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# Hedging House Price Risk: Portfolio Choice with Housing Futures

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**Abstract**

We assess the economic benefits of having access to housing futures for home-owning investors, using a model for the portfolio choice between stock, bonds of various maturity (including mortgages) and the housing futures. We compare the utility gains of housing futures with the economic benefits of two other important housing-related portfolio decisions: (i) incorporating the housing exposure in financial portfolio choice and (ii) mortgage choice. Our analysis indicates that the portfolio implications and welfare improvements of the housing futures are small. This is mainly due to the large remaining idiosyncratic house price risk which cannot be hedged using futures written on a city-level house price index.

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# Hedging House Price Risk: Portfolio Choice with Housing Futures


**Abstract**

We assess the economic benefits of having access to housing futures for home-owning investors, using a model for the portfolio choice between stock, bonds of various maturity (including mortgages) and the housing futures. We compare the utility gains of housing futures with the economic benefits of two other important housing-related portfolio decisions: (i) incorporating the housing exposure in financial portfolio choice and (ii) mortgage choice. Our analysis indicates that the portfolio implications and welfare improvements of the housing futures are small. This is mainly due to the large remaining idiosyncratic house price risk which cannot be hedged using futures written on a city-level house price index.

# 1 Introduction

House price risk is one of the largest financial risks that homeowning households face. How to manage and hedge this risk is therefore a crucial question in financial decision making. Clearly, the extent to which house price risk can be hedged has implications for financial portfolio choice and mortgage choice. Recently, the Chicago Mercantile Exchange introduced housing futures for 10 US cities with as prospective users "Real estate owners who wish to hedge risk...".[1] In this paper we investigate financial portfolio choice for homeowners, including housing futures in the menu of available assets.

We assess the economic benefits of having access to housing futures in a portfolio choice context for an investor who is exposed to house price risk and has access to stocks, bonds, and mortgages. To put the utility gains of housing futures in perspective, we compare these gains with the economic benefits of two other important housing-related portfolio decisions: (i) incorporating the housing exposure in financial portfolio choice, and (ii) mortgage choice.

Estimating our model using house price data and financial asset return data, the main result is that having access to these housing futures generates small utility gains and has little impact on portfolio choice for stocks and bonds. In contrast, mortgage choice and incorporating housing in financial portfolio choice turn out to be economically very important for financial decision making. We find that optimal positions in housing futures are (close to) zero in most cases. Only a very risk averse investor with a large housing position shorts significant amounts of housing futures, but even in this case the economic benefits of housing futures are relatively small. The main reason for this result is that a large part of housing risk is idiosyncratic, so that housing futures hedge only a small part of total housing risk. In addition, the positive expected return on housing creates a positive speculative demand for housing futures which partially offsets the negative hedging demand. We show that a hypothetical futures contract that fully hedges house price risk would have much higher economic value. We also document that stocks and bonds have low correlations with house prices, so that these assets only provide a limited hedge of house price risk.

[1] Futures on a national house price index were also introduced. See the May 2006 presentation by the Chicago Mercentile exchange, slide 22, http://www.cme.com/files/CmeCsiHousing.pdf.

Our setup is as follows. We take a long-term investment perspective, where the investor derives utility from terminal wealth. The housing investment is taken as fixed and given, while positions in financial assets are rebalanced dynamically. At the end of the horizon the investor will liquidate the housing position, so that the investor is long house price risk. This setup applies to households that have a large investment in housing and a low present value of future housing consumption.[2] Many older households fit into this category because they plan to move to a smaller house and have short expected remaining lifetime. In a life-cycle model, Van Hemert (2007) indeed shows that many investors end up in this situation around age 65. In the absence of a bequest motive, the value of the house is used for consumption, creating a potential desire to hedge house price risk. This class of investors is an obvious candidate to potentially benefit from housing futures.

Another important part of our setup is a realistic model for the term structure of interest rates, with expected inflation and real interest rate as factors. This allows us to assess the economic benefits of different types of mortgage loans, which we model as a short position in fixed income securities. In addition to the two term-structure factors, we model unexpected inflation, house price risk and stock market risk, leading to a total of five sources of uncertainty. This structure enables us to realistically examine the interaction of financial asset prices and the house price. This way, we can analyze the extent to which financial assets hedge house price risk.

Besides the main result on the usefulness of housing futures, this paper also derives an insightful analytical expression for the investor's optimal financial portfolio. This portfolio is composed of positions in (i) the nominal mean-variance tangency portfolio; (ii) a portfolio that most closely resembles an inflation-indexed bond; and (iii) a portfolio that best offsets the risk of the illiquid house. The first two portfolios were also derived by Brennan and Xia (2002). The expressions show that investors can partially hedge house price risk using housing futures. However, since the housing futures are based on city-level indices, the fixed housing position cannot be hedged fully. This market incompleteness reduces the effective value of the house that is relevant for financial decision making and thus impacts financial portfolio choice. This effect is related to the impact of undiversifiable labor income risk on portfolio choice (see Koo (1998) and

[2]Sinai and Souleles (2005) stress the contribution of future housing consumption to the overall exposure to house price risk.

Munk (2000)).[3] Also, since a house is not a pure financial asset, but also provides housing services, it has a lower expected return than a housing futures contract. The difference in returns is often called market-imputed rent. Our analytical results show that this market-imputed rent affects portfolio choice in a similar way as the market incompleteness does. More specifically, both mechanisms reduce the effective value of the house, thus lowering total effective wealth of the investor and decreasing the optimal investment in financial assets. We find that these effects of housing on portfolio choice increase with the investor horizon.

We estimate the model parameters using data on equity, bond, and house prices, and study the optimal financial portfolios for different investor horizons and house sizes. Besides the main results on hedging with housing futures, we also document interesting effects for the portfolio choice of stocks and bonds and mortgage choice. First, in order to maintain an approximately constant absolute stock market exposure, the financial portfolio weights are levered up. Second, since the risk-averse investor is exposed to undiversified risk of the fixed house position, she will decrease her exposure to stock and bond market risk. In case of short-sale constraints, an additional effect is that stock and bond positions compete in terms of their hedging and return benefits. An investor with moderate risk aversion focuses on capturing risk premia and thus optimally buys both short-term and long-term bonds, and has a large stock position. In contrast, an investor with high risk aversion is more concerned about hedging real rate risk and expected inflation risk, and therefore buys short-term bonds and sells long-term bonds.

We also study optimal mortgage choice in this framework and find that in most cases investors prefer adjustable-rate mortgages (ARM) to fixed-rate mortgages (FRM), which is broadly in line with results of Campbell and Cocco (2003) and Van Hemert (2007). More specifically, we find that a moderately risk-averse investor always prefers an ARM, in order to avoid paying the risk premium on long-term bonds. A very risk-averse investor is relatively more concerned about hedging inflation and interest rate risk, and thus optimally chooses a combination of an ARM and FRM. We show that choosing a suboptimal mortgage can lead to a utility losses, expressed as certainty-equivalent wealth

[3] More generally, several papers have studied the impact of background risk on portfolio choice. For example, Gollier and Pratt (1996) charactize utility functions for which background risk makes investors behave in a more risk-averse way, and this includes our CRRA preferences. In our case, the effect of housing risk (in the role of background risk) is more complicated due to (i) correlations between asset and housing returns and (ii) the market-imputed rent.

reduction, of up to 3% for long horizons, which is much larger than the economic benefits of housing futures. This illustrates that the mortgage choice should play a central role in a household's financial planning.

Our work contributes to a growing literature that studies housing and portfolio choice. Brueckner (1997) and Flavin and Yamashita (2002) study portfolio choice and housing in a static one-period mean-variance setting. Cocco (2005), Hu (2005), and Yao and Zhang (2005) incorporate housing in life-cycle portfolio choice. Our paper complements this literature in several ways. We have a much richer asset menu, incorporate housing futures, study the choice for mortgage type, and implement more sophisticated modelling of the interaction of the return on the house with financial asset returns. We do not, however, model life-cycle features like the labor income profile or labor income risk.

To the best of our knowledge, independent work by Voicu (2007) is the only existing paper that studies the role of housing futures for hedging house price risk. He studies the case of an investor who considers moving to a different city. Instead, our setup is more suitable for older investors who consider moving to a smaller house around retirement. Voicu (2007) does not compare the utility gains of using housing futures with other housing-related decisions, as we do. Also, Voicu (2007) assumes deterministic interest rates while we incorporate a detailed two-factor model of interest rates, which allows us to study mortgage choice.

The structure of the paper is as follows. Section 2 presents the investor's portfolio allocation problem, describes the price processes of the available assets, and discusses the optimal portfolio choice. In section 3 we discuss the estimation of the model parameters. Section 4 contains our main results for unconstrained investors as well as for investors with short-sale constraints. We study the economic benefits of housing futures and mortgages, and consider several robustness checks. Section 5 concludes.

# 2 Optimal Asset Allocation

In this section we present the homeowner's portfolio allocation problem. Our setup incorporates that housing differs from financial assets in several respects. First, the total amount of housing is often dictated by consumption motives rather than investment motives. Second, the housing investment is far less liquid than financial investments because of high monetary and effort costs involved with moving. Third, the expected housing return will be lower than on a hypothetical pure financial asset with comparable risk characteristics, because the market will recognize that a house also provides housing services.

The structure of this section is as follows. We first describe the economy and the price processes of the available assets, and discuss the optimal portfolio choice. For the case with no constraints on the size of positions in financial assets, we are able to provide an implicit analytical expression for the optimal investment in stocks, bonds with different maturities, and cash. In the special case that housing risk is perfectly hedgeable we can solve for the optimal investment explicitly in closed form. Finally, we discuss the numerical techniques used to analyze the model with short sale constraints.

## 2.1 The Investor's Optimization Problem

We consider optimal financial portfolio choice for an investor from time 0 until time $T$, her horizon. We assume that besides financial assets, the investor owns the house she lives in, which has a given size $H$. The house size is interpreted as a one-dimensional representation of the quality of the house. At the investment horizon $T$, which could be interpreted as the moment of retirement, the house is sold and the proceeds, together with the financial assets, are used for consumption. The possibility to sell her house or buy a second house before time $T$ is ignored. The nominal price of a unit of housing at time $t$ is denoted $Q_t$, where $Q_0$ is normalized to 1. Nominal housing wealth is denoted $W_t^H \equiv Q_t H$. We define $W_t^F$ as nominal financial wealth. Initial financial wealth $W_0^F$ excludes housing wealth but includes human capital; labor income risk and moral hazard issues involved in capitalizing labor income are ignored. Recall, however, that we have in mind an older investor who is close to retirement, for whom these assumptions may not be too strong. We also make the simplifying assumption that maintenance costs are capitalized and paid in advance, which means they do not play an explicit role in our analysis. Taking into account that labor income is capitalized, we like to think of the housing to total wealth ratio, $h$, as being in the order of magnitude of 0.3, and in our tables it typically ranges from 0 to 0.6.[4]

Total nominal wealth is denoted $W_t \equiv W_t^F + W_t^H = W_t^F + Q_t H$. At time $T$, the investor uses her total wealth for consumption of other goods. The real price of these consumption goods is chosen to be the numeraire. The nominal price level at time $t$ is denoted as $\Pi_t$ and we normalize $\Pi_0 = 1$. We use uppercase letters for nominal variables and the corresponding lowercase letter for their real counterpart, so total real wealth is $w_t = W_t/\Pi_t$. Following Cocco (2005) and Yao and Zhang (2005) we represent preferences over housing consumption to other goods by the Cobb-Douglas function

$$u\left(w_T, H\right) = \frac{\left(w_T^{\psi} H^{1-\psi}\right)^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} = \frac{w_T^{1-\gamma}}{1-\gamma}\nu_H \tag{1}$$

with $\nu_H \equiv \psi H^{(1-\psi)(1-\tilde{\gamma})}$, $\gamma \equiv 1 - \psi\left(1-\tilde{\gamma}\right)$. We have $\gamma = -w_T u_{ww}/u_w$, which is the coefficient of relative risk aversion given a fixed position in housing.

At $t \in [0, T]$ the investor solves the dynamic portfolio allocation problem

$$\max_{\substack{x(\tau)\in A,\\ t\leq\tau\leq T}} E_t\left[u\left(w_T, H\right)\right], \tag{2}$$

where the financial portfolio weights $x\left(\tau\right)$, $t \leq \tau \leq T$, determine the dynamics of nominal financial wealth $W_t^F$, which in turn affects total real wealth $w_t = \left(W_t^F + W_t^H\right)/\Pi_t$, and where $A$ is the set of admissible financial portfolio weights. Throughout the paper, we assume that this set $A$ is independent of total wealth and the real interest rate. We also assume that starting financial wealth is nonnegative, $W_0^F \geq 0$, implying that it is always possible to have nonnegative financial wealth at any time, $W_t^F \geq 0$ for all $t \in [0, T]$, which ensures that the investor's problem is well defined.

[4]Heaton and Lucas (2000, Table V) report housing to total wealth ratios in the range 0.1 to 0.3 while including capitalized labor income, social security and pension benefits in the total wealth measure. In contrast, Flavin and Yamashita (2002) ignore human capital as part of total wealth. Their housing to total wealth ratio ranges from 0.65 to 3.51.

## 2.2 Asset Price Dynamics

We consider an economy similar to Brennan and Xia (2002) in order to model the price behavior of stocks and bonds, and add two sources of uncertainty to capture aggregate and idiosyncratic house price risk. As stated earlier, we focus on the financial portfolio choice of a single investor who takes price processes as given. Furthermore, we assume that the risk premia on the sources of uncertainty are constant.

The dynamics for the nominal stock price, $S$, real interest rate, $r$, expected inflation, $\pi$, housing futures price, $G$, nominal house price, $Q$, and the price level, $\Pi$, are respectively given by

$$\frac{dS}{S} = [R_f + \sigma_S \lambda_S] \, dt + \sigma_S dz_S, \tag{3}$$

$$dr = \kappa_r (\bar{r} - r) \, dt + \sigma_r dz_r, \tag{4}$$

$$d\pi = \kappa_\pi (\bar{\pi} - \pi) \, dt + \sigma_\pi dz_\pi, \tag{5}$$

$$\frac{dG}{G} = \theta' \lambda dt + \theta' dz, \tag{6}$$

$$\frac{dQ}{Q} = \left[R_f + \theta' \lambda - r^{imp}\right] dt + \theta' dz + \theta_\varepsilon dz_\varepsilon, \tag{7}$$

$$\frac{d\Pi}{\Pi} = \pi dt + \xi' dz + \xi_u dz_u, \tag{8}$$

where $R_f$ is the nominal interest rate, $\sigma$'s capture the volatility, $\lambda$'s are the nominal prices of risk, $dz$ the vector of innovations in Brownian motions (discussed in more detail below), $\kappa$ the mean reversion parameters, $\bar{r}$ and $\bar{\pi}$ unconditional means, and $r^{imp}$ is the imputed rent on owner-occupied housing.

The first three equations are taken from the Brennan and Xia (2002) model, and have shocks to stock prices, real interest rates and inflation $dz_S$, $dz_r$, $dz_\pi$, with covariance matrix $\rho_{S,r,\pi}$.

For the house price equations, we define a vector $z = (z_S, z_r, z_\pi, z_v)$ where the element $dz_v$ is the unexpected shock to the general house price index, which is uncorrelated with $dz_S$, $dz_r$, $dz_\pi$. The correlation matrix of $dz$ therefore is

$$\rho = \begin{pmatrix} \{\rho_{S,r,\pi}\}_{3\times 3} & 0_{3\times 1} \\ 0_{1\times 3} & 1 \end{pmatrix}.$$

In addition, $\theta = (\theta_S, \theta_r, \theta_\pi, \theta_v)'$ is the vector of loadings of the house price index changes on the various Brownian motions, and $\lambda = (\lambda_S, \lambda_r, \lambda_\pi, \lambda_v)'$ is the vector of nominal prices of risk associated with $z$. The housing futures contract is written on this index, and therefore follows the dynamics given in equation (6).

The return on an individual house is the sum of the risk free rate, the housing futures return, minus the imputed rent,[5] and an idiosyncratic component, $z_\varepsilon$. We assume that idiosyncratic housing risk is orthogonal to the other sources of risk; total house price volatility therefore is $\theta'\rho\theta + \theta_\varepsilon^2$. In our calibration we assume that idiosyncratic house price risk is not priced. We also assume that there are no tradable financial assets available whose nominal return have a non-zero loading on $dz_\varepsilon$, i.e. there are no contracts on idiosyncratic house price risk.

In equation (8) for the price level, $\xi = (\xi_S, \xi_r, \xi_\pi, \xi_v)'$ and $dz_u$ is orthogonal to $dz$. Throughout the paper, we assume that there are no assets available whose nominal return have non-zero loading on $dz_u$, and therefore that there are no inflation-indexed assets available.

The processes of the real interest rate and inflation rate imply a two-factor affine term-structure model, and Brennan and Xia (2002) show that the nominal price at time $t$ of a discount bond with a maturity $\tau$, denoted as $P_t(\tau)$, satisfies

$$\begin{aligned}
\frac{dP(\tau)}{P(\tau)} &= \left[R_f - B_r(\tau)\,\sigma_r\lambda_r - B_\pi(\tau)\,\sigma_\pi\lambda_\pi\right]dt - B_r\sigma_r dz_r - B_\pi\sigma_\pi dz_\pi, && (9)\\
B_r(\tau) &= \kappa_r^{-1}\left(1 - e^{-\kappa_r\tau}\right), && (10)\\
B_\pi(\tau) &= \kappa_\pi^{-1}\left(1 - e^{-\kappa_\pi\tau}\right), && (11)\\
R_f &= r + \pi - \xi'\lambda - \xi_u\lambda_u, && (12)
\end{aligned}$$

where $R_f$ is the nominal return on the instantaneous nominal risk-free asset (cash). Notice that the return processes of bonds with different maturities differ only in their loadings on $dz_r$ and $dz_\pi$. When there are no constraints on position size, any desired combination of loadings on $dz_r$ and $dz_\pi$ can be accomplished by positions in any two

[5] Flavin and Yamashita (2002) also specify the imputed rent as a constant fraction of the house value, but it has a slightly different interpretation. In their mean-variance set-up it reflects the monetary value of the utility an individual derives from the housing services. In contrast, in our case it represents the expected excess return differential between the house and the housing futures.

bonds with different maturities. Finally, the nominal pricing kernel, $M$, evolves as

$$\frac{dM}{M} = -R_f dt + \varphi' dz. \tag{13}$$

with $\varphi = -\rho^{-1}\lambda$.

### 2.3 Wealth Dynamics and the Indirect Utility Function

To find the solution to the dynamic asset allocation and utility maximization problem (2), we define the state variables total real wealth $w_t$ and the housing-to-wealth ratio $h_t \equiv w_t^H / w_t$. The dynamics of these variables are given in the following theorem:

**Theorem 1 (Dynamics Total Real Wealth and Housing-to-Wealth Ratio)** *We have*

$$\begin{aligned} \frac{dw}{w} &= \left[r + \sigma_w'(\lambda - \rho\xi) - \xi_u(\lambda_u - \xi_u) - hr^{imp}\right] dt + \sigma_w' dz - \xi_u dz_u, & (14) \\ \frac{dh}{h\,(1-h)} &= \left[\sigma_h'(\lambda - \rho\xi) - \sigma_h' \rho \sigma_w - r^{imp}\right] dt + \sigma_h' dz. & (15) \end{aligned}$$

*with $\sigma_w = (1-h)\sigma_F(x) + h\theta - \xi$, $\sigma_h = \theta - \sigma_F(x)$, and $\sigma_F$ is a function of financial portfolio choice $x$.*

**Proof.** See appendix A. ■

The real wealth return and the dynamics of $h$ do not depend on the expected inflation rate, $\pi$. This implies that the investor's indirect utility function will depend on the real riskless interest rate, $r$, but not on the current expected rate of inflation, $\pi$. We therefore have as four state variables total real wealth, $w$, the housing-to-wealth ratio, $h$, the real interest rate, $r$, and time, $t$. Notice that the dynamics for the housing-to-wealth ratio, $h$, are also independent of $dz_u$.

We now can prove the following useful theorem on the indirect utility function:

**Theorem 2 (Indirect Utility Function)** *If the set of admissible portfolio weights, A,* *is independent of $w_t$ and $r_t$, then the indirect utility function can be written as*

$$\begin{aligned} J(w,h,r,t) \equiv\; & \frac{w_t^{1-\gamma}}{1-\gamma}\nu_H \\ & *\exp\left\{(1-\gamma)(r_t-\bar{r})B_r(T-t)\right\} \\ & *\exp\left\{(1-\gamma)\left(-\xi_u(\lambda_u-\xi_u)-\frac{\gamma}{2}\xi_u^2\right)(T-t)\right\} \\ & *I(h_t,t), \end{aligned} \tag{16}$$

**Proof.** See appendix A. ■

The fact that indirect utility is separable in wealth is a well-known consequence of power utility. It is more surprising that it is also separable in the real interest rate. The assumption that the variance of increments in $r$ is independent of the level of $r$ is key for this to hold. Notice that we have not yet specified whether short positions in available assets are possible or not. We only assume that portfolio restrictions do not depend on $w_t$ and $r_t$.

Theorem 2 has two important implications for financial portfolio choice. First, financial portfolio choice is independent of the current value of real wealth, $w_t$, and the current value of the real interest rate, $r_t$. Second, it implies that the market incompleteness caused by the lack of financial assets that load on unexpected inflation shocks $dz_u$ has no impact on the financial asset allocation. The reason is that $dz_u$ is orthogonal to $dz$ and that the financial asset allocation does not influence the future degree of market incompleteness due to the lack of inflation-indexed assets.

## 2.4 Asset Allocation without Housing Futures

When there are no constraints on the size of the position in the different financial assets, we are able to derive some insightful analytical results on optimal asset allocation. We consider both the case without housing futures (this subsection) and the case with housing futures (next subsection).

We assume that the available financial assets are nominal bonds with different maturities (including an instantaneous bond which will be referred to as cash), stocks and (possibly) housing futures. The investor also owner-occupies a house. Recall from subsection 2.2 that the return processes of bonds with different maturities differ only in their loadings on $dz_r$ and $dz_\pi$. When there are no constraints on position size, any desired combination of loadings on $dz_r$ and $dz_\pi$ can be accomplished by positions in any two bonds with different maturities. In the unconstrained case we therefore characterize optimal portfolio choice by optimal allocation to factor assets, whose nominal return has a nonzero loading on exactly one factor (source of uncertainty).

In this subsection we assume that there are no nominal aggregate housing futures. With the available assets we can then obtain any combination of loadings on $dz_S$, $dz_r$ and $dz_\pi$. If we assume that the investor can unconstrained allocate fractions $x_S$, $x_r$ and $x_\pi$ of her financial wealth to three factor assets whose nominal returns have stochastic components $\sigma_S dz_S$, $\sigma_r dz_r$, and $\sigma_\pi dz_\pi$ respectively, and allocate a fraction $1-x_S-x_r-x_\pi$ to cash, then we can derive an implicit expression for the optimal asset allocation.

**Theorem 3 (Factor Asset Allocation without Housing Futures)** *Let the set of admissible portfolio weights, $A$, be such that the investor can allocate unconstrained fractions $x_S$, $x_r$ and $x_\pi$ of her financial wealth to three factor assets whose nominal returns have stochastic components $\sigma_S dz_S$, $\sigma_r dz_r$ and $\sigma_\pi dz_\pi$ respectively, and allocate a fraction $1-x_S-x_r-x_\pi$ to cash. Then the optimal fractions are given by*

$$x_S = \left(\frac{1-(1-\omega)h}{1-h}\right)\left[-\frac{1}{\gamma}\frac{\varphi_S}{\sigma_S}+\left(1-\frac{1}{\gamma}\right)\frac{\xi_S}{\sigma_S}\right]-\frac{\omega h}{1-h}\frac{\theta_S}{\sigma_S}, \tag{17}$$

$$x_r = \left(\frac{1-(1-\omega)h}{1-h}\right)\left[-\frac{1}{\gamma}\frac{\varphi_r}{\sigma_r}+\left(1-\frac{1}{\gamma}\right)\left(\frac{\xi_r}{\sigma_r}-B_r\left(T-t\right)\right)\right]-\frac{\omega h}{1-h}\frac{\theta_r}{\sigma_r}, \tag{18}$$

$$x_\pi = \left(\frac{1-(1-\omega)h}{1-h}\right)\left[-\frac{1}{\gamma}\frac{\varphi_\pi}{\sigma_\pi}+\left(1-\frac{1}{\gamma}\right)\frac{\xi_\pi}{\sigma_\pi}\right]-\frac{\omega h}{1-h}\frac{\theta_\pi}{\sigma_\pi}, \tag{19}$$

*where*

$$\omega\left(h,\tau\right)=1+\frac{\gamma I_h+hI_{hh}}{\gamma\left(1-\gamma\right)I-2\gamma hI_h-h^2I_{hh}}. \tag{20}$$

**Proof.** See appendix A. ■

The intuition for these portfolio weights is fairly simple. The expression in square brackets is exactly the same as the long-term investment portfolio derived by Brennan and Xia (2002). The first term of this portfolio can be seen as a position in the nominal mean-variance tangency portfolio. The second term in the Brennan-Xia portfolio is the projection of an inflation-indexed bond with maturity $T-t$ on $dz$, which is the best possible hedge against unexpected inflation plus a hedge against real interest changes, captured by $B_r(T-t)$. The Brennan-Xia portfolio is pre-multiplied by the term $\frac{1-(1-\omega)h}{1-h}$. The denominator of this term can be understood by noticing that financial wealth is a fraction $1-h$ of total wealth, so that the financial portfolio should be leveraged up by a factor $1/(1-h)$ to get the desired exposure for the total portfolio. The numerator $1-(1-\omega)h$ takes into account that increases in total and financial wealth have different consequences for the housing-to-wealth ratio, $h$, and therefore different utility implications.[6] For our calibrated parameter values, the correction factor $\omega$ will generally be smaller than one.

The last term in (17)–(19) represents a house price hedge term. Taking into account the relative size of financial and housing wealth, a one-for-one hedge against house price risk would give $\frac{h}{1-h}\frac{\theta_i}{\sigma_i}$. However, changes in financial and housing wealth have different utility implications. The appendix shows that $\omega = J_{w^F w^H}/J_{w^F w^F}$. In our calibrations we find $\omega < 1$ which implies that a marginal change in financial wealth has a bigger impact on $J_{w^F}$ than the same marginal change in housing wealth.

Together, equations (17)–(19) show that the investor behaves *as if* the value of the house is smaller than the prevailing price in the market whenever $\omega < 1$. We refer to $1-\omega$ as the (percentage) reduction in effective housing wealth. There are two sources for this reduction in effective housing wealth. First the house is not only an investment good, but also a consumption good, which is reflected in the fact that the expected excess return on housing is lower than the expected housing futures return, and the difference is equal to $r^{imp}$. The part of the house value that merely reflects the (market imputed) value of the stream of housing services until time $T$ should play no direct role in the financial investment decisions. This has a close analogy: when hedging a futures position on an asset with a positive convenience yield, one uses a hedge ratio smaller than one. Second, the fixed housing position leaves the investor exposed to house price risk that cannot be fully hedged. A risk-averse investor takes this into account by effectively lowering the value associated with the housing investment.

[6]Here a change in total wealth means a wealth change leaving the housing to total wealth ratio, $h$, the same. That is, a \$1 increase in $w$ corresponds to a \$$h$ increase $w^H$ and a \$$1-h$ increase $w^F$. In contrast, a change in financial wealth does affect $h$.

This interpretation of the asset allocation results is slightly different from the usual interpretation that in the presence of an illiquid asset an investor effectively behaves more risk averse in her liquid asset allocation, see e.g. Grossman and Laroque (1990) or Faig and Shum (2002). Indeed, appendix A, proof of theorem 3, shows that the term $\frac{1-(1-\omega)h}{1-h}$ can be interpreted as the ratio of the coefficient of relative risk aversion associated with total wealth changes, $\gamma$, to the coefficient of relative risk aversion associated with financial wealth changes, $-w^F J_{w^F w^F}/J_{w^F}$. However, this is not equivalent to assuming a larger value for the risk aversion coefficient $\gamma$ in the utility function, as the interest rate hedge (the second component of the Brennan-Xia portfolio) is still determined by $\gamma$.

Returning to the asset allocation equations (17)–(19), we observe that there are two distinct horizon effects. First, $B_r(T-t)$ captures the horizon-dependent hedge against changes in the real interest rate. Second, as we will show below, the reduction in the effective housing wealth, $1-\omega$, changes substantially with horizon. Both effects make the asset allocation horizon dependent. In addition, with a fixed position in the house, the housing to total wealth ratio $h$ is stochastic and generates time-varying asset allocations.

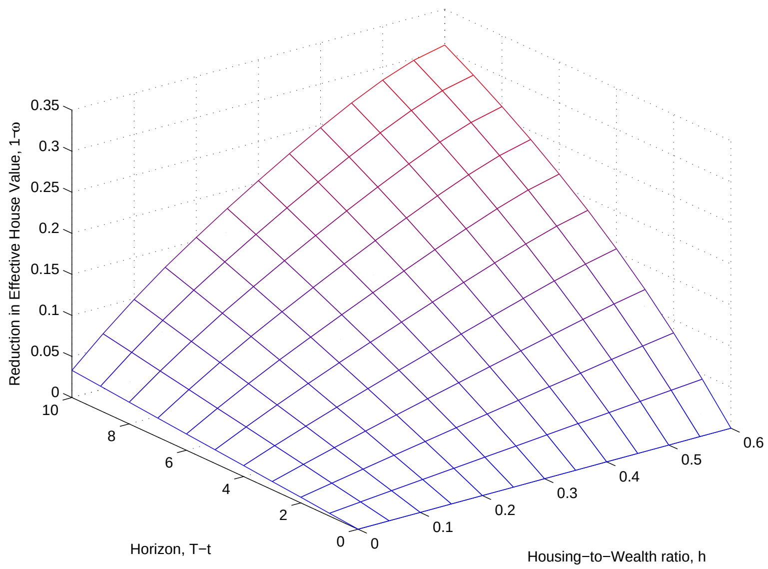
Figure 1 shows the reduction in effective housing wealth, $1-\omega$, for the calibrated parameter values that will be presented in Table 1. There are two sources of market incompleteness in this case: (i) no housing futures on aggregate house price risk are traded, and (ii) the house is subject to idiosyncratic house price risk.

We see that the longer the horizon, the larger the reduction in effective housing wealth. In the limiting case of a zero horizon, the reduction is zero. We also see that the larger the house, the larger is the (percentage) reduction in effective housing wealth. At a horizon of $T = 10$ years and a housing-to-wealth ratio of $h = 0.6$ this amounts to a reduction of 30%.

## 2.5 Asset allocation with Housing Futures

Now we turn to the case that nominal housing futures are available. Housing futures are a zero-investment position with size as a fraction of financial wealth denoted by $x^G$. At time $t$ the investor receives (if she is long) or pays (if she is short) $x^G dG/G$ as a fraction of financial wealth. Housing futures allow for exposure to general house prices shocks

Figure 1: Reduction in Effective Housing Wealth (No Housing Futures).
The figure plots the reduction in effective housing wealth, $1-\omega$, for a $\gamma = 5$ investor. No futures on aggregate house price risk are available. The calibrated parameter values of Table 1 are used.




uncorrelated with financial asset price shocks, denoted by $dz_v$.

**Theorem 4 (Factor Asset Allocation with Housing Futures)** *Let the set of admissible portfolio weights, $A$, be such that the investor can allocate unconstrained fractions $x_S$, $x_r$, $x_\pi$, and $x_v$ of her financial wealth to four factor assets whose nominal returns have stochastic components $\sigma_S dz_S$, $\sigma_r dz_r$, $\sigma_\pi dz_\pi$, and $\theta_v dz_v$ respectively, and allocate a fraction $1 - x_S - x_r - x_\pi - x_v$ to cash. Then optimal fractions for are $x_S$, $x_r$, $x_\pi$ are still given by (17)–(19), and*

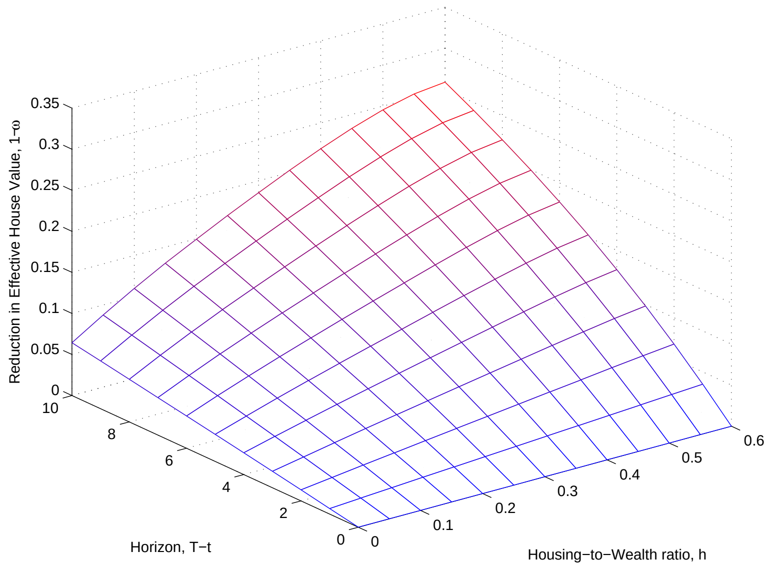
$$x_v = \left(\frac{1-(1-\omega)h}{1-h}\right)\left[-\frac{1}{\gamma}\frac{\varphi_v}{\theta_v} + \left(1-\frac{1}{\gamma}\right)\frac{\xi_v}{\theta_v}\right] - \frac{\omega h}{1-h}, \tag{21}$$

*where as before* $\omega(h,\tau) = 1 + \frac{\gamma I_h + h I_{hh}}{\gamma(1-\gamma)I - 2\gamma h I_h - h^2 I_{hh}}$.

**Proof.** See appendix A. ■

In figure 2 we present the reduction in effective housing wealth when futures on aggregate house price risk are available, again using the calibrated parameter values.

Figure 2: Reduction in Effective Housing Wealth (With Housing Futures).
The figure plots the reduction in effective housing wealth, $1-\omega$, for a $\gamma = 5$ investor. Futures on aggregate house price risk are available. The calibrated parameter values of Table 1 are used.




We see that for a long horizon ($T = 10$) and a large housing-to-wealth ratio ($h = 0.6$), the reduction in effective housing wealth is smaller when no futures are present (compare 26% in figure 2 with 31% in figure 1). Interestingly for a small housing-to-wealth ratio, e.g. consider the limiting case $h = 0$, the reduction in effective housing wealth is larger in the presence of housing futures. The reason is that in the absence of housing futures, the investor actually prefers a (small) positive aggregate house price risk exposure through owner-occupied housing to exploit the associated (positive) risk premium. This has a positive impact on effective housing wealth. In the presence of housing futures the owner-occupied house has no added value in terms of reaping risk premiums; recall that idiosyncratic house price risk is not priced.

To further illustrate the effective housing to total wealth ratio we provide an explicit closed-form solution for $\omega$ in the special case that there is no idiosyncratic house price risk, i.e. $\theta_\varepsilon = 0$. In this case, all the house price risk can be hedged using the housing futures contract.

**Theorem 5 (No Idiosyncratic House Price Risk)** *If (i) it is possible to perfectly* *hedge house price risk with futures, i.e.* $\theta_\varepsilon = 0$*, and (ii) the investment opportunity set,* $A$*, is such that the investor can allocate unconstrained fractions* $x_S$*,* $x_r$*,* $x_\pi$*, and* $x_v$ *of her financial wealth to four factor assets whose nominal returns have stochastic components* $\sigma_S dz_S$*,* $\sigma_r dz_r$*,* $\sigma_\pi dz_\pi$*, and* $\theta_v dz_v$ *respectively, and allocate a fraction* $1 - x_S - x_r - x_\pi - x_v$ *to cash, then*

$$I(h,t) = [1 - (1-\omega)h]^{1-\gamma} \exp\{(1-\gamma) f(\tau)\} \tag{22}$$

*with* $\omega = e^{-r^{imp}\tau}$ *and the expression for* $f(\tau)$ *is given in the appendix.*

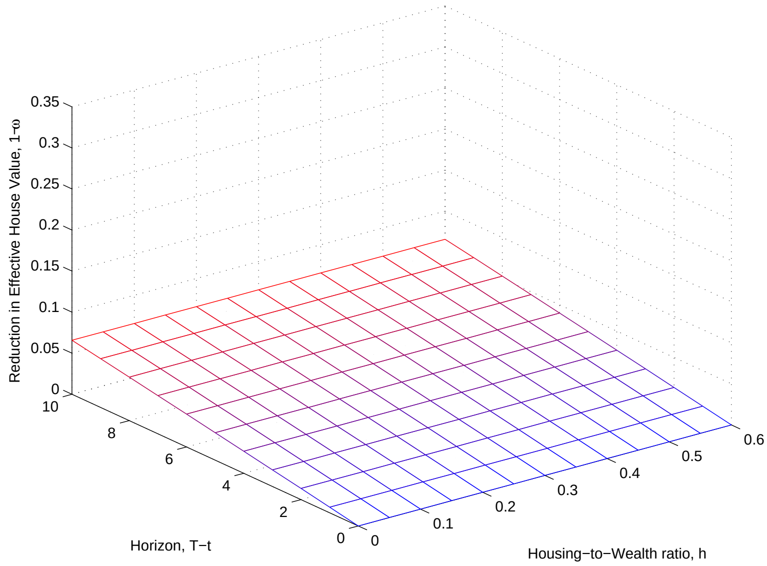
**Proof.** See appendix A. ■

Figure 3 shows that with $\theta_\varepsilon = 0$ the reduction in effective housing wealth is independent of the housing-to-wealth ratio, $h$; it is solely due to the market imputed value of the housing services up until time $T$. Notice that the reduction in effective housing wealth for $T = 10$ and $h = 0.6$ is only 6%, whereas in the presence of idiosyncratic house price risk, figure 2, it is was 26%. These numbers can be understood from the size of the idiosyncratic and systematic house risk, which are equal to 9.00% and 5.89%, respectively. 'Removing' idiosyncratic risk has a larger impact on $\omega$ than hedging systematic house price risk using a housing futures contract has. Of course, introducing a futures contract that simultaneously hedges $dz_v$ and the idiosyncratic house price risk $dz_\varepsilon$ will have the same effect.

## 2.6 Numerical Solution Technique

For the numerical results we continue to assume that the investment opportunity set, $A$, is independent of $w_t$ and $r_t$. In this case, we can use Theorem 2, and the only part of the indirect utility function that is not known in closed form is $I(h,t)$. We know that $I(h,T) = 1$ for all housing-to-wealth ratios $h$. A grid over $h$ and $t$ is chosen and we solve for $I(h,t)$ and the optimal asset allocation backwards in time. More precisely, without loss of generality at node $(h,t)$ we normalize $w_t = 1$ and $r_t = \bar{r}$. Given the separability of the idiosyncratic price risk of consumption goods, we can ignore $\xi_u$ and $\lambda_u$, which reduces the number of Brownian motions from six to five in the numerical procedure.

Figure 3: Reduction in Effective Housing Wealth (With Futures, No Idiosyncratic House Price Risk).
The figure plots the reduction in effective housing wealth, $1-\omega = 1-e^{-r^{imp}\tau}$, when futures on aggregate house price risk are available and there is no idiosyncratic house price risk ($\theta_\varepsilon = 0$). We have $r^{imp} = 0.68\%$, as in Table 1. The reduction in effective housing wealth is independent of the risk aversion parameter $\gamma$.




Thus we determine $I(h,t)$ by solving

$$I(h_t,t) = \max_{x\in A} E\left[w_{t+dt}^{1-\gamma}(x)\, e^{(\tilde{r}_{t+dt}-\bar{r})B_r(T-t-dt)} I(h_{t+dt}(x), t+dt)\,|w_t = 1, \tilde{r}_t = \bar{r}, h_t\right] \tag{23}$$

where $dt$ is the step size of the grid over time.[7]

[7]To determine $I(h, t+dt)$ for values of $h$ that are not on the grid, we use cubic spline interpolation. The expectation is evaluated using Gaussian quadrature with 3 points. Increasing the number of points did not alter results in the presented precision. For the optimization over $x$ we combine a search algorithm that uses numerically-determined derivative information with one that doesn't. The methods are found to be complementary and together well capable of finding the optimum. The grid on $h$ and $t$ is chosen fine enough to ensure precision up to the presented number of decimals.

# 3 Calibration

In this section we report the calibrated parameter values (table 1) and provide some details on the data and calibration procedure.

### Term structure of interest rates

We first estimate a term structure model on quarterly data on nominal interest rates and inflation from 1973Q1 to 2003Q4. We use a Kalman filter to extract the real interest rate and expected inflation rate from the data, and estimate the model by Quasi Maximum Likelihood.[8] This procedure provides estimates of the mean reversion parameters and provides time series of innovations in the real interest rate and expected inflation, and a time series of unexpected inflation. The values for the mean reversion parameters of real interest and expected inflation rate, $\kappa_r = 0.6501$ and $\kappa_\pi = 0.0548$, imply half-lives of 1.1 and 12.6 years respectively.[9]

For the other parameters of interest we use quarterly data from 1980Q2 until 2003Q4, which is motivated by the availability of house price data. The reason to estimate the mean reversion parameters over a longer sample period than the other parameters is that we need a long sample to obtain good estimates of the mean reversions; all the other parameters are best fitted to the more recent common sample period, taking the estimated mean reversions from the first step as given. The mean expected inflation is estimated by the mean increase of the CPI. For the mean real interest rate we take the difference between the means of the T-bill rate and the expected inflation, minus a 0.5% correction to reflect the premium on unexpected inflation.[10] The standard deviations of the real interest rate, expected inflation and unexpected inflation are determined using the time series generated by the Kalman filter.[11] We estimate the market price of risk parameters $\lambda_r$ and $\lambda_\pi$ by matching the average yields of two bond portfolios with a

[8]Details on the procedure are provided in Appendix B.

[9]Using different sample periods, Brennan and Xia (2002) and Campbell and Viceira (2001) also find a half-life of around 1 year for innovations in the real rate and a much longer half-life for expected inflation.

[10]The unexpected inflation risk premium is based on the estimate of Campbell and Viceira (2002, p.72)). With this assumption there is no further need to estimate the market price of risk for unexpected inflation, $\lambda_u$, because it does not influence the asset allocation in our set-up with only nominal securities.

[11]The discrete-time standard deviations are converted to the continuous-time counterparts, incorporating the effect of mean reversion in the processes.

Table 1: Calibrated Parameter Values for Asset Price Dynamics

This table presents the calibrated parameter values for the asset price dynamics. Parameter values for the real interest, expected inflation, and unexpected inflation rate are calibrated to quarterly US data on nominal interest rates and inflation from 1973Q1 to 2003Q4. The house price dynamics are calibrated to OFHEO repeated-sales index data for 10 US cities from 1980Q2 to 2003Q4. For stocks an index comprising all NYSE, AMEX, and NASDAQ firms over the same sample period as the house price data is used. All parameters values are annualized.

| Parameter | Estimate |
|---|---|
| Stock return process: $dS/S = (R_f + \sigma_S \lambda_S)\, dt + \sigma_S dz_S$ | |
| $\sigma_S$ | 0.1748 |
| $\lambda_S$ | 0.3633 |
| Real riskless interest rate process: $dr = \kappa_r (\bar{r} - r)\, dt + \sigma_r dz_r$ | |
| $\bar{r}$ | 0.0226 |
| $\kappa_r$ | 0.6501 |
| $\sigma_r$ | 0.0183 |
| $\lambda_r$ | $-0.3035$ |
| Expected inflation process: $d\pi = \kappa_\pi (\bar{\pi} - \pi)\, dt + \sigma_\pi dz_\pi$ | |
| $\bar{\pi}$ | 0.0351 |
| $\kappa_\pi$ | 0.0548 |
| $\sigma_\pi$ | 0.0191 |
| $\lambda_\pi$ | $-0.1674$ |
| House price process: $dQ/Q = \left(R_f + \theta'\lambda - r^{imp}\right) dt + \theta' dz$ | |
| $\theta_S$ | $-0.0087$ |
| $\theta_r$ | $-0.0017$ |
| $\theta_\pi$ | 0.0080 |
| $\theta_v$ | 0.0589 |
| $\theta_\varepsilon$ | 0.0900 |
| $\lambda_v$ | 0.0761 |
| $r^{imp}$ | 0.0068 |
| Realized inflation process: $d\Pi/\Pi = \pi dt + \xi' dz + \xi_u dz_u$ | |
| $\xi_S$ | $-0.0033$ |
| $\xi_r$ | 0.0067 |
| $\xi_\pi$ | 0.0012 |
| $\xi_v$ | $-0.0048$ |
| $\xi_u$ | 0.0527 |
| Correlations | |
| $\rho_{Sr}$ | $-0.1643$ |
| $\rho_{S\pi}$ | 0.0544 |
| $\rho_{r\pi}$ | $-0.2323$ |

constant time to maturity of 3.4 and 10.4 years. For this we use formulas derived by Brennan and Xia (2002), Appendix A.

In the second step of the calibration, we fit the means, standard deviations and correlations of stock returns, real interest rates, expected and unexpected inflation and house prices, and the market prices of risk. Table 1 provides all the (annualized) parameter estimates.

### Stocks

To estimate the stock return process parameters we use quarterly stock returns for the period 1980Q2 until 2003Q4 on an index comprising all NYSE, AMEX and NASDAQ firms.[12]

### House prices

The data on the house price index are obtained from the US Office of Federal Housing Enterprise Oversight (OFHEO). We focus on house price data of cities for which housing futures are currently traded at Chicago Mercantile Exchange (CME).[13] The data consist of quarterly house price indices for each city.[14] Our sample period is 1980Q2 trough 2003Q4. For each city, we transform these data to overlapping annual returns (observed on a quarterly frequency) and we correlate these returns with the unexpected shocks in stock returns, interest rates and inflation. These correlations are then averaged over the 10 cities. The standard deviation of annual house index returns equals 5.99% (averaged across 10 cities). Combining this number with the average correlations, we calculate the coefficients $\theta$ of the housing return process, equation (7). From 1997, OFHEO also reports each quarter an estimate for the idiosyncratic house price risk, as obtained from their repeat-sales regressions at the state level. This estimate is on average 9.00% (aver-

[12]The authors would like to thank Kenneth R. French for making this data available at his website.

[13]These cities are: Boston, Chicago, Denver, Las Vegas, Los Angeles, Miami, New York, San Diego, San Francisco, and Washington D.C.

[14]The CME housing futures are based on the Case-Shiller house price indices. We use the OFHEO data in this paper since OFHEO also provides estimates of idiosyncratic house price risk, which is crucial for our analysis. In Leventis (2007) the Case-Shiller and OFHEO indices for New York and Los Angeles are compared in detail, and a correlation of 93% between the OFHEO and Case-Shiller indices is reported.

aged across states and over time).[15] We use this value as an estimate of the idiosyncratic house price risk parameter, $\theta_\varepsilon$. As we will see below, $\theta_\varepsilon$ is a key parameter for our analysis. It is thus important to mention that across all quarters in our sample period and across all regions used for our analysis, the lowest value for idiosyncratic house price risk equals 6.3%, and the highest value equals 12.0%. These numbers show that idiosyncratic house price risk is consistently large over time and across states. The fact that we use idiosyncratic risk at the state level implies that we consider an investor who lives on an 'average' location in a state and attempts to hedge using a city-level house price futures contract. It is of course possible that for some home-owners the house price is more strongly related to the city-level index. The 9% idiosyncratic risk standard deviation represents the idiosyncratic risk for an 'average' investor. We assume that idiosyncratic house price risk is not priced, i.e. $\lambda_\varepsilon = 0$.

**Imputed rent**

Finally, we need to specify a value for the imputed rent $r^{imp}$ and the market price of systematic house price risk, $\lambda_v$. The average excess house price return is equal to $-0.53\%$ in our sample (averaged over time and across cities). Notice that given the average house price return, $\lambda_v$ and $r^{imp}$ are linked through $E[R_{house}] - R_f = \theta'\lambda - r^{imp}$, and we have to determine only one of both parameters. In principle, $r^{imp}$ could be calculated from the convenience yield in housing futures prices. Voicu (2007, Table 2) calculates this convenience yield and finds very large discounts on futures prices compared to the current house price index (early 2007) and imply an unrealistically high imputed rent (averaged across all 10 cities) of 8.21% per year.[16]

In the absence of good data on imputed rents, we choose to fix the market price of systematic house price risk, $\lambda_v$, as follows. Aggregate data on household wealth in the US show that housing wealth is about as large as stock market wealth.[17] To set a reasonable value for the market price of house risk, we use the asset demand from our model, equations (17) and (21), and find a market price of housing risk that equates

[15] We include here only states which have cities for which housing futures contracts are traded.

[16] This very high value may reflect expected declines in house prices, the illiquidity of the housing futures market, and the breakdown of arbitrage pricing in the futures market due to short-sales constraints.

[17] See the Federal Reserve Board's Flow of Fund Accounts of the United States, release March 2007, Table B.100.

the speculative demand for stocks and the speculative demand for housing. Given the Sharpe ratio of stocks $\lambda_S = 0.0366$, the stock price volatility $\sigma_S = 0.1748$ and the standard deviation of country-wide systematic house price risk, 3.66%, [18] this implies $\lambda_v = (0.0366/0.1748)0.0366 = 0.0761$. With this parameter, and given the average excess house price return $\overline{R_{house} - R_f} = -0.53\%$, it follows that $r^{imp} = 0.6771\%$. We use this value in our calibrations.

**Correlations**

Finally, we estimate the correlation matrix $\rho$ and the coefficient vectors $\xi$ and $\theta$ using stock returns, house price returns, the innovations in the real interest rate, expected inflation, and unexpected inflation. We calculate correlations with the house price innovation on a yearly instead of a quarterly basis. We do so because house prices adjust more slowly to news than financial assets. We find that extending the calibration horizon beyond one year makes little difference: for example, using 2-year housing returns instead of annual housing returns increases the standard deviation from 5.99% to 6.74%.

# 4 Asset Allocation Results

In this section, we present the results for the optimal portfolio allocation with and without short-sales constraints. We consider two different values for the coefficient of relative risk aversion, a moderately risk averse investor with $\gamma = 5$ and a more conservative investor with $\gamma = 10$. For these investors, we also calculate the utility gains of having access to housing futures and mortgage markets.

[18] This is the annual standard deviation of an equally-weighted index of the house price indices of the 10 cities. Due to the imperfect correlation between house prices in these 10 cities, the country-wide index has a lower standard devation (3.66%) than the city-level standard deviation (5.99%). Clearly, only systematic house price risk should be priced and hence we use the country-wide index to calibrate the risk premium.

## 4.1 No short-sales constraints: the $\gamma = 5$ Investor

We start with a setup without short-sale constraints. Even though this is not the most realistic case, it will provide intuition for the effects that also play a role in the more relevant setup with short-sales constraints.

Table 2 shows the asset allocation without (Panel A) and with (Panel B) house price futures.[19] We choose two particular bond maturities, 3 and 10 years, and describe the optimal portfolio choice in terms of the weights on these bonds. Denoting the fraction of financial wealth allocated to stocks, the 3-year bond, the 10-year bond, and cash by $x^s$, $x^{b3}$, $x^{b10}$, and $x^c$, we have the following budget constraint.

$$x^s + x^{b3} + x^{b10} + x^c = 1.$$

We convert the factor asset allocation into portfolio choice using

$$\begin{pmatrix} x^s \\ x^{b3} \\ x^{b10} \\ x^G \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \theta_S \\ 0 & -B_r(3) & -B_r(10) & \theta_r \\ 0 & -B_\pi(10) & -B_\pi(10) & \theta_\pi \\ 0 & 0 & 0 & \theta_v \end{pmatrix}^{-1} \begin{pmatrix} x_S \\ x_r \\ x_\pi \\ x_v \end{pmatrix}, \tag{24}$$

where the fourth row and column vanish in the absence of housing futures.

We see in Table 2 that the mean-variance component, $C_1$, involves large positions in the two bonds. Because the 3-year and 10-year bond returns are quite highly correlated, creating the appropriate exposure to real interest rate risk and inflation risk leads to rather extreme positions in these bonds. The horizon effect is governed by the $\frac{1-(1-h)\omega}{1-h}$ term that appears in theorems 3 and 4. Because the effective housing wealth $\omega$ is smaller than one, and decreasing in horizon for our parameter values, the risk taking is somewhat depressed. The denominator of this term, $1-h$, is a simple leverage effect. The investor ultimately cares about her total wealth, but can attain the desired risk exposure only in

[19] We compute the total portfolio choice using (i) the numerically computed (percentage) effective housing wealth, $\omega$, and then applying theorems 3 and 4 for the portfolio choice, or (ii) using the numerically computed portfolio shares. The results are the same up to the reported precision. It is likely that an error in either the theorem or the code would have resulted in the two methods yielding different values. Hence, this can be interpreted as a check on both the theoretical formula and the numerical procedure.

Table 2: Unconstrained Portfolio Choice for Different Housing-to-Wealth Ratios

This table reports the unconstrained optimal portfolio choice without (Panel A) and with (Panel B) futures for different housing-to-wealth rations, $h$ . The investor has a horizon of $T = 10$ years and a coefficient of relative risk aversion of $\gamma = 5$. The variables $x^s$, $x^{b10}$, $x^{b3}$, and $x^G$ denote the allocation to stocks, 3-year bonds, 10-year bonds, and housing futures. The components of the portfolio weights are denoted as $C_1$ (mean-variance component), $C_2$ (inflation and interest rate hedge component), and $C_3$ (house price hedge component). In addition, this table presents the utility gain of having access to housing futures, $UG$, measured as the certainty-equivalent gain in percentage points.

Panel A: without housing futures

| | $h = 0.0$ | | | | $h = 0.3$ | | | | $h = 0.6$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $C_1$ | $C_2$ | $C_3$ | **total** | $C_1$ | $C_2$ | $C_3$ | **total** | $C_1$ | $C_2$ | $C_3$ | **total** |
| $x^s$ | 0.37 | −0.02 | 0.00 | **0.36** | 0.50 | −0.02 | 0.02 | **0.50** | 0.76 | −0.03 | 0.05 | **0.78** |
| $x^{b3}$ | 3.71 | 1.23 | 0.00 | **4.94** | 4.98 | 1.65 | −0.08 | **6.55** | 7.57 | 2.51 | −0.24 | **9.84** |
| $x^{b10}$ | −0.98 | −0.45 | 0.00 | **−1.43** | −1.32 | −0.60 | 0.05 | **−1.87** | −2.01 | −0.92 | 0.14 | **−2.78** |

Panel B: with housing futures

| | $h = 0.0$ | | | | $h = 0.3$ | | | | $h = 0.6$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $C_1$ | $C_2$ | $C_3$ | **total** | $C_1$ | $C_2$ | $C_3$ | **total** | $C_1$ | $C_2$ | $C_3$ | **total** |
| $x^s$ | 0.39 | −0.02 | 0.00 | **0.37** | 0.52 | −0.02 | 0.00 | **0.49** | 0.82 | −0.04 | 0.00 | **0.78** |
| $x^{b3}$ | 3.65 | 1.25 | 0.00 | **4.90** | 4.92 | 1.68 | 0.00 | **6.61** | 7.71 | 2.63 | 0.00 | **10.34** |
| $x^{b10}$ | −0.95 | −0.46 | 0.00 | **−1.41** | −1.28 | −0.62 | 0.00 | **−1.90** | −2.00 | −0.97 | 0.00 | **−2.97** |
| $x^G$ | 0.26 | −0.07 | 0.00 | **0.19** | 0.35 | −0.09 | −0.35 | **−0.09** | 0.55 | −0.14 | −1.11 | **−0.70** |
| $UG$ | | | | **0.32%** | | | | **0.09%** | | | | **1.08%** |

her financial portfolio. The smaller the financial wealth compared to total wealth, the larger the desired leverage in risk taking. Depending on the house size, this generates economically large effects on the optimal portfolio weights.[20]

The second component $C_2$ is the portfolio that most closely resembles the long-term risk free asset, which is an inflation-indexed bond. The largest position in this component is to the 3-year bonds that provide a hedge against changes in the real interest rate. A 10-year bond position would also provide this hedge, but has a much larger exposure to inflation risk.

The house price hedge component, $C_3$, is zero for $h = 0$. We see that in the absence of housing futures, panel A, stocks and bonds serve as a partial hedge against aggregate

[20]Notice that the stock and bond allocation differs between panel A and B, even for $h = 0$ when $\omega = 1$. At first sight this might seem in contrast with theorem 3 and 4, stating that the same expression for the stock and bond allocation holds with and without housing futures. However these statements are on factor asset allocation. When converting to actual portfolio choice, the investor takes into account that the return on housing futures is correlated with the return on stocks and bonds ($\theta_S, \theta_r, \theta_\pi \neq 0$), affecting the actual position in the stocks and bonds.

house price risk for $h > 0$. In the presence of housing futures, stocks and bonds have no added value for hedging aggregate house price risk, resulting in a zero value for the $C_3$ component in panel B. Instead, households short housing futures, as can be seen by the negative value for $x^G$ in component $C_3$.

For $h = 0$ the investor desires to take a long position in housing futures to exploit the positive risk premium ($C_1$ component) and a short position for the portfolio most closely resembling a long-term inflation-indexed bond ($C_2$ component). For $h > 0$, the hedge against house price risk is a third motivation to hold housing futures. For our calibrated parameter values and a coefficient of relative risk aversion of $\gamma = 5$, at $h = 0.3$ the different motivations to hold housing futures largely offset, leading to a small net short position, in turn leading to a small utility gain of having access to housing futures of 0.09%. Moving to the $h = 0.6$ case, the hedge motive increases disproportionately, with a moderate utility gain of 1.08%. In the next subsection we discuss the utility gains in more detail, when we focus on the more relevant case of constrained portfolio choice.

## 4.2 Constrained Portfolio Choice: Imposed Constraints

In this subsection we discuss the imposed constraints on the size of financial positions. In the next subsections we discuss the results for the baseline case and do several sensitivity analyses.

Recall that $x^i$ denotes the position in financial asset $i$ as a fraction of financial wealth $w^F$. We impose the following constraints:

$$\begin{aligned} x^s, x^{b3} &\geq 0 \text{ (short-sale constraints)} \\ x^{b10}, x^c - |x^G|, x^{b10} + x^c - |x^G| &\geq -(1-\delta)\, h/(1-h) \text{ (mortgage constraint)} \end{aligned}$$

We assume that any position in a housing futures $x^G$ should be accompanied with a margin position of $|x^G|$ in cash. The total cash position, $x^c$, can thus be expressed as the sum of the cash that is tied up as margin for the housing futures, $|x^G|$, and free cash $x^c - |x^G|$.

A mortgage is modelled as a negative position in cash and/or 10-year bonds, proxying for an adjustable-rate mortgage (ARM) and a fixed-rate mortgage (FRM) respectively.

We allow for a mortgage up to the value of the house minus a downpayment, $(1-\delta)\,w^H$, where $\delta$ is the downpayment ratio. This translates to a constraint $x^i > -(1-\delta)\,h/(1-h)$ on the 10-year bond weight, free cash, and the sum of both. Note that in the setup without portfolio constraints, mortgage choice was irrelevant since it can always be 'undone' by appropriate positions in the bond market.

Also note that it is the free cash that enters the constraints for mortgage choice above. When the investor is borrowing constrained, e.g. $x^c - |x^G| = -(1-\delta)\,h/\,(1-h)$, then a \$1 increase in the margin requirement, $|x^G|$, must lead to a \$1 increase in the cash position, $x^c$, which through the budget constraint $x^s + x^{b3} + x^{b10} + x^c = 1$ must lead to a \$1 decrease in the risky asset allocation $x^s + x^{b3} + x^{b10}$. Given the one-for-one margin requirement, it is straightforward to show that stock and bond futures would be redundant assets. In our model, the only way to obtain a leveraged portfolio is by taking out a mortgage on the house.

## 4.3 Constrained Portfolio Choice: Results

Table 3 shows the optimal portfolio choice under the portfolio constraints for a $\gamma = 5$ (panel A) and $\gamma = 10$ (panel B) investor, both for the cases with and without housing futures. We cannot decompose the total portfolio choice into three components, as we did for the unconstrained case presented in Table 2. In the presence of constraints, the allocation to the different asset classes need to be rationed. Considering the strict positive downpayment ratio $\delta = 20\%$, the larger $h$, the lower the funds available to invest in financial assets, the greater the rationing.

For risk aversion $\gamma = 5$, the stock allocation is hardly affected by the rationing, even at larger housing-to-wealth ratios, and is very similar to the unconstrained counterpart. We see that the allocation to 10-year bonds is strictly positive for all $h$, while it was always negative for the unconstrained case. This illustrates the trade-off the investor needs to make between the speculative and hedging demands for real interest rate risk and inflation risk. For the $\gamma = 5$ investor, risk premia are relatively more important than hedging, leading to a positive weight for 3-year and 10-year bonds.

The demand for housing futures is zero for all values of $h$: the $h = 0$ investor does not go long in the futures since the expected equity return is more attractive, while the

$h > 0$ investor does not care about hedging enough to take a short futures position. Recall that any futures position is costly in the sense that the cash margin requirement restricts the investor to exploit stock and bond risk premia.

The associated utility gains of housing futures are calculated as

$$UG = \left\{ \left( J^{with} / J^{without} \right)^{1/(1-\gamma)} - 1 \right\} * 100\%, \tag{25}$$

where $J^{with}$ and $J^{without}$ denote the indirect utility with and without futures respectively. For $h = 0$ and $h = 0.3$, these utility gains are equal to 0.00%. For $h = 0.6$, the utility gains of housing futures are slightly positive at 0.02%, even though the initial portfolio weight for housing futures equals zero. This can be explained by the fact that in certain future scenarios the housing futures allocation is nonzero (in particular if $h$ increases), which already impacts today's asset allocation and utility. This also explains why the allocation to to 10-year bonds changes a bit when housing futures are included in the asset menu.

The possibility to borrow against the value of the house is taken only in the form of cash, i.e. an adjustable rate mortgage (ARM). An ARM is less 'costly' in terms of expected returns than an FRM, and this is the main reason to choose an ARM for the $\gamma = 5$ investor.

The results for a more risk averse investor, $\gamma = 10$, are quite different. Naturally, the demand for stocks is lower, but the more interesting differences are in the bond and futures allocations. The demand for 3-year bonds is increased compared to the $\gamma = 5$ case, and the demand for 10-year bonds is negative (for $h > 0$). Recall that the hedge portfolio for interest rate and inflation risk ($C_2$ in Table 2) is long in the 3-year bond and short in the 10-year bond. The $\gamma = 10$ investor cares more about this hedging, which explains the bond positions. The fact that the demand for 10-year bonds is negative, indicates that these investors partially take out a fixed rate mortgage (FRM).

Turning to housing futures, we see that the $h = 0$ investor is long in the futures contract, to exploit the housing expected return. Note that the short-sales constraint on cash is not binding for this investor, so that he does not have to choose between equity, bonds, and housing futures, in contrast to the $\gamma = 5$ investor. For $h = 0.3$ the position in housing futures is a tiny $-3\%$, with small associated utility gains of futures. Only for

Table 3: Constrained Portfolio Choice for Different Housing-to-Wealth Ratios

This table reports the unconstrained optimal portfolio choice without /with housing futures for different housing-to-wealth ratios, $h$ . The investor has a horizon of $T = 10$ years and a coefficient of relative risk aversion of $\gamma = 5$ or $\gamma = 10$. The variables $x^s$, $x^{b10}$, $x^{b3}$, and $x^G$ denote the allocation to stocks, 3-year bonds, 10-year bonds, and housing futures. The utility gain is computed as $UG = \left\{ \left( J^{with} / J^{without} \right)^{1/(1-\gamma)} - 1 \right\} * 100\%$

| | $\gamma = 5$ | | | $\gamma = 10$ | | |
|---|---|---|---|---|---|---|
| | $h = 0.0$ | $h = 0.3$ | $h = 0.6$ | $h = 0.0$ | $h = 0.3$ | $h = 0.6$ |
| $x^s$ | 0.37 / 0.37 | 0.52 / 0.52 | 0.81 / 0.81 | 0.21 / 0.21 | 0.26 / 0.26 | 0.38 / 0.37 |
| $x^{b3}$ | 0.57 / 0.57 | 0.71 / 0.71 | 1.25 / 1.25 | 0.63 / 0.65 | 1.08 / 1.05 | 1.82 / 1.35 |
| $x^{b10}$ | 0.06 / 0.06 | 0.11 / 0.11 | 0.14 / 0.15 | 0.00 / 0.00 | −0.17 / −0.16 | −0.32 / −0.17 |
| $x^G$ | 0.00 / 0.00 | 0.00 / 0.00 | 0.00 / 0.00 | 0.00 / 0.11 | 0.00 / −0.03 | 0.00 / −0.47 |
| $UG$ | 0.00% | 0.00% | 0.02% | 0.19% | 0.06% | 1.19% |

$h = 0.6$ we find that housing futures matter to some extent, with a position of $-47\%$ and utility gain of 1.19%. In this case we also see that access to housing futures has an impact on financial portfolio choice: especially the optimal bond positions change quite a bit when housing futures are included. Below we discuss the utility gains in more detail.

Figure 4 shows the portfolio and house price future position for the $\gamma = 10$ investor at different values of the housing-to-wealth-ratio $h$ (for an investment horizon $T = 10$) and for different investment horizons (for a housing-to-wealth ratio of $h = 0.3$). The figures show the housing futures plus margin account, and the cash position that is plotted is actually the free cash position $x^c - |x^G|$.

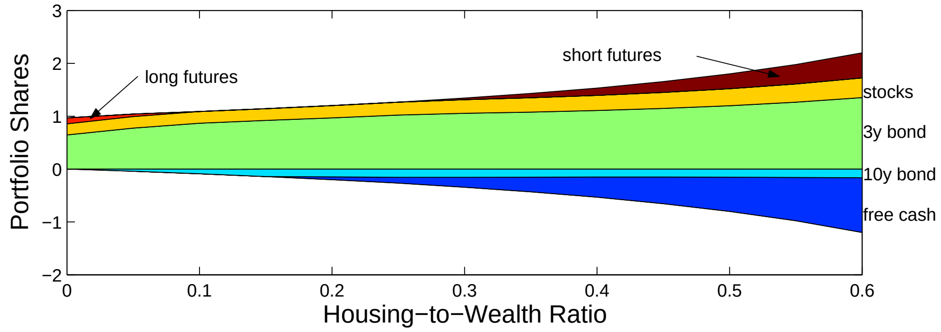
As a function of $h$, the demand for housing futures is positive but small for low housing-to-wealth-ratio's $h$, and increasingly negative for high (roughly $h > 0.3$) housing-to-wealth-ratio's. The most pronounced horizon effect is in the size of the futures position. The reduction in effective housing wealth $1 - \omega$ increases with horizon, leading to a smaller hedge demand for housing futures and a less negative futures position.

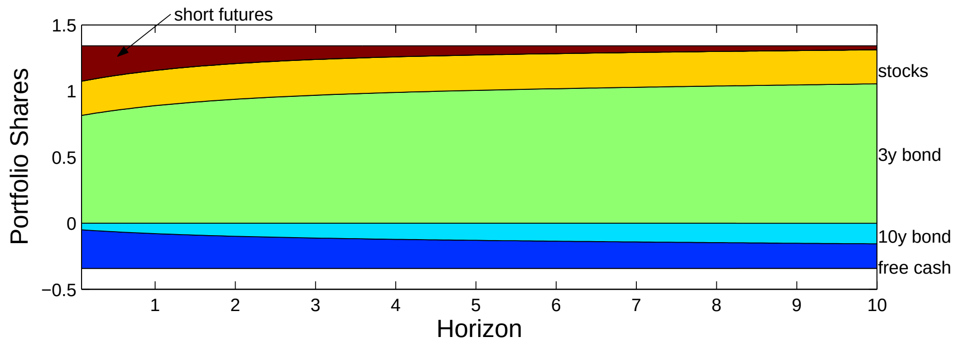
## 4.4 Suboptimal Portfolio Choice

So far our results show mostly small economic benefits of having access to housing futures. In this subsection we put these benefits into perspective, by comparing them to the utility gains/losses associated with two other important housing-related decisions. First, we look at the utility loss of a limited mortgage market, having access only to

Figure 4: Constrained Portfolio Choice for $\gamma = 10$ (With Futures)

The figure plots portfolio choice for different housing-to-wealth ratios $h$ and fixed time horizon $T = 10$ (upper panel) and for different horizon $T$ and fixed housing-to-wealth ratio $h = 0.3$ (lower panel). The investor has a coefficient of relative risk aversion equal to $\gamma = 10$. The parameter values presented in Table 1 are used.



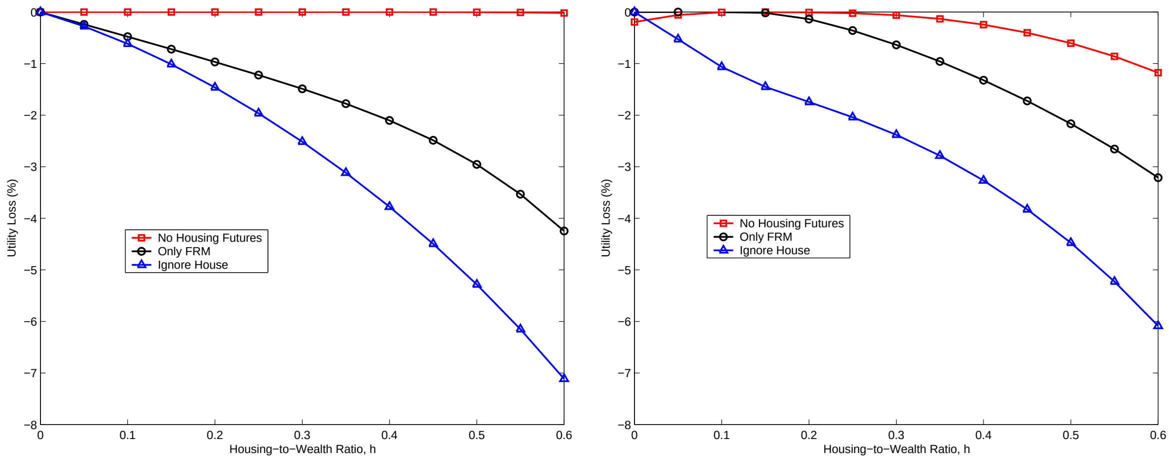

a fixed-rate mortgage but no adjustable-rate mortgage. Second, we consider a more extreme case where the investor fully neglects the house in his financial portfolio choice, and thus behaves as an investor with $h = 0$. This automatically means that the investor does not take a mortgage on the house. The utility losses of this latter case are therefore always larger than those associated with having only access to FRMs.

Figure 5 depicts the respective utility losses for the $\gamma = 5$ and $\gamma = 10$ investor for different levels of $h$. Consistent with Table 3, the utility gains of having access to housing futures are essentially zero. In contrast, the utility losses of having only access to FRMs are large, about 1.5% for $h = 0.3$ and more than 4% for $h = 0.6$. This is consistent with the ARM being the optimal mortgage type for the $\gamma = 5$ investor. Fully neglecting housing for financial portfolio choice leads to even larger utility losses, up to about $-7\%$ for $h = 0.6$.

Turning to the $\gamma = 10$ investor, we see that housing futures generate no utility gains

Figure 5: Suboptimal Portfolio Choice

The figure plots suboptimal portfolio choice for a $\gamma = 5$ (left figure) and a $\gamma = 10$ (right figure) investor.




for values of $h$ between about 0.1 and 0.2. In this case, the speculative and hedging demands fully offset each other. For small values of $h$, having access to housing futures is beneficial since Table 3 shows that investors are optimally long in housing futures in this case. However, the size of the utility gains is small (below 0.2%). For large values of $h$ utility losses up to 1% occur, due to the hedging properties of the housing futures.

Focusing on the case of limited mortgage choice, we see that the line "only FRM" is zero until $h = 0.15$. This is consistent with Figure 4 for constrained portfolio choice for $\gamma = 10$, where until $h = 0.15$ the optimal mortgage is a pure FRM. For large house sizes the utility losses increase to about 3%, since the optimal mortgage is a combination of an ARM and FRM in this case. Finally, if the investor fully neglects housing, large utility losses up to 6% occur.

In sum, these results show that mortgage choice and the incorporation of housing exposure for financial portfolio choice are crucial decisions for investors, and suboptimal behavior is typically very costly. In contrast, hedging house price risk using housing futures is of minor importance to investors, and only leads to small utility gains at best.

Table 4: Constrained Portfolio Choice with zero housing futures risk premium

This table reports the unconstrained optimal portfolio choice without /with housing futures for different housing-to-wealth ratios, $h$, under the assumption $\lambda_v = 0$. The investor has a horizon of $T = 10$ years and a coefficient of relative risk aversion of $\gamma = 5$ or $\gamma = 10$. The variables $x^s$, $x^{b10}$, $x^{b3}$, and $x^G$ denote the allocation to stocks, 3-year bonds, 10-year bonds, and housing futures. The utility gain is computed as $UG = \left\{ \left(J^{with}/J^{without}\right)^{1/(1-\gamma)} - 1 \right\} * 100\%$

| | $\gamma = 5$ | | | $\gamma = 10$ | | |
|---|---|---|---|---|---|---|
| | $h = 0.0$ | $h = 0.3$ | $h = 0.6$ | $h = 0.0$ | $h = 0.3$ | $h = 0.6$ |
| $x^s$ | 0.37 / 0.37 | 0.52 / 0.52 | 0.81 / 0.81 | 0.21 / 0.20 | 0.26 / 0.25 | 0.38 / 0.37 |
| $x^{b3}$ | 0.57 / 0.57 | 0.71 / 0.71 | 1.25 / 0.73 | 0.63 / 0.63 | 1.08 / 0.89 | 1.82 / 1.11 |
| $x^{b10}$ | 0.06 / 0.06 | 0.11 / 0.12 | 0.14 / 0.31 | 0.00 / 0.00 | −0.17 / −0.11 | −0.32 / −0.09 |
| $x^G$ | 0.00 / 0.00 | 0.00 / 0.00 | 0.00 / −0.35 | 0.00 / −0.02 | 0.00 / −0.20 | 0.00 / −0.72 |
| $UG$ | 0.00% | 0.01% | 0.34% | 0.01% | 0.50% | 2.58% |

## 4.5 Additional Analyses

In order to better understand the main result of this paper, the small utility gains of housing futures, we perform two additional analyses. First, we set the risk premium of housing risk $\lambda_v = -\varphi_v$ equal to 0. This knocks out the speculative demand for housing futures, and makes hedging house price risk using futures effectively cheaper. This allows us to assess the sensitivity of our results to the calibrated value of $\lambda_v$. Keeping $\theta'\lambda - r^{imp}$ constant at the empirical estimate of −0.53%, the market-imputed rent changes to $r^{imp} = 0.23\%$.

Table 4 reports the results. For the $\gamma = 5$ investor and $h = 0.6$, the investors shorts housing futures with a weight of −35%, leading to a utility gain of 0.34%, which is still much smaller than the utility gains of optimal mortgage choice and incorporating housing in portfolio choice. Only for the $\gamma = 10$ investor and $h = 0.6$ do we obtain larger utility gains of 2.58%, again smaller than the gains of mortgage choice for this case.[21]

The second additional analysis we perform is to consider a hypothetical futures contract that hedges all of the house price risk of the house, i.e. both the systematic risk and the idiosyncratic house price risk. Clearly, actual implementation of such a contract is nontrivial because of measurement and moral hazard issues. Our goal here is to assess the potential benefits of such a contract. In this case, the futures price follows the process $dG/G = \theta'\lambda dt + \theta' dz + \theta_\varepsilon dz_\varepsilon$. Notice that the case without housing

[21] Table 4 also shows that setting $\lambda_v = 0$ does not change the portfolio choice when there are no housing futures. Theoretically, even in the absence of housing futures, $\lambda_v$ influences the wealth dynamics and thus marginally affects the optimal financial asset allocation. Numerically however, this does not lead to a change in the optimal weights at the reported precision.

Table 5: Constrained Portfolio Choice with a perfect futures contract

This table reports the unconstrained optimal portfolio choice without /with perfect housing futures for different housing-to-wealth ratios, $h$. The investor has a horizon of $T = 10$ years and a coefficient of relative risk aversion of $\gamma = 5$ or $\gamma = 10$. The variables $x^s$, $x^{b10}$, $x^{b3}$, and $x^G$ denote the allocation to stocks, 3-year bonds, 10-year bonds, and housing futures. The utility gain is computed as $UG = \left\{ \left( J^{with} / J^{without} \right)^{1/(1-\gamma)} - 1 \right\} * 100\%$

| | $\gamma = 5$ | | | $\gamma = 10$ | | |
|---|---|---|---|---|---|---|
| | $h = 0.0$ | $h = 0.3$ | $h = 0.6$ | $h = 0.0$ | $h = 0.3$ | $h = 0.6$ |
| $x^s$ | 0.37 / 0.37 | 0.52 / 0.52 | 0.81 / 0.85 | 0.21 / 0.21 | 0.26 / 0.26 | 0.38 / 0.42 |
| $x^{b3}$ | 0.57 / 0.57 | 0.71 / 0.52 | 1.25 / 0.00 | 0.63 / 0.64 | 1.08 / 0.80 | 1.82 / 0.71 |
| $x^{b10}$ | 0.06 / 0.06 | 0.11 / 0.18 | 0.14 / 0.57 | 0.00 / 0.00 | $-0.17$ / $-0.06$ | $-0.32$ / 0.01 |
| $x^G$ | 0.00 / 0.00 | 0.00 / $-0.12$ | 0.00 / $-0.78$ | 0.00 / 0.03 | 0.00 / $-0.28$ | 0.00 / $-1.06$ |
| $UG$ | 0.00% | 0.32% | 3.46% | 0.06% | 2.21% | 11.51% |

futures remains unaltered compared to the baseline case. For comparison we do repeat the results without housing futures from Table 3.

Interestingly, Table 5 shows that this hypothetical futures contract generates much larger utility gains than the existing futures contract. Even for the $\gamma = 5$ investor, we obtain gains up to 3.5% for $h = 0.6$, while for $\gamma = 10$ and $h = 0.6$ the utility gains are 11.51%, much larger than the benefits from mortgage choice and housing-corrected portfolio choice. These results can be understood from the fact that most of the house price risk is idiosyncratic: the standard deviation of idiosyncratic shocks equals 9.00%, while the systematic orthogonal part has a standard deviation of 5.89% per year. As discussed in section 3, this high level of idiosyncratic house price risk is a robust feature of the OFHEO house price data.

A related analysis is performed by Cauley, Pavlov, and Schwartz (2007), who assess the economic benefits of having the possibility to freely trade (part of) an investor's house, without explicitly modeling housing futures contracts. They report large utility gains for a home-owner who has the possibility to sell (without frictions) a fractional interest of her house. Our analysis of the hypothetical futures contract differs from Cauley, Pavlov, and Schwartz (2007) because in our long-term model the investor only cares about hedging the effective house value, which is substantially lower than the market value of the house for long horizons. Also, our model includes a speculative mean-variance demand for housing futures. These two effects imply that the investor optimally never fully hedges the house position.

In sum, the results in this subsection show that (i) lowering the expected return on housing futures leads to a small increase in the hedging attractiveness of housing futures, and (ii) a hypothetical futures contract that fully hedges the house price would be much more beneficial to investors than the existing city-level housing futures contract.

# 5 Conclusion

The main result of this paper is that city-level housing futures, as recently introduced on the Chicago Mercantile Exchange, generally have small economic value to homeowning investors. In order of importance, this is due to (i) large idiosyncratic variation in house prices and (ii) the fact that hedging using house price futures is costly in terms of expected returns, as housing futures have positive expected returns. We also show that the economic benefits of other housing-related choices are much larger. Suboptimal mortgage choice leads to substantial utility losses, and fully neglecting the housing exposure in financial portfolio choice generates even larger utility losses. Finally, we find that a hypothetical futures contract that fully hedges house price risk would be beneficial for investors, and in some cases even more important than having access to appropriate mortgage contracts.

These results are obtained in a long-horizon portfolio choice setup, where investors derive utility from housing services and from consumption of other goods. The housing investment is taken as fixed and given, while positions in financial assets are rebalanced dynamically. At the end of the horizon the investor liquidates the housing position, so that the investor is long house price risk. To be able to study mortgage choice, we use a realistic model for the term structure of interest rates, with expected inflation and real interest rate as factors.

The interpretation of our results is enhanced by an analytical expression for the investor's optimal financial portfolio in the absence of short-sales constraints. This portfolio is composed of positions in (i) the nominal mean-variance tangency portfolio; (ii) a portfolio that most closely resembles an inflation-indexed bond; and (iii) a portfolio that best offsets the risk of the illiquid house. We show that the unhedgeable part of housing risk and the market-imputed rent reduce the effective value of the house and in this way decrease the optimal investment in financial assets.

# Appendix A: Proof of Theorems

In this appendix we provide the proof to the several theorems.

## Proof Theorem 1 (Dynamics Total Wealth and Housing-to-Wealth Ratio)

Denote the (nominal) housing wealth by $W^H \equiv Q_t H$; its dynamics are given by

$$dW^H/W^H = [R_f + \theta'\lambda - r^{imp}]dt + \theta' dz, \tag{26}$$

Financial wealth is denoted by $W^F$ and its dynamics are given by

$$dW^F/W^F = [R_f + \sigma_F'(x)\lambda]\,dt + \sigma_F'(x)\,dz, \tag{27}$$

where $\sigma_F(x)$ is the vector of risk exposures to $dz$ for the nominal financial return. Total wealth is defined as $W = W^F + W^H$ and evolves as

$$dW/W = [R_f + ((1-h)\sigma_F(x) + h\theta)'\lambda]\,dt + ((1-h)\sigma_F(x) + h\theta)' dz, \tag{28}$$

The dynamics for the real wealth components $w^H = W^H/\Pi$ and $w^F = W^F/\Pi$ are

$$\frac{dw^H}{w^H} = [r + (\theta' - \xi')(\lambda - \rho\xi) - \xi_u(\lambda_u - \xi_u) - r^{imp}]dt + (\theta' - \xi')\,dz - \xi_u dz_u, \tag{29}$$

and

$$\frac{dw^F}{w^F} = [r + (\sigma_F'(x) - \xi')(\lambda - \rho\xi) - \xi_u(\lambda_u - \xi_u)]\,dt + (\sigma_F'(x) - \xi')\,dz - \xi_u dz_u, \tag{30}$$

The dynamics for total real wealth $w = w^H + w^F$ then follow directly as

$$\frac{dw}{w} = \left[r + \sigma_w'(\lambda - \rho\xi) - \xi_u(\lambda_u - \xi_u) - hr^{imp}\right]dt + \sigma_w' dz - \xi_u dz_u, \tag{31}$$

with $\sigma_w = (1-h)\sigma_F(x) + h\theta - \xi$.

For the housing-to-wealth ratio we have

$$dh = w^H d\left[\frac{1}{w}\right] + \frac{1}{w}dw^H + dw^H \rho d\left[\frac{1}{w}\right], \tag{32}$$

where

$$\begin{aligned} d\left[\frac{1}{w}\right] &= -\frac{1}{w^2}dw + \frac{1}{2}\frac{2}{w^3}dw\rho dw, \\ &= -\frac{1}{w^2}dw + \frac{1}{w}\frac{dw}{w}\rho\frac{dw}{w}. \end{aligned} \tag{33}$$

Simplifying and deleting terms of order less than $dt$ yields

$$\begin{aligned} \frac{dh}{h} &= -\frac{dw}{w} + \frac{dw}{w}\rho\frac{dw}{w} + \frac{dw^H}{w^H} - \frac{dw^H}{w^H}\rho\frac{dw}{w} \\ &= \frac{dw^H}{w^H} - \frac{dw}{w} - \left(\frac{dw^H}{w^H} - \frac{dw}{w}\right)\rho\frac{dw}{w}. \end{aligned} \tag{34}$$

Using the dynamics for $dw/w$ we get

$$\frac{dh}{h\,(1-h)} = \frac{dw^H}{w^H} - \frac{dw^F}{w^F} - \left(\frac{dw^H}{w^H} - \frac{dw^F}{w^F}\right)\rho\frac{dw}{w} \tag{35}$$

Substituting equations (29) and (30) gives the result

$$\frac{dh}{h\,(1-h)} = \left[\sigma_h'(\lambda - \rho\xi) - \sigma_h'\rho\sigma_w - r^{imp}\right]dt + \sigma_h' dz. \tag{36}$$

with $\sigma_h = \theta - \sigma_F\,(x)$.

## Proof Theorem 2 (Indirect Utility Function)

The indirect utility function is given by

$$J\left(w,h,r,t\right)=\max_{x\in A}E_{t}\left[\frac{\left(w_{T}\right)^{1-\gamma}}{1-\gamma}\right]\nu_{H}, \tag{37}$$

subject to the dynamics for $w$, $h$, and $r$ as given in equations (14), (15), and (4) respectively. Because $A$ is assumed to be independent of $w_t$, we can write (for $\gamma > 1$)

$$\max_{x\in A}E_{t}\left[\frac{\left(w_{T}\right)^{1-\gamma}}{1-\gamma}\right]=\frac{w_{t}^{1-\gamma}}{1-\gamma}\min_{x\in A}E_{t}\left[\left(\frac{w_{T}}{w_{t}}\right)^{1-\gamma}\right] \tag{38}$$

For a given strategy for financial portfolio choice $x$ we have

$$\begin{aligned}\frac{w_{T}}{w_{t}} &= \exp\left\{\int_{t}^{T}\left(-\xi_{u}\left(\lambda_{u}-\xi_{u}\right)-\frac{1}{2}\xi_{u}^{2}\right)ds+\int_{t}^{T}-\xi_{u}dz_{u}\right\} \\ &\quad *\exp\left\{\int_{t}^{T}\left(r_{s}+\mu^{e}-\frac{1}{2}\sigma_{w}'\rho\sigma_{w}\right)ds+\int_{t}^{T}\sigma_{w}'dz\right\},\end{aligned} \tag{39}$$

with $\mu^e = \sigma_w'(\lambda - \rho\xi) - hr^{imp}$ and as before $\sigma_w = h\theta + (1-h)\,\sigma_F(x) - \xi$.

Now define $\tilde{r}$ by $\tilde{r}_t = \bar{r}$ and $d\tilde{r} = \kappa_r\left(\bar{r} - \tilde{r}\right)dt + \sigma_r dz_r$. Then we can write

$$\int_{t}^{T}r_{s}ds=\int_{t}^{T}\tilde{r}_{s}ds+\left(r_{t}-\bar{r}\right)B_{r}\left(T-t\right), \tag{40}$$

with $B_r(T-t)$ defined in equation (10). Using this we can write

$$\begin{aligned}\frac{w_{T}}{w_{t}} &= \exp\left\{\left(r_{t}-\bar{r}\right)B_{r}\left(T-t\right)\right\} \\ &\quad *\exp\left\{\int_{t}^{T}\left(-\xi_{u}\left(\lambda_{u}-\xi_{u}\right)-\frac{1}{2}\xi_{u}^{2}\right)ds+\int_{t}^{T}-\xi_{u}dz_{u}\right\} \\ &\quad *\exp\left\{\int_{t}^{T}\left(\tilde{r}_{s}+\mu^{e}-\frac{1}{2}\sigma_{w}'\rho\sigma_{w}\right)ds+\int_{t}^{T}\sigma_{w}'dz\right\}.\end{aligned} \tag{41}$$

Notice that the last exponential does not depend on $w$ and $r$, and, by assumption, neither does $A$. The expression does depend on $h$ however.

Taking expectations, we can now write the indirect utility function as

$$\begin{aligned} J\left(w,h,r,t\right) &\equiv \frac{w_t^{1-\gamma}}{1-\gamma}\nu_H \\ &* \exp\left\{\left(1-\gamma\right)\left(r_t-\bar{r}\right)B_r\left(T-t\right)\right\} \\ &* \exp\left\{\left(1-\gamma\right)\left(-\xi_u\left(\lambda_u-\xi_u\right)-\frac{\gamma}{2}\xi_u^2\right)\left(T-t\right)\right\} \\ &* I\left(h_t,t\right). \end{aligned} \tag{42}$$

where $I$ satisfies

$$I\left(h_t,t\right)=\min_{x\in A}E_t\left[\left(\exp\left\{\int_t^T\left(\tilde{r}_s+\mu^e-\frac{1}{2}\sigma_w'\rho\sigma_w\right)ds+\int_t^T\sigma_w'dz\right\}\right)^{1-\gamma}\right]. \tag{43}$$

with $I\left(h,T\right)=1$ for all $h$.

## Proof Theorem 3 (Factor Asset Allocation without Housing Futures)

The HJB equation for $J(w,r,h,t)$ is

$$\begin{aligned} 0 &= \max_x\{J_t+J_wE_t[dw]+J_rE_t[dr]+J_hE_t[dh] \\ &+\frac{1}{2}\left[J_{ww}E_t[(dw)^2]+J_{rr}E_t[(dr)^2]+J_{hh}E_t[(dh)^2]\right] \\ &+J_{wr}E_t[dwdr]+J_{wh}E_t[dwdh]+J_{rh}E_t[drdh]\} \end{aligned} \tag{44}$$

Using that $J_t$, $E_t[dr]$ and $E_t[(dr)^2]$ are independent of $x$ and that equation (16) implies

$$\begin{aligned} J_w/J &= (1-\gamma)/w_t \\ J_{ww}/J &= -\gamma(1-\gamma)/w_t^2 \\ J_{wr}/J &= (1-\gamma)^2B_r(T-t)/w_t \\ J_h/J &= I_h/I \\ J_{hh}/J &= I_{hh}/I \\ J_{wh}/J &= (I_h/I)(1-\gamma)/w_t \\ J_{rh}/J &= (I_h/I)(1-\gamma)B_r(T-t) \end{aligned}$$

we find using equations (14) and (15)

$$\begin{aligned} 0 &= \min_x\{(1-\gamma)\,\mu^e - \frac{1}{2}\gamma\,(1-\gamma)\,\sigma_w'\rho\sigma_w + (1-\gamma)^2\,B_r(T-t)\sigma_w'\rho e_2\sigma_r \\ &\quad + \frac{I_h}{I}\mu_h + \frac{1}{2}\frac{I_{hh}}{I}\sigma_h'\rho\sigma_h + \frac{I_h}{I}\,(1-\gamma)\,\sigma_w'\rho\sigma_h + \frac{I_h}{I}\,(1-\gamma)\,B_r(T-t)\sigma_h'\rho e_2\sigma_r\}, \end{aligned} \tag{45}$$

where $\sigma_w = h\theta + (1-h)\,\sigma_F\,(x) - \xi$, $\mu^e = \sigma_w'(\lambda - \rho\xi) - hr^{imp}$, $\sigma_h = h(1-h)\,[\theta - \sigma_F(x)]$, $\mu_h = \sigma_h'\,(\lambda - \rho\xi) - \sigma_h'\rho\sigma_w - h(1-h)r^{imp}$, and $e_2 \equiv (0,1,0,0,0)'$.

Defining $\sigma_F\,(x) = [x_S, x_r, x_\pi, 0, 0]'$, the three first order conditions for $x_S$, $x_r$ and $x_\pi$ form a system of three linear equations in three unknowns. Solving this system gives the proportional asset allocations in the factor assets as given in equations (17)–(19).

Applying the chain rule we can straightforwardly determine $J_{w^F}$, $J_{w^F w^F}$ and $J_{w^F w^H}$ in terms of partial derivatives of $J$ to $w$ and $h$. For example

$$J_{w^F} \equiv J_w \frac{dw}{dw^F} + J_h \frac{dh}{dw^F} = J_w - \frac{h}{w} J_h. \tag{46}$$

Using the functional form for $J\,(w, h, r, t)$ as given in equation (16) we get

$$-\frac{J_{w^F}}{w^F J_{w^F w^F}} = \frac{1}{1-h}\frac{(1-\gamma)\,I - hI_h}{\gamma\,(1-\gamma)\,I - 2\gamma h I_h - h^2 I_{hh}} = \frac{1-(1-\omega)h}{\gamma(1-h)} \tag{47}$$

$$\frac{J_{w^F w^H}}{J_{w^F w^F}} = 1 + \frac{\gamma I_h + h I_{hh}}{\gamma\,(1-\gamma)\,I - 2\gamma h I_h - h^2 I_{hh}} = \omega. \tag{48}$$

## Proof Theorem 4 (Factor Asset Allocation with Housing Futures)

Now we have $\sigma_F\,(x) = [x_S, x_r, x_\pi, x_v, 0]'$. The HJB equation is similar to (45), but with minimization over $x_S$, $x_r$, $x_\pi$, and $x_v$, yielding four first-order conditions. The rest of the proof is similar to the proof of Theorem 3.

## Proof Theorem 5 (No Idiosyncratic House Price Risk)

We have $\sigma_\varepsilon = 0$. Again we use HJB equation (45), but with minimization over $x_S$, $x_r$, $x_\pi$, and $x_v$, and with $\sigma_F'(x) = [x_S, x_r, x_\pi, x_v, 0]'$. We conjecture the functional form $I(h,t) = \left[1 - \left(1 - e^{-r^{imp}\tau}\right) h\right]^{1-\gamma} \hat{I}(t)$, and explicitly solving the three first order conditions for $x_S$, $x_r$, $x_\pi$, and $x_v$, gives the presented proportional asset allocations in the factor assets. Substituting these values in equation (45), changing variables from $t$ to $\tau = T - t$, and simplifying yields

$$\frac{\hat{I}_\tau}{\hat{I}} = (1-\gamma)\left[\bar{r} + \frac{1}{2}\frac{1}{\gamma}\phi'\rho\phi + \left(1 - \frac{1}{\gamma}\right) B_r(\tau)\sigma_r\phi'\rho e_2 - \frac{1}{2}\left(1 - \frac{1}{\gamma}\right) B_r(\tau)^2\sigma_r^2\right], \quad (49)$$

where $\phi = \varphi + \xi$ are the parameters of the real pricing kernel. Since no terms involving $h$ remain, our conjecture is proven. Solving the differential equation, using that $\hat{I}(0) = 1$, gives the solution

$$I(h,t) = [1 - (1-\omega)h]^{1-\gamma} \exp\{(1-\gamma) f(\tau)\}, \quad (50)$$

with

$$\begin{aligned} \omega &= e^{-r^{imp}\tau}, & (51) \\ f(\tau) &= \bar{r}\tau + \frac{1}{2}\frac{1}{\gamma}\phi'\rho\phi\tau + \frac{1}{\kappa}\left(1 - \frac{1}{\gamma}\right)\sigma_r\phi'\rho e_2\left(\tau - B_r(\tau)\right) & (52) \\ &\quad - \frac{1}{4\kappa^3}\left(1 - \frac{1}{\gamma}\right)\sigma_r^2\left(2\kappa\tau - 3 + 4e^{-\kappa\tau} - e^{-2\kappa\tau}\right), \\ \phi &= \varphi + \xi. & (53) \end{aligned}$$

# Appendix B: Calibration of the Term Structure Model

The continuous-time model equations for long-term interest rates, short-term interest rates and inflation can be discretized as follows

$$\begin{aligned}
y_t &= a + b(r_t - \bar{r}) + c(\pi_t - \bar{\pi}) + u_{yt} && (54)\\
R_t^f &= d + (r_t - \bar{r}) + (\pi_t - \bar{\pi}) + u_{ft} && (55)\\
\Delta \ln \Pi_{t+1} &= \bar{\pi} + (\pi_t - \bar{\pi}) + \epsilon_{t+1} && (56)\\
r_t - \bar{r} &= a_r(r_{t-1} - \bar{r}) + \eta_t^r && (57)\\
\pi_t - \bar{\pi} &= a_\pi(\pi_{t-1} - \bar{\pi}) + \eta_t^\pi && (58)
\end{aligned}$$

where $y_t$ is a vector of long-term coupon bond yields (which we approximate by zero-coupon yields with constant durations of 3.4 and 10.4 years), $R_t^f$ the 3-month t-bill rate, $r_t$ the real interest rate, $\pi_t$ the expected inflation, and $\Delta \ln \Pi_{t+1}$ the actual inflation. The error terms $\epsilon$, $\eta^\pi$ and $\eta^r$ are discretized versions of $\sigma_\Pi dZ_\Pi$, $\sigma_\pi dZ_\pi$, and $\sigma_r dZ_r$ respectively. The terms $u_{yt}$ and $u_{ft}$ are measurement error terms, assumed to be i.i.d with mean zero and variance $\sigma^2$. The parameters $b$, $c$, $a_r$ and $a_\pi$ are functions of the mean reversion parameters, as follows

$$b = \frac{1 - \exp(-\kappa_r T)}{\kappa_r T}, \quad c = \frac{1 - \exp(-\kappa_\pi T)}{\kappa_\pi T} \tag{59}$$

where $T$ is the maturity of the bond, and

$$a_r = \exp(-\kappa_r \Delta t), \quad a_\pi = \exp(-\kappa_\pi \Delta t) \tag{60}$$

where $\Delta t$ is the period of the observations (0.25 for our quarterly observations).

In the estimation, we first remove the intercepts $a$, $d$, and $\bar{\pi}$ by fitting them to the sample mean of the observed yields, short rates, and actual inflation. There is no need to estimate $\bar{r}$ since we take $r_t - \bar{r}$ and $\pi_t - \bar{\pi}$ as zero-mean state variables. This leaves six parameters to be estimated: $(\kappa_\pi, \kappa_r, \sigma_\eta^\pi, \sigma_\eta^r, \sigma_\epsilon, \sigma)$. The estimation of these parameters is done using the Kalman filter based Quasi Maximum Likelihood method described in detail in De Jong (2000).

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