implications of a different implicit value of human capital.

Correlations between labor income and financial risks are likely to depend on the industry, profession, and individual characteristics, like education level, age, and gender. Most empirical research has been concentrated on the correlation between labor income risk and equity risk. Cocco et al. (2005) estimate the correlation between permanent labor income shocks and stock correlations between -1% and 2%. Heaton and Lucas (2000) report estimates between -7% and 14%. Munk and Sorensen (2005) provide an estimate for this correlation 17%. Finally, Davis and Willen (2000) report estimates between -25% and 30% for the correlations between a broad equity index and labor income innovations. Regarding the correlation between labor income risk and industry-specific equity risk, the correlation ranges between -10% and 40%, depending on the individual's education level, age, and gender. Munk and Sorensen (2005) provide an estimate of the correlation between labor income innovations and real interest rates of 26%.

As correlation estimates are likely to vary across different industries and among individuals with different characteristics, we determine the optimal portfolios for a range of correlations. We consider a conservative investor with an investment horizon of 10 years and low initial wealth. Table 11 indicates that this investor optimally invests 22% in 3Y nominal bonds and 78% in 10Y nominal bonds when the asset menu is restricted to nominal bonds in absence of any correlation. When inflation-linked bonds are added to the menu of assets, the investor allocates 30% to 10Y inflation-linked bonds and 70% to 10Y nominal bonds. Concerning the stock-bond mix, the investor allocates 66% to stocks and 34% to 10Y nominal bonds, or 71% to stocks and 29% to 10Y inflation-linked bonds, see Table 13.

We consider correlations that range between -60% and 60% for either the correlation with real interest rates, expected inflation, or equity risk. The results are presented in Figures 2 and 3.

Insert Figure 2

Insert Figure 3

The upper left graph in Figure 2 depicts the optimal portfolio choice between either 3Y and 10Y nominal bonds or 10Y nominal and 10Y inflation-linked bonds for different correlations between real interest rates and labor income innovations. As noted before, an increase in the correlation between

labor income innovations and the real interest rate increases the implicit value of human capital. As Section 5.1 shows, this causes a shift towards 10Y nominal bonds in both asset menus. On the other hand, long positions in 3Y nominal and, in particular, 10Y inflation-linked bonds provide a better hedge against real interest rate risk the larger the correlation. This is clearly perceptible in case inflation-linked bonds are present in the asset menu. After all, inflation-linked bonds have only exposure to the real interest rate, which causes inflation-linked bonds to be the prime instrument to hedge labor income uncertainty. This is reflected in the upper left graph of Figure 2. When the asset menu contains only nominal bonds, the optimal portfolio is tilted towards 10Y nominal bonds when the correlation increases, albeit that the effects are weak. Hence, the value effect dominates in this case. When the asset menu contains 10Y nominal and 10Y inflation-linked bonds, the optimal portfolio gradually shifts towards inflation-linked bonds as the correlation increases. Thus, in this case the hedging effect is dominating.

The latter result suggests that the desire to increase the real interest rate exposure is larger, the larger the correlation between labor income risk and real interest rates. This implies that the value added of inflation-linked bonds is increasing in the correlation between labor income and real interest rate innovations, see the bottom left graph of Figure 2.

Next, in the upper right graph of Figure 2, we consider the effect of correlation between labor income and expected inflation innovations. Up to now, labor income served as a excellent hedging instrument against inflation risk for long-term investors. However, it may very well be that for certain industries labor income is not always fully indexed with inflation, or with a different index. Therefore, we are interested in the sensitivity of our results for different correlations between labor income innovations and expected inflation. The two right graphs of Figure 2 present the results for the optimal portfolios and the value added of inflation-linked bonds for correlations in a range from -60% to 60%.

As the price of expected inflation risk is negative, the value of human capital is increasing with the correlation between labor income and expected inflation innovations. As mentioned before, this triggers a shift towards long-term nominal bonds for both asset menus. Moreover, as 10Y nominal bonds have a larger exposure to expected inflation, this asset also forms the best hedge against labor income risk. Hence, the value effect and the hedging effect work in the same direction in this case. This results in the optimal asset allocation as depicted in the upper right graph of Figure 2. For negative correlations, the optimal portfolio shifts towards 3Y nominal bonds in case of the nominal asset menu. When inflation-linked bonds are present in the asset menu, the optimal portfolio is tilted towards real bonds. As correlations increase, we find that the investor wants to increase the exposure to expected inflation. Hence the optimal portfolio allocates a larger fraction of wealth to 10Y nominal bonds. Comparing the two top graphs in Figure 2, we find that the effects of correlation between labor income and expected inflation innovations has a larger impact than the effect of correlation between labor income and real interest rate innovations on the composition of the optimal portfolio. Finally, the bottom right graph of Figure 2 depicts the value added of inflation-linked bonds for different values of the correlation between labor income and expected inflation innovations. In this case, the value and hedging effect point into the same direction and we find that the value added on inflation-linked bonds is decreasing in the correlation between labor income and expected inflation innovations.

Figure 3 considers effect of correlations between labor income risk and equity risk on either the

stock-bond mix or the demand for long-term bonds. Since the equity risk premium is estimated to be positive, the value effect implies that the implicit value of labor income is decreasing in the correlation between labor income risk and equity risk. Consider first of all the effect on the stock-bond mix. A lower value of human capital implies a shift towards long-term bonds, see Table 13. On the other hand, in case of a positive correlation between labor income risk and equity risk, a short position in equities can be used to hedge labor income risk. As such, the value and hedging effect are aligned. We indeed find that the optimal ratio of stocks to bonds is decreasing in the correlation between labor income risk and equity risk, both for nominal and inflation-linked bonds. It turns out that the optimal portfolio is rather sensitive to this parameter. Secondly, we consider the effect of correlations on the demand for long-term bonds. The value effect implies that the optimal portfolio is tilted towards 3Y nominal bonds or 10Y inflation-linked bonds, depending on the asset menu available to the investor. Table 3 indicates that all bond returns considered in the investor's asset menu are weakly positively correlated with equity risk. However, Figure 3 indicates that the correlations between labor income risk and equity risk hardly have an effect on the optimal demand for long-term bonds.

7 Conclusions

In this paper, we consider the impact of labor income on the optimal demand for long-term bonds. The effects of labor income on the investment problem are well-understood in simple financial markets as is the demand for long-term bonds in more realistic financial markets when the investor does not receive any form of income. However, their interplay is largely unexplored. Since labor income is often indexed with inflation, riskless labor income can be considered as a particular fixed investment in inflation-linked bonds. As such, it is likely that incorporation of labor income into the investment problem has important consequences for the optimal portfolio composition, even when labor income is not riskless in real terms. Apart from the optimal portfolio composition, accounting for labor income during the investment period is also likely to alter the conclusions regarding the enormous utility gains provided by having access to inflation-linked bonds. After all, these conclusions have been reached in models where the investor is not endowed with any form of income.

In the baseline case we consider, real labor income risk is fully idiosyncratic. We find indeed that accounting for labor income in the investment problem reduces the prominent role of inflation-linked bonds considerably. The optimal portfolio is tilted towards long-term nominal bonds and the utility gains of having access to inflation-linked bonds decline by 30-40% when the investor's initial wealth is relatively low. However, the utility gains remain sizeable and inflation-linked remain an important asset class, especially for conservative long-term investors. When the investor's asset menu contains only nominal bonds, we find that the duration of the optimal nominal bond portfolio is lengthened due to the incorporation of labor income into the investment problem.

Apart from the optimal allocation among nominal and inflation-linked bonds, we consider as well the optimal allocation to stocks and either nominal or inflation-linked bonds. Accounting for labor income during the investment period implies that the optimal portfolio is in many cases fully invested in stocks, especially for investors who are not too risk averse. Importantly, we do find that the ratio

of long-term bonds to stocks is increasing in the investor's risk aversion, which holds true for both nominal and inflation-linked bonds. This implies that we are, at least qualitatively, able to rationalize the investment advises as summarized in Canner et al. (1997).

We perform several robustness checks to verify our results. First of all, our results have been reached in a model which accommodates time-variation in bond risk premia. We find that the optimal portfolio allocations as well as the utility gains of having access to inflation-linked bonds are strongly affected by different values of the current bond risk premia. However, the main conclusions that follow from the comparison of the investment problem with and without labor income remain valid. Therefore, we conclude that the results are robust to time-variation in bond risk premia. Secondly, we vary the amount of idiosyncratic real labor income risk. This has hardly an effect on the results and we thus report that our results are robust to perturbations of idiosyncratic labor income risk. Thirdly, we introduce correlations between labor income risk and either the real interest rate, expected inflation or equity risk. We find that the fraction invested in inflation-linked bonds is positively related to correlation between labor income risk and real interest rate risk. As a consequence, the value added of inflation-linked bonds is increasing in the correlation between labor income risk and real interest rate risk. However, the effects are quantitatively modest. This contrasts sharply the implications of introducing correlation between labor income risk and expected inflation risk. Modifying this correlation may erode or strengthen the inflation hedging potential of labor income. We find dramatic portfolio implications for small changes in this correlation. As the correlation between labor income risk and expected inflation is negative, i.e. a deterioration of labor income risk as a hedge against inflation, we find that the value added of inflation-linked bonds increases. On the other hand, a positive correlation between labor income risk and expected inflation, i.e. an improvement of labor income as a hedge against inflation risk, leads to a decrease in the value added of inflation-linked bonds. However, for the correlations we have considered (from -60% up to 60%), the value added of inflation-linked bonds remains smaller than the utility gains in the investment problem where labor income is absent. Hence, we conclude that the composition of the optimal portfolio is sensitive to the correlation between labor income risk and expected inflation risk, but the value added of inflation-linked bonds is in all cases reduced.

Concerning the correlation between labor income risk and equity risk, we find that the optimal ratio of stocks to either nominal or inflation-linked bonds is increasing in this correlation. Quantitatively, the effects are strong. Finally, we find that this correlation has hardly an effect on the demand for long-term bonds, when equities are not part of the investor's asset menu.

The results in this paper have been derived by extending a recently developed simulation-based approach by Brandt et al. (2005). We illustrate how to account for short-sale and borrowing constraints in the investment problem. Apart from these extensions, we modify a particular approximation that has been criticized in the recent literature. We show that our approximation overcomes the shortcomings mentioned in DeTemple et al. (2003, 2005), delivers a simple optimization problem that can be solved fast under constraints, and we provide evidence that the accurateness is improved in the same example used in the original paper of Brandt et al. (2005). Apart from these extensions, we illustrate how the method can be used to decompose the optimal portfolio into the myopic demand and hedging demands induced by time-variation in the real rate and expected inflation.

This paper can be extended along different lines. First of all, we abstract from predictability in stocks returns as in Campbell and Viceira (1999) and Wachter (2002), despite the increasing evidence that stock returns are to some extent predictable, see for instance Ang and Bekaert (2003), Campbell and Yogo (2005), and Brennan and Xia (2005). This may provide a more conclusive answer on whether or not finance theory can rationalize popular investment advises and which variables are important to account for in the optimal portfolio composition. Secondly, we abstract from parameter uncertainty within this model. It is well-known from, for instance, Barberis (2000) and Wachter and Warusawitharana (2005) that accounting for parameter uncertainty may have a substantial influence on the optimal portfolio allocation. Finally, we abstract in this paper from intermediate consumption and the potential impact of housing. If these results are to be used within a life-cycle perspective, endogenous savings and consumption decisions become relevant, as well the ability to borrow against future labor income using a mortgage to finance a house. This may be an important step towards a life-cycle model that incorporates both flexible portfolio constraints, labor income, and a realistic model for the financial market.

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A Pricing nominal and inflation-linked bonds

We derive the nominal prices of both nominal and inflation-linked bonds in the financial market described in Section 2.1, following the results on affine term structure models in, for instance, Duffie and Kan (1996) and Sangvinatsos and Wachter (2005).

To that extent, we assume that both nominal and inflation-linked bond prices are smooth functions of time and the term structure factors X. Denote the price of a nominal bond at time t that matures at time T by $P(X_t, t, T)$. Since nominal bonds are traded assets, we must have that $\phi_t^{\$} P(X_t, t, T)$ is a martingale, where $\phi^{\$}$ is given in (10). This implies

$$-P_X K_X X + P_t + \frac{1}{2} tr\left(\Sigma_X^\top P_{XX} \Sigma_X\right) - RP - P_X^\intercal \Sigma_X \Lambda = 0, \tag{A.1}$$

where the subscripts of P denote partial derivatives with respect to the different arguments. Subsequently, Duffie and Kan (1996) have shown that in this case, when the diffusion of the state variables under the risk neutral measure is affine in the state variables and the instantaneous nominal short rate is affine in the state variables, we obtain nominal bond prices that are exponentially affine in the state variables, i.e.,

$$P(X, t, t+\tau) = \exp\left(A(\tau) + B(\tau)^\top X\right). \tag{A.2}$$

Substituting this expression in (A.1) and matching the coefficients on the constant and the state variables X, we obtain the following set of ordinary differential equations

$$\begin{aligned} A'(\tau) &= -B(\tau)^\top \Sigma_X \Lambda_0 + \frac{1}{2} B(\tau)^\top \Sigma_X \Sigma_X^\top B(\tau) - \delta_R, & \text{(A.3)} \\ B'(\tau) &= -\left(K_X^\top + \Lambda_1^\intercal \Sigma_X^\intercal\right) B(\tau) - \left(\iota_2 - \sigma_\Pi^\intercal \Lambda_1\right), & \text{(A.4)} \end{aligned}$$

where ι_2 denotes a two dimensional vector of ones. We also have the boundary conditions

$$A(0) = 0, \; B(0) = 0. \tag{A.5}$$

The ODEs can be solved in closed form, see for instance Dai and Singleton (2002). This leads to

$$\begin{aligned} B(\tau) &= \left(K_X^\top + \Lambda_1^\intercal \Sigma_X^\intercal\right)^{-1} \left[\exp\left(-\left(K_X^\top + \Lambda_1^\intercal \Sigma_X^\intercal\right)\tau\right) - I_{2\times 2}\right] \left(\iota_2 - \sigma_\Pi^\intercal \Lambda_1\right), & \text{(A.6)} \\ A(\tau) &= \int_0^\tau A'(s) ds, & \text{(A.7)} \end{aligned}$$

where $I_{2\times 2}$ denotes the two by two identity matrix.

For inflation-linked bonds, the derivation is slightly more involved. In this case, the nominal price of a real bond is denoted by the product $\Pi_t P^R(X, t, T)$. The martingale property of $\phi_t^{\$} \Pi_t P^R(X_t, t, T)$ leads to

$$-P_X^R K_X X + P_t^R + \frac{1}{2} tr\left(\Sigma_X^\top P_{XX}^R \Sigma_X\right) - (R - \pi + \sigma_\Pi^\intercal \Lambda) P^R + P_X^{R\intercal} \Sigma_X \left(\sigma_\Pi - \Lambda\right) = 0, \tag{A.8}$$

Since we postulate that the instantaneous expected inflation is affine in the state variables, the price process corresponding to holding a real bond is also affine under the risk-neutral measure and we conjecture

$$P^R(X, t, t+\tau) = \exp\left(A^R(\tau) + B^R(\tau)^\top X\right), \tag{A.9}$$

implying that (A.8) boils down to

$$-B^R(\tau)^\top K_X X - A'^R(\tau) - B'^R(\tau)^\top X + \frac{1}{2}B^R(\tau)^\top \Sigma_X \Sigma_X^\top B^R(\tau) - r + B^R(\tau)^\intercal \Sigma_X \left(\sigma_\Pi - \Lambda\right) = 0. \quad \text{(A.10)}$$

We again match the coefficients on the constant and the state variables X, which leads to the following set of ordinary differential equations

$$\begin{aligned} A'^R(\tau) &= \frac{1}{2}B^R(\tau)^\top \Sigma_X \Sigma_X^\top B^R(\tau) - \left(\delta_R - \delta_\pi + \sigma_\Pi^\intercal \Lambda_0\right) + \left(B^R(\tau)^\top \Sigma_X\right)\left(\sigma_\Pi - \Lambda_0\right); \quad \text{(A.11)} \\ B'^R(\tau) &= -\left(K_X^\top + \Lambda_1^\intercal \Sigma_X^\intercal\right) B^R(\tau) - e_1, \end{aligned}$$

where e_i denotes the i-th unit vector. Again we can find easily an expression for $B^R(\tau)$, i.e.,

$$B^R(\tau) = \left(K_X^\top + \Lambda_1^\top \Sigma_X^\intercal\right)^{-1} \left(\exp\left(-\left(K_X^\top + \Lambda_1^\intercal \Sigma_X^\intercal\right)\tau\right) - I_{2\times 2}\right) e_1. \quad \text{(A.12)}$$

B Continuous time optimal portfolio choice

We first of all summarize the result of Sangvinatsos and Wachter (2005) concerning the optimal portfolio. Next, we address the decomposition proposed in (20). Sangvinatsos and Wachter (2005) show for any affine strategy[10]

$$x_t(\zeta_0, \zeta_1, X_t) = \zeta_0(\tau) + \zeta_1(\tau) X_t,$$

where τ indicates the investor's investment horizon. The expected utility of following this strategy is exponentially quadratic in the state variables, i.e.

$$\begin{aligned} V_1\left(w_t, X_t, t, T\right) &= \mathbb{E}_t\left(\frac{\left(w_T\right)^{1-\gamma}}{1-\gamma}\right) \quad \text{(B.1)} \\ &= \frac{\left(w_t\right)^{1-\gamma}}{1-\gamma} \exp\left(\left[\frac{1}{2}X_t^\top \Gamma_3(\tau) X_t + \Gamma_2^\top(\tau) X_t + \Gamma_1(\tau)\right]\right) \end{aligned}$$

with $w_t = W_t/\Pi_t$ denotes real wealth and $\tau = T - t$. The parameters Γ_2 and Γ_3 satisfy[11] the system of differential equations, where we omit the argument τ and the subscript of V for notational convenience

$$\begin{aligned} \Gamma_3' &= \left(\Gamma_3 + \Gamma_3^\top\right)\left(\Sigma_X \Sigma^\intercal \zeta_1 (1-\gamma) - K_X\right) + \frac{1}{4}\left(\Gamma_3 + \Gamma_3^\top\right) \Sigma_X \Sigma_X^\top \left(\Gamma_3 + \Gamma_3^\top\right) \quad \text{(B.2)} \\ &\quad 2(1-\gamma)\zeta_1^\intercal \Sigma \Lambda_1 - \gamma(1-\gamma)\zeta_1^\intercal \Sigma \Sigma^\intercal \zeta_1; \\ \Gamma_2'^\top &= \Gamma_2^\top \left((1-\gamma)\Sigma_X \Sigma^\intercal \zeta_1 + \frac{1}{2}\Sigma_X \Sigma_X^\top \left(\Gamma_3 + \Gamma_3^\top\right) - K_X\right) + \frac{1}{2}(1-\gamma)\left(\zeta_0^\intercal \Sigma - \sigma_\Pi^\intercal\right) \Sigma_X^\top \left(\Gamma_3 + \Gamma_3^\top\right) \\ &\quad +(1-\gamma)\left(e_1 + \zeta_0^\intercal \Sigma \Lambda_1 + \Lambda_0^\top \Sigma^\top \zeta_1\right) - \gamma(1-\gamma)\zeta_0^\intercal \Sigma \Sigma^\intercal \zeta_1 - (1-\gamma)^2 \sigma_\Pi^\intercal \Sigma^\top \zeta_1. \end{aligned}$$

In addition, we have the boundary equations

$$\Gamma_3(0) = 0, \ \Gamma_2(0) = 0. \quad \text{(B.3)}$$

[10] We confine ourselves to affine strategies since Sangvinatsos and Wachter (2005) have shown that the optimal strategy belongs to this class of portfolio strategies.

[11] The expression for Γ_1 is not required for the optimal portfolio and hence we omit it. We refer those interested in the value function to Sangvinatsos and Wachter (2005).

In order to derive the optimal portfolio, we solve the Hamilton-Jacobi-Bellman (HJB) equation for this problem. Therefore, we need the dynamics of real wealth, which follows using Ito's lemma applied to (19), i.e.

$$\frac{dw_t}{w_t} = \left[(r_t + x_t(\zeta_0, \zeta_1, X_t)^\intercal \Sigma (\Lambda_t - \sigma_\Pi) + \sigma_\Pi^\top \sigma_\Pi - \sigma_\Pi^\top \Lambda_t\right] dt + (x_t(\zeta_0, \zeta_1, X_t)^\intercal \Sigma - \sigma_\Pi^\intercal) dZ_t. \tag{B.4}$$

The HJB equation reads subsequently as

$$0 = \max_x \begin{bmatrix} V_t + wV_w \left[(r + x^\intercal \Sigma (\Lambda - \sigma_\Pi) + \sigma_\Pi^\top \sigma_\Pi - \sigma_\Pi^\top \Lambda_t\right] + \\ \frac{1}{2} w^2 V_{ww} \left(x^\intercal \Sigma - \sigma_\Pi^\top\right)\left(x^\intercal \Sigma - \sigma_\Pi^\top\right)^\intercal - V_X^\intercal K_X + \frac{1}{2} tr\left(V_{XX} \Sigma_X \Sigma_X^\intercal\right) \\ + w \left(x^\intercal \Sigma - \sigma_\Pi^\top\right) \Sigma_X^\intercal V_{wX} \end{bmatrix}, \tag{B.5}$$

where the subscripts of V denote partial derivatives with respect to the different arguments. Consequently, the optimal portfolio should satisfy the first order condition

$$wV_w \Sigma (\Lambda - \sigma_\Pi) + w^2 V_{ww} \Sigma \Sigma^\intercal x^* - w^2 V_{ww} \Sigma \sigma_\Pi + w \Sigma \Sigma_X^\intercal V_{wX} = 0. \tag{B.6}$$

This results in the following expression for the optimal portfolio

$$\begin{aligned} x_t^* &= -\frac{V_w}{wV_{ww}} (\Sigma\Sigma^\intercal)^{-1} \Sigma (\Lambda_t - \sigma_\Pi) + (\Sigma\Sigma^\intercal)^{-1} \Sigma \sigma_\Pi - (\Sigma\Sigma^\intercal)^{-1} \Sigma \Sigma_X^\intercal \frac{V_{wX}}{wV_{ww}} \qquad \text{(B.7)} \\ &= \frac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \Lambda_t + \left(1 - \frac{1}{\gamma}\right) (\Sigma\Sigma^\intercal)^{-1} \Sigma \sigma_\Pi + \frac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \Sigma_X^\intercal \left(\frac{1}{2}\left(\Gamma_3 + \Gamma_3^\top\right) X_t + \Gamma_2\right) \\ &= \zeta_0^* + \zeta_1^* X_t, \end{aligned}$$

with

$$\zeta_0^* = \frac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \Lambda_0 + \left(1 - \frac{1}{\gamma}\right) (\Sigma\Sigma^\intercal)^{-1} \Sigma \sigma_\Pi + \frac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \Sigma_X^\intercal \Gamma_2; \tag{B.8}$$

$$\zeta_1^* = \frac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \Lambda_1 + \frac{1}{2} \frac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \Sigma_X^\intercal \left(\Gamma_3 + \Gamma_3^\top\right), \tag{B.9}$$

where Γ_2 and Γ_3 solve (B.2). This summarizes the results derived in Sangvinatsos and Wachter (2005). However, due to the fact that we interpret the factors to be the real interest rate and expected inflation, we can conveniently decompose the optimal portfolio, i.e.

$$\begin{aligned} x_t^* &= \tfrac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \Lambda_t + \left(1 - \tfrac{1}{\gamma}\right) (\Sigma\Sigma^\intercal)^{-1} \Sigma \sigma_\Pi + \tfrac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \Sigma_X^\top \left(\tfrac{V_X}{V}\right) \\ &= \tfrac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \Lambda_t + \left(1 - \tfrac{1}{\gamma}\right) (\Sigma\Sigma^\intercal)^{-1} \Sigma \sigma_\Pi \qquad \text{(B.10)} \\ &\quad + \tfrac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \sigma_1 \left(\tfrac{V_r}{V}\right) + \tfrac{1}{\gamma} (\Sigma\Sigma^\intercal)^{-1} \Sigma \sigma_2 \left(\tfrac{V_\pi}{V}\right). \end{aligned}$$

C Hedging demands in discrete time

In the unconstrained and continuous time investment problem, it is possible to disentangle the myopic demand and hedging demands induced by time-variation in either the real interest rate and expected inflation, see the results of Sangvinatsos and Wachter (2005) and in particular (20). This appendix proposes an extension of this concept to a discrete time setting, possibly in the presence of labor income.

First, consider the investment problem in absence of labor income, i.e. (28). In this case, the myopic demand is generally defined as the solution to a single period investment problem. As shown by

Samuelson (1969), the myopic demand is also the optimal portfolio strategy in a multi-period investment problem as long as interest rates are constant and asset returns are i.i.d. This implies that the myopic demand equals the multi-period demand in a world where state variables are, each period, reset to their initial values. More precisely, using the fact that our state variables follow a Markov process, this world uses the one-period transition density $p(X_{t+1} \mid X_t = x_0)$ instead of $p(X_{t+1} \mid X_t = x_t)$. The myopic demand can be derived, therefore, also by simulating the state variables for a single period, determining the asset returns and subsequently resetting the state variables to their initial values. Remark that it is not possible to assume that the factors remain constant, as this implies that bond returns are deterministic. This market induces i.i.d. asset returns, which implies that the optimal strategy in the multi-period problem and the single period problem coincide.

Given the myopic demand, the total hedging demand is defined as the difference between the optimal multi-period demand and the myopic demand. In order to decompose the total hedging demand into a hedging demand caused by time-variation in either the real rate or expected inflation, we recall given the structure of the financial market model in Section 2.1

$$\begin{bmatrix} X_{1,t+1} \\ X_{2,t+1} \end{bmatrix} = \begin{bmatrix} \exp(\kappa_1) & 0 \\ 0 & \exp(\kappa_2) \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix}.$$

As a consequence, we have, with $i = 1, 2$,

$$\mathcal{L}(X_{i,t+1} \mid \mathcal{F}_t) = \mathcal{L}(X_{i,t+1} \mid X_{it}),$$

where $\mathcal{F}_t$ denotes the natural filtration generated by $(\varepsilon_t)_{t\geq 0}$. Next, in order to assess the impact on portfolio choice of time-variation in the real rate, X_1, alone, we follow the ideas above. More precisely, reset only X_2 to its initial value in each period. This implies that asset returns, conditional on X_1, become i.i.d. Moreover, the term structure of expected inflation rates, defined as $\mathbb{E}_t(\Pi_{t+\tau}/\Pi_t)$ for $\tau \geq 0$, is time-invariant. The difference between the solution to this multi-period problem and the single period problem will be called the hedging demand induced by time-variation in the real interest rate. Along similar lines, we can reset X_2 to its initial value. This implies that asset returns, conditional on X_2, become i.i.d. In that case, the real term structure becomes time-invariant. We refer to the hedging demand arising in this investment problem as the hedging demand induced by time-variation in expected inflation.

Decompositions of the total portfolio into myopic and hedging demands have generally two shortcomings. First of all, they account only for single period correlation between the real interest rate and expected inflation. If multi-period correlations between real interest rates and expected inflation rates have strong implications for the hedging demands, the two components of the hedging demand will not sum to the total hedging demand. Secondly, as we consider constrained portfolio problems, short-sale and borrowing constraints may hamper additivity of the different components of the hedging portfolio to sum to the total hedging portfolio. However, in our applications it turns out that these shortcomings hardly constitute a problem from an empirical perspective.

Next, we consider the investment problem in the presence of labor income, i.e. (32). In this investment problem, the conventional definition of hedging demands, namely the optimal strategy that solves the single period problem, cannot be used. After all, an investor with multiple periods ahead has a different entitlement to labor income than an individual that faces a single period investment problem.

However, in absence of labor income, resetting the state variables results in the same optimal portfolio, independent of the investment horizon, as the single period (myopic) portfolio. Therefore, we extend the concept to the case with labor income and reset the state variables after a single period simulation in order to determine the myopic demand in the presence of labor income. This enables a decomposition of the optimal portfolio choice into the myopic and total hedging demand. As before, the total hedging demand can be decomposed into a hedging demand induced by time-variation in the real rate and time-variation in expected inflation rates. It is noteworthy that our empirical results indicate that the difference between the total demand and the sum of the myopic demand and the different hedge demands as constructed, as discussed before, is negligible.

D Estimation procedure

Our estimation procedure in closely related to Sangvinatsos and Wachter (2005). The main difference is that we allow all yields to be measured with error, following De Jong (2000), Brennan and Xia (2002), and Campbell and Viceira (2001). However, we assume that the measurement errors are independent, both sequentionally and cross-sectionally. The continuous time equations underlying the financial market in Section 2.1 can be written as

$$\begin{aligned} d\begin{bmatrix} X_t \\ \log \Pi_t \\ \log S_t \end{bmatrix} &= \left(\begin{bmatrix} 0_{2\times 1} \\ \delta_\pi - \frac{1}{2}\sigma_\Pi^\intercal \sigma_\Pi \\ \delta_R + \eta_S - \frac{1}{2}\sigma_S^\intercal \sigma_S \end{bmatrix} + \begin{bmatrix} -K_X & 0_{2\times 2} \\ e_2^\top & 0_{1\times 2} \\ \left(\iota_2^\top - \sigma_\Pi^\intercal \Lambda_1\right) & 0_{1\times 2} \end{bmatrix} \begin{bmatrix} X_t \\ \log \Pi_t \\ \log S_t \end{bmatrix} \right) dt + \begin{bmatrix} \Sigma_X \\ \sigma_\Pi^\intercal \\ \sigma_S^\intercal \end{bmatrix} dZ_t \\ &= \left(\Theta_0 + \Theta_1 K_t\right) dt + \Sigma_K dZ_t. \end{aligned} \tag{D.1}$$

with

$$K_t = \begin{bmatrix} X_t \\ \log \Pi_t \\ \log S_t \end{bmatrix}. \tag{D.2}$$

As Y_t follow a standard multivariate multivariate Ornstein-Uhlenbeck process, we may write the exact discretization (see, e.g., Bergstrom (1984) and Sangvinatsos and Wachter (2005))

$$K_{t+h} = \mu^{(h)} + \Gamma^{(h)} K_t + \varepsilon_{t+h}, \tag{D.3}$$

where $\varepsilon_{t+h} \overset{i.i.d.}{\sim} N\left(0_{4\times 1}, \Sigma^{(h)}\right)$ for appropriate $\mu^{(h)}$, $\Gamma^{(h)}$, and $\Sigma^{(h)}$ which we derive below. To derive the discrete time parameters, we consider the eigenvalue decomposition[12] $\Theta_1 = UDU^{-1}$. The parameters in the VAR(1) - model relate to the structural parameters via

$$\begin{aligned} \Gamma^{(h)} &= \exp\left(\Theta_1 h\right) = U \exp\left(Dh\right) U^{-1}; \\ \mu^{(h)} &= \left[\int_t^{t+h} \exp\left(\Theta_1 \left[t+h-s\right]\right) ds\right] \Theta_0 \\ &= UFU^{-1}\Theta_0, \end{aligned} \tag{D.4}$$

[12] Note that, since K_X is a diagonal matrix, the eigenvalues of Θ_1 are given by κ_1, κ_2, and 0 (with multiplicity two). Recall that a square matrix is diagonalizable if and only if the dimension of the eigenspace of every eigenvalue equals the multiplicity of the eigenvalue. This condition is satisfied for Θ_1.

where F is a diagonal matrix with elements $F_{ii} = h\alpha(D_{ii}h)$, with

$$\alpha(x) = \frac{\exp(x) - 1}{x},$$

and $\alpha(0) = 1$. The derivation of $\Sigma^{(h)}$ is a bit more involved. We have

$$\begin{aligned} \Sigma^{(h)} &= \int_t^{t+h} \exp\left(\Theta_1 \left[t + h - s\right]\right) \Sigma_K \Sigma_K^\top \exp\left(\Theta_1 \left[t + h - s\right]\right) ds \\ &= UVU^\top, \end{aligned} \tag{D.5}$$

where V is a matrix with elements

$$\begin{aligned} V_{ij} &= \left[\int_t^{t+h} \exp\left(D\left[t + h - s\right]\right) U^{-1} \Sigma_K \Sigma_K^\top \left(U^{-1}\right)^\top \exp\left(D\left[t + h - s\right]\right) ds\right]_{ij} \\ &= \left[U^{-1}\Sigma_K\Sigma_K^\top\left(U^{-1}\right)^\top\right]_{ij} \int_t^{t+h} \exp\left(\left[D_{ii} + D_{jj}\right]\left[t + h - s\right]\right) ds \\ &= \left[U^{-1}\Sigma_K\Sigma_K^\top\left(U^{-1}\right)^\top\right]_{ij} h\alpha\left(\left[D_{ii} + D_{jj}\right]h\right). \end{aligned} \tag{D.6}$$

Using data on six yields, stock returns, and inflation, we estimate the model using the Kalman filter. The transition equation is given by (D.3). We assume that all yields are measured with measurement error, in line with De Jong (2000), Brennan and Xia (2002), and Campbell and Viceira (2001). On the other hand, Duffee (2002) and Sangvinatsos and Wachter (2005) select certain maturities and fit these exactly, which is tantamount to identifying the factors. In line with all these papers[13], we assume the measurement to be Gaussian and independent of the innovations in the transition equation. The likelihood can subsequently be constructed using the error-prediction decomposition, see for instance Harvey (1989).

E Simulation-based portfolio choice

We extend the simulation-based approach to portfolio choice as it has been introduced by Brandt et al. (2005) along two lines. First of all, we incorporate short-sale and borrowing constraints. Secondly, we show how to account for the income stream. Apart from both extensions, we address the criticism raised by DeTemple et al. (2003, 2005).

Consider first of all an investor whose portfolio choice is subject to short-sale and borrowing constraints. We abstract initially from labor income. The problem is then given by

$$V_2(t, T, X_t, w_t) = \max_{(x_s)_{s=t}^{T-1} \in \mathcal{K}} \mathbb{E}_t\left(\frac{1}{1-\gamma}\left(w_T\right)^{1-\gamma}\right), \tag{E.1}$$

with w_t indicating real wealth, subject to

$$w_{t+1} = w_t\left(x_t^\top r_{t+1}^e + r_{t+1}^f\right), \tag{E.2}$$

[13] De Jong (2000) is a notable exception to the extent that he allows for cross-sectional correlation between the measurement errors.

with

$$\mathcal{K} = \{x \mid x \geq 0, x^\top \iota \leq 1\}. \tag{E.3}$$

The principle of dynamic programming is used to determine the optimal portfolio strategy. Starting at time $T-1$, we first solve

$$\max_{x \in \mathcal{K}} \mathbb{E}_{T-1} \left(\frac{1}{1-\gamma} \left(x^\top r_T^e + r_T^f \right)^{1-\gamma} \right), \tag{E.4}$$

where the homogeneity of the power utility index is exploited. The main complication is that this conditional expectation cannot be calculated analytically. In line with Brandt et al. (2005) and Longstaff and Schwartz (2001), we approximate the conditional expectation via a projection on a set of basis functions in the state variables, i.e.

$$\mathbb{E}_{T-1} \left(\frac{1}{1-\gamma} \left(x^\top r_T^e + r_T^f \right)^{1-\gamma} \right) \simeq \zeta(x)^\top f\left(X_{T-1}\right), \tag{E.5}$$

where $\zeta(x)$ denote the projection coefficients, which are functions of the portfolio choice, x. In order to estimate the projection coefficients, $\zeta(x)$, we simulate M paths of both state variables and asset returns on the basis of the discretized model. We indicate the paths by $(\omega_1, ..., \omega_M)$. Next, the projection coefficients are estimated via a cross-sectional regression across all paths, which results in the following estimator

$$\hat{\zeta}(x) = \left(\sum_{i=1}^{M} f\left(X_{T-1}(\omega_i)\right) f\left(X_{T-1}(\omega_i)\right)^\top \right)^{-1} \left(\frac{1}{1-\gamma} \sum_{i=1}^{M} f\left(X_{T-1}(\omega_i)\right) \left(x^\top r_T^e(\omega_i) + r_T^f(\omega_i) \right)^{1-\gamma} \right), \tag{E.6}$$

where $f\left(X_{T-1}(\omega_i)\right)$ is a column vector containing the values of the basis functions, evaluated at the state variables in branch ω_i. Regarding the choice of the basis functions, details are provided at the end of this section.

Next, we determine the optimal portfolio in every branch ω_i

$$x_{T-1}^*(\omega_i) = \arg\max_{x \in \mathcal{K}} \hat{\zeta}(x)^\top f\left(X_{T-1}(\omega_i)\right), \tag{E.7}$$

$i = 1, ..., M$. It is important to note that when we determine the optimal portfolio choice in a certain branch, say ω_i, then for every function evaluation for a different choice of x, the projection coefficients need to be recalculated, which requires a cross-sectional regression. As M is typically large, this procedure turns out to be extremely time-consuming.

To accelerate the latter step, Brandt et al. (2005) suggest to determine a fourth order expansion of the utility index and solve the optimal portfolio from this expansion. Solving this fourth order expansion is done using an iterative procedure that is initiated in the solution to the second order expansion[14]. The solution based on a second order expansion can be determined in closed-form. The main advantage of this approximation is that the optimization can be done simultaneously over all paths, which makes

[14] More precisely, Brandt et al. (2005) propose to approximate

$$\begin{aligned} \mathbb{E}_t \left(u\left(x^\top R_{t+1} + R_t^f \right) \right) \simeq\ & u(c) + u'(c)\, \mathbb{E}_t \left(x^\top R_{t+1} + R_t^f \right) + \frac{1}{2} u''(c)\, \mathbb{E}_t \left(x^\top R_{t+1} + R_t^f \right)^2 \\ & + \frac{1}{6} u'''(c)\, \mathbb{E}_t \left(x^\top R_{t+1} + R_t^f \right)^3 + \frac{1}{24} u''''(c)\, \mathbb{E}_t \left(x^\top R_{t+1} + R_t^f \right)^4. \end{aligned}$$

the problem computationally feasible. As remarked and illustrated in DeTemple et al. (2003, 2005), this recursion is not guaranteed to converge. We propose an alternative approximation that has three advantages. First of all, we avoid the iterative procedure. Secondly, the resulting optimization problem has a quadratic form, which can therefore be solved fast, even under constraints. Thirdly, we assess the accuracy of our approximation in an example proposed by Brandt et al. (2005) . It turns out that our approximation is in all cases at least as accurate as the approximation of Brandt et al. (2005), while more accurate for low rebalancing frequencies.

We propose to approximate the projection coefficients in (E.5) in terms of the portfolio choice, x. We project the projection coefficients on a second set of basis function in the portfolio weights, i.e.

$$\zeta(x) = \Psi^{\top} h(x), \tag{E.8}$$

where $h(\cdot)$ represents a set of basis functions in the portfolio choices. In applying this approximation, the first step is to take a set of N_1 test portfolios, $x^{(1)}, .., x^{(N_1)}$, and determine the projection coefficients, $\hat{\zeta}\left(x^{(1)}\right), .., \hat{\zeta}\left(x^{(N_1)}\right)$. Next, we estimate the projection coefficients Ψ using OLS. The basis functions have been chosen to be complete polynomials up to the second order, see Judd (1998) for further details. The main advantage is that this results in a quadratic optimization problem, which can be solved easily under constraints. However, when the risk aversion becomes extremely high, like $\gamma > 10$, this approximation may require a larger number of basis functions. We solve this problem by considering in these cases a local rather than a global approximation. This means that we use the test portfolios $x^{(1)}, ..., x^{(N_1)}$ and determine the value function for these portfolios. Subsequently, we select the k portfolios that maximize the expected utility, with $k < N_1$, and estimate the parameterization for these projection coefficients. The intuition is that the curvature is globally too high to be properly approximated by a small number of basis functions. Locally, on the other hand, a quadratic approximation turns out to be sufficient.

In order to assess the accuracy of our approximation, we use exactly the same example as provided in Brandt et al. (2005), Table 1. We consider a single period problem in which asset returns are log-normally distributed and the investor allocates wealth between stocks and a money market account, which earns a fixed rate of interest. Table D.1 summarizes the results for different investment horizons and thus trading frequencies. The problem is solved exactly using grid search and approximately using the second and fourth order approximation of Brandt et al. (2005), as well as our global and local

Then the first order condition reads as

$$x = -\left[u''(c)\,\mathbb{E}_t\left(\left(R_{t+1} + R_t^f\right) R_{t+1}^{\top}\right)\right]^{-1} \left[\begin{array}{l} u'(c)\,\mathbb{E}_t\left(R_{t+1} + R_t^f\right) + \frac{1}{2}u'''(c)\,\mathbb{E}_t\left(\left(x^{\top}R_{t+1} + R_t^f\right)^2 R_{t+1}\right) \\ + \frac{1}{6}u''''(c)\,\mathbb{E}_t\left(x^{\top}R_{t+1} + R_t^f\right)^3 \end{array}\right],$$

which is used in an iterative procedure. The starting point originates from the second order approximation

$$x^{(1)} = -\left[u''(c)\,\mathbb{E}_t\left(\left(R_{t+1} + R_t^f\right) R_{t+1}^{\top}\right)\right]^{-1} \left[u'(c)\,\mathbb{E}_t\left(R_{t+1} + R_t^f\right)\right].$$

As pointed out by DeTemple et al. (2003, 2005), this scheme is not guaranteed to converge.

approximation.

	Exact	2nd order	4th order	Global approximation	Local approximation
Monthly					
$\gamma = 5$	0.7567	0.7310	0.7591	0.7506	0.7512
$\gamma = 10$	0.3777	0.3655	0.3797	0.3756	0.3756
$\gamma = 20$	0.1888	0.1828	0.1898	0.2055	0.1880
Quarterly					
$\gamma = 5$	0.7439	0.6623	0.7299	0.7292	0.7307
$\gamma = 10$	0.3725	0.3312	0.3662	0.3656	0.3655
$\gamma = 20$	0.1849	0.1656	0.1833	0.2336	0.1833
Semi-annually					
$\gamma = 5$	0.8051	0.6456	0.7571	0.7879	0.7937
$\gamma = 10$	0.4037	0.3228	0.3842	0.3964	0.3969
$\gamma = 20$	0.2018	0.1614	0.1933	0.2679	0.1999
Annually					
$\gamma = 5$	1.1273	0.7113	0.8731	1.0514	1.1199
$\gamma = 10$	0.5737	0.3557	0.4561	0.5639	0.5627
$\gamma = 20$	0.2848	0.1778	0.2332	0.3107	0.2820

Table D.1: Comparison of four different solution approximations to the exact solution. The problem considered is a single period investment problem in which wealth is allocated to stocks and a money market account which earns a fixed rate of interest of six percent. Stock returns are i.i.d. distributed according to the means and volatilities that are mentioned in Brandt et al. (2005). The first column provides the exact solution, which is determined using grid search methods. The second and third columns present the results based on the second and fourth order approximations proposed in Brandt et al. (2005). The fourth column present our global approximation method, using parameterized regression coefficients. The fifth column presents the results for the local approximation, using parameterized regression coefficients. The approximations are determined for different investment horizons (monthly, quarterly, semi-annually, and annually) and risk preferences (γ).

The numbers reported for the first three columns are close to the results reported in Brandt et al. (2005). Their fourth order approximation works well for monthly and quarterly rebalancing frequencies. However, on a semi-annual and in particular on an annual investment horizon, their approximation tends to be inaccurate, especially for modest levels of risk aversion. Our approximation tends to perform properly in all cases, and especially the local approximation is in all cases within one percent of the exact solution. In terms of calculation time, the time required to solve the problem for 100 batches of 10,000 simulations, is approximately one hour for an investment horizon of 20 years for our approximation[15]. In absence of labor income, the calculation time is approximately ten minutes. However, the approach is ideally suited for parallel computing and the computation time is therefore less of a problem. In terms of accuracy, there are several ways to enhance the results. We have experimented with both antithetic variables and control variates in a regression model. Especially the latter variance reduction technique turns out to be useful as we can easily calculate moments of asset returns and state variables analytically.

[15] The computer used is equipped with a 3.06MHz processor and 512MB of RAM.

Hence, we use this modified approximation to approximate the conditional expectations. In that case, the optimization in every branch reduces to

$$x^*_{T-1}(\omega_i) = \arg\max_{x\in\mathcal{K}} h(x)^\top \hat{\Psi} f\left(X_{T-1}(\omega_i)\right), \tag{E.9}$$

$i = 1, ..., M$, which is quadratic when we confine ourselves to complete polynomials of the second order.

Once the optimal portfolio at time $T-1$ has been determined, we proceed backwards. Since we have determined the optimal portfolio in every branch at time $T-1$, the portfolio problem at time $T-2$ reduces to a single period problem, and the same comments apply. Repeating these steps up to time t provides the optimal portfolio strategy. As a by-product, we can estimate the expected utility via

$$\frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{1-\gamma}\left(w^*_T(\omega_i)\right)^{1-\gamma}\right), \tag{E.10}$$

where $w^*_T(\omega_i)$ has been determined using the optimal portfolio strategy. In sum, the four approximations applied here are respectively replacing conditional expectations by a projection on a finite set of basis functions, estimating the projection coefficients via cross-sectional regressions, parameterizing the projection coefficients in the portfolio weights, and finally estimating this parameterization using test portfolios. The test portfolios can be chosen on a course grid of the set of feasible portfolio choices. We selected portfolios on a grid with step sizes of 10%. Any further refinement of this grid does not alter the results for the reported precision.

So far we restricted attention to the case without an income stream. A similar approach can be used when the investor is entitled to labor income. The problem then reads as

$$V_3(t, T, X_t, w_t, Y_t) = \max_{(x_s)_{s=t}^{T-1}\in\mathcal{K}} \mathbb{E}_t\left(\frac{1}{1-\gamma}\left(w_T\right)^{1-\gamma}\right), \tag{E.11}$$

subject to

$$w_{t+1} = w_t\left(x_t^\top r^e_{t+1} + r^f_{t+1}\right) + Y_{t+1}, \tag{E.12}$$

with $\mathcal{K}$ as in (E.3).

Starting again at time $T-1$ and exploiting the homogeneity of the power utility index, the problem simplifies to

$$\begin{aligned} &\max_x \mathbb{E}_{T-1}\left(\frac{1}{1-\gamma}\left(\bar{w}_T\right)^{1-\gamma}\exp(g+\xi_T)^{1-\gamma}\right) \qquad (E.13)\\ = \;&\max_x \mathbb{E}_{T-1}\left(\frac{1}{1-\gamma}\left(\bar{w}_{T-1}\left(x^\top r^e_T + r^f_T\right)\exp(-g-\xi_T)+1\right)^{1-\gamma}\exp(g+\xi_T)^{1-\gamma}\right), \end{aligned}$$

with $\bar{w}_t = w_t/Y_t$.

The second equation illustrates the main problem once we account for the labor income stream. At time $T-1$, $\bar{w}_{T-1}$ depends on the portfolio decisions that have been made before. As a consequence, this state variable cannot be simulated. As suggested in Brandt et al. (2005), we construct a grid for real normalized wealth, where the grid points are indicated by $\bar{w}^{(1)}_{T-1}, ..., \bar{w}^{(N_2)}_{T-1}$[16]. Subsequently,

[16] The grid points have been selected time-dependently to ensure that the grid is more dispersed as the investment horizon increases. The grid points have been selected as the quantiles of simulated wealth under risky portfolios, so for instance 100% stocks.

the same procedure applies as in the case without labor income. The sole complication that arises is that when we solve for the optimal portfolio at a certain grid point of normalized real wealth, say $\bar{w}_t^{(j)}$, then the value of normalized real wealth in consecutive time periods will probably not lie on the grid. In life-cycle models, it is common practice to interpolate the value functions at these points, see for instance Campbell and Cocco (2003), Cocco et al. (2005), and Woolley (2004). As remarked by Cochrane (1989), the utility cost of suboptimal strategies have only a second order effect on the indirect utility function. Therefore, we choose to interpolate the optimal portfolio strategy rather than the indirect utility function, in line with the suggestion in Brandt et al. (2005). We interpolate the optimal policy using polynomials in inverse wealth, thereby ensuring that the optimal portfolio becomes independent of wealth as wealth tends to infinity.

Regarding the basis functions in the state variables, we use second-order polynomials, including cross-terms. For parameterizing the projection coefficients in the portfolio weights, second-order complete polynomials have been used, see for instance Judd (1998).

F Tables and figures

Parameter	Estimate	(Standard error)
Expected inflation: $\pi_t = \delta_\pi + X_{2t}$		
δ_π	3.65%	(1.30%)
Nominal interest rate: $R_t = \delta_R + (\iota_2^\intercal - \sigma_\Pi^\intercal \Lambda_1) X_t$		
δ_R	5.43%	(1.36%)
Process real interest rate and expected inflation: $dX_t = -KX_t dt + \Sigma_X dZ_t$		
κ_1	1.2271	(0.2023)
κ_2	0.1564	(0.0679)
σ_1	2.00%	(0.08%)
σ_{12}	-0.19%	(0.04%)
σ_2	1.26%	(0.04%)
Realized inflation process: $d\Pi_t/\Pi_t = \pi_t dt + \sigma_\Pi^\intercal dZ_t$		
$\sigma_{\Pi(1)}$	0.16%	(0.05%)
$\sigma_{\Pi(2)}$	0.16%	(0.05%)
$\sigma_{\Pi(3)}$	1.08%	(0.03%)
Stock return process: $dS_t/S_t = (R_t + \eta_S)dt + \sigma_S^\intercal dZ_t$		
η_S	5.36%	(2.63%)
$\sigma_{S(1)}$	-1.57%	(0.58%)
$\sigma_{S(2)}$	-2.73%	(0.70%)
$\sigma_{S(3)}$	-1.71%	(0.67%)
$\sigma_{S(4)}$	14.89%	(0.32%)
Prices of risk: $\Lambda_t = \Lambda_0 + \Lambda_1 X_t$		
$\Lambda_{0(1)}$	-0.3445	(0.1118)
$\Lambda_{0(2)}$	-0.1687	(0.0345)
$\Lambda_{1(1,1)}$	-28.3879	(11.1977)
$\Lambda_{1(2,2)}$	-11.2458	(5.3363)
Standard errors of yield measurement error: $\sigma_{u_1}, ..., \sigma_{u_6}$		
σ_{u_1}	0.46%	(0.02%)
σ_{u_2}	0.22%	(0.01%)
σ_{u_3}	0.05%	(0.01%)
σ_{u_4}	0.11%	(0.00%)
σ_{u_5}	0.03%	(0.02%)
σ_{u_6}	0.19%	(0.01%)

Table 1: **Estimation results for the financial market in Section 2.1** The two factor model described in Section 2.1 is estimated using monthly data on 6 bond yields, inflation, and stock returns over the period from January 1959 up to May 2002. The standard errors are determined using the outer product gradient estimator.

Maturity	Estimate	Maturity	Estimate
Risk premium nominal bonds: $B(\tau)^\intercal \Sigma_X \Lambda_0$		Risk premium real bonds: $\left(B^R(\tau)^\intercal \Sigma_X + \sigma_\Pi^\top\right) \Lambda_0$	
1Y	0.64%	1Y	0.42%
3Y	1.25%	3Y	0.82%
10Y	1.98%	10Y	0.97%
Volatility nominal bonds: $\sqrt{B(\tau)^\intercal \Sigma_X \Sigma_X^\intercal B(\tau)}$		Volatility real bonds: $\sqrt{\left(B^R(\tau)^\intercal \Sigma_X + \sigma_\Pi^\top\right)\left(B^R(\tau)^\intercal \Sigma_X + \sigma_\Pi^\top\right)^\top}$	
1Y	1.76%	1Y	1.70%
3Y	4.11%	3Y	2.70%
10Y	11.71%	10Y	3.08%
Inflation risk premium: $\left[B(\tau)^\intercal \Sigma_X - B^R(\tau)^\intercal \Sigma_X - \sigma_\Pi^\top\right] \Lambda_0$			
1Y	22bp		
3Y	43bp		
10Y	101bp		

Table 2: **Risk premia and volatilities** Implied risk premia on both nominal and real bonds using the estimation results in Table 1 when the factors equal their unconditional expectation. In addition, we provide the corresponding volatilities of bond returns and the inflation risk premium for these maturities. The inflation risk premium has been defined as the difference between the risk premia on nominal and real bonds with a particular maturity.

Panel A							
	Stocks	N1Y	N3Y	N10Y	R1Y	R3Y	R10Y
Stocks	1.0000						
N1Y	0.1988	1.0000					
N3Y	0.2052	0.9470	1.0000				
N10Y	0.1785	0.7107	0.8991	1.0000			
R1Y	-0.0087	0.4743	0.2527	-0.0929	1.0000		
R3Y	0.0384	0.6017	0.3478	-0.0584	0.9616	1.0000	
R10Y	0.0475	0.6217	0.3637	-0.0509	0.9450	0.9985	1.0000

Panel B				
	Stocks	N3Y	N10Y	R10Y
μ_3-R	-0.1475	-0.6872	-0.2909	-0.8892
μ_{10}-R	-0.1643	-0.8458	-0.9950	0.1310
μ_{10}^R-R	-0.1023	-0.4534	-0.0080	-0.9404

Table 3: **Correlations between asset returns and risk premia** Panel A reports the implied instantaneous correlations between stock returns and returns on both nominal and real bond returns with maturities 1Y, 3Y, and 10Y on the basis of the parameter estimates that have been reported in Table 1. The abbreviation NτY refers to a nominal bond with τ years to maturity. Similarly, RτY refers to a real bond with τ years to maturity. Panel B provides the implied instantaneous correlation between the risk premia on 3Y nominal bonds, $(\mu_3 - R)$, 10Y nominal bonds, $(\mu_{10} - R)$, and 10Y real bonds, $(\mu_{10}^R - R)$ and the traded assets. For instance, the correlation between stocks and the risk premium on τY nominal bonds is given by $B(\tau)^\intercal \Sigma_X \Lambda_1 \Sigma_X \sigma_S \left[\sqrt{B(\tau)^\intercal \Sigma_X \Lambda_1 \Sigma_X \Sigma_X^\intercal \Lambda_1^\intercal \Sigma_X^\intercal B(\tau)} \sqrt{\sigma_S^\intercal \sigma_S}\right]^{-1}$.

	Stock returns	Inflation	N3M	N1Y	N5Y	N10Y
Average						
Data	0.83%	0.35%	5.93%	6.38%	6.97%	7.17%
Model	0.80%	0.30%	5.53%	5.77%	6.38%	6.58%
Volatility						
Data	4.41%	0.33%	2.69%	2.70%	2.49%	2.44%
Model	4.42%	0.38%	2.46%	2.36%	2.19%	2.11%

Table 4: **Comparison of sample moments with model implied moments** Comparison of the means and volatilities of stock returns, inflation, and nominal yields with maturities 3M, 1Y, 5Y, and 10Y that follow from the data and the model, where the parameter estimates used are reported in Table 1. The abbreviation NτY refers to a nominal bond with τ years to maturity. N3M refers to a nominal bond with three months to maturity.

$\gamma = 3$	Myopic		Hedge		Hedge		Total portfolio		Exposure	Exposure
(in %)			real rate		exp. infl.				real rate (r)	expected inflation (π)
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y		
1	631	-152	229	-72	-4	6	856	-219	-8.14	-4.80
5	631	-152	401	-127	-21	28	1012	-251	-9.69	-6.37
10	631	-152	400	-126	-37	50	994	-228	-9.67	-7.94
20	631	-152	392	-124	-56	75	968	-201	-9.61	-9.71
30	631	-152	389	-123	-63	85	957	-190	-9.58	-10.44
$\gamma = 5$	Myopic		Hedge		Hedge		Total portfolio		Exposure	Exposure
(in %)			real rate		exp. infl.				real rate (r)	expected inflation (π)
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y		
1	375	-91	198	-63	-3	4	570	-149	-5.39	-2.88
5	375	-91	371	-117	-16	21	731	-187	-6.95	-4.10
10	375	-91	371	-117	-30	41	716	-167	-6.95	-5.49
20	375	-91	360	-114	-51	68	685	-136	-6.85	-7.42
30	375	-91	354	-112	-61	82	669	-120	-6.80	-8.42
$\gamma = 7$	Myopic		Hedge		Hedge		Total portfolio		Exposure	Exposure
(in %)			real rate		exp. infl.				real rate (r)	expected inflation (π)
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y		
1	266	-64	174	-55	-2	3	437	-116	-4.11	-2.03
5	266	-64	336	-106	-12	17	589	-154	-5.57	-2.99
10	266	-64	337	-106	-25	33	578	-138	-5.58	-4.15
20	266	-64	325	-103	-43	58	547	-109	-5.48	-5.96
30	266	-64	318	-100	-55	74	529	-91	-5.41	-7.04

Table 5: **Continuous time optimal portfolio choice on the basis of nominal bonds** Optimal portfolio choice between 3Y nominal bonds, 10Y nominal bonds, and a nominal money market account. The investor can trade continuously and is not subject to trading constraints. The optimal portfolio composition is determined on the basis of (20). The optimal portfolio is decomposed into the myopic demand, the real interest rate hedge, and the expected inflation hedge. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The state variables are set to their unconditional expectations. The portfolio weights are presented in percentages. The last two columns show the implied exposures of the optimal portfolio to the real interest rate and expected inflation.

$\gamma = 3$ (in %)	Myopic			Hedge real rate			Hedge exp. infl.			Total portfolio			Exposure real rate (r)	Exposure expected inflation (π)
T	N3Y	N10Y	R10Y	N3Y	N10Y	R10Y	N3Y	N10Y	R10Y	N3Y	N10Y	R10Y		
1	525	-118	67	229	-72	0	-4	6	0	750	-184	67	-8.20	-4.88
5	525	-118	67	401	-127	0	-21	28	0	905	-216	67	-9.75	-6.46
10	525	-118	67	400	-126	0	-37	50	0	887	-194	67	-9.73	-8.02
20	525	-118	67	392	-124	0	-56	75	0	861	-166	67	-9.67	-9.80
30	525	-118	67	389	-123	0	-63	85	0	850	-155	67	-9.64	-10.53
$\gamma = 5$ (in %)	Myopic			Hedge real rate			Hedge exp. infl.			Total portfolio			Exposure real rate (r)	Exposure expected inflation (π)
T	N3Y	N10Y	R10Y	N3Y	N10Y	R10Y	N3Y	N10Y	R10Y	N3Y	N10Y	R10Y		
1	247	-49	80	198	-63	0	-3	4	0	442	-108	80	-5.47	-2.98
5	247	-49	80	371	-117	0	-16	21	0	603	-145	80	-7.02	-4.20
10	247	-49	80	371	-117	0	-30	41	0	588	-125	80	-7.02	-5.59
20	247	-49	80	360	-114	0	-51	68	0	557	-95	80	-6.92	-7.52
30	247	-49	80	354	-112	0	-61	82	0	540	-79	80	-6.87	-8.52
$\gamma = 7$ (in %)	Myopic			Hedge real rate			Hedge exp. infl.			Total portfolio			Exposure real rate (r)	Exposure expected inflation (π)
T	N3Y	N10Y	R10Y	N3Y	N10Y	R10Y	N3Y	N10Y	R10Y	N3Y	N10Y	R10Y		
1	128	-20	86	174	-55	0	-2	3	0	300	-72	86	-4.19	-2.14
5	128	-20	86	336	-106	0	-12	17	0	452	-109	86	-5.65	-3.10
10	128	-20	86	337	-106	0	-25	33	0	441	-93	86	-5.66	-4.26
20	128	-20	86	325	-103	0	-43	58	0	410	-64	86	-5.56	-6.07
30	128	-20	86	318	-100	0	-55	74	0	392	-47	86	-5.49	-7.15

Table 6: **Continuous time optimal portfolio choice on the basis of nominal and inflation-linked bonds** Optimal portfolio choice between 3Y nominal bonds, 10Y nominal bonds, 10Y inflation-linked bonds, and a nominal money market account. The investor can trade continuously and is not subject to trading constraints. The optimal portfolio composition is determined on the basis of (20). The optimal portfolio is decomposed into the myopic demand, the real interest rate hedge, and the expected inflation hedge. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The state variables are set to their unconditional expectations. The portfolio weights are presented in percentages. The last two columns show the implied exposures of the optimal portfolio to the real interest rate and expected inflation.

	Baseline case				Robustness checks				
$\gamma = 3$					$\sigma_\xi = 0$	High real rate (r)		High exp. infl. (π)	
T	CT	DT	$\bar{w}_0 = 1$	$\bar{w}_0 = 25$	$\bar{w}_0 = 1$	DT	$\bar{w}_0 = 1$	DT	$\bar{w}_0 = 1$
1	0.01%	0.08%	0.06%	0.08%	0.06%	0.44%	0.35%	0.00%	0.00%
5	0.04%	0.86%	0.58%	0.84%	0.57%	1.28%	0.81%	0.07%	0.06%
10	0.08%	1.85%	1.22%	1.76%	1.20%	2.22%	1.37%	0.45%	0.40%
20	0.16%	3.80%	2.52%	3.62%	2.46%	4.15%	2.62%	1.93%	1.64%
30	0.23%	5.77%	3.80%	5.49%	3.66%	6.11%	3.88%	3.76%	3.07%
$\gamma = 5$									
T	CT	DT	$\bar{w}_0 = 1$	$\bar{w}_0 = 25$	$\bar{w}_0 = 1$	DT	$\bar{w}_0 = 1$	DT	$\bar{w}_0 = 1$
1	0.02%	0.12%	0.08%	0.12%	0.08%	0.53%	0.41%	0.00%	0.00%
5	0.09%	1.05%	0.68%	1.02%	0.67%	1.51%	0.92%	0.12%	0.09%
10	0.19%	2.08%	1.34%	2.00%	1.27%	2.46%	1.50%	0.50%	0.45%
20	0.37%	3.97%	2.80%	3.89%	2.52%	4.32%	2.89%	1.97%	1.79%
30	0.56%	6.05%	4.11%	5.73%	3.72%	6.36%	4.17%	3.97%	3.27%
$\gamma = 7$									
T	CT	DT	$\bar{w}_0 = 1$	$\bar{w}_0 = 25$	$\bar{w}_0 = 1$	DT	$\bar{w}_0 = 1$	DT	$\bar{w}_0 = 1$
1	0.03%	0.18%	0.12%	0.18%	0.12%	0.61%	0.47%	0.05%	0.01%
5	0.15%	1.23%	0.79%	1.20%	0.77%	1.76%	1.04%	0.19%	0.14%
10	0.30%	2.39%	1.56%	2.27%	1.39%	2.78%	1.73%	0.60%	0.53%
20	0.60%	4.75%	3.03%	4.27%	2.71%	5.12%	3.12%	2.64%	1.89%
30	0.90%	7.68%	4.34%	6.44%	3.84%	8.00%	4.40%	5.36%	3.46%

Table 7: **Utility gains from enriching the asset menu with inflation-linked bonds** Utility gains from enriching the investor's asset menu, which contains a 3Y, a 10Y nominal bond, and a nominal money market account, with a 10Y inflation-linked bond. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). In the baseline case, the state variables equal their unconditional expectations and the volatility of log labor income growth is set to ten percent. In addition, labor income risk is independent of financial risks. The abbreviation CT indicates the unconstrained, continuous time problem. The abbreviation DT refers to the constrained, discrete time investment problem. When the investor is entitled to a labor income stream, the amount of initial wealth is given. Initial wealth is either low, $\bar{w}_0 = 1$, or high, i.e. $\bar{w}_0 = 25$. The robustness checks consider the effect of zero labor income risk ($\sigma_{\xi=0}$) and the effect of different states of the economy as characterized by the real interest rate and expected inflation, which are set two unconditional standard deviations above their unconditional expectation.

	Asset menu: 3Y, 10Y nominal bonds, and cash										Asset menu: 10Y inflation-linked and 10Y nominal bonds, and cash										
$\gamma = 3$	Myopic		Hedge		Hedge		Total		Total		Myopic		Hedge		Hedge		Total		Total		
(in %)			real rate		exp. infl.		portfolio DT		portfolio CT				real rate		exp. infl.		portfolio DT		portfolio CT		
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	N3Y	N10Y	R10Y
1	94	6	0	0	0	0	94	6	856	-219	70	30	0	0	0	0	70	30	750	-184	67
5	94	6	2	-2	-21	21	75	25	1012	-251	70	30	2	-2	-12	12	60	40	905	-216	67
10	94	6	2	-2	-37	37	59	41	994	-228	70	30	2	-2	-22	22	50	50	887	-194	67
20	94	6	2	-2	-50	50	45	55	968	-201	70	30	2	-2	-30	30	42	58	861	-166	67
30	94	6	2	-2	-54	54	42	58	957	-190	70	30	2	-2	-33	33	39	61	850	-155	67
$\gamma = 5$	Myopic		Hedge		Hedge		Total		Total		Myopic		Hedge		Hedge		Total		Total		
(in %)			real rate		exp. infl.		portfolio DT		portfolio CT				real rate		exp. infl.		portfolio DT		portfolio CT		
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	N3Y	N10Y	R10Y
1	100	0	0	0	0	0	100	0	570	-149	81	19	0	0	0	0	81	19	442	-108	80
5	100	0	0	0	-8	8	94	6	731	-187	81	19	2	-2	-10	10	73	27	603	-145	80
10	100	0	0	0	-27	27	75	25	716	-167	80	20	2	-2	-22	22	61	39	588	-125	80
20	100	0	0	0	-45	45	57	43	685	-136	80	20	2	-2	-34	34	50	50	557	-95	80
30	100	0	0	0	-49	49	54	46	669	-120	80	20	2	-2	-37	37	47	53	540	-79	80
$\gamma = 7$	Myopic		Hedge		Hedge		Total		Total		Myopic		Hedge		Hedge		Total		Total		
(in %)			real rate		exp. infl.		portfolio DT		portfolio CT				real rate		exp. infl.		portfolio DT		portfolio CT		
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	N3Y	N10Y	R10Y
1	100	0	0	0	0	0	100	0	437	-116	85	15	0	0	0	0	85	15	300	-72	86
5	100	0	0	0	0	0	100	0	589	-154	85	15	3	-3	-8	8	80	20	452	-109	86
10	100	0	0	0	-17	17	86	14	578	-138	85	15	3	-3	-19	19	69	31	441	-93	86
20	100	0	0	0	-33	33	70	30	547	-109	85	15	3	-3	-32	32	57	43	410	-64	86
30	100	0	0	0	-34	34	70	30	529	-91	85	15	3	-3	-33	33	55	45	392	-47	86

Table 8: **Constrained, discrete time portfolio choice on the basis of nominal bonds and (possibly) inflation-linked bonds in absence of labor income** Optimal portfolio choice between either 3Y nominal bonds, 10Y nominal bonds, and cash or the same asset menu, but enriched with 10Y inflation-linked bonds. The optimal portfolio is decomposed into the myopic demand, the real interest rate hedge, and the expected inflation hedge, as outlined in Appendix C. In order to draw a comparison with the unconstrained, continuous time problem, these results are reproduced. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The state variables are set to their unconditional expectations. The portfolio weights are presented in percentages. The abbreviation CT indicates the unconstrained, continuous time problem. The abbreviation DT refers to the constrained, discrete time investment problem. We use 100 batches of $M = 10000$ simulations and report the average portfolio weights across all simulations.

	Asset menu: stocks, 10Y nominal bonds, and cash								Asset menu: stocks, 10Y inflation-linked bonds, and cash							
$\gamma = 3$	Myopic		Hedge		Hedge		Total portfolio		Myopic		Hedge		Hedge		Total portfolio	
(in %)			real rate		exp. infl.						real rate		exp. infl.			
T	Stocks	N10Y	Stocks	N10Y	Stocks	N10Y	Stocks	N10Y	Stocks	R10Y	Stocks	R10Y	Stocks	R10Y	Stocks	R10Y
1	71	31	1	0	0	3	72	33	72	327	1	106	0	0	74	433
5	71	31	1	0	0	16	72	46	72	327	2	181	0	2	74	509
10	71	31	1	0	0	31	72	61	72	327	2	183	-1	3	74	513
20	71	31	1	0	-1	49	71	79	72	327	2	183	-1	4	74	514
30	71	31	1	0	-1	57	71	87	72	327	2	183	-1	4	74	514
$\gamma = 5$	Myopic		Hedge		Hedge		Total portfolio		Myopic		Hedge		Hedge		Total portfolio	
(in %)			real rate		exp. infl.						real rate		exp. infl.			
T	Stocks	N10Y	Stocks	N10Y	Stocks	N10Y	Stocks	N10Y	Stocks	R10Y	Stocks	R10Y	Stocks	R10Y	Stocks	R10Y
1	42	18	1	0	0	2	43	20	43	199	1	94	0	0	44	294
5	42	18	1	0	0	13	43	30	43	199	2	169	0	2	45	370
10	42	18	1	0	0	27	43	45	43	199	2	172	0	3	45	374
20	42	18	1	0	-1	49	43	67	43	199	2	173	-1	3	45	375
30	42	18	1	0	-1	61	43	79	43	199	2	173	-1	3	44	375
$\gamma = 7$	Myopic		Hedge		Hedge		Total portfolio		Myopic		Hedge		Hedge		Total portfolio	
(in %)			real rate		exp. infl.						real rate		exp. infl.			
T	Stocks	N10Y	Stocks	N10Y	Stocks	N10Y	Stocks	N10Y	Stocks	R10Y	Stocks	R10Y	Stocks	R10Y	Stocks	R10Y
1	30	12	1	0	0	2	31	14	30	145	1	84	0	0	31	229
5	30	12	1	0	0	10	31	22	30	145	2	155	0	1	32	301
10	30	12	1	0	0	23	31	35	30	145	2	158	0	2	32	305
20	30	12	1	0	-1	45	30	57	30	145	2	159	0	3	32	306
30	30	12	1	0	-1	59	30	71	30	145	2	159	-1	3	32	306

Table 9: **Continuous time optimal portfolio choice on the basis of stocks and either nominal or inflation-linked bonds** Optimal portfolio choice between either stocks, 10Y nominal bonds, and cash or stocks, 10Y inflation-linked bonds, and cash. The investor can trade continuously and is not subject to trading constraints. The optimal portfolio composition is determined on the basis of (20). The optimal portfolio is decomposed into the myopic demand, the real interest rate hedge, and the expected inflation hedge. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The state variables are set to their unconditional expectations. The portfolio weights are presented in percentages.

	Asset menu: stocks, 10Y nominal bonds, and cash										Asset menu: stocks, 10Y inflation-linked bonds, and cash									
$\gamma = 3$	Myopic		Hedge		Hedge		Total		Total		Myopic		Hedge		Hedge		Total		Total	
(in %)			real rate		exp. infl.		portfolio DT		portfolio CT				real rate		exp. infl.		portfolio DT		portfolio CT	
T	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y
1	69	30	0	0	0	0	69	30	72	33	63	37	0	0	0	0	63	37	74	433
5	69	30	0	-1	-2	3	67	33	72	46	63	37	0	0	0	0	63	37	74	509
10	69	30	0	-1	-4	5	65	35	72	61	63	37	0	0	0	0	63	37	74	513
20	69	30	0	-1	-6	7	63	37	71	79	63	37	0	0	0	0	63	37	74	514
30	69	30	0	-2	-6	8	63	37	71	87	63	37	0	0	0	0	63	37	74	514
$\gamma = 5$	Myopic		Hedge		Hedge		Total		Total		Myopic		Hedge		Hedge		Total		Total	
(in %)			real rate		exp. infl.		portfolio DT		portfolio CT				real rate		exp. infl.		portfolio DT		portfolio CT	
T	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y
1	41	15	0	0	0	0	41	15	43	20	38	62	0	0	0	0	38	62	44	294
5	41	15	1	-1	0	11	41	26	43	30	38	62	0	0	0	0	38	62	45	370
10	41	15	1	0	0	24	41	38	43	45	38	62	0	0	0	0	38	62	45	374
20	41	15	0	0	0	34	41	49	43	67	38	62	0	0	0	0	38	62	45	375
30	41	15	1	0	0	36	41	51	43	79	38	62	0	0	0	0	38	62	44	375
$\gamma = 7$	Myopic		Hedge		Hedge		Total		Total		Myopic		Hedge		Hedge		Total		Total	
(in %)			real rate		exp. infl.		portfolio DT		portfolio CT				real rate		exp. infl.		portfolio DT		portfolio CT	
T	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y
1	29	9	0	0	0	0	29	9	31	14	27	73	0	0	0	0	27	73	31	229
5	29	9	1	-1	0	10	29	18	31	22	27	73	0	0	0	0	27	73	32	301
10	29	9	1	0	0	23	29	32	31	35	27	73	0	0	0	0	27	73	32	305
20	29	9	0	0	0	34	29	42	30	57	27	73	0	0	0	0	27	73	32	306
30	29	9	1	0	0	34	29	42	30	71	27	73	0	0	0	0	27	73	32	306

Table 10: **Constrained, discrete time portfolio choice on the basis of stocks and either nominal or inflation-linked bonds in the absence of labor income** Optimal portfolio choice between either 10Y nominal bonds, stocks, and cash or 10Y inflation-linked bonds, stocks, and cash. The optimal portfolio is decomposed into the myopic demand, the real interest rate hedge, and the expected inflation hedge, as outlined in Appendix C. In order to draw a comparison with the unconstrained, continuous time problem, these results are reproduced. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The state variables are set to their unconditional expectations. The portfolio weights are presented in percentages. The abbreviation CT indicates the unconstrained, continuous time problem. The abbreviation DT refers to the constrained, discrete time investment problem. We use 100 batches of $M = 10000$ simulations and report the average portfolio weights across all simulations.

	Asset menu: 3Y, 10Y nominal bonds, and cash										Asset menu: 10Y inflation-linked and 10Y nominal bonds, and cash									
$\gamma = 3$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y
1	86	14	0	0	0	0	86	14	94	6	65	35	0	0	0	0	65	35	70	30
5	52	48	2	-2	-29	29	25	75	75	25	44	56	2	-2	-17	17	29	71	60	40
10	14	86	0	0	-14	14	0	100	59	41	20	80	0	0	-20	20	0	100	50	50
20	0	100	0	0	0	0	0	100	45	55	0	100	0	0	0	0	0	100	42	58
30	0	100	0	0	0	0	0	100	42	58	0	100	0	0	0	0	0	100	39	61
$\gamma = 5$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y
1	100	0	0	0	0	0	100	0	100	0	77	23	0	0	0	0	77	23	81	19
5	87	13	3	-3	-31	31	58	42	94	6	65	35	3	-3	-18	18	50	50	73	27
10	66	34	2	-2	-63	63	4	96	75	25	52	48	2	-2	-39	39	16	84	61	39
20	27	73	-1	1	-27	27	0	100	57	43	29	71	0	0	-29	29	0	100	50	50
30	9	91	-2	2	-9	9	0	100	54	46	15	85	-2	2	-15	15	0	100	47	53
$\gamma = 7$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y
1	100	0	0	0	0	0	100	0	100	0	83	17	0	0	0	0	83	17	85	15
5	99	1	1	-1	-26	26	76	24	100	0	75	25	3	-3	-15	15	63	37	80	20
10	88	12	2	-2	-68	68	22	78	86	14	66	34	2	-2	-39	39	30	70	69	31
20	59	41	0	0	-57	57	2	98	70	30	50	50	0	0	-45	45	5	95	57	43
30	39	61	0	0	-35	35	0	100	70	30	36	64	-1	1	-31	31	4	96	55	45

Table 11: **Constrained, discrete time portfolio choice on the basis of nominal bonds and (possibly) inflation-linked bonds in the presence of labor income and low initial wealth** Optimal portfolio choice between either 3Y nominal bonds, 10Y nominal bonds, and cash or the same asset menu, but enriched with 10Y inflation-linked bonds. The optimal portfolio is decomposed into the myopic demand, the real interest rate hedge, and the expected inflation hedge, as outlined in Appendix C. In order to draw a comparison with the constrained, discrete time problem without labor income, these results are reproduced. Investors are endowed with labor income. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The amount of initial wealth is low, i.e. $\bar{w}_0 = 1$. The state variables are set to their unconditional expectations. The portfolio weights are presented in percentages. The abbreviation DT refers to the constrained, discrete time investment problem. We use 100 batches of $M = 10000$ simulations and report the average portfolio weights across all simulations.

	Asset menu: 3Y, 10Y nominal bonds, and cash										Asset menu: 10 inflation-linked and 10Y nominal bonds, and cash									
$\gamma = 3$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y
1	94	6	0	0	0	0	94	6	94	6	70	30	0	0	0	0	70	30	70	30
5	93	7	2	-2	-21	21	73	27	75	25	69	31	2	-2	-12	12	59	41	60	40
10	91	9	2	-2	-38	38	55	45	59	41	68	32	2	-2	-23	23	47	53	50	50
20	87	13	2	-2	-51	51	38	62	45	55	66	34	2	-2	-31	31	37	63	42	58
30	82	18	2	-2	-53	53	32	68	42	58	63	37	2	-2	-32	32	33	67	39	61
$\gamma = 5$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y
1	100	0	0	0	0	0	100	0	100	0	81	19	0	0	0	0	81	19	81	19
5	100	0	0	0	-9	9	93	7	94	6	80	20	2	-2	-11	11	72	28	73	27
10	100	0	0	0	-31	31	72	28	75	25	80	20	2	-2	-23	23	59	41	61	39
20	100	0	0	0	-52	52	51	49	57	43	78	22	2	-2	-35	35	46	54	50	50
30	100	0	0	0	-60	60	42	58	54	46	76	24	2	-2	-39	39	41	59	47	53
$\gamma = 7$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y	R10Y	N10Y
1	100	0	0	0	0	0	100	0	100	0	85	15	0	0	0	0	85	15	85	15
5	100	0	0	0	0	0	100	0	100	0	85	15	2	-2	-9	9	79	21	80	20
10	100	0	0	0	-20	20	83	17	86	14	85	15	2	-2	-21	21	67	33	69	31
20	100	0	0	0	-41	41	61	39	70	30	83	17	3	-3	-34	34	53	47	57	43
30	100	0	0	0	-52	52	52	48	70	30	82	18	3	-3	-39	39	47	53	55	45

Table 12: **Constrained, discrete time portfolio choice on the basis of nominal bonds and (possibly) inflation-linked bonds in the presence of labor income and high initial wealth** Optimal portfolio choice between either 3Y nominal bonds, 10Y nominal bonds, and cash or the same asset menu, but enriched with 10Y inflation-linked bonds. The optimal portfolio is decomposed into the myopic demand, the real interest rate hedge, and the expected inflation hedge, as outlined in Appendix C. In order to draw a comparison with the constrained, discrete time problem without labor income, these results are reproduced. Investors are endowed with labor income. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The amount of initial wealth is low, i.e. $\bar{w}_0 = 25$. The state variables are set to their unconditional expectations. The portfolio weights are presented in percentages. The abbreviation DT refers to the constrained, discrete time investment problem. We use 100 batches of $M = 10000$ simulations and report the average portfolio weights across all simulations.

	Asset menu: stocks, 10Y nominal bonds, and cash										Asset menu: stocks, 10Y inflation-linked bonds, and cash									
$\gamma = 3$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y
1	77	23	0	0	0	0	77	23	69	30	75	25	0	0	0	0	75	25	63	37
5	100	0	0	0	0	0	100	0	67	33	100	0	0	0	0	0	100	0	63	37
10	100	0	0	0	0	0	100	0	65	35	100	0	0	0	0	0	100	0	63	37
20	100	0	0	0	0	0	100	0	63	37	100	0	0	0	0	0	100	0	63	37
30	100	0	0	0	0	0	100	0	63	37	100	0	0	0	0	0	100	0	63	37
$\gamma = 5$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y
1	49	20	0	0	0	0	49	20	41	15	45	55	0	0	0	0	45	55	38	62
5	75	25	1	-1	-4	4	71	29	41	26	72	28	0	0	0	0	72	28	38	62
10	95	5	1	-1	-7	7	88	12	41	38	99	1	0	0	0	0	99	1	38	62
20	100	0	0	0	-1	1	100	0	41	49	100	0	0	0	0	0	100	0	38	62
30	100	0	0	0	0	0	100	0	41	51	100	0	0	0	0	0	100	0	38	62
$\gamma = 7$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y
1	35	12	0	0	0	0	35	12	29	9	32	68	0	0	0	0	32	68	27	73
5	56	24	1	0	0	16	56	40	29	18	51	49	-1	1	0	0	51	49	27	73
10	75	25	1	-1	-9	9	66	34	29	32	71	29	0	0	0	0	71	29	27	73
20	89	11	-3	3	-10	10	80	20	29	42	93	7	-2	2	-2	2	90	10	27	73
30	92	8	-4	4	-17	17	77	23	29	42	96	4	-2	2	-6	6	93	7	27	73

Table 13: **Constrained, discrete time portfolio choice on the basis of stocks and either nominal or inflation-linked bonds in the presence of labor income and low initial wealth** Optimal portfolio choice between either 10Y nominal bonds, stocks, and cash or 10Y inflation-linked bonds, stocks, and cash. The optimal portfolio is decomposed into the myopic demand, the real interest rate hedge, and the expected inflation hedge, as outlined in Appendix C. In order to draw a comparison with the constrained, discrete time problem, these results are reproduced. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The amount of initial wealth is, i.e. $\bar{w}_0 = 1$. The state variables are set to their unconditional expectations. The portfolio weights are presented in percentages. The abbreviation DT refers to the constrained, discrete time investment problem. We use 100 batches of $M = 10000$ simulations and report the average portfolio weights across all simulations.

	Asset menu: stocks, 10Y nominal bonds, and cash										Asset menu: stocks, 10Y inflation-linked bonds, and cash									
$\gamma = 3$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y
1	69	30	0	0	0	0	69	30	69	30	63	37	0	0	0	0	63	37	63	37
5	70	29	0	0	-2	3	68	32	67	33	65	35	0	0	0	0	64	36	63	37
10	72	28	0	0	-5	5	68	32	65	35	67	33	0	0	0	0	67	33	63	37
20	75	25	0	0	-6	6	69	31	63	37	72	28	0	0	0	0	72	28	63	37
30	79	21	0	0	-6	6	73	27	63	37	77	23	1	-1	1	-1	77	23	63	37
$\gamma = 5$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y
1	41	15	0	0	0	0	41	15	41	15	38	62	0	0	0	0	38	62	38	62
5	42	16	0	0	0	12	42	28	41	26	39	61	0	0	0	0	39	61	38	62
10	44	17	0	0	0	24	45	41	41	38	41	59	0	0	0	0	40	60	38	62
20	47	20	1	-1	-2	31	46	51	41	49	43	57	0	0	0	0	43	57	38	62
30	52	23	3	0	-4	26	49	49	41	51	46	54	1	-1	0	0	46	54	38	62
$\gamma = 7$	Myopic		Hedge		Hedge		Total DT		Total DT		Myopic		Hedge		Hedge		Total DT		Total DT	
(in %)			real rate		exp. infl.		income		no income				real rate		exp. infl.		income		no income	
T	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	N10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y	Stock	R10Y
1	29	9	0	0	0	0	29	9	29	9	27	73	0	0	0	0	27	73	27	73
5	30	9	0	0	0	10	30	19	29	18	28	72	0	0	0	0	28	72	27	73
10	32	10	0	0	0	24	32	34	29	32	29	71	0	0	0	0	29	71	27	73
20	34	13	1	-1	-1	37	33	49	29	42	31	69	0	0	0	0	30	70	27	73
30	37	16	2	-1	-1	40	37	54	29	42	33	67	1	-1	0	0	33	67	27	73

Table 14: **Constrained, discrete time portfolio choice on the basis of stocks and either nominal or inflation-linked bonds in the presence of labor income and high initial wealth** Optimal portfolio choice between either 10Y nominal bonds, stocks, and cash or 10Y inflation-linked bonds, stocks, and cash. The optimal portfolio is decomposed into the myopic demand, the real interest rate hedge, and the expected inflation hedge, as outlined in Appendix C. In order to draw a comparison with the constrained, discrete time problem, these results are reproduced. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The amount of initial wealth is, i.e. $\bar{w}_0 = 25$. The state variables are set to their unconditional expectations. The portfolio weights are presented in percentages. The abbreviation DT refers to the constrained, discrete time investment problem. We use 100 batches of $M = 10000$ simulations and report the average portfolio weights across all simulations.

	Baseline case				Robustness checks									
$\gamma = 3$					$\sigma_\xi = 0$		High real rate (r)				High expected inflation (π)			
(in %)	DT		$\bar{w}_0 = 1$		$\bar{w}_0 = 1$		DT		$\bar{w}_0 = 1$		DT		$\bar{w}_0 = 1$	
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y
1	94	6	86	14	86	14	100	0	100	0	0	100	0	100
5	75	25	25	75	24	76	100	0	100	0	0	100	0	100
10	59	41	0	100	0	100	99	1	78	22	0	100	0	100
20	45	55	0	100	0	100	87	13	71	29	0	100	0	100
30	42	58	0	100	0	100	83	17	74	26	0	100	0	100
$\gamma = 5$														
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y
1	100	0	100	0	100	0	100	0	100	0	0	100	0	100
5	94	6	58	42	56	44	100	0	100	0	0	100	0	100
10	75	25	4	96	0	100	100	0	69	31	0	100	0	100
20	57	43	0	100	0	100	84	16	39	61	0	100	0	100
30	54	46	0	100	0	100	79	21	33	67	0	100	0	100
$\gamma = 7$														
T	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y	N3Y	N10Y
1	100	0	100	0	100	0	100	0	100	0	27	73	6	94
5	100	0	76	24	73	27	100	0	100	0	0	100	0	100
10	86	14	22	78	9	91	100	0	71	29	0	100	0	100
20	70	30	2	98	0	100	89	11	33	67	0	100	0	100
30	70	30	0	100	0	100	88	12	27	73	0	100	0	100

Table 15: **Robustness of the results concerning optimal long-term bond demand with respect to time-variation in bond risk premia and the amount of idiosyncratic labor income** Optimal portfolio choice between 3Y nominal bonds, 10Y nominal bonds, and cash. Investors are potentially endowed with labor income. The investors are distinguished on the basis of the investment horizon (T) and preferences regarding risk (γ). The state variables are set to their unconditional expectations in the benchmark case. The robustness checks consider the effect of zero labor income risk ($\sigma_{\xi=0}$) and the effects of time-variation in the state variables, i.e. the real rate and expected inflation, which are set two unconditional standard deviations above the unconditional expectation. The portfolio weights are presented in percentages. The abbreviation DT indicates the constrained, discrete time investment problem without labor income. The abbreviation $\bar{w}_0 = 1$ refers to the investment problem with labor income and low initial wealth. We use 100 batches of $M = 10000$ simulations and report the average portfolio weights across all simulations.

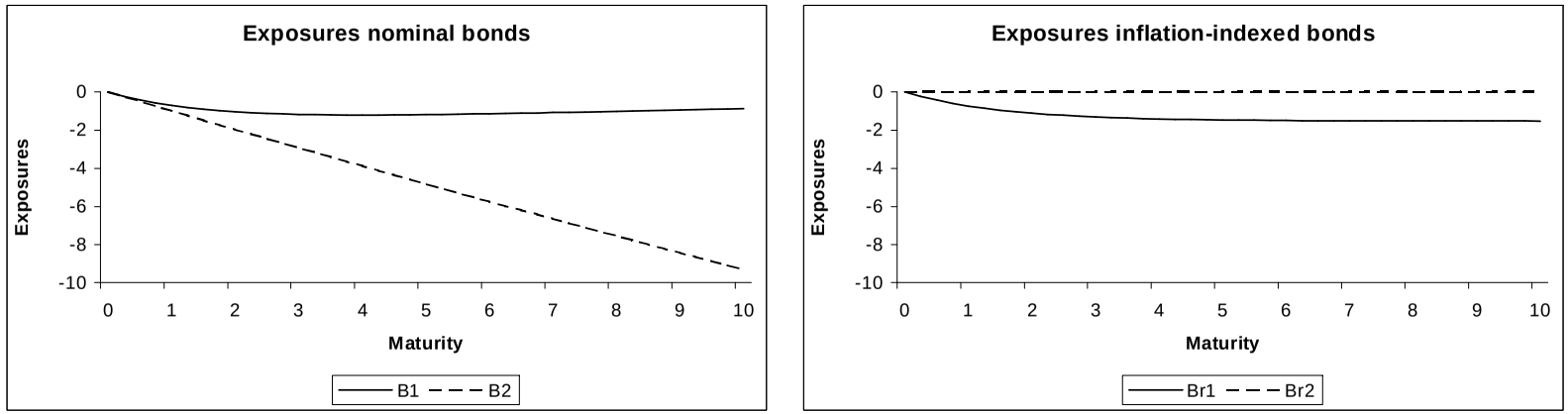

Figure 1: **Exposures of nominal and inflation-linked bonds to the real interest rate and expected inflation** Exposures of both nominal (left graph) and inflation-linked (right graph) bonds to the real interest rate and expected inflation. The horizontal axis depicts the maturity of the bond and the vertical axis represents the exposure. B1 indicates the exposure of a nominal bond to the real interest rate and B2 denotes the exposure of a nominal bond to expected inflation. Br1 and Br2 have the same interpretation, but for inflation-linked rather than nominal bonds.

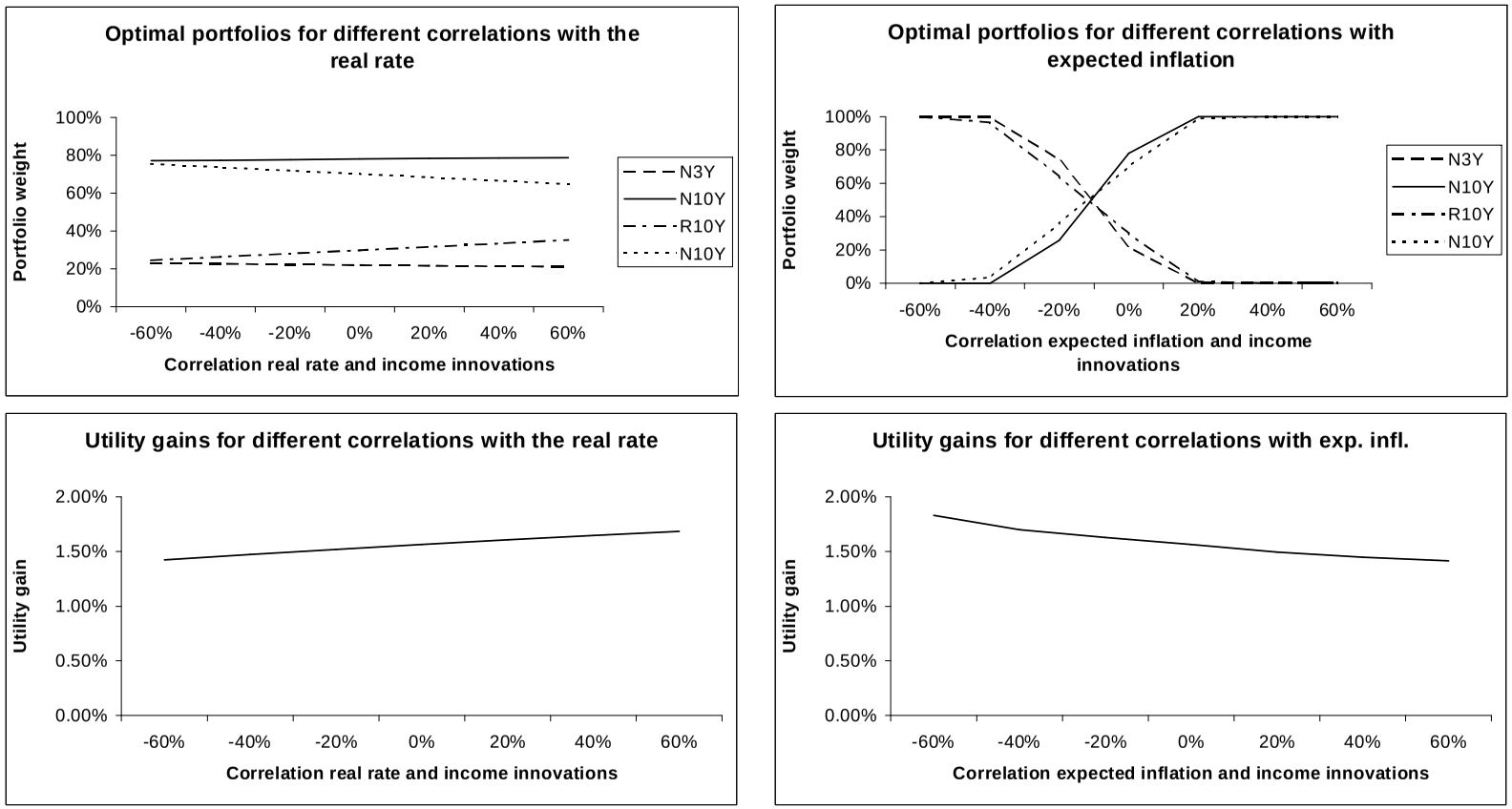

Figure 2: **Impact of correlations between labor income risk and either real interest rate or expected inflation risk on the optimal demand for long-term bonds and the value added of inflation-linked bonds** The two top graphs depict the optimal allocation between either 3Y and 10Y nominal bonds or 10Y nominal and 10Y inflation-linked bonds for different correlations between either labor income and real interest rate innovations (left) or labor income and expected inflation innovations (right). In both top graphs, the portfolios are grouped in descending order. The two bottom graphs depict the utility gains of having access to inflation-linked bonds for different correlations between either labor income and real interest rate innovations (left) or labor income and expected inflation innovations (right). The correlations considered vary between -60% and 60%.

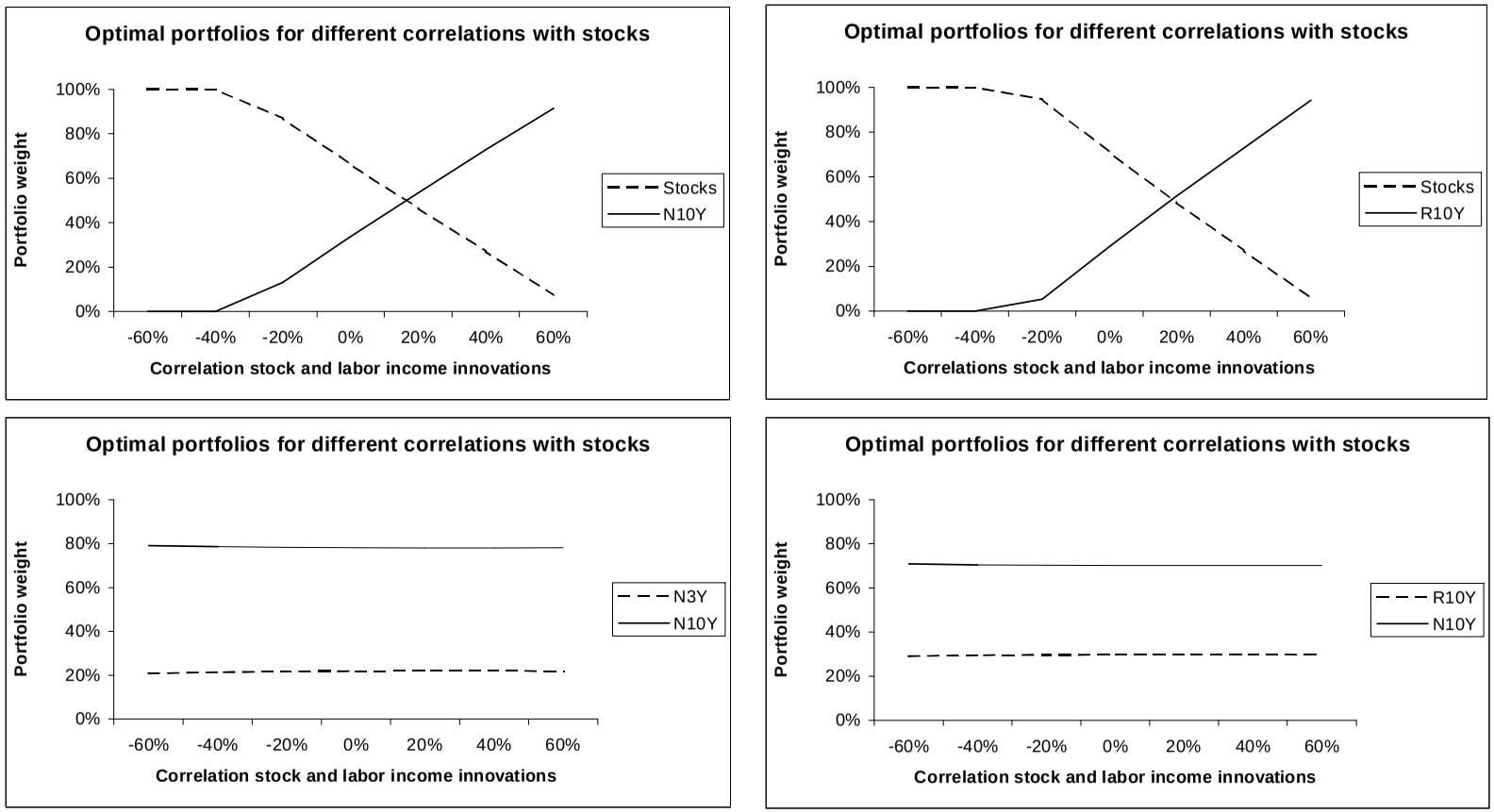

Figure 3: **Impact of correlations between labor income risk and stock risk on either the optimal demand for long-term bonds or the optimal stock-bond mix** The two top graphs depict the optimal allocation between either stocks and 10Y nominal bonds (left) and stocks and 10Y inflation-linked bonds (right) for different correlations between labor income and stock innovations. The two bottom graphs depict the optimal allocation between either 3Y and 10Y nominal bonds (left) and 10Y inflation-linked bonds and 10Y nominal bonds (right) for different correlations between labor income and stock innovations. The correlations considered vary between -60% and 60%.