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Johannes Gierlinger
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Contents

Preface

General Introduction

This thesis consists of three research papers which have been conceived together with my coauthors Christian Gollier and Sarolta Laczó. All of the papers have been written while pursuing my PhD at the Toulouse School of Economics in the years 2006–2010. While each is self-contained, the papers share a unifying methodological theme, which is to be incremental on the standard model of decision making under risk –the expected utility model– in allowing for small departures from along the lines suggested by data in two ways.

On the one hand, we allow economic agents to be sensitive to the presence of ambiguity as suggested in Ellsberg (1961). More precisely, our notion of ambiguity describes a situation in which decision makers do not know which of several plausible probability models correctly describes an economic risk.

On the other hand, the way heterogeneous agents interact to share risk or ambiguity depends on the contractual framework at hand. We investigate how predictions from the standard model change once we acknowledge that standard complete and competitive financial markets, as in Arrow (1964), may be either inadequate to reach an efficient outcome or unavailable altogether.

The economic applications we consider are threefold. The first chapter, entitled *Socially efficient discounting under ambiguity aversion*, is joint work with Christian Gollier. It studies how the critical rate of return for desirable policy projects depends on the presence of both ambiguity and ambiguity aversion.

The second chapter, entitled *Hedging priors*, turns to financial markets. I study how insurance allocations under risk and ambiguity vary with the sophistication of possible insurance transfers. In contrast to existing literature, I introduce a class of securities which are inspired by recent innovations on financial markets.

Finally, the chapter *Matching and self-enforcing insurance*, written together with Sarolta Laczó, studies questions related to family economics and household behavior. Our paper considers how couples sort themselves on a marriage market if the benefit from a social union is to engage in mutual insurance. As opposed to existing literature we acknowledge that instead of binding legal contracts, agents may need to engage in self-sustaining risk sharing arrangements.

Socially efficient discounting under ambiguity aversion

Institutions with concerns for sustainability –public policy makers in particular– may be confronted with competing projects whose benefits and costs will be borne by distinct or distant generations. For instance, resources spent on greenhouse gas reduction could otherwise be used to finance immediate social good like employee benefits or vaccinations in developing countries.[1]

While financial markets may be helpful to value short term projects, they will be less informative about maturities of several decades and beyond.[2] Alternatively, the social discount rate needs to be determined by feeding

[1] Take the public and scientific debate spurred by the *Copenhagen consensus* (Lomborg, 2004) or the *Stern review* (Stern, 2008).

[2] The longest maturity of so-called *Methuselah* bonds is 50 years.

parameters into an economic model.

The most widely used set-up is a representative agent model under constant relative risk aversion and stationary output growth which prescribes to discount sure costs and benefits at a constant annualized rate. However, the suggested time-invariant discount rate has been debated, most fiercely on ethical grounds, as it is based on an exponentially declining concern for the well-being of future generations.

Our paper retains the classical model for concerns about the future but instead it deviates by allowing for ambiguity about the wealth of future generations. By doing so we add to a class of recent models which acknowledge the potentially dominant role of economic uncertainty in determining the social discount rate.[3] In addition, we also allow for concerns about ambiguity in the decision maker's welfare function to answer to the natural question: Does not knowing the true risk on future wealth result in a greater willingness to postpone consumption?

We will show that it is not true in general. At the same time, we will also identify conditions on the social welfare function and the nature of economic uncertainty to guarantee that rates should indeed go down. In particular, we will argue that a time-invariant rate in the presence of ambiguity is generally not efficient. While our calibrations suggest the effect of ambiguity aversion to be of second-order for the short and medium run, we show that the standard expected utility benchmark strongly undervalues distant future benefits even under moderate concerns for ambiguity.

Hedging Priors

While Chapter 1 considers optimal consumption profiles of a single decision maker under ambiguity, I determine how to allocate resources in a diverse

[3] Consult for instance Weitzman (2007).

population in Chapter 2. Retaining the presence of ambiguity in an economy I study how agents trade off hedging risk against hedging ambiguity.

This research question becomes meaningful in the face of recent innovations on financial markets like *VIX options*. These are options which have the *Chicago Board Options Exchange's Market Volatility Index* (VIX) as an underlying to satisfy the demand for trading pure exposure to volatility.[4] That is, actors on financial markets seem to be willing to trade securities whose payoffs profiles are designed so as not to depend on the *level* of an index like the *S&P 500*, but rather to trade pure exposure on its statistical properties, like volatility of the *S&P 500*.

Conceptually this is related to recent literature in monetary economics where ambiguity is considered to play a role in a recent *flight towards insurance* during the last financial crisis (consult Caballero and Krishnamurthy (2008) or Caballero and Simsek (2009)). In particular, Caballero and Kurlat (2009) propose a policy paper where they put forward *tradable insurance credits* in response to ambiguity. Such credits are assumed to be convertible into an insurance policy (credit default swaps) if and only if the central bank declares markets to be in a *panic mode*, which happens precisely when the unknown risk on financial markets turns out to be *surprisingly* unfavorable.

Similar in spirit, chapter 2 allows for assets whose payoffs correlate with a purely statistical property of economic risk. If, say, the volatility in the economy is unknown, agents may trade securities which pay contingent on a signal which correlates with a measure of volatility, similar to VIX options.

I will show that, in an economy of heterogeneous ambiguity-averse agents, a social planner would ideally want to make individual levels of consumption dependent on the unknown parameter in the risk model, say the true level of

[4]By the beginning of 2010, the *iPath S&P 500 VIX Short-Term Futures* had \$1.3 Billion in assets despite losing 82% of its since its introduction in 2009, the *iPath S&P 500 VIX Mid-Term Futures* had \$680 Million in assets while losing 52% since its introduction in 2009.

volatility. I will then go on to characterize the efficient allocation of resources when the social planner is able to verify the true unknown parameter ex post. Finally, I am going to study a competitive equilibrium on markets where enough assets are traded to decentralize the optimal allocation with a verifiable parameter.

The results in this paper complements existing literature on risk sharing under ambiguity as in Billot, Chateauneuf, Gilboa, and Tallon (2000), Chateauneuf, Dana, and Tallon (2000) or Rigotti, Shannon, and Strzalecki (2008). These papers consider optimal allocations and equilibrium allocations of state-contingent claims alone, as opposed to a combination of state-contingent claims and "bets" on the unknown parameter in the *Hedging priors*.

Methodolgy in the first two chapters: Smooth ambiguity preferences

The first two papers acknowledge the presence of risks which are difficult to quantify and the empirical evidence against the expected utility hypothesis in such situations. Arguably, risks on financial markets, hazards to life, and risks which materialize in the distant future, may be difficult to quantify precisely.

Among the many competing models of ambiguity aversion, we chose to use the smooth ambiguity specification as proposed by Klibanoff, Marinacci, and Mukerji (2005). Simply put, in the smooth model, the expected utility theorem is applied twice: once for preferences over lotteries, and another time for preferences over *lotteries over lotteries*.

This framework allows us to separate ambiguity attitudes from the degree of ambiguity and from beliefs. Had we used the family of max-min preferences, as introduced by Gilboa and Schmeidler (1989), attitudes and beliefs would have needed to be jointly represented by the relevant set of priors.

Moreover, our choice is motivated by the model's tractability in comparative statics analyses since it is differentiable. Finally, the proximity of smooth preferences to the standard expected utility model facilitates to generalize well-known results from the literature on risk.

Matching and self-enforcing insurance

Similarly to the *Hedging priors*, the final chapter investigates how the standard expected utility benchmark changes with the menu of possible insurance contracts. This will be studied in the context of a matching game among standard risk-averse agents who may wish to engage in mutual insurance.

The Economics literature on matching typically determines how partners sort themselves in order to overcome market frictions. [5]. In like manner, we consider an economy where risk sharing in couples is a substitute for insurance markets. More precisely, we follow existing literature in considering risks for which competitive insurance markets do not exist, as is the case for some risks on labor income.[6]

We consider our matching predictions complementary to literature on financial decision making as they can be linked immediately to representative household preferences. That is, if the household composition is indeed the outcome of a matching game, then two ways to sort a diverse population of risk-averse agents may generate different predictions for aggregate data on financial markets.

Our paper builds on pioneering works by Legros and Newman (2007a) and Chiappori and Reny (2006) who investigate this question in a static

[5]Notably the jobs markets (Kelso and Crawford (1982), or Rogerson, Shimer, and Wright (2005) for a survey), or how to assign children to scarce spots in schools (see e.g. Abdulkadiroglu and Sonmez, 2003)

[6]This is subject of models of financial decision making under background risk (Gollier and Pratt (1996), Heaton and Lucas (2005)).

framework. In particular, they consider the case where couples will be able to commit to binding insurance schemes. Their models predict efficient risk sharing within households and, under fairly general conditions, negative assortative matching with respect to risk preferences. That is, the most risk-averse woman is predicted to form a couple with the least risk-averse man, and so forth.

At the same time, we are not aware of data in support of the negative assortative matching prediction. Instead, Dohmen, Falk, Huffman, and Sunde (2008) find that the correlation among appetites for risk within households is positive on average.

We are going to propose a model where positive assortative matching may indeed occur in equilibrium thanks to the lack of commitment among spouses. More precisely, we consider a pool of infinitely lived agents, each of them endowed with a risky income process. Crucially, promises to share risk will only be credible if these are self-enforcing. in the sense that agents cannot be kept from breaking up a match and divorcing their spouses if it is individually rational. This implies that any credible arrangement needs to sustain future states of the world in which one spouse is hit by an adverse income shock while the income of his or her partner is high.[7]

We are going to show that both positive and negative assortative matching may occur, depending on the correlation of shocks among spouses. If risk is perfectly negatively correlated among men and women, we can show that any stable equilibrium must be payoff equivalent to positive assortative matching. This is due to the comparatively low incentive of very risk-averse spouses to divorce when they face an outside option in which they cannot pool negatively correlated risks with their spouses.

However, we then go on to show that it is not true in general that match-

[7]Our model relies heavily on the literature on dynamic insurance under limited commitment, initiated by Kocherlakota (1996) and Ligon et al. (2002a).

ing will be always positive assortative under limited commitment. In particular, if incomes are not correlated, we can show that any equilibrium must be negative assortative. This is due to the comparatively high incentive of very risk-averse spouses to divorce since their outside option is one where they do not need to bear some of the risk of their partners.

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Chapter 1

Socially efficient discounting under ambiguity aversion

1.1 Introduction

The emergence of public policy problems associated with the sustainability of our development has raised considerable interest for the determination of a socially efficient discount rate. This debate has culminated in the publication of two reports about the evaluation of different public investments. The Copenhagen Consensus (Lomborg, 2004) put top priority on public programs yielding immediate benefits, like fighting AIDS and malnutrition and rejected measures to fight climate change as not being cost-effective. The Stern Review, on the other hand, (Stern, 2007) argues for decisive and immediate action against climate change.

Because global warming will really affect our economies in a relatively distant time horizon, the choice of the rate at which these costs are discounted plays a key role in reaching either conclusion. While Stern applies an implicit rate of 1.4% per year, the Copenhagen Consensus is based on a rate around 5%. For the sake of illustrating the power of discounting, consider a project which yields its benefits in t years time. For a horizon $t = 100$ the Copenhagen Consensus would require a rate-of-return 36 times higher than Stern.

As stated by the well-known Ramsey rule (Ramsey (1928)), the socially efficient discount rate (net of the rate of pure preference for the present) is equal to the product of relative risk aversion and the growth rate of consumption. With future generations likely being richer, the return on investment needs to be large enough to compensate for the increased intertemporal inequality that it generates. If we assume that the growth rate of wealth is 2% and relative risk aversion equals 2, this yields a discount rate of 4%.

However, this basic reasoning does not take into account the riskiness affecting the long-term growth of consumption. Hansen and Singleton (1983), Gollier (2002) and Weitzman (2007a), among others, have extended the Ramsey rule in this spirit. An exogenously given stochastic growth process adds a precautionary term to the Ramsey rule which reduces the discount rate under prudence: The willingness to save increases with risk since marginal utility is convex (Leland, 1968; Drèze and Modigliani, 1972).

The present paper goes one step further in recognizing the potential uncertainty on the growth process itself. Such parameter uncertainty on priors is typically referred to as statistical ambiguity or Knightian uncertainty. We believe that this assumption is realistic, especially for long-term forecasts.

Departing from the standard Subjective Expected Utility paradigm (SEU, Savage (1954)), we also assume that the representative agent is ambiguity-averse, i.e., that she dislikes mean-preserving spreads over prior beliefs. Indeed, starting with the pioneering work by Ellsberg (1961), ample evidence in favor of this hypothesis has been accrued.[1] All of which suggests that

[1] The *Ellsberg Paradox* goes back to a thought experiment by Daniel Ellsberg (1961). Suppose two urns containing 90 balls of three colors. The first contains 30 red balls, with the remaining black and yellow in unknown proportion. The second one contains 30 balls of each color. Imagine a bet on drawing the color black. Ellsberg conjectured that a

it is behaviorally meaningful to distinguish lotteries over prior distributions from lotteries over final outcomes. In what follows, we will consider a representative agent who displays "smooth ambiguity preferences", as recently proposed by Klibanoff, Marinacci and Mukerji (KMM, 2005, 2009). Accordingly, the agent computes the expected utility of future consumption conditional on each possible value of the uncertain parameter. She then evaluates her future felicity by computing the certainty equivalent of these conditional expected utilities, using an increasing and concave function ϕ. The concavity of this function implies that she dislikes any mean-preserving spread in the set of plausible beliefs, i.e. that she is ambiguity-averse. It can be shown that the smooth ambiguity family entails the well-known max-min criterion as a special case.

Intuitively, one might expect that ambiguity aversion should raise the agent's willingness to save in order to compensate for its adverse effect on the present value of future welfare. In this paper we show that this is not true in general: ambiguity aversion may increase the socially efficient discount rate. This is connected to two, possibly opposing, effects of ambiguity aversion on marginal utility. On the one hand, there is an ambiguity prudence effect, similar to the prudence effect in the expected utility framework. Independent of risk preferences, decreasing absolute ambiguity aversion (DAAA) has a positive effect on the willingness to save for the ambiguous future. This effect is related to future felicity being measured by the ϕ-certainty equivalent rather than the expectation of ϕ.

On the other hand ambiguity aversion acts like an implicit pessimistic shift in beliefs with respect to the expected utility benchmark. This has

majority of bettors prefers to place it on the unambiguous urn. However, when asked to bet on the contrary (*not black*), they would also prefer to place the opposite bet on the unambiguous urn. This pattern would be incompatible with any stable belief in the SEU model. Experimental data confirms Ellsberg's "paradoxical" predictions (see e.g. Camerer and Weber, 1992).

been observed by KMM (2005, 2007). It is as if probability weights were shifted towards unfavorable priors in the sense of the Monotone Likelihood Ratio order (MLR). However, pessimism does not in general imply a reduction of the interest rate. We derive pairs joint conditions on the risk attitude and the stochastic ordering of plausible distributions to guarantee that, under DAAA, the socially efficient discount rate will indeed be lower under ambiguity aversion.

This paper is related to Weitzman (2007a) and Gollier (2007b) in recognizing uncertainty as a determinant feature of the discounting problem. Weitzman (2007a) shows that uncertainty about the volatility of the growth process may yield a term structure of the discount rate that tends to minus infinity for very long time horizons. Gollier (2007b) provides a general typology for parameter uncertainty. He shows that the sign of the third or fourth derivative of the utility function are necessary to sign the effect on the efficient discount rate, depending upon its type. The present paper departs strongly from these works in acknowledging evidence in favor of ambiguity-averse preferences.

A similar approach has been taken by Traeger (2010). He investigates how preferences for early resolution of uncertainty and ambiguity aversion affect the level of the social discount rate. The present paper provides comparative statics results in for a general class of utility functions, Traeger (2010) focusses on the case of power-utility functions. Moreover, his paper considers a fixed time-horizon of one period while ours one of our main objects of interest is the effect of time on the discount rate.

Jouini, Marin and Napp (2008) and Gollier (2007a) consider the related question of how to aggregate diverging beliefs in a SEU framework. Jouini, Marin and Napp show that an aggregation bias might cause a rich evolution of the discount rate than in the representative agent models. In particular, the discount rate might be first increasing and only then approach its limit, namely the smallest individual rate.

The most active branch of the literature on uncertainty studies phenomena on financial markets. Methodologically, our paper is most closely related to Gollier (2006). He investigates the effect of ambiguity aversion on the demand for risky assets. Like in the present model, the pessimism effect on beliefs is shown to affect behavior differently than risk aversion.

Ju and Miao (2007) and Collard, Mukerji, Sheppard and Tallon (2008) investigate the quantitative effects of KMM preferences on asset prices. Using tractable functional forms, they are able to replicate several empirical phenomena, like low risk-free rates, which are difficult to explain within the standard expected utility model. The present paper shows, however, that the relation between the degree of ambiguity aversion and the risk-free rate need not be negative in the general case.

The remainder of the paper is organized as follows. Section 2 introduces the basic model and presents the equilibrium pricing formula. In Section 3 an analytical example yields an adapted Ramsey-rule for the interest rate under ambiguity. We decompose the effect of ambiguity aversion into its two components in Section 4, whereas Sections 5 and 6 are devoted to respectively the ambiguity prudence effect and the pessimism effect. Section 7 investigates under which conditions our findings extend to any increase in ambiguity aversion. Finally, before concluding, we calibrate the model using two different specifications in Section 8.

1.2 The model

We consider an economy à la Lucas (1978). Each agent in the economy is endowed with a tree which produces $\widetilde{c}_t$ fruits at date t, $t = 0, 1, 2, ...$ There is a market for zero-coupon bonds at date 0 in which agents may exchange the delivery of one fruit today against the delivery of $e^{r_t t}$ fruits for sure at date t.

Thus, the real interest rate associated to maturity t is r_t. The distribution of $\widetilde{c}_t$ is a function of a parameter θ, $\theta = 1, 2, ..., n$. This parametric uncertainty takes the form of a random variable $\widetilde{\theta}$ whose probability distribution is a vector $q =(q_1, ..., q_n)$, where q_θ is the probability that $\widetilde{\theta}$ takes value θ. Let the cumulative distribution function of $\widetilde{c}_t$ conditional on θ be denoted by $F_{t\theta}$. The crop conditional to θ is denoted $\widetilde{c}_{t\theta}$. An ambiguous environment for $\widetilde{c}_t$ is thus fully described by $\widetilde{c}_t \sim (\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$. Conditional on θ, the expected utility of an agent who purchases α zero-coupon bonds with maturity t equals

$$U_t(\alpha, \theta) = Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t}) = \int u(c + \alpha e^{r_t t}) dF_{t\theta}(c).$$

We assume that u is three times differentiable, increasing and concave, so that $U(\cdot, \theta)$ is concave in the investment α, for all θ.

Following Klibanoff, Marinacci and Mukerji (2005) and its recursive generalization (Klibanoff, Marinacci and Mukerji, 2009), we assume that the preferences of the representative agent exhibit smooth ambiguity aversion. Ex ante, for a given investment α, her welfare is measured by $V_t(\alpha)$, which is the certainty equivalent of the conditional expected utilities:

$$\phi(V_t(\alpha)) = \sum_{\theta=1}^{n} q_\theta \phi(U_t(\alpha, \theta)) = \sum_{\theta=1}^{n} q_\theta \phi\left(Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})\right). \qquad (1.1)$$

Function ϕ describes the investor's attitude towards ambiguity (or parameter uncertainty). It is assumed to be three times differentiable, increasing and concave. A linear function ϕ means that the investor is neutral to ambiguity as her preferences simplify to the subjective expected utility functional $V_t^{SEU}(\alpha) = Eu(\widetilde{c}_t + \alpha e^{r_t t})$. In contrast, a concave ϕ is equivalent to ambiguity aversion. In other words, she dislikes mean-preserving spreads over candidate levels of $U_t(\alpha, \theta)$.

An interesting particular case arises when absolute ambiguity aversion $A(U) = -\phi''(U)/\phi'(U)$ is constant, so that $\phi(U) = -A^{-1}\exp(-AU)$. As proven by Klibanoff, Marinacci and Mukerji (2005), the ex-ante welfare $V_t(\alpha)$ tends to the max-min expected utility functional $V_t^{MEU}(\alpha) = \min_\theta Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})$ when the degree of absolute ambiguity aversion ϕ tends to infinity. Thus, the max-min criterion à la Gilboa and Schmeidler (1989) is a special case of this model.

The optimal investment α^* maximizes the intertemporal welfare of the investor,

$$\alpha^* \in \arg\max_{\alpha} \qquad u(c_0 - \alpha) + e^{-\delta t} V_t(\alpha), \tag{1.2}$$

where parameter δ is the rate of pure preference for the present.

At this stage, it is important to point out that the basic assumptions underlying KMM models do not guarantee that the maximization problem (1.2) is convex. To see why, it suffices to recall that certainty equivalent functions need not be concave. Indeed, even if we imposed ϕ and u to be strictly concave, the solution to program (1.2), when it exists, need not be unique. To handle this problem we prove the following.

Proposition 1 *Suppose that ϕ has a concave absolute ambiguity tolerance, i.e., $-\phi'(U)/\phi''(U)$ is concave in U. This implies that V_t is concave in α.*

Proof. Relegated to the Appendix.

If the inverse of absolute ambiguity aversion increases at a linear or decreasing rate in U, then the KMM functional is concave in α. The above proposition includes the specifications which are most widely used in the literature, in particular the families of exponential and power functions.

Henceforward we will consider the following assumption satisfied.

Assumption 1 *The function ϕ exhibits a concave absolute ambiguity tolerance, i.e. $-\phi'(U)/\phi''(U)$ is concave in U everywhere.*

Thanks to Assumption 1, the necessary and sufficient condition to solve program (1.2) can be written as

$$u'(c_0 - \alpha^*) = e^{-\delta t} V_t'(\alpha^*).$$

Fully differentiating equation (1.1) with respect to α yields

$$V_t'(\alpha) = e^{r_t t} \frac{\sum_{\theta=1}^{n} q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t}) \right) Eu'(\widetilde{c}_{t\theta} + \alpha e^{r_t t})}{\phi'(V_t(\alpha))}.$$

Because we assume that all agents have the same preferences and the same stochastic endowment, the equilibrium condition on the market for the zero-coupon bond associated to maturity t is $\alpha^* = 0$. Combining the above two equations implies the following equilibrium condition:

$$r_t = \delta - \frac{1}{t} \ln \left[\frac{\sum_{\theta=1}^{n} q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right) Eu'(\widetilde{c}_{t\theta})}{\phi'(V_t(0)) u'(c_0)} \right]. \tag{1.3}$$

This is also the socially efficient rate at which sure benefits and costs occurring at date t must be discounted in any cost-benefit analysis at date 0.

Under ambiguity-neutrality, the standard bond pricing formula $r_t = \delta - t^{-1} \ln \left[Eu'(\widetilde{c}_t)/u'(c_0) \right]$obtains.[2] The riskiness of future consumption reduces the social discount rate if and only if $Eu'(\widetilde{c}_t)$ is larger than $u'(E\widetilde{c}_t)$,, that is to say, if and only if u' is convex and the agent displays *prudence* (see Leland, 1968; Drèze and Modigliani, 1972; or Kimball, 1990).

Our goal in this paper is to determine the conditions under which ambiguity aversion reduces the discount rate. An ambiguous environment $(\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$ is said to be acceptable if the respective supports of the $\widetilde{c}_{t\theta}$ are in the domain of u, and if all $Eu'(\widetilde{c}_{t\theta})$ are in the domain of ϕ. The set of acceptable

[2]See for example Cochrane (2001).

ambiguous environments is denoted Ψ.

1.3 An analytical solution

Let us consider the following specification:

- The plausible distributions of $\ln \widetilde{c}_{t\theta}$ are all normal with the same variance $\sigma^2 t$, and with mean $\ln c_0 + \theta t$.[3]
- The parameter θ is normally distributed with mean μ and variance σ_0^2.[4]
- The representative agent's preferences exhibit constant relative risk aversion $\gamma = -cu''(c)/u'(c)$, such that $u(c) = c^{1-\gamma}/(1-\gamma)$.
- The representative agent's preferences exhibit constant relative ambiguity aversion $\eta = -|u|\,\phi''(u)/\phi'(u) \geq 0$. This means that $\phi(U) = k(kU)^{1-\eta k}/(1-\eta k)$, where $k = sign(1-\gamma)$ is the sign of u.

As is well-known, the Arrow-Pratt approximation is exact under CRRA and lognormally distributed consumption. Therefore, conditional to each θ, we have that

$$Eu(\widetilde{c}_{t\theta}) = (1-\gamma)^{-1}\exp(1-\gamma)(\ln c_0 + \theta t + 0.5(1-\gamma)\sigma^2 t).$$

We can again use the same trick to compute the ϕ-certainty equivalent V_t, since $\phi(Eu(\widetilde{c}_{t\theta}))$ is an exponential function and the random variable $\widetilde{\theta}$ is normal, which is another case where the Arrow-Pratt approximation is exact.

[3]In continuous time, this would mean that the consumption process is a geometric Brownian motion $d\ln c_t = \theta dt + \sigma dw$.

[4]We consider the natural continuous extension of our model with a discrete distribution for $\widetilde{\theta}$.

It yields

$$V_t(0) = (1-\gamma)^{-1}\exp(1-\gamma)\Big(\ln c_0+\mu t+0.5(1-\gamma)\sigma^2 t+0.5(1-\gamma)(1-k\eta)\sigma_0^2 t^2\Big).$$

However, in order to solve for the pricing rule (1.3) we are really interested in $V_t'(0)$. A convenient way to structure the algebra is to decompose $V_t'(0)$ in the following way: again exploiting the Arrow-Pratt approximation, we have on the one hand

$$\frac{E\phi'\left(Eu(\widetilde{c}_{t\theta})\right)}{\phi'(V_t(0))} = \exp\left(\frac{1}{2}(1-\gamma)^2 k\eta\sigma_0^2 t^2\right), \tag{1.4}$$

and on the other hand

$$\begin{aligned}\frac{E[\phi'\left(Eu(\widetilde{c}_{t\theta})\right) Eu'(\widetilde{c}_{t\theta})]}{E\phi'\left(Eu(\widetilde{c}_{t\theta})\right)} = \quad & \exp -\Big(\gamma(\ln c_0+\mu t) \quad -\frac{1}{2}\gamma^2(\sigma^2 t+\sigma_0^2 t^2) - \\ & -(\gamma(1-\gamma)k\eta)\sigma_0^2 t^2\Big)\end{aligned} \tag{1.5}$$

Finally, multiplying (1.4) by (1.5) and plugging the result into (1.3), yields the desired analytical expression:

$$r_t = \delta + \gamma\mu - \frac{1}{2}\gamma^2(\sigma^2+\sigma_0^2 t) - \frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2 t. \tag{1.6}$$

Let g be the growth rate of expected consumption. It is easy to check that $g = \mu + 0.5(\sigma^2+\sigma_0^2 t)$. It implies that the above equation can be rewritten as

$$r_t = \delta + \gamma g - \frac{1}{2}\gamma(\gamma+1)(\sigma^2+\sigma_0^2 t) - \frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2 t. \tag{1.7}$$

The first two terms on the right-hand side of this equation correspond to the classical Ramsey rule. The interest rate is increasing in the growth rate of expected consumption g. When g is positive, decreasing marginal utility implies that the marginal utility of consumption is expected to be smaller in the future than it is today. This yields a positive interest rate. The third term expresses prudence. Because the riskiness of future consumption increases the expected marginal utility $Eu'(\widetilde{c}_t)$ under prudence, this has a negative impact on the discount rate.[5] Notice that the variance of consumption at date t equals $\sigma^2 t + \sigma_0^2 t^2$, so that it increases at an increasing rate with respect to the time horizon. Therefore, the precautionary effect has a relatively larger impact on the discount rate for longer horizons. This argument has been developed in Weitzman (2007a) and Gollier (2007b) to justify a decreasing discount rate in an expected utility framework.

The final term reduces the discount rate under positive ambiguity aversion ($\eta > 0$). It is increasing with ambiguity aversion η, the degree of uncertainty σ_0, and with the time horizon t.

Thanks to the above specifications, absent ambiguity the term structure is flat. The mere presence of ambiguity (i.e. $\sigma_0^2 > 0$) causes the rates to decrease linearly over time. If in addition, the agent displays ambiguity aversion ($\eta > 0$), this decline steepens.

The following sections investigate whether it is true in general, that ambiguity aversion decreases the socially efficient discount rate for any maturity. Contrary to the example presented above, the next section reveals that ambiguity aversion might even decrease the willingness to save.

[5]This precautionary effect is equivalent to reducing the growth rate of consumption g by the precautionary premium (Kimball (1990)) $0.5(\gamma + 1)(\sigma^2 + \sigma_0^2 t)$. Indeed, $\gamma + 1 = -cu'''(c)/u''(c)$ is the index of relative prudence of the representative agent.

1.4 The two effects of ambiguity aversion

The usual bond-pricing formula under SEU yields a benchmark expression for the social discount rate

$$r_t = \delta - \frac{1}{t} \ln \left[\frac{Eu'(\widetilde{c}_t)}{u'(c_0)} \right], \tag{1.8}$$

where the random variable $\widetilde{c}_t$ describes future consumption, distributed as $(\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$. Like in the analytical example from above, the effect of ambiguity-aversion on $V_t'(0)$ can be decomposes such that

$$r_t = \delta - \frac{1}{t} \ln \left[a \frac{Eu'(\overset{\circ}{\widetilde{c}_t})}{u'(c_0)} \right], \tag{1.9}$$

where the constant a is defined as

$$a = \frac{\sum_{\theta=1}^{n} q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right)}{\phi'(V_t(0))}, \tag{1.10}$$

and where $\overset{\circ}{\widetilde{c}_t}$ is a distorted probability distribution $(\widetilde{c}_{t1}, \overset{\circ}{q}_1; ...; \widetilde{c}_{tn}, \overset{\circ}{q}_n)$ with the property that for any $\theta = 1, ..., n$,

$$\overset{\circ}{q}_\theta = \frac{q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right)}{\sum_{\tau=1}^{n} q_\tau \phi' \left(Eu(\widetilde{c}_{t\tau}) \right)}. \tag{1.11}$$

Notice the similarity between pricing formula (1.8) and (1.9). It implies that ambiguity aversion reduces the discount rate if

$$aEu'(\overset{\circ}{\widetilde{c}_t}) \geq Eu'(\widetilde{c}_t). \tag{1.12}$$

Moreover, Observe that this condition simplifies to $a \geq 1$ when the agent is risk neutral. Because we don't constrain the risk attitude in any way except risk aversion, condition $a \geq 1$ is necessary to guarantee that ambiguity aversion reduces the discount rate. For reasons that will be clarified in the next section, we will refer to $a \geq 1$ as the ambiguity prudence effect.

In the absence of an ambiguity prudence effect ($a = 1$), condition (1.12) becomes $Eu'(\widetilde{c}_t^{\circ}) \geq Eu'(\widetilde{c}_t)$, which is referred to as the pessimism effect. At this stage, it is enough to say that it comes from a distortion of the beliefs $(q_1, ..., q_n)$ on the likelihood of the different plausible probability distributions $(\widetilde{c}_1, ..., \widetilde{c}_n)$.

1.5 The ambiguity prudence effect

In this section, we focus on whether the constant a, defined by equation (1.10), is larger than unity. As stated above, this is necessary to guarantee that the discount rate is reduced and it becomes necessary and sufficient in the special case of risk-neutrality. Notice that in the latter case, a can be interpreted as the sensitiveness of the $\phi-$certainty equivalent of $\overline{c}_{\widetilde{\theta}} = E\left[\widetilde{c}_{t\widetilde{\theta}} \mid \widetilde{\theta}\right]$ with respect to an increase in saving.[6] We seek to determine whether one more unit saved yields an increase in the $\phi-$certainty equivalent future consumption. More generally, condition $a \geq 1$ can be rewritten as

$$\sum_{\theta} q_\theta \phi(u_\theta) = \phi(V_t) \quad \Longrightarrow \quad \sum_{\theta=1}^{n} q_\theta \phi'(u_\theta) \geq \phi'(V_t). \tag{1.13}$$

Similar questions have been raised in risk theory: Do expected-utility-preserving risks raise expected marginal utility? Along the same lines, we are able to conclude from expected utility theory that condition (1.13) requires ϕ to satisfy decreasing absolute ambiguity aversion (see e.g. Gollier, 2001, Section 2.5). Indeed, defining function ψ such that $\psi(\phi(U)) = \phi'(U)$ for all U, the

[6]Define $V(s, \overline{c}_{\widetilde{\theta}})$ such that $\phi(s + V) = E\phi(s + \overline{c}_{\widetilde{\theta}})$. We have that $a = \partial V(s, \overline{c}_{\widetilde{\theta}})/\partial s$ at $s = 0$.

above condition can be rewritten as

$$\sum_{\theta=1}^{n} q_\theta \psi\left(\phi_\theta\right) \geq \psi(\Sigma_\theta q_\theta \phi_\theta),$$

where $\phi_\theta = \phi(u_\theta)$ for all θ. This is true for all distributions of $(\phi_1, q_1; ...; \phi_n, q_n)$ if and only if ψ is convex. Because $\psi'(\phi(U)) = \phi''(U)/\phi'(U)$, this is true iff $A(U) = -\phi''(U)/\phi'(U)$ be non-increasing. This proves the following results.

Lemma 1 *$a \geq 1$ (resp. $a \leq 1$) for all acceptable ambiguous environments $\widetilde{c} \in \Psi$ if and only if absolute ambiguity aversion is non-increasing (resp. non-decreasing).*

Proposition 2 *Suppose that the representative agent is risk-neutral. The socially efficient discount rate is smaller (resp. larger) than under ambiguity neutrality for all ambiguous environments $\widetilde{c}$ if and only if ϕ exhibits non-increasing (resp. non-decreasing) absolute ambiguity aversion.*

Rather than the extent of ambiguity aversion itself, what drives the result is how the degree of ambiguity aversion relates to conditional expected utility U. For instance, in the limit case with constant absolute ambiguity aversion, ambiguity has *no* effect on the equilibrium interest rate. The intuition for these results is easy to derive from the observation that the period-t felicity V_t is approximately equal to expected consumption minus the ambiguity premium. Moreover, the premium is itself proportional to ambiguity aversion A, which makes the willingness to save decreasing in A'. Thus, ambiguity aversion raises the willingness to save – therefore reducing the equilibrium interest rate – if absolute ambiguity aversion is decreasing.

Exactly as decreasing absolute risk aversion is unanimously accepted as a natural assumption for risk preferences, we believe that decreasing absolute ambiguity aversion (DAAA) is a reasonable property of uncertainty preferences. It means that a local mean-preserving spread in conditional expected utility has an impact on welfare that is decreasing in the level of utility where this spread is realized.

We call this the *ambiguity prudence effect* because it emerges as a consequence of the uncertainty of the future conditional expected utility. This raises the willingness to save exactly as the risk on future income raises savings in the standard expected utility model under "risk prudence". But contrary to risk prudence, which is characterized by $u''' \geq 0$, ambiguity prudence is described by decreasing absolute uncertainty aversion, which is weaker than $\phi''' \geq 0$. This is because, in the intertemporal KMM model, the future felicity is represented by the $\phi-$certainty equivalent of the conditional expected utilities, rather than by the expected $\phi-$valuation of the conditional expected utilities. Had we used this alternative model, ϕ' convex (concave) would have been the property to determine the sign of the effect.

However, once we allow for risk aversion, non-increasing ambiguity aversion is no longer sufficient to sign the impact of ambiguity on the discount rate, as shown by the following counterexample.

Counterexample 1 Let c_0 equal 2 and $\widetilde{c}_t$ shall take two plausible distributions $\widetilde{c}_{t1} \sim (1, 1/3; 4, 1/3; 7, 1/3)$ or $\widetilde{c}_{t2} \sim (3, 2/3; 4, 1/3)$, both equally plausible, i.e. $q_1 = q_2 = 1/2$. Further, preferences shall display constant relative risk aversion (CRRA) with $\gamma = 2$ such that $u(c) = -c^{-1}$ and $\delta = 0$. It is easy to check that absent ambiguity aversion the interest rate equals 9.24%. However, under constant (and therefore non-increasing) absolute ambiguity aversion of $A = 2.11$, i.e. $\phi(U) = -\exp(-2.11U)$, tedious computations yield a discount rate of exactly zero: $r_t = 0$!■

1.6 The pessimism effect

Counter-example 1 can be explained by the presence of a second effect, the pessimism effect. In the pricing formula (1.9), ambiguity has an effect which is equivalent to compute marginal expected utility using the distorted random variable $\widetilde{c}_t^{\circ}$ instead of $\widetilde{c}_t$. The distortion of these implicit beliefs depends upon the degree of ambiguity aversion and is governed by rule (1.11). This section is devoted to characterize how the distortion affects the discount rate. If we find that it is pessimistic in the sense of FSD, then we are able to unambiguously sign the effect on the discount rate.

To examine this question we begin by comparing of the distorted probabilities $q^{\circ} = (q_1^{\circ}, ..., q_n^{\circ})$ to the original probabilities $q = (q_1, ..., q_n)$.

Let the priors θ be ordered such that $Eu(\widetilde{c}_{t1}) \leq Eu(\widetilde{c}_{t2}) \leq ... \leq Eu(\widetilde{c}_{tn})$ and the agent prefers θ to be large. We hereafter show that ambiguity aversion is equivalent to a distortion of the prior beliefs on parameter $\widetilde{\theta}$ in the sense of the Monotone Likelihood Ratio Order (MLR). By definition, a shift of beliefs from q to q° entails a deterioration in the sense of the *monotone likelihood ratio ordering* (MLR) if $q^{\circ}_{\widetilde{\theta}}/q_{\widetilde{\theta}}$ and $\tilde{\theta}$ are anti-comonotonic. Observe from (1.11) that $q^{\circ}_{\theta}/q_{\theta}$ is proportional to $\phi'(Eu(\widetilde{c}_{t\theta}))$. Thus, since ϕ' is decreasing, we know that $q^{\circ}_{\widetilde{\theta}}/q_{\widetilde{\theta}}$ and $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ are anti-comonotonic, establishing the following property.

Lemma 2 *The subsequent conditions are equivalent:*

1. *Beliefs q° are dominated by q in the sense of the monotone likelihood ratio order for any set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ such that $Eu(\widetilde{c}_{t1}) \leq Eu(\widetilde{c}_{t2}) \leq ... \leq Eu(\widetilde{c}_{tn})$.*

2. *ϕ is concave.*

The intuitive interpretation is that ambiguity aversion is characterized by an MLR-dominated shift in the prior beliefs. In other words, it biases beliefs by favoring the worse marginals in a very specific sense: if the agent prefers marginal $\widetilde{c}_{t\theta}$ to marginal $\widetilde{c}_{t\theta'}$, then, the ambiguity-averse representative agent increases the implicit prior probability q°_θ relatively more than the implicit prior probability $q^\circ_{\theta'}$. This result gives some flesh to our pessimism terminology. It also generalizes – and builds a bridge to – the max-min case where all the weight is transferred to the worst θ.[7]

However, the MLR deterioration of the distribution $\widetilde{\theta}$ of priors is not enough to ensure a negative pessimism effect on the discount rate, as shown by Counterexample 1. Instead, the crucial requirement would be that the distortion overweights scenarios which yield larger conditional expected *marginal* utility. The above lemma says something different, namely that the distortion overweights scenarios which yield larger conditional expected utility. Therefore, to obtain the desired result, we need to find conditions such that u and $-u'$ agree on the ranking of scenarios.

Lemma 3 *The following two conditions are equivalent:*

1. *The pessimism effect reduces the discount rate, i.e.* $Eu'(\widetilde{c}^\circ_t) \geq Eu'(\widetilde{c}_t)$, *for all* ϕ *increasing and concave;*

2. $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ *and* $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ *are anti-comonotonic.*

Proof: To prove that $2 \Rightarrow 1$, suppose that $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ be anti-comonotonic. Since ϕ' is decreasing, our assumption implies that $\phi'(E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right])$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are comonotonic. By the covariance rule,

[7]This equivalence between concave transformation functions and stochastic orderings is well known (see Lehmann, 1955). Economic applications notably include rank-dependent utility models (see Quiggin, 1995).

it implies that

$$\begin{aligned}
Eu'(\overset{\circ}{\widetilde{c}_t}) &= \frac{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right) Eu'(\widetilde{c}_{t\theta})}{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)} \\
&\geq \frac{\left[\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)\right]\left[\sum_{\theta=1}^{n} q_\theta Eu'(\widetilde{c}_{t\theta})\right]}{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)} \\
&= \sum_{\theta=1}^{n} q_\theta Eu'(\widetilde{c}_{t\theta}) = Eu'(\widetilde{c}_t).
\end{aligned}$$

In order to prove that $1 \implies 2$, suppose by contradiction that $Eu(\widetilde{c}_{t1}) < Eu(\widetilde{c}_{t2}) < ... < Eu(\widetilde{c}_{tn})$, but there exists $\theta \in [1, n-1]$ such that $Eu'(\widetilde{c}_{t\theta}) \leq Eu'(\widetilde{c}_{t\theta+1})$. Then, consider any increasing and concave ϕ that is locally linear for all $U \leq Eu(\widetilde{c}_{t\theta})$ and for all $U \geq Eu(\widetilde{c}_{t\theta+1})$, and has a strictly negative derivative in between these bounds. For any such function ϕ, we have that $\phi'(E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right])$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are anti-comonotonic. Using the covariance rule as above, that implies that $Eu'(\overset{\circ}{\widetilde{c}_t}) < Eu'(\widetilde{c}_t)$, a contradiction.■

1.6.1 The CARA case

By consequence of Lemma 3, in order to sign the pessimism effect, we need to look for conditions such that u and $-u'$ indeed "agree" on a ranking of lotteries $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$. Consider first an agent who satisfies constant absolute risk aversion (CARA), i.e.

$$u(c) = -\frac{1}{A}\exp(-Ac)$$

Since $u'(c) = \exp(-Ac)$, functions u and $-u'$ represent the same preferences over priors. Hence the following result.

Proposition 3 *Under CARA preferences, the pessimism effect always reduces the socially efficient discount rate.*

1.6.2 The general case

One might conjecture that the previous result extends to any concave u. That is, if both u and $-u'$ are increasing, it may seem natural that their expectations agree on the ranking of lotteries. Yet, the theory of stochastic dominance tells us that the solution is not that simple. Indeed, a necessary and sufficient condition for any two increasing utility functions on agree on the ranking of two lotteries is that they be ranked along first-degree stochastic dominance (FSD).

However, ranking the priors according to FSD is rather restrictive. It would be desirable to extend this result to a weaker stochastic order. For instance, consider the second-degree stochastic dominance order (SSD). It guarantees that $Ef(\widetilde{c}_{t\theta})$ is increasing in θ for all increasing and concave functions f. Indeed, prudence means precisely that $f = u$ and $f = -u'$ are increasing and concave. Using these conditions, we are able to once more apply Lemma 3 to obtain the following.

Proposition 4 *The pessimism effect reduces the socially efficient discount rate if*

1. *The set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to FSD.*
2. *The set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to SSD and u exhibits prudence.*

Essentially, the previous result exploits results on how changes in risk affect savings decisions under ambiguity-neutrality. Indeed, being risk-averse (prudent) means that an SEU agent would like to save more if an unfair (zero-mean) risk is added to her wealth (Leland, 1968; Drèze and Modigliani, 1972).

In the following proposition, we put forward a third pair of sufficient conditions. Compared to the SSD/prudence requirement, we impose a stronger restriction on utility functions as we replace prudence by the stronger DARA condition. In return we are able to relax SSD to the weaker stochastic order introduced by Jewitt (1989).

Definition 1 *We say that $\widetilde{c}_{\theta'}$ dominates $\widetilde{c}_{\theta}$ in the sense of Jewitt if the following condition is satisfied: for all increasing and concave u, if agent u prefers $\widetilde{c}_{\theta'}$ to $\widetilde{c}_{\theta}$, then all agents more risk-averse than u also prefer $\widetilde{c}_{\theta'}$ to $\widetilde{c}_{\theta}$.*

Of course, from the definition itself, if $\widetilde{c}_{\theta'}$ dominates $\widetilde{c}_{\theta}$ in the sense of SSD, this preference order also holds in the sense of Jewitt, thereby showing that this order is weaker than SSD. Jewitt (1989) shows that distribution function $F_{t\theta'}$ dominates $F_{t\theta}$ in this sense if and only if there exists some w in their joint support $[a,b]$, such that

$$\begin{aligned}
&\int_a^x (F_{t\theta'}(z) - F_{t\theta}(z))dz \geq 0 \qquad \text{for all} \qquad x \in [a,w], \\
&\int_a^w (F_{t\theta'}(z) - F_{t\theta}(z))dz = 0, \\
&\int_a^x (F_{t\theta'}(z) - F_{t\theta}(z))dz \qquad \text{is non-increasing} \quad \text{on } [w,b].
\end{aligned}$$

Hence the result.[8]

Proposition 5 *The pessimism effect reduces the socially efficient discount rate if the set of marginals $(\widetilde{c}_{t1},...,\widetilde{c}_{tn})$ can be ranked according to Jewitt's stochastic order and u exhibits decreasing absolute risk aversion.*

Proof: Decreasing absolute risk aversion means that $v = -u'$ is more concave than u in the sense of Arrow-Pratt. By definition of Jewitt's stochastic order, it implies that $Eu(\widetilde{c}_{t\theta'}) \geq Eu(\widetilde{c}_{t\theta})$ implies that $Ev(\widetilde{c}_{t\theta'}) \geq Ev(\widetilde{c}_{t\theta})$,

[8]Two random variables fulfill Definition 1 if there exists a consumption level w in their support such that, conditional on the outcome being lower than w, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of SSD, whereas conditional on the outcome being higher than w, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of FSD.

or equivalently, that $Eu'(\widetilde{c}_{t\theta'}) \leq Eu'(\widetilde{c}_{t\theta})$. Using Lemma 3 concludes the proof. ■

Finally, combining Lemma 1 with Propositions 3, 4 and 5 yields our main result.

Proposition 6 *Suppose that the representative agent exhibits non increasing absolute ambiguity aversion (DAAA). Then, ambiguity aversion reduces the socially efficient discount rate if one of the following conditions holds:*

1. *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to FSD and* u *is increasing and concave.*
2. *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to SSD and* u *is increasing, concave, and exhibits prudence.*
3. *The set of marginals* $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *can be ranked according to Jewitt (1989) and* u *is increasing and concave, and exhibits DARA.*
4. u *exhibits constant absolute risk aversion.*

Observe that the result in our analytical example in Section 1.3 fits condition *1*: A mere translation in the distribution constitutes a first-degree stochastic dominance. Yet, in many circumstances, the degrees of riskiness also differ across the plausible distributions, usually implying that the plausible prior distributions cannot be ranked according to FSD. Condition *2* provides a sufficient condition on risk attitudes if marginals can only be ranked according to second-degree stochastic dominance, which contains Rothschild-Stiglitz's increases in risk as a particular case. It turns out that in this case, in addition to risk-aversion, the representative agent should also be prudent. Note that even the weaker Jewitt-ordering from condition *3* only requires decreasing absolute risk aversion. This property is widely accepted in the economic literature. It is in particular compatible with the observation that more wealthy individuals tend to take more portfolio risk.[9]

1.7 The comparative statics of an increase in ambiguity aversion

Our results up to now characterize the effect of smooth ambiguity aversion on the equilibrium interest rate, starting from the ambiguity-neutral benchmark. A natural question to ask is whether our results hold for any increase in ambiguity aversion.

For this purpose, consider two economies, $i = 1, 2$, identical up to the level of ambiguity aversion, with the agent in $i = 2$ more ambiguity-averse. That is to say that $\phi_2(U) = \psi(\phi_1(U))$ for all U, with the property that ψ is increasing and concave. According to the adjusted pricing formula in (1.9) an increase in ambiguity aversion decreases the social discount rate if and only if

$$a_2 E u'(\tilde{c}_t^2) \geq a_1 E u'(\tilde{c}_t^1), \tag{1.14}$$

where a_i is defined as in (1.10) with ϕ being replaced by ϕ_i, and where $\tilde{c}_t^i$ is random future consumption distorted by weights q_θ^i, as in (1.11). Naturally, taking ϕ_1 linear, we retrieve condition (1.12) from the SEU benchmark.

At the outset, we are able to generalize our findings about the pessimism effect to any increase in ambiguity aversion.

Lemma 4 *The following two conditions are equivalent:*

1. *Beliefs q^2 are dominated by q^1 in the sense of the monotone likelihood*

[9]In counterexample 1, the two random variables $\widetilde{c}_{t1}$ and $\widetilde{c}_{t2}$ cannot be ranked according to SSD. This is why we obtain that ambiguity aversion raises the interest rate in spite of the fact that $u'(c) = c^{-2}$ is convex.

ratio order for any set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ *such that* $Eu(\widetilde{c}_{t1}) \leq ... \leq Eu(\widetilde{c}_{tn})$.

2. *Agent* $i = 2$ *is more ambiguity-averse than* $i = 1$.

Proof: Note that we need to find that $q^2_{\tilde{\theta}}/q^1_{\tilde{\theta}}$ and $\tilde{\theta}$ are anti-comonotonic. Using (1.11), we can rewrite the ratio as

$$\frac{q^2_\theta}{q^1_\theta} = \psi'\left(\phi_1(Eu(\widetilde{c}_{t\theta}))\right) \frac{\sum_{\tau=1}^n q_\tau \phi'_1\left(Eu(\widetilde{c}_{t\tau})\right)}{\sum_{\tau=1}^n q_\tau \phi'_2\left(Eu(\widetilde{c}_{t\tau})\right)}.$$

The fraction on the RHS does not change with θ. Furthermore, ψ' is decreasing in its argument. Finally, since the argument $\phi_1(Eu(\widetilde{c}_{t\theta}))$ is itself increasing with θ by assumption, we get the desired result.■

Hence, under the stochastic order conditions from Proposition 6, more ambiguity aversion reinforces the pessimism effect, which makes saving more attractive. However, it is clear from section 1.5 that an increase of ambiguity aversion need not reinforce the ambiguity *prudence* effect a.[10] For small amounts of ambiguity, the relation between ambiguity aversion to ambiguity on a can be approximated in the following way.

Lemma 5 *Consider a family of ambiguous environments parametrized by* $k \in R$ *and a vector* $(u_1, ..., u_n) \in R^n$ *such that* $Eu(\widetilde{c}_{t\theta}(k)) = u_0 + ku_\theta$ *for all* θ. *Let us define* $a(k) = \Sigma_\theta q_\theta \phi'(Eu(\widetilde{c}_{t\theta}(k)))/\phi'(V(k))$, *where* $\phi(V(k)) = \Sigma_\theta q_\theta \phi(Eu(\widetilde{c}_{t\theta}(k)))$. *We have that*

$$a(k) = 1 - \frac{1}{2} Var(ku_{\tilde{\theta}}) \frac{\partial}{\partial u_0}\left(\frac{-\phi''(u_0)}{\phi'(u_0)}\right) + o(k^2), \tag{1.15}$$

where $\lim_{k \to 0} o(k^2)/k^2 = 0$.

[10]For instance, introducing *increasing* absolute ambiguity aversion will in fact raise the interest rate if the representative agent is risk neutral.

Proof: Observe first that $V(0) = u_0$, $V'(0) = Eu_{\widetilde{\theta}}$, and $V''(0) = Var(u_{\widetilde{\theta}})\phi''(u_0)/\phi'(u_0)$. Notice also that $a(0) = 1$. We have in turn that

$$a'(k) = \frac{E\left[u_{\widetilde{\theta}}\phi''(u_0 + ku_{\widetilde{\theta}})\right]\phi'(V(k)) - E\left[\phi'(u_0 + ku_{\widetilde{\theta}})\right]\phi''(V(k))V'(k)}{\phi'(V(k))^2}.$$

It implies that $a'(0) = 0$. Differentiating again the above equality at $k = 0$ yields

$$\begin{aligned} {\phi'_0}^2 a''(0) &= E\left[u_{\widetilde{\theta}}^2\right]\phi'_0\phi'''_0 + \left(Eu_{\widetilde{\theta}}\right)^2\phi''^2_0 - \left(Eu_{\widetilde{\theta}}\right)^2\phi''^2_0 - \\ &\quad - \left(Eu_{\widetilde{\theta}}\right)^2\phi'_0\phi'''_0 - \phi'_0\phi''_0V''(0) \\ &= \left(E\left[u_{\widetilde{\theta}}^2\right] - \left(Eu_{\widetilde{\theta}}\right)^2\right)\left(\phi'_0\phi'''_0 - \phi''^2_0\right), \end{aligned}$$

where $\phi_0^{(i)} = \phi^{(i)}(u_0)$. This implies that

$$a''(0) = -Var(u_{\widetilde{\theta}})\frac{\partial}{\partial u_0}\left(\frac{-\phi''(u_0)}{\phi'(u_0)}\right).$$

The Taylor expansion of a yields $a(k) = a(0) + ka'(0) + 0.5k^2a''(0) + o(k^2)$. Collecting the successive derivatives of a concludes the proof.■

Accordingly, for small degrees of ambiguity, a_2 is larger than a_1 if and only if

$$\frac{\partial}{\partial u_0}\left(\frac{-\phi''_2(u_0)}{\phi'_2(u_0)}\right) \geq \frac{\partial}{\partial u_0}\left(\frac{-\phi''_1(u_0)}{\phi'_1(u_0)}\right). \tag{1.16}$$

In others words, if locally, at the ambiguity-free expected utility level u_0, absolute ambiguity aversion decreases more rapidly under ϕ_2 than under ϕ_1 the ambiguity prudence effect is more negative in economy 2.

Unfortunately, as the following example shows, even if condition (1.16) holds for all u_0, this is not sufficient to guarantee $a_2 \geq a_1$.

Counterexample 2. Let $\phi(U) = U^{1-\eta}/(1-\eta)$ be defined on R^+. Observe that $-\phi''(U)/\phi'(U) = \eta/U$ is positive and decreasing in its domain. Moreover, an increase in η raises both ambiguity aversion, and the speed at which absolute ambiguity aversion decreases with U. Proposition 5, yields that a is increasing in η when the risk on U is small. To show that this is not true for large degrees of ambiguity suppose risk-neutrality $u(c) = c$, $n = 2$ equally likely plausible probability distributions with $\overline{c}_1 = 0.5$ and $\overline{c}_2 = 1.5$, and let $\delta = 0.25$. In Figure 1, we draw the socially efficient discount rate r_t for $t = 1$ as a function of the degree of relative ambiguity aversion η. As stated in Proposition 2, we see that the discount rate $r_1(\eta)$ under ambiguity aversion is always smaller than under ambiguity neutrality ($r(0)$). However, the relationship between the discount rate and the degree of ambiguity aversion is not monotone. For example, increasing relative ambiguity aversion from $\eta = 3$ to any larger level *raises* the discount rate. ■

With a counter-example based on the most common family of utility functions $\phi(U) = U^{1-\eta}/(1-\eta)$, there is no hope for convincing sufficient conditions to guarantee an increase in savings. To summarize, we are left with three special cases where signing the effect on a is possible:

i) The degree of ambiguity aversion is small and condition (1.16) is satisfied;

ii) The initial degree of ambiguity aversion is small, so that Proposition 2 can be used as an approximation;

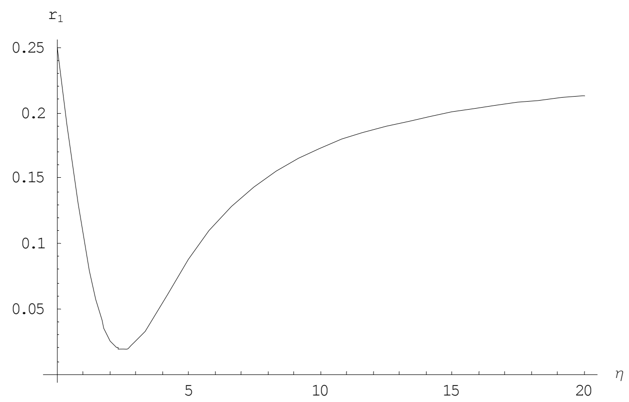

Figure 1.1: The discount rate as a function of relative ambiguity aversion. We assume that $\phi(U) = U^{1-\eta}/(1-\eta)$, $u(c) = c$, $\delta = 0.25$, $\bar{c}_1 = 0.5$, $\bar{c}_2 = 1.5$ and $p = 0.5$.

iii) The initial ϕ_1 function exhibits non decreasing ambiguity aversion, whereas the final ϕ_2 function exhibits non increasing ambiguity aversion. This implies that $a_1 \leq 1 \leq a_2$.

Combining any of these conditions with any of the three conditions from Proposition 6 is sufficient to guarantee that a marginal increase in ambiguity aversion reduces the socially efficient discount rate.

1.8 Numerical illustrations

1.8.1 The power-power normal-normal case

As observed in Section 1.3, we can solve analytically for the socially efficient discount rate by taking a "power-power" specification. That is, CRRA risk preferences and CRAA ambiguity preferences allow for an exact solution if both ambiguity and the logarithm of consumption are normally distributed. For our quantitative analysis, we parametrize the model according to the *quartet of Twos*, as put forward by Weitzman (2007b). We assume a rate of pure preference for the present $\delta = 2\%$, a degree of relative risk aversion $\gamma = 2$, a mean growth rate of consumption $g = 2\%$, and standard deviation of growth $\sigma = 2\%$. We introduce ambiguity by assuming that the growth-trend has a normal distribution with standard deviation $\sigma_0 = 1\%$. In other words, consumers believe that with a 95% probability, the growth trend lies between 0% and 4%. The Ramsey rule (1.7) implies

$$r_t = 5.88\% - 3\sigma_0^2 t(1 + \eta/2). \tag{1.17}$$

As usual, in the absence of ambiguity, the Ramsey rule prescribes a flat discount rate of 5.88%. This is no longer true under ambiguity, even for SEU agents ($\eta = 0$), as shown by Weitzman (2007a) and Gollier (2007b).

This is because ambiguity creates fatter tails in the distribution of future consumption. Indeed, ambiguity increases the volatility of log-consumption at date t by $\sigma_0^2 t^2$. Accordingly, the prudent agent wants to save more for the remote future, and the interest rate should fall with the time-horizon. If in addition, the agent exhibits ambiguity aversion, the social discount rate decreases more quickly, as seen in equation (1.17).

In order to calibrate the model, one needs to evaluate the degree of relative ambiguity aversion η. Consider therefore the following thought experiment.[11] Suppose that the growth rate of the economy over the next 10 years is either 20% – with probability π–, or 0%. Further, suppose that the true value of π is unknown. Rather, it is uniformly distributed on $[0, 1]$, as in the Ellsberg game in which the player has no information on the proportion of black and white balls in the urn.

Let us define the certainty equivalent growth rate $CE(\eta)$ as the sure growth rate of the economy that yields the same welfare as the ambiguous environment described above. It is implicitly defined by the following condi-

[11] This is based on a 10-year version of the calibration exercise performed by Collard, Mukerji, Sheppard and Tallon (2008), who considered a power-exponential specification.

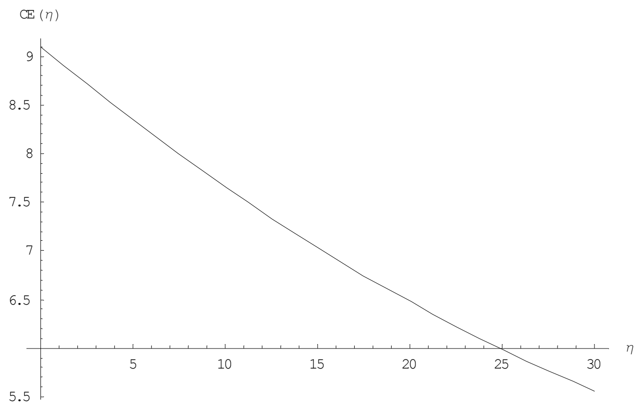

Figure 1.2: The certainty equivalent growth rate CE (in %) as a function of relative ambiguity aversion η. We assume that the growth rate is either 20% or 0% respectively with probability π and $1-\pi$, with $\pi \sim U(0,1)$. Relative risk aversion equals $\gamma = 2$.

tion:

$$\left(k\frac{(1+CE)^{1-\gamma}}{1-\gamma}\right)^{1-k\eta} = \int_0^1 \left(k\left(\pi\frac{1.2^{1-\gamma}}{1-\gamma} + (1-\pi)\frac{1^{1-\gamma}}{1-\gamma}\right)\right)^{1-k\eta} d\pi,$$

where γ is set at $\gamma = 2$. In Figure 1.2, we plot the certainty equivalent as a function of the degree of relative ambiguity aversion. In the absence of ambiguity aversion (or if π is known to be equal to 50%), the certainty equivalent growth rate equals $CE(0) = 9.1\%$. Surveying experimental studies, Camerer (1999) reports ambiguity premia $CE(0)-CE(\eta)$ in the order of magnitude of 10% of the expected value for such an Ellsberg-style uncertainty. This environment yields a reasonable ambiguity premium of 10%, i.e., a 1% reduction in the growth rate. Thus, ambiguity aversion should reduce the certainty equivalent from 9.1% to around 8%. From Figure 2, this is compatible with a degree of relative ambiguity aversion between $\eta = 5$ and $\eta = 10$.

Table 1 reports the values of efficient rates for projects with maturity 10 and 30 respectively.

Table 1.1: The social discount rate at the benchmark "quartet of twos", with $\sigma_0 = 1\%$.

t	$\eta = \mathbf{0}$	$\eta = \mathbf{5}$	$\eta = \mathbf{10}$
10	5.58%	4.83%	4.08%
30	4.98%	2.73%	0.48%

While ambiguity aversion has no effect on the short term interest rate, its effect on the long rate is important. The discount rate for a cash flow occurring in 30 years is reduced from 4.98% to 2.73% when relative ambiguity aversion goes from $\eta = 0$ to $\eta = 5$.

The discrepancies between the settings call for an empirical separation between standard risk and ambiguity in an economy. While the former shifts the level of the yield curve, the latter determines its slope. A negative slope increases the relative importance of long-term costs and benefits.

1.8.2 An AR(1) process for log consumption with an ambiguous long-term trend

Clearly, in our benchmark economy, we abstract from rich consumption dynamics, notably any serial correlation. It is thus not surprising that our predictions do not fare well when confronted with the term structure of interest rates observed on financial markets. Thus, we will relax the assumption of uncorrelated growth rates and allow for persistence of shocks, as in Collard, Mukerji, Sheppard and Tallon (2008) and Gollier (2008). We hereafter show that this model can produce the desired non-linear term structure in the short run and the medium run. While, in the limit, it generates a linearly decreasing term structure in the long run.

Consider first an auto-regressive consumption process of order 1 à la Vasicek (1977), but in which the long-term growth μ of log consumption around which the actual growth mean-reverts is uncertain:

$$\begin{aligned}
\ln c_{t+1} &= \ln c_t + x_t \\
x_t &= \xi x_{t-1} + (1-\xi)\mu + \varepsilon_t \\
\varepsilon_t &\sim N(0,\sigma^2),\ \ \varepsilon_t \perp \varepsilon_{t'} \\
\mu &\sim N(\mu_0,\sigma_0^2),
\end{aligned} \tag{1.18}$$

where $0 \leq \xi \leq 1$. That is, system (1.18) describes an AR(1) consumption process with unknown trend. The polar case without persistence ($\xi = 0$), amounts to the discrete time equivalent of the geometric Brownian motion considered in Section 1.3 and calibrated here above. In contrast, $\xi = 1$ describes shocks on the growth of log consumption that are fully persistent. Using the same techniques which led us to equation (1.6), we obtain the following generalization:

$$r_t = \delta + \gamma \frac{EX_t}{t} - \frac{1}{2}\gamma^2 \frac{Var\left[X_t \mid \mu\right] + Var\left[E[X_t \mid \mu]\right]}{t} - \frac{1}{2}\eta \left|1-\gamma^2\right| \frac{Var\left[E[X_t \mid \mu]\right]}{t}, \tag{1.19}$$

where X_t is defined as

$$X_t = \ln c_t - \ln c_0 = \mu t + (x_{-1} - \mu)\frac{\xi(1-\xi^t)}{1-\xi} + \sum_{\tau=1}^{t} \frac{1-\xi^\tau}{1-\xi}\varepsilon_{t-\tau}.$$

It yields

$$\frac{EX_t}{t} = \mu_0 + (x_{-1} - \mu_0)\frac{\xi(1-\xi^t)}{t(1-\xi)},$$

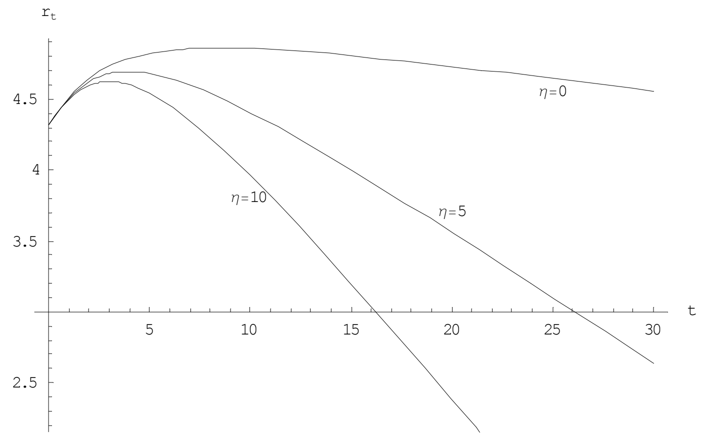

Figure 1.3: The term structure of discount rates in the case of an AR(1) with an ambiguous long term trend, with $\delta = 2\%$, $\gamma = 2$, $\mu_0 = 2\%$, $\sigma = 2\%$, $\sigma_0 = 1\%$, $x_{-1} = 1\%$, and $\xi = 0.7$.

$$\frac{Var\left[X_t \mid \mu\right]}{t} = \frac{\sigma^2}{(1-\xi)^2} + \sigma^2 \frac{\xi(1-\xi^t)}{t(1-\xi)^3}\left[\frac{\xi(1+\xi^t)}{1+\xi} - 2\right],$$

and

$$\frac{Var\left[E[X_t \mid \mu]\right]}{t} = \frac{\sigma_0^2}{t}\left(t - \frac{\xi(1-\xi^t)}{1-\xi}\right)^2.$$

To illustrate, suppose that $\delta = 2\%$, $\gamma = 2$, $\mu_0 = 2\%$, $\sigma = 2\%$, $\sigma_0 = 1\%$,and $x_{-1} = 1\%$. Following Backus, Foresi and Telmer (1998), suppose also that $\xi = 0.7$ year^{-1}, such that a shock has a half-life of 3.2 years. In Figure 3, we have drawn the term structure of discount rates for 3 different degrees of ambiguity aversion: $\eta = 0$, 5, and 10. We can see that, as in the absence of persistence, the role of ambiguity aversion is to force a downward slope on the yield curve for long time horizons. This is confirmed by the following observation:

$$\lim_{t\to\infty} \frac{\partial r_t}{\partial t} = -\frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2.$$

1.8.3 An AR(1) process for log consumption with an ambiguous degree of mean reversion

Consider alternatively an auto-regressive consumption process of order 1 with a known long-term trend, but in which the coefficient of mean reversion is unkown:

$$
\begin{aligned}
\ln c_{t+1} &= \ln c_t + x_t \\
x_t &= \xi x_{t-1} + (1-\xi)\mu + \varepsilon_t \\
\varepsilon_t &\sim N(0,\sigma^2),\ \varepsilon_t \perp \varepsilon_{t'} \\
\xi &\sim U(\underline{\xi},\overline{\xi}).
\end{aligned}
$$

There is no analytical solution for the discount rate, which must be computed numerically by estimating the following two terms, deduced from equation (1.3) (we normalized $c_0 = 1$):

$$\frac{E\phi'\left(Eu\right)Eu'}{u'(c_0)} = b(E\exp(G))$$

and

$$\phi'(V_t(0)) = b\left(E\exp\left(H\right)\right)^{\frac{-k\eta}{1-k\eta}},$$

with

$$
\begin{aligned}
G &= -(\gamma + k\eta(1-\gamma))E\left[X_t \mid \xi\right] + \frac{1}{2}\left(\gamma^2 - k\eta(1-\gamma)^2\right)Var\left[X_t \mid \xi\right], \\
H &= (1-k\eta)(1-\gamma)E[X_t \mid \xi] + \frac{1}{2}(1-k\eta)(1-\gamma)^2 Var[X_t \mid \xi].
\end{aligned}
$$

In Figure 4 we draw the term structure of the discount rate with the same parameter values as in the previous section, except that $\mu = 2\%$ and $\xi \sim U(0.5, 0.9)$. As before, longer time horizons yields more ambiguity in the set of plausi-

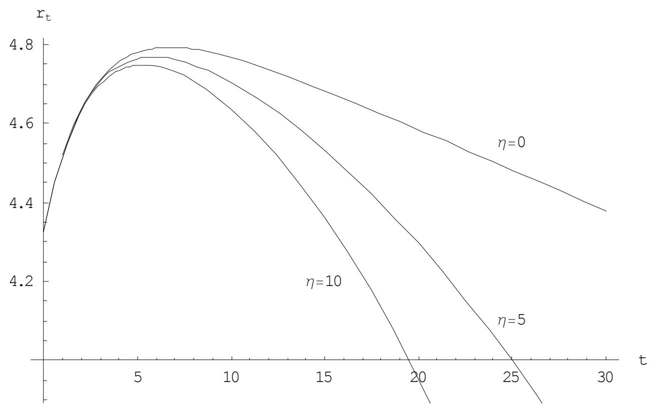

Figure 1.4: The term structure of discount rates in the case of an AR(1) with an ambiguous mean reversion coefficient, with $\delta = 2\%$, $\gamma = 2$, $\mu = 2\%$, $\sigma = 2\%$, $x_{-1} = 1\%$, and $\xi \sim U(0.5, 0.9)$.

ble distributions of consumption, which implies that ambiguity aversion has a stronger negative impact on the discount rates associated to these longer durations.

1.9 Conclusion

The present paper has shown how ambiguity-aversion affects the efficient rate to discount future costs and benefits of investment projects. In line with recent literature, our analysis suggests that parameter uncertainty might be decisive for long-term policy appraisals. We found that, in general, it is not true that ambiguity aversion decreases the discount rate. However, we identified moderate requirements on risk-attitudes and the statistical relation among prior distributions, such that decreasing ambiguity aversion should induce us to use a smaller discount rate. Our numerical illustrations suggest the effect of ambiguity aversion on the discount rate to be large, in particular for longer time horizons.

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Appendix

Proof of Proposition 1. In order to prove this result, we need the following Lemma, which is Theorem 106 in Hardy, Littlewood and Polya (1934), Proposition 1 in Polak (1996), and Lemma 8 in Gollier (2001).

Lemma 6 *Consider a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, twice differentiable, increasing and concave. Consider a vector $(q_1, ..., q_n) \in \mathbb{R}^n_+$ with $\sum_{j=1}^n q_j = 1$, and a function f from $\mathbb{R}^n$ to $\mathbb{R}$, defined as*

$$f(U_1, ..., U_n) = \phi^{-1}(\sum_{\theta=1}^{n} q_\theta \phi(U_\theta)).$$

Define function T such that $T(U) = -\frac{\phi'(U)}{\phi''(U)}$. Function f is concave in $\mathbb{R}^n$ if and only if T is weakly concave in $\mathbb{R}$.

Having established the above, consider two scalars α_1 and α_2 and let us denote $U_{i\theta} = Eu(\widetilde{c}_{t\theta} + \alpha_i e^{r_t t})$. Using the notation introduced in the Lemma, it implies that $V_t(\alpha_i) = f(U_{i1}, ..., U_{in})$. Because u is concave, we have that, for any (λ_1, λ_2) such that $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 = 1$,

$$\begin{aligned} \lambda U_{1\theta} + \lambda_2 U_{2\theta} &= E\left[\lambda_1 u(\widetilde{c}_{t\theta} + \alpha_1 e^{r_t t}) + \lambda_2 u(\widetilde{c}_{t\theta} + \alpha_2 e^{r_t t})\right] \\ &\leq Eu(\widetilde{c}_{t\theta} + \alpha_\lambda e^{r_t t}) =_{def} U_{\lambda\theta}, \end{aligned}$$

for all θ, where $\alpha_\lambda = \lambda_1\alpha_1 + \lambda_2\alpha_2$. Because f is increasing in R^n, this inequality implies that

$$\begin{aligned} V_t(\lambda_1\alpha_1 + \lambda_2\alpha_2) &= f(U_{\lambda 1}, ..., U_{\lambda n}) \\ &\geq f(\lambda_1 U_{11} + \lambda_2 U_{21}, ..., \lambda_1 U_{1n} + \lambda_2 U_{2n}). \qquad (1.20) \end{aligned}$$

Suppose that $-\phi'/\phi''$ be concave. By the Lemma, it implies that

$$\begin{aligned} f(\lambda_1 U_{11} + \lambda_2 U_{21}, ..., \lambda_1 U_{1n} + \lambda_2 U_{2n}) &\geq \lambda_1 f(U_{11}, ..., U_{1n}) + \lambda_2 f(U_{21}, ..., U_{2n}) \\ &= \lambda_1 V_t(\alpha_1) + \lambda_2 V_t(\alpha_2). \end{aligned} \tag{1.21}$$

Combining equations (1.20) and (1.21) yields $V_t(\lambda_1\alpha_1 + \lambda_2\alpha_2) \geq \lambda_1 V_t(\alpha_1) + \lambda_2 V_t(\alpha_2)$, i.e., V_t is concave in α. ■

Chapter 2

Hedging priors

2.1 Introduction

Throughout the public debate on the subprime mortgage crisis the conceptual distinction between risk and Knightian uncertainty –"known unknowns" versus "unknown unknowns"– has been emphasized:

"So what are policymakers to do? First and foremost, reduce uncertainty.[...] *Second, undo the effects of uncertainty on the portfolio side, and help recycle the funds towards risky assets."* Olivier Blanchard, IMF chief economist.[1]

Policy responses like recapitalization efforts or the purchase of "troubled" assets had to be enacted in the absence of a complete efficiency benchmark under uncertainty. It is therefore not clear which is the optimal amount of funds to be "recycled towards risky assets" or whether an uncertainty-neutral agent –possibly the "state"– should bear all uncertainty.

This paper characterizes all risk sharing allocations which satisfy the criterion of Pareto optimality. In particular, we acknowledge laboratory evi-

[1] "(Nearly) nothing to fear but fear itself", Olivier Blanchard, 01/29/2009. http://www.imf.org/external/np/vc/2009/012909.htmhttp://www.imf.org/external/np/vc/2009/012909.htm
.

dence on behavior under uncertainty which cannot be reconciled with the expected utility (EU) hypothesis.[2] This is frequently illustrated in variants of Ellsberg's renowned two-urn example.

> **Ellsberg urns** (Ellsberg, 1961). Two urns contain 90 balls, colored in black, red, or yellow. The only additional information about Urn I is that it contains exactly 30 black balls. Urn II contains 30 balls of each color.

When asked to bet on *red*, most participants prefer urn II. If they were EU maximizers this would suggest a belief that urn I contains less than 30 red balls. However, when asked to bet on *not red*, the median choice is still urn II. This cannot be reconciled with any stable belief under EU.

The evidence suggests a decision process with a particular concern about how choices fare in worse risk models. Indeed, such a pattern can be rationalized by allowing for preferences which are no longer liner in beliefs, like the (max-min) multiple priors model due to Gilboa and Schmeidler (1989), the smooth ambiguity model due to Klibanoff, Marinacci, and Mukerji (2005), or the theory of robust control developed by Hansen and Sargent (see e.g. Hansen and Sargent, 2008).

Indeed, allowing for an uncertainty-averse representative investor predicts higher equity premia and lower risk-free rates alongside moderate degrees of risk aversion.[3] Moreover, concerns about uncertainty affecting portfolio choice are substantiated in Gollier (2008).

These models make the simplifying assumption of a single investor, which is particularly useful to study prices. However, since trade does not occur in equilibrium they are not informative about how uncertainty affects risk sharing. Instead, the current paper considers a group of heterogeneous agents

[2]For a survey on the empirical evidence consult e.g. Camerer and Weber (1992).

[3]See e.g. works by Epstein and Schneider (2008), Ju and Miao (2007) or Hansen, Sargent, and Tallarini (1999)

who interact in a pure exchange economy. Similar to Wilson (1968), but allowing for smooth ambiguity aversion as proposed by Klibanoff, Marinacci, and Mukerji (2005). Our notion of ambiguity –to be made precise below– acknowledges that agents may be confronted with more than one plausible probability model, as in Ju and Miao (2007), Collard, Mukerji, Sheppard, and Tallon (2009) or Gollier (2008).

The first part of the paper highlights a profound difference between Pareto optima under ambiguity aversion and expected utility that has not been addressed in the literature: While state-contingent transfers are sophisticated enough to reach the utility possibility frontier under EU, this is no longer true under ambiguity aversion. To structure the argument, reconsider the Ellsberg game.

> **Correlated bets.** There are two participants, both endowed with bets on urn I. Each of them wins if and only if *red* is drawn. To simplify, agents know that the urn contains at least one ball of each color. Agent 1 is EU and agent 2 is ambiguity-averse.

Suppose that after observing the draw, agents could also observe the true *composition* of the urn. Can such a piece of information be valuable? Within the EU paradigm the answer must be negative. As long as *draws* can be learnt and contracted upon, any additional information can be safely ignored.

This is no longer true under ambiguity aversion. Say agent 2 was supposed to receive x units if *red* was drawn. Learning the composition of the urn could be used to specify $x+\varepsilon$ if and only if *red* was drawn from the urn containing only 1 red ball. Whereas in return she could reduce her share to $x–2\varepsilon$ if *red* was drawn from the urn with 59 red balls. Clearly, for a small enough ε, the EU agent accepts the change since her marginal expected utility is independent from the urn. At the same time, agent 2 is mainly concerned about how she fares if the worst odds turn out to be true. For high enough ambiguity aversion of agent 2 *both* of them will be better off.

What makes ambiguity aversion different from EU is not the concern about the least favorable outcome. This could be captured by increased risk aversion. Rather it is a concern about *any* outcome which was generated by unfavorable probability models.

Similarly, suppose the volatility of the *S&P 500* index were difficult to quantify. Investors with different ambiguity preferences could bet on a measure which covaries with volatility, like the volatility index VIX. Under EU, there is no reason why variance swaps, correlation swaps, and the like should be traded if not for purely speculative purposes or to exploit market imperfections. In contrast, the present paper suggests that some of those trades could also be used to hedge exposure to worse probability models.

Acknowledging the importance of risk-contingent transfers, we characterize what would be an unconstrained optimal allocation. The structure of this Pareto problem is dramatically simpler than encountered in existing literature; to the extent that the optimum can be fully characterized.

Regarding the allocation of risk, it is optimal to make sharing rules contingent on probability models – hence the deviation from EU. As if the social planner cared more about ambiguity-averse individuals in bad probability models. Importantly, for each probability candidate, the sharing rule must still be efficient for EU maximizers.

Regarding the optimal allocation of ambiguity, exposure to ambiguity should be proportional to risk tolerance. Therefore, it cannot be efficient for a single agent to bear all aggregate ambiguity unless she is not only ambiguity-neutral but also risk-neutral.

The main focus of this paper is to provide a normative benchmark. Its descriptive power is limited. Most importantly, implementation demands that the true probability model can be learnt ex post and contracted upon ex ante. In this respect the present framework is more demanding than, and highly complementary to, existing literature (Billot, Chateauneuf, Gilboa, and Tallon (2000), Chateauneuf, Dana, and Tallon (2000), or Dana (2004)).

This branch of the literature analyzes classical state-contingent consumption rules. For instance, in a framework of Choquet-expected utility preferences (see Schmeidler, 1989), the efficient state-contingent allocation of consumption will be like under EU, unless ambiguity attitudes or beliefs are too dispersed. If there is too much heterogeneity, however, the Pareto frontier cannot be characterized. Equilibria may not even be determinate (see Dana, 2004). Rigotti, Shannon, and Strzalecki (2008) extend this analysis for a general class of preferences in the special case without aggregate risk. Like under EU they find full insurance to be optimal. Rigotti and Shannon (2007) relax the assumption of zero-sum risks to prove the existence of a competitive equilibrium even under aggregate risk. A full characterization of the Pareto efficient allocations has not been achieved yet. The present paper is the first to acknowledge that the difficulties are due to the incompleteness of markets for risk-contingent claims.

The paper is structured as follows: Section 2.2 describes the economy. Section 2.3 introduces signals which foster ambiguity insurance. Section 2.4 characterizes the Pareto frontier and describes the optimal allocation of risk and ambiguity. Before concluding, implementation on competitive securities markets is discussed together with effects of ambiguity on asset prices.

2.2 The Economy

Consider an exchange economy à la Lucas (1978) in which each agent $i \in \{1, 2, \ldots, N\}$ is endowed with an uncertain quantity $\omega_i(s)$ of the single consumption good. $\boldsymbol{S} = \{1, 2, \ldots, S\}$ is an exhaustive list of realizations s. We refer to $\boldsymbol{S}$ as the set of (payoff-relevant) states .

The representation of smooth ambiguity preferences in Klibanoff, Marinacci, and Mukerji (2005) specifies two distinct notions of beliefs. First-order beliefs $p^\theta : S \to [0;1]$ assign probability weight $p^\theta(s)$ to state s, with $\sum_{s=1}^{S} p^\theta(s) = 1$, where $\theta \in \boldsymbol{\Theta} = \{1, 2, \ldots, \Theta\}$ is a parameter which identifies a plausible probability model. In other words, if θ were the true model, the objective probability of state s would be $p^\theta(s)$.

Moreover, we need to specify a set of *second-order beliefs* $q : \boldsymbol{\Theta} \to [0;1]$, assigning weight $q(\theta)$ to model θ, with $\sum_{\theta=1}^{\Theta} q(\theta) = 1$.[4] As for first-order beliefs, these beliefs may be subjective. But in order to separate mutual insurance motives from other reasons for trade, we assume that all agents perceive uncertainty in the same way.

Assumption 2 *Agents hold common beliefs in the sense that* $\{p^\theta\}_{\theta=1}^{\Theta}$ *and* q *are agreed upon among all* i.

Agents will be heterogeneous in all other dimensions, i.e. their risk preferences, ambiguity preferences and their endowments. To introduce notation and the smooth ambiguity model we will consider a particular agent i for the remainder of this section, i.e. preference parameters and endowments are understood to depend on i.

Let $\succeq^1$ be the preference over risk represented by payoff functions g with $g : \boldsymbol{S} \to \mathbb{R}$. These preferences are assumed to satisfy the usual expected utility axioms such that we can find a von Neumann-Morgenstern function u such that

$$g^+ \succeq^1 g \iff \sum_{s=1}^{S} p^\theta(s) u(g^+(s)) \geq \sum_{s=1}^{S} p^\theta(s) u(g(s)). \tag{2.1}$$

[4]In the Ellsberg game, the state corresponds to a draw $s \in \{black, yellow, red\}$, while model θ must capture the properties of urn I. For instance, $\theta = 10$ could describe an urn with 10 red balls such that e.g. $p^{10}(red) = 1/9$. The diffused prior in the Ellsberg game is such that $q(\theta) = 1/59$ for all θ.

We will consider only strictly risk-averse agents, with u three times differentiable and $u' \geq 0$, $u'' < 0$.

Similarly, $\succeq^2$ describes preferences over payoffs $h : \boldsymbol{\Theta} \to \mathbb{R}$, i.e. a prospect of $h(\theta)$ units of the consumption good contingent on the true probability model being θ. Again, like for $\succeq^1$, the expected utility axioms apply, this time using q and payoffs $h(\theta)$. We can find a function v such that

$$h^+ \succeq^2 h \Longleftrightarrow \sum_{\theta=1}^{\Theta} q(\theta)v(h^+(\theta)) \geq \sum_{\theta=1}^{\Theta} q(\theta)v(h(\theta)). \tag{2.2}$$

The "paradoxical" behavior with respect to an EU requires v to be different from u. In particular, if v is a concave transformation of u, with $\phi(u(x)) = v(x)$ and $\phi' \geq 0, \phi'' \leq 0$, then her preferences can be shown to display ambiguity aversion. As a polar case, EU preferences can be nested by taking ϕ linear.[5]

Klibanoff, Marinacci, and Mukerji (2005) show that preferences (2.1) and (2.2) translate into preferences over general payoffs $x : \boldsymbol{S} \times \boldsymbol{\Theta} \to \mathbb{R}$ as

$$x^1 \succeq x^2 \Longleftrightarrow V(x^1) \geq V(x^2),$$

with

$$V(x) = \sum_{\theta=1}^{\Theta} q(\theta)\phi\left(\sum_{s=1}^{S} p^\theta(s)u(x(s,\theta))\right). \tag{2.3}$$

Finally, we restrict the class of functions u further by imposing Inada conditions u:

$$\lim_{x\to 0} \frac{\partial}{\partial x} u(x) = +\infty, \qquad \lim_{x\to +\infty} \frac{\partial}{\partial x} u(x) = 0.$$

[5] Intuitively, to rationalize observed choices in the Ellsberg game, agents need to dislike "mean preserving probability spreads", i.e. whenever $\sum_{\theta=1}^{\Theta} qp^\theta(s) = \sum_{\theta=1}^{\Theta} q'p^\theta(s)$ and q is less dispersed than q', then being exposed to q yields greater felicity than being exposed to q'.

As usual, this assumption will make sure that it cannot be a competitive equilibrium to deprive one or more agents from consumption.

The timing of the transactions is straightforward: At $t = 0$ agents may engage in trades for the purpose of improving upon their felicity $V(\omega)$ under autarky. At $t = 1$ uncertainty resolves and ex-ante trades are honored.

2.3 Informative signals

To study interactions in the heterogeneous group $\{1, \ldots, i, \ldots, N\}$, we are going to do away with a tacit restriction in previous literature. Namely that transfers be contingent on the elements in $\mathbf{S}$ –the state space– alone. Within the expected utility paradigm such a restriction could be made without loss of generality as long as beliefs are homogeneous. We show that this is no longer true under ambiguity aversion: increasing the *span* of traded assets beyond $\mathbf{S}$ will improve welfare, even if Assumption 2 is satisfied.

To elaborate, we examine local properties of agents' preferences in the spirit of Yaari's dual theory (Yaari, 1969) and its application to ambiguity aversion in Rigotti and Shannon (2007) or Rigotti, Shannon, and Strzalecki (2008). Consider first a standard risky payoff $x = (x(1), \ldots, x(s), \ldots, x(S))$. We follow Rigotti and Shannon (2007) to call preferences for component-wise changes in x and agent's *effective prior*.

Definition 2 *Call* $\widehat{p}_{i,x} \in \Delta S$ *the agent's* effective prior *on payoffs, measured at* x*. It satisfies* $\widehat{p}_{i,x} \cdot x^+ \geq \widehat{p}_{i,x} \cdot x$ *for all* $x^+ \succeq x$ *with* $x^+ \in \mathbb{R}^S_+$*.*

Essentially, $\widehat{p}_{i,x}$ describes the hyperplane which supports an agent's upper contour set at x. If she were EU this would correspond to marginal utility-weighted probabilities

$$\widehat{p}_{i,x_i}(s) = (n_{i,x_i})p(s)u_i'(x_i(s)), \tag{2.4}$$

where $n_{i,x_i} = (\sum_{s=1}^{S} p(s)u_i'(x_i(s)))^{-1}$ is a normalizing constant.

There is a well known link between homogeneous effective priors in a heterogeneous group and *Pareto optimality* (PO). As usual, $\boldsymbol{x}$ is (PO) if there is no feasible allocation which makes at least one agent strictly better off without making any other agent worse off.[6]

Lemma 7 *Consider a group of ambiguity-neutral agents. An allocation* $\boldsymbol{x}$ *is PO if and only if there exists a vector* $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_i, \ldots, \lambda_I\}$*, strictly positive, with:*

$$\frac{u_i'(x_i(s))}{u_j'(x_j(s))} = \frac{\lambda_j}{\lambda_i}, \qquad \forall s \in \boldsymbol{S}, \forall (i,j) \in \{1, 2, \ldots, N\}^2.$$

The strict concavity of u_i together with the Inada conditions guarantee an interior solution. The result follows directly from Theorem 1 in Wilson (1968). Lemma 7 states that if agents are EU and Assumption 2 holds, then an allocation is PO if and only if it implies a common effective prior

$$\widehat{p}_{i,x_i} = \widehat{p}_{j,x_j} \equiv \widehat{p}_{\boldsymbol{x}}. \tag{2.5}$$

Consider now effective priors under ambiguity and let them be denoted by $\widehat{\nu}_{i,x_i}$. In the EU benchmark economy, ambiguity is no different than a two-stage lottery such that

$$\widehat{\nu}^{EU}_{i,x_i}(s) = \sum_{\theta=1}^{\Theta} q(\theta)(n_{i,x_i^\theta})p^\theta(s)u_i'(x_i(s)) = \sum_{\theta=1}^{\Theta} q(\theta)\widehat{p}^\theta_{i,x_i}(s).$$

In contrast, probabilities enter in a nonlinear way as soon as the agent ex-

[6] Below we will call an allocation feasible if all resource constraint are binding, i.e. individual consumption levels sum exactly to $\sum_{i=1}^{N} \omega_i(s)$ in each s. Due to the increasingness of u_i this is without loss of generality.

hibits ambiguity aversion

$$\widehat{\nu}_{i,x_i}(s) = (m_{i,x_i}) \sum_{\theta=1}^{\Theta} \left(\phi_i'(U_i^\theta(x_i))q(\theta)p^\theta(s)\right) u_i'(x_i(s)), \tag{2.6}$$

where, again, m_{i,x_i} is a normalizing constant and the remainder on the RHS is the well-known expression for marginal utility under smooth ambiguity preferences, known from Gollier (2008) or Rigotti et al. (2008).[7]

Suppose for a moment that we were to proceed as usual to determine efficient allocations. Consider therefore all $\boldsymbol{x}$ which satisfy

$$\widehat{\nu}_{i,x_i} = \widehat{\nu}_{j,x_j} \equiv \widehat{\nu}_{\boldsymbol{x}}. \tag{2.7}$$

Rigotti, Shannon, and Strzalecki (2008) show that if there is no aggregate risk ($\sum_{i=1}^{N} \omega_i(s) = cst$), then Assumption 2 and (2.7) imply full insurance, like under EU. For the general case with aggregate risk we lack a definitive characterization of optimal allocations. Rigotti and Shannon (2007) show the existence of a finite number of allocations $\boldsymbol{x}$ which satisfy (2.7) under variational ambiguity preferences, a general class introduced in Maccheroni, Marinacci, and Rustichini (2006), as well as in the KMM model.[8]

In the following we argue that allocations, even if they satisfy property (2.7), may be improved upon. Enriching the contract space in ways which would be redundant in an expected utility economy may allow mutually beneficial transfers. Consider therefore Figures 2.1 and 2.2 and the following

[7]Importantly, thanks to the differentiability of the smooth model in (2.3), there exists a unique effective prior at each x_i.In general, ambiguity preferences involve "kinks" such that, instead of holding a single effective prior, agents may hold a set of effective priors. See Rigotti et al. (2008) for a characterization of effective priors for various models of ambiguity aversion.

[8]Outside the KMM model, Chateauneuf et al. (2000) show that for special cases in the Choquet-expected utility model – which can be embedded in the class of variational preferences – such allocations look like PO under expected utility.

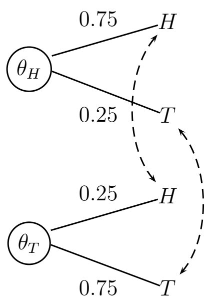

Figure 2.1: *Fundamentals setting.* Unfair coin with unknown bias.

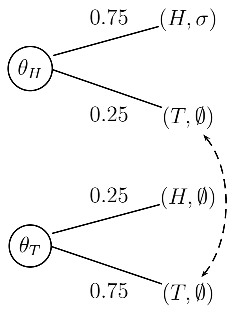

Figure 2.2: *Sunspot setting.* If heads come up, the bias can be inferred.

illustration.

> **Example. (Unfair coins)**. Let agent $i = 1$ be EU with constant income $\omega_1 = 2$ and let agent $i = 2$ be ambiguity-averse and with an income determined by a coin toss: $\omega_2(T) = 3$ or $\omega_2(H) = 1$. The coin is known to show one side three times more often, but the actual bias is unknown ex ante. Denote θ_H (θ_T) the scenario where heads is more likely (tails is more likely) and let the common second-order beliefs be $q(\theta_H) = q(\theta_T) = 0.5$. The *sunspot setting* differs from the *fundamentals setting* in the availability of a signal $\widetilde{\sigma} \in \{\sigma, \emptyset\}$, which is not observed until the actual state $s \in \{H, T\}$ is known.$\square$

Under expected utility and common beliefs, the set of PO allocations should be identical in both settings. In particular, the welfare gain from observing σ would be Zero. However, this reasoning does not extend to the case of ambiguity aversion.

Lemma 8 *Let $\boldsymbol{x}$ be an allocation which satisfies $\widehat{\nu}_{i,x_i} = \widehat{\nu}_{j,x_j} \equiv \widehat{\nu}_{\boldsymbol{x}}$. In the fundamentals setting $\boldsymbol{x}$ must be PO. If $i = 2$ is EU (strictly ambiguity-averse) then $\boldsymbol{x}$ must be (cannot be) PO in the sunspot setting.*

Since $\widetilde{\sigma}$ does not affect aggregate resources the EU claim follows directly from Borch (1962). To prove the result for ambiguity aversion it suffices to show

$$\widehat{\nu}_{1,x_1}(H) = \widehat{\nu}_{2,x_2}(H) \Longrightarrow \widehat{\nu}_{1,x_1}(H,\sigma) \neq \widehat{\nu}_{2,x_2}(H,\sigma).$$

Since agent 1 is EU, we have $\widehat{\nu}_{1,x_1}(H,\sigma) = 0.75\widehat{\nu}_{\boldsymbol{x}}(H)$, while $\widehat{\nu}_{2,x_2}(H,\sigma) = \psi 0.75\widehat{\nu}_{\boldsymbol{x}}(H)$ with $\psi = \Big(\phi_2'(U_2^{\theta_H}(x_2))/(\phi_2'(U_2^{\theta_H}(x_2))0.75 + \phi_2'(U_2^{\theta_T}(x_2))0.25)\Big)$. Since, by assumption, ϕ_2 is strictly concave, ϕ_2' will be strictly decreasing. To show $\psi \neq 1$ it suffices to show that $U_2^{\theta_T}(x_2) = U_2^{\theta_H}(x_2)$ is impossible. In a binary lottery, this requires either agent 2 being risk-neutral, which is ruled out by assumption. If u_2 is strictly concave, then $U_2^{\theta_T}(x_2) = U_2^{\theta_H}(x_2)$ requires full insurance $x_2(H) = x_2(T)$, which cannot be optimal alongside strict concavity of u_1.

To relate Lemma 8 to previous literature interpret signal $\widetilde{\sigma}$ as a means to refine the partition of states and replace the payoff-state H by two states (H,σ) and $(H,\emptyset)$.[9] If s is contractible then a group of EU agents which anticipates to observe $\widehat{\sigma}$ simultaneously with (or after) s cannot gain from $\widehat{\sigma}$ if already $\widehat{p}_{i,x_i}(H) = \widehat{p}_{\boldsymbol{x}}(H)$ for all i since this implies

$$\widehat{\nu}_{i,x_i}(H,\sigma) = 0.75\widehat{p}_{\boldsymbol{x}}(H), \qquad \widehat{\nu}_{i,x_i}(H,\emptyset) = 0.25\widehat{p}_{\boldsymbol{x}}(H), \qquad \text{for } i = 1,2.$$

This is the the classical *mutuality principle* at work (see e.g. Wilson (1968)): if average per-capita resources are independent of $\widetilde{\sigma}$, then individual consumption levels should not vary with $\widetilde{\sigma}$ either.

However, if ϕ_i is nonlinear for one or more agents then there may be disagreement on the subjective value of increasing consumption whenever $\widetilde{\sigma} = \sigma$. In particular, if ϕ_1 is linear and ϕ_2 is strictly concave then $x_2(H,\sigma) =$

[9]Consider the information gain through the signal as a refinement of the partition $\mathcal{I}_{fund} = \{\{(H,\sigma),(H,\emptyset)\},\{T,\emptyset\}\}$ and $\mathcal{I}_{sun} = \{\{(H,\sigma)\},\{(H,\emptyset)\},\{T,\emptyset\}\}$, with $\mathcal{I}_{sun} \leq \mathcal{I}_{fund}$.

$x_2(H, \emptyset)$ cannot be PO, hence the following Proposition.

Proposition 7 *Even if there exist no feasible gains from trade across states s, an allocation need not be Pareto-optimal under ambiguity aversion.*

2.4 Efficient allocations of ambiguity and risk

2.4.1 Exposure to risk

In the following we will operate under Assumption 3 in order to separate implementation issues from efficiency considerations.

Assumption 3 *There exist fully informative signals $\widetilde{\sigma}$ which allow to infer the true model θ at each s.*

An informative signal allows us to study the (unconstrained) Pareto frontier under ambiguity aversion subject to aggregate resources alone. Thanks to our assumptions on the utility function, we are able to express any PO allocation $\boldsymbol{x}^{\boldsymbol{\lambda}}$ as the solution of a utilitarian welfare problem with social weights $\boldsymbol{\lambda}$

$$\mathcal{P}^{\boldsymbol{\lambda}} = \begin{cases} \max\limits_{\boldsymbol{x}} \sum\limits_{i=1}^{N} \lambda_i V_i(x_i) \\ s.t. \sum\limits_{i=1}^{N} x_i(s,\theta) = \sum\limits_{i=1}^{N} \omega_i(s), \quad \forall s, \forall \theta, \end{cases} \tag{2.8}$$

with the necessary and sufficient conditions for optimality for any i, s, θ

$$\lambda_i q(\theta) \phi_i'(U_i^\theta(x_i^{\boldsymbol{\lambda}})) p^\theta(s) u'(x_i^{\boldsymbol{\lambda}}(s,\theta)) = \kappa^{\boldsymbol{\lambda}}(s,\theta), \tag{2.9}$$

where Lagrange multiplier $\kappa^{\boldsymbol{\lambda}}(s,\theta)$ is associated to the planner's resource constraint.

Consider now an arbitrary allocation $\boldsymbol{x}^{\boldsymbol{\lambda}} \equiv \boldsymbol{x}^*$ on the Pareto frontier and its properties. To simplify further, rewrite the first-order condition (2.9) as

$$\frac{\lambda_i}{m_i^*}\widehat{q}_i^*(\theta)p^\theta(s)u_i'(x_i^*(s,\theta)) = \kappa^*(s,\theta), \tag{2.10}$$

with the second-order prior satisfying

$$\widehat{q}_i^*(\theta) = (m_i^*)q(\theta)\phi'(U_i^\theta(x_i^*)), \tag{2.11}$$

and a normalizing constant $m_i^* = (\sum_{\theta=1}^{\Theta} q_\theta \phi_i'(U_i^\theta(x_i^*)))^{-1}$. We will call $\widehat{q}_i^*$ the agent's *ambiguity-neutral* (second-order) *probabilities* at x_i^*. [10]

Condition (2.10) depends on s in two ways: the left-hand side increases with probability p^θ while the right-hand side increases with the shadow price $\kappa^*(\cdot,\theta)$. Importantly, the state does not affect ambiguity-neutral probabilities, which allows to state the following result.

Proposition 8 *A Pareto-optimal consumption rule must replicate an optimal expected utility risk sharing rule for each plausible probability model θ.*

It is sufficient to show that for all (s,θ), there will be common risk-neutral probabilities $\widehat{p}_i^{\theta *}(s) = \widehat{p}_j^{\theta *}(s)$ for all i, j, s. The first-order conditions (2.10) imply that at the PO, for each agent i

$$\widehat{p}_i^{\theta *}(s) = \frac{\kappa(s,\theta)}{\sum_{s=1}^{S} p^\theta(s)\kappa(s,\theta)}.$$

[10]These can be expressed as effective priors when applying Definition 2 to functional $W(U) = \sum_{\theta=1}^{\Theta} q_\theta \phi_i(U(\theta))$, where $U(\theta) \equiv U_i^\theta(x_i^*)$ is the conditional expected utility at θ. Notice that we take indeed ϕ rather than v, from functional $\sum_{\theta=1}^{\Theta} q_\theta v(f(\theta))$ in (2.2). That is $\widehat{q}_{i,x_i}$ describes the hyperplane which supports the lottery over conditional levels of utility U, induced by x_i, rather than the second-order lottery induced by x_i.

The fact that the RHS does not depend on i completes the proof. Like under EU, risk-neutral probabilities will be proportional to a shadow price. The fact that this shadow price varies with θ has no effect on how risk should be allocated contingent on θ. The following restates well-known properties from EU which remain valid in the presence ambiguity aversion.

Corollary 1 *An optimal allocation $\boldsymbol{x}$ must satisfy the following properties.*

(i) ***Conditional mutuality principle.*** *Consumption of individual i in contingency (s, θ) only depends on the average per-capita endowment $z(s) = \frac{1}{N}\sum_{i=1}^{N} \omega_i(s)$. Write it as $c_i(z(s), \theta) \equiv x_i(s, \theta)$.*

(ii) ***Conditional risk tolerance rule.*** *Agent i's share of an incremental increase in z will be proportional to her risk tolerance. If the distributions p^θ over z are continuous, then*

$$\frac{\partial c_i}{\partial z}(z, \theta) = \frac{\rho_i(c_i(z, \theta))}{\frac{1}{N}\sum_{j=1}^{N} \rho_j(c_j(z, \theta))} \geq 0.$$

Consider first the conditional mutuality principle (i). It says that for each θ all idiosyncratic –hence fully insurable– risks must be washed out (see e.g. Gollier (2001)). It suffices to consider aggregate risk alone which takes on states $z \in \boldsymbol{Z} = \{1, 2, \ldots, Z\}$.

Dividing the first-order condition (2.9) by its derivative and summing over individuals yields the risk tolerance rule (ii). Like in the EU case it implies that i's share of an incremental risk must be proportional to her risk tolerance $\rho_i(c_i(z, \theta))$. Unless there is a risk-neutral agent, even the most risk-averse individual will have to bear a share of the social risk on z. This, in turn, implies that a pure bet on s cannot be Pareto-efficient, such that consumption paths must be comonotonic:

$$z \geq z' \quad \Longrightarrow \quad c_i(z, \theta) \geq c_i(z', \theta) \qquad \forall i, \forall\theta.$$

Unlike previous literature we allow variation in consumption across θ to account for heterogeneous concerns about particular probability models in an optimal way. This has been ruled out in the literature, in particular this additional variation cannot be (ex-ante) optimal under EU. From an ex-post perspective the prescriptions look similar to Billot et al. (2000), Chateauneuf et al. (2000) and Rigotti et al. (2008). They prove the optimality of expected utility risk sharing allocations under ambiguity absent aggregate risk under some conditions of minimal agreement on priors among agents.

2.4.2 Exposure to ambiguity

To better comprehend optimal allocations to ambiguity consider for a moment risk neutrality. Each agent disposes of a one-to-one technology to convert the consumption good into "utils" $U_i^\theta(x_i) = \sum_{s=1}^S p^\theta(s) x_i(s,\theta) = \overline{x}_i(\theta)$ such that the first-order conditions from (2.10) reduce to

$$\frac{\phi_i'(\overline{x}_i(\theta))}{\phi_j'(\overline{x}_j(\theta))} = \frac{\lambda_j}{\lambda_i}. \tag{2.11}$$

This is a natural equivalent to the Borch-condition from Lemma 7 but applied to transfers across probability models.

In contrast, risk aversion renders the technology to transfer utils nonlinear. To simplify the exposition, consider the case where the distribution of risk models θ is continuous. The parameter θ could stand for unknown moments of the distribution which generates z, say the hidden variance.

Thanks to the conditional mutuality principle from Proposition 8, it suffices to analyze simple sharing rules of the kind $\boldsymbol{c}^\theta = \{c_1(z,\theta), \ldots, c_N(z,\theta)\}$. Our object of interest in this section is therefore the monotonicity of $\frac{\partial c_i}{\partial \theta}(z,\theta)$.

Definition 3 *An agent is a net-receiver (net-provider) of ambiguity insurance against θ, if her share of aggregate resources increases (decreases) in θ*

for every aggregate state z.

Beforehand, recall that any Pareto problem under expected utility can be solved by splitting it into cake sharing problems $\mathcal{C}^{\boldsymbol{\lambda}}(z)$ under certainty, one for each possible aggregate state z, with

$$\mathcal{C}^{\boldsymbol{\lambda}}(z) = \left\{ \max_{\boldsymbol{c}} \sum_{i=1}^{N} \lambda_i u_i(c_i(z)), \quad s.t. \sum_{i=1}^{N} c_i(z) = zN \right\}. \tag{2.11}$$

Using our efficient risk sharing result in Proposition 8 we can directly state the following.

Corollary 2 *For each $\boldsymbol{c}^\theta = \{c_1(z,\theta), \ldots, c_N(z,\theta)\}$ there exists a vector $\widehat{\boldsymbol{\lambda}}^\theta$ such that $\boldsymbol{c}^\theta$ solves $\mathcal{C}^{\widehat{\boldsymbol{\lambda}}^\theta}(z)$.*

By our assumptions, each problem $\mathcal{C}^{\boldsymbol{\lambda}}(z)$ has the properties that the sets of first-order conditions are necessary and sufficient with

$$\widehat{\lambda}_i^\theta u_i'(c_i(z,\theta)) = \xi(z,\theta), \tag{2.11}$$

where $\xi(z,\theta)$ is the Lagrange multiplier associated to z, conditional on θ. Normalizing the vector of Pareto-weights and using the first-order conditions under ambiguity (2.10) implies that the distorted weights $\widehat{\lambda}_i^\theta$ which satisfy

$$\frac{\widehat{\lambda}_i^\theta}{\lambda_i} = l_i(\theta) \equiv \phi_i'(U_i^\theta(x_i^*))$$

generate the PO. As we order models according to i's preferences ($\theta' \geq \theta \Longrightarrow U_i^{\theta'}(x_i) \geq U_i^\theta(x_i)$) a concave ϕ_i implies $l_i'(\theta) \leq 0$. This is related to the equivalence result in Gierlinger and Gollier (2008) between the statement that i exhibits smooth ambiguity aversion and the property that q dominates $\widehat{q}_i^*$ in the sense of the monotone-likelihood-ratio ordering (MLR) since (2.11) implies $l_i(\theta) \propto \widehat{q}_i^*(\theta)/q(\theta)$.

Thus, individually, an agent's distorted *as-if* Pareto-weight for consumption in model θ reflects her pessimism about θ. However, an agreed-upon notion of pessimism requires agents to agree on a ranking of models θ, which is not possible in general.

Before stating the result we need to define the risk tolerance-weighted average operator

$$A_z^{\theta*}[b_i] = \sum_{j=1}^{N} \left(\frac{\rho_j(c_j(z,\theta))}{\sum_{k=1}^{N} \rho_k(c_k(z,\theta))} b_j \right). \tag{2.11}$$

Expression (2.4.2) is well known in general equilibrium theory. The weight associated to i corresponds to the amount of social risk she bears at z in model θ, which is why it is sometimes called a share-weighted average.

Finally, we refer to the expression $-d\log l_i(\theta) \equiv -l_i'(\theta)/l_i(\theta)$ as an agent's *subjective distortion* around θ. If it is positive everywhere, then the higher θ the less weight is assigned to the model.

Proposition 9 *Agent* i *receives ambiguity insurance against* θ *if her subjective distortion is greater than a risk tolerance-weighted average in the economy:*

$$\frac{\partial c_j}{\partial \theta}(z,\theta) = \rho_j(c_j(z,\theta))\Big(d\log l_j(\theta) - A_z^{\theta*}[d\log l_i(\theta)]\Big).$$

Differentiating the first-order condition (2.4.2) and dividing it once more by (2.4.2) implies that

$$\frac{\partial c_i}{\partial \theta}(z,\theta) = \rho_i(c_i(z,\theta))\Big(d\log l_i(\theta) - \frac{\frac{\partial}{\partial\theta}\xi(z,\theta)}{\xi(z,\theta)}\Big).$$

Taking resource constraint $\sum_{i=1}^{N} \frac{\partial c_i}{\partial \theta}(z,\theta) = 0$, then summing over i, the Lagrange multiplier's sensitivity to a change in θ can be replaced by

$$\frac{\frac{\partial}{\partial\theta}\xi(z,\theta)}{\xi(z,\theta)} = A_z^{\theta*}[d\log l_i(\theta)].$$

with $A_z^{\theta*}$ defined in (2.4.2).

Similar results on optimal side-bets can be found within the expected utility framework when Assumption 2 does not hold and agents disagree on probabilities, as in Gollier (2007). Importantly, the degree of "disagreement" in the present case is endogenous. Moreover, if two agents were to put distort q in the same way around θ but one was more risk-averse, then the latter's consumption would fluctuate less across probability models. In particular, the expression for $\frac{\partial c_i}{\partial \theta}(z, \theta)$ implies the following.

Corollary 3 *It cannot be Pareto-optimal for an ambiguity-neutral agent to bear all ambiguity in the economy unless she is also risk-neutral.*

That is, the presence of an ambiguity-neutral agent –possibly "the state"– does not necessarily mean that welfare is maximized if none of the remaining agents be exposed to ambiguity since any risk-averter dislikes consumption fluctuations, be they with s, or by providing ambiguity insurance, that is with θ.

2.4.3 Implementation: Competitive securities markets

Like under expected utility, decentralizing the PO allocation generally requires complete asset markets. Under ambiguity aversion this means that next to the usual trades to hedge exposure to risk (variations with s), investors may also wish to hedge against variations with θ. Hence the following assumption.

Assumption 4 *There exist enough linearly independent assets to span* $\mathbb{R}^{\boldsymbol{Z}\times\boldsymbol{\Theta}}$.

To fix ideas, we interpret θ as an unknown moment of market risk, say the variance of the distribution over $\boldsymbol{Z}$. Assumption 4 requires that there exist enough assets like variance swaps, options on volatility indexes like the VIX, or similar instruments which pay contingent on statistical properties of market risk. This assumption allows us to only consider contingent unit claims on (z, θ) without loss of generality.[11] By our assumptions the first welfare theorem applies such that there exists a vector $\boldsymbol{\lambda}$ for each model θ with the property that equilibrium portfolios c_i^* of unit claims solve problems $C^{\boldsymbol{\lambda}}$ from (2.4.2).

Again, we use the results in Proposition 8 and reduce the number of assets to replicate sharing rules of the kind $c_i(z, \theta)$. Denote by $\Pi(z, \theta)$ the price of a unit claim on (z, θ). An agent's portfolio c_i^* must solve

$$\mathcal{S}_i(\Pi) = \begin{cases} \max\limits_{c_i} \quad V_i(c_i) \\ s.t. \sum\limits_{\theta=1}^{\Theta} \sum\limits_{s=1}^{N} \Pi(z, \theta) \left(c_i(z, \theta) - \omega_i(s|z(s) = z) \right) = 0 \end{cases} \tag{2.11}$$

with first-order conditions

$$m_i^* l_i(\theta) u_i'(c_i^*(z, \theta)) = \psi_i^* \pi(z, \theta), \tag{2.11}$$

where ψ_i^* is the Lagrange multiplier associated to the budget constraint of individual i and where the *pricing kernel* π denotes asset prices Π per unit of probability $\pi(z, \theta) = \Pi(z, \theta)/q(\theta)p^\theta(z)$.

Definition 4 *A market equilibrium is a collection of portfolios and a price vector* $(c_1^*, \ldots, c_I^*; \Pi)$ *with*

1. $\forall i : c_i^*$ solves $\mathcal{S}_i(\Pi)$.

2. Prices Π clear markets: $1/N \sum_{i=1}^{N} c_i^*(z(s), \theta) = z(s) \quad \forall s$.

[11] That is, the pay dividends $D_{e,f}$ in the transaction period with $D_{e,f} = 1$ if $z = e$ and $\theta = f$, and nothing otherwise.

Notice again the parallels to the simple cake sharing problem $\mathcal{C}^{\boldsymbol{\lambda}}(z)$. In particular, the clearing condition for market (z, θ) is equivalent to the associated resource constraint in the cake sharing problem (2.4.2) which implies

$$\pi(z, \theta) = \xi^*(z, \theta), \tag{2.11}$$

where multiplier $\xi^*(z, \theta)$ represents the shadow value of capital under Pareto weights $\boldsymbol{\lambda}^* = \{\frac{1}{\psi_1} \ldots, \frac{1}{\psi_i}, \ldots, \frac{1}{\psi_N}\}$.

The first-order conditions (2.4.3) reveal that basic equilibrium properties from the EU model extend to the ambiguous case. Consider the sensitivity of the pricing kernel to changes in z and θ respectively. Using the risk tolerance rule, we recover the standard wealth-sensitivity of asset prices from the EU model

$$\frac{\partial \pi(z, \theta)/\partial z}{\pi(z, \theta)} = \frac{N}{\sum_{i=1}^{N} \rho_i(c_i^*(z, \theta))}. \tag{2.11}$$

Expression (2.4.3) says that the pricing kernel π is decreasing in z and its slope is inversely related to the average degree of risk tolerance. As a polar case, if there exists a risk-neutral agent, then the pricing kernel will be flat, i.e. actuarially fair, such that $\pi(z, \theta) = 1$ for all z and θ.

Can we retrieve an analogous result with respect to the unknown moment θ for a given state z? Before answering this question, consider for a moment alternative benchmarks. Take the representative agent (ra) economy with $N = 1$ as in Ju and Miao (2007) or Gierlinger and Gollier (2008), and introduce a complete market for securities. The no-trade condition yields equilibrium prices $\pi^{ra}(z, \theta) = l(\theta)u'(z)$. Risk-aversion is reflected in the sensitivity of asset prices $\frac{\partial \pi^{ra}(z,\theta)/\partial \theta}{\pi^{ra}(z,\theta)} = \frac{1}{\rho(z)}$, pessimism is reflected by the subjective distortion $\frac{\partial \pi^{ra}(z,\theta)/\partial \theta}{\pi^{ra}(z,\theta)} = d \log l(\theta)$.

In the disaggregated economy from above the distortion of the social value of capital is captured by the sensitivity of the Lagrange multiplier $\xi^*(z, \theta)$ to a change in the risk model θ. Indeed, taking the derivative of (2.4.3) yields

$$\frac{\partial \pi(z,\theta)/\partial \theta}{\pi(z,\theta)} = A_z^{\theta *}\left[d \log l_i(\theta)\right]. \tag{2.11}$$

This means that the θ-sensitivity of the price of capital in a given aggregate state z corresponds to the risk tolerance-weighted average subjective distortion.

Indeed, there is a direct relation between this result and (2.4.3) since we are able to express the wealth sensitivity as a risk tolerance-weighted average

$$\frac{\partial \pi(z,\theta)/\partial z}{\pi(z,\theta)} = A_z^{\theta *}\left[\frac{1}{\rho_i(c_i^*(z,\theta))}\right]. \tag{2.11}$$

While the Arrow-Pratt measure of risk aversion ($1/\rho_i$) describes the sensitivity of marginal utility u_i' with respect to changes in wealth, the distortion captures the sensitivity of ϕ_i' with respect to changes in θ.

In a representative agent economy, the premium on a security reflects pessimism through a higher premium for less desirable θ. In a disaggregated economy we cannot make statements about the monotonicity without imposing more structure on the economic environment.

Indeed, as stated above, the sign of the right-hand-side of (2.4.3) is ambiguous since agents might disagree on the ranking of θ. One special case where there is unanimity, arises when risk models can be ordered according to first-degree stochastic dominance (FSD). To see why recall that for every function $\overline{u}(z)$ which is weakly increasing in z we have

$$\theta \succeq_{FSD} \theta' \quad \Longrightarrow \quad \sum_{z=1}^{Z} p_z^{\theta}\overline{u}_i(z) \geq \sum_{z=1}^{Z} p_z^{\theta'}\overline{u}_i(z).$$

Define

$$\overline{u}_i^{\theta}(z) \equiv u_i(c_i^*(z,\theta)).$$

Indeed, the comonotonicity condition in Proposition 8 together with the assumption that u_i are increasing implies that $\overline{u}_i^\theta(z)$ is indeed increasing. This yields the desired result.[12]

Another special case where we can sign (2.4.3) is when all agents display the same degree of risk tolerance $\rho(x) = \beta + \gamma x$, i.e. agents share the same vNM utility function with the property of *harmonic absolute risk aversion* (HARA). Prominent examples of this class include all functions which display constant absolute risk aversion (CARA) or constant relative risk aversion (CRRA). The unanimity follows from the property that agents with identical HARA preferences always share the same attitude towards market risk, independently of wealth (see e.g. Gollier (2001)).

Our final object of interest will be portfolio separation properties which allow to reduce the minimum number of assets to decentralize the PO. We can once more import equilibrium properties from the EU framework. In particular, recall that the class of identically sloped harmonic risk aversion preferences (ISHARA) allow for a two-fund separation between a risk-less asset and a market portfolio (Rubinstein (1974)). The following definition will be useful.

Definition 5 *A risk contingent swap (α, β) with $\alpha, \beta : \Theta \to \mathbb{R}$ is a contract which obliges the holder to exchange the payoff-streams of $\alpha(\theta)$ units of security A against $\beta(\theta)$ units of security B, if θ obtains.*

Variance swaps, for instance, are derivatives by which agents trade pure exposure to statistical properties of an underlying, e.g. the S&P 500 index. The difference between a strike level of variance (fixed ex-ante) and the actual realized variance is then settled in cash, i.e a risk-less asset.

[12] Examples where models can be ordered along FSD include situations distribution functions with an unknown mean, in particular the Ellsberg game, where θ corresponds to the number of winning balls in the urn.

Proposition 10 *If preferences are of the ISHARA class, then any complete market equilibrium can be replicated by a mutual fund, a risk-less asset and risk contingent swaps.*

Under EU the ISHARA property allows to replicate any complete markets equilibrium by trading only two funds: a riskless asset and a mutual fund which replicates market risk z. Depending on θ, the agent's positions in the two assets should vary, which will be achieved through risk contingent swaps.

This result exploits the monetary separation theorem under EU due to Rubinstein (1974), which says that two funds (one of them "money") are enough to replicate any Arrow-Debreu complete markets equilibrium if agents are ISHARA. Reshuffling the positions appropriately allows to reach any point on the Pareto frontier, which is achieved by trading risk-contingent swaps.

In particular, if all individuals are CARA, the equilibrium amount invested in the market portfolio will be independent of wealth. This implies the following.

Corollary 4 *In the competitive equilibrium under ambiguity aversion and CARA utilities. Each agent i holds $\rho_i / \sum_{j=1}^{N} \rho_j$ units of the market portfolio which pays off zN. Ambiguity insurance solely crowds out the risk-less asset.*

Ambiguity insurance under CARA is like a side-bet on θ which pays out money. That is, the exposure to aggregate risk is no different under risk than under ambiguity. Analogously, if all individuals were CRRA with the same degree of relative risk aversion, then ambiguity insurance could be replicated by betting a certain amount of shares of the mutual fund on θ. Hence, the degree of leverage should be unaffected by ambiguity.

2.5 Conclusion

This paper provided a benchmark on the socially efficient allocation of risk and ambiguity in an economy with heterogeneous agents. If preferences do not satisfy the expected utility hypothesis then state-contingent transfers are an imperfect means to share resources efficiently. The distinguishing feature of the paper is to specify transfers to be made contingent on potential future information about the true probability model. If enough information is available, then despite non-expected utility preferences, any plausible risk should still be shared in an expected utility-optimal way.

This benchmark should be confronted with data from the laboratory and from financial markets. The former may corroborate our predictions for individual ambiguity insurance demand. Whereas our predictions for the aggregate could be confronted with realized premia and volumes of trade in volatility futures.

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Chapter 3

Matching and self-enforcing insurance

3.1 Introduction

Seminal works in economic theory characterize how a group of expected utility maximizers ideally share their individual risks (Borch, 1962; Arrow, 1964; Wilson, 1968): each member of the group should take on a share of the aggregate risk which is proportional to her degree of risk tolerance.

At the same time there is evidence that capital markets fall short of this efficiency benchmark (e.g. Cochrane, 1991; Mace, 1991) through market frictions like asymmetric information.

As economic agents are potentially exposed to non-tradable risk, they might instead turn to informal risk sharing institutions which are less susceptible to market frictions. Be it arrangements between employers and employees, crop sharing within a village or the absorption of income shocks within families.

Recent literature investigates whether such risk sharing motives might help to explain the composition of households when interpreted as a risk sharing institution. Schulhofer-Wohl (2006), Chiappori and Reny (2006), and Legros and Newman (2007b) consider two kinds of agents – *men* and *women*– who might choose to form a social union to share a one-shot risk. Under fairly general conditions it can be shown that any stable matching equilibrium among agents who are heterogeneous in their risk attitudes will be negative assortative. That is, when agents are ordered according to risk aversion in the sense of Arrow and Pratt, the least risk-averse woman is predicted to from a couple with the most risk-averse man, continuing through along the index until the most risk-averse woman is matched with the least risk-averse man.

As, say, risk-averse men are willing to give up a relatively large share of their income for insurance, they become attractive partners. In particular to less risk-averse women. The latter are always willing and able to outbid more risk-averse women as they are willing to take on more risk in return for the same transfer.

Beyond the theoretical importance, preferences for risk attitudes of potential spouses are key to link individual data with decisions on the household level, and, ultimately, with the aggregate. For instance, the outcome of a risk-tolerance weighted aggregation of individual characteristics may depend on the composition of households. That is, the risk aversion of the *representative agent* may depend on whether household members tend to be similar or dissimilar in risk aversion themselves.

To our knowledge there is little empirical support for the prediction of negative correlation of risk attitudes within households. Using survey data from the *German Socio-Economic Panel* (SOEP) Dohmen, Falk, Huffman, and Sunde (2008) find indeed positive correlation within couples. Moreover, the aforementioned literature predicts risk to be shared efficiently, as prescribed by (Wilson, 1968). However, Townsend (1994), for instance, finds that even for communities as small as villages, farmers do not share agricultural risks according to the efficient risk sharing predictions.

The phenomenon that despite low monitoring costs even smallest institutions fail to reach the Pareto frontier has been of interest in literature on dynamic risk sharing with limited commitment, initiated by Ligon, Thomas, and Worrall (2002b). Problems of commitment arise from the lack of institutions to enforce promises since lucky agents may refuse to make agreed-upon transfers, exploiting their threat to revert to autarky.

Importantly, under full commitment, each spouse requires at least the utility he or she would get when single *ex ante*. In contrast, under limited commitment, any candidate rule is required to be sustainable in every possible future contingency.

This paper acknowledges that even in small entities like households, risk sharing arrangements may need to account for the evolution of future bargaining power. Such as the threat of breaking up when income shocks are favorable to only one party. Our model is complementary to existing literature in introducing repeated interactions as opposed to modeling household risk as a static lottery. The prospect of being on the receiving side in the future may provide the necessary incentives for a lucky individual to live up to his or her promise and make a positive transfer.

The object of interest below will be assortative matching predictions. In particular whether the negative assortative matching equilibria from static models will be robust to limited commitment.

We investigate the question studying how couples are formed, using the interpretation of marriage markets throughout. While this is limiting the scope of the paper, our interest in small groups is in line with recent literature on commitment issues in larger groups. It has been shown that as soon as it is subgroups rather than individuals who may deviate, there will be an upper bound on the size of sustainable risk sharing groups (Genicot and

Ray, 2003).[1]

The scope of our paper is to propose a simple model of dynamic risk sharing where the sole source of heterogeneity in the economy is a group-specific –i.e. gender-specific– non-tradable risk and the individuals' risk attitudes. In particular, we will focus on situations where the full commitment prediction of negative assortative matching may fail due to the absence of efficient risk sharing.

The subsequent section introduces the economic environment and the rules of the game. Section 3.3 characterizes equilibria in a repeated game with full commitment, followed by a section 3.4 on dynamic risk-sharing under limited commitment. We then characterize stable matches in two specific cases, absent aggregate risk (Section 3.4.1) and with uncorrelated incomes (Section 3.4.2), before discussing the general case (Section 3.4.3). The final section concludes.

3.2 The model

Suppose $2N$ infinitely-lived agents, each belonging to one of two disjoint sub-populations. One half of the population belongs to the group of *women* $i \in \{1, 2, \ldots, N\}$ and the other half belongs to the group of *men* $j \in \{1, 2, \ldots, N\}$.

Each agent receives a random income of a perishable consumption good which is not tradable on financial markets. For simplicity let the economic risk be binary with two possible states of the world $s_\tau \in \{s, t\}$, where s_τ is the state of the world which realizes at date $\tau = 1, 2, \ldots$. The amount of uninsurable risk can be described by the vectors $W_i = (w_{is}, w_{it})$ and $W_j = (w_{js}, w_{jt})$, identical for all women $i = \in \{1, 2, \ldots, N\}$ and for all men $j \in$

[1] See also Bold (2009), who characterizes the solution of the model of risk sharing with limited commitment with group deviations.

$\{1, 2, \ldots, N\}$. The state space $\{s, t\}$, state-probabilities $\Pr(s_\tau = s) = p$ and $\Pr(s_\tau = t) = 1 - p$ and the income vectors are assumed to be common knowledge and independent of history $s^{\tau-1} = (s_1, \ldots, s_{\tau-1})$. That is, the individual income W_i and W_j, and the aggregate income $W_i + W_j = Z = (z_s, z_t)$ follow *iid* processes.

The binary *iid* case is arguably the simplest possible stochastic process. It is particularly appealing through its tractability in the present context. It is well known since Ligon et al. (2002b) that for more than two states of the world optimal contracts will be history-dependent and numerical methods would be needed to solve for the equilibrium. In contrast, our environment will allow us to characterize optimal contracts analytically and to perform the comparative statics analyses we are interested in.

Suppose further that there is an instantaneous utility function $(1 - \beta)u$ with $u : \{1, \ldots, N\} \times \mathbb{R} \to \mathbb{R}$, normalized by the discount factor $0 < \beta < 1$, which describes preferences over uninsurable risk. Further, let individuals $\{1, \ldots, N\}$ be described by their risk aversion.[2] That is, we assume that agents can be ranked in terms of their risk aversion in the sense of Arrow and Pratt, such that for every i' and i'' with $i'' > i'$, there exists an increasing and concave function ϕ whith $u(i'', x) = \phi(u(i', x))$. Analogously, for all $j', j'' \in \{1, \ldots, N\}$ with $j'' > j'$, Mr j'' is more risk-averse than Mr j'.

Throughout the paper we assume strict monotonicity and strict risk aversion with $\frac{\partial}{\partial c} u = u_c > 0$ and $\frac{\partial^2}{\partial c^2} u = u_{cc} < 0$ for all i, j. We assume further that u satisfies Inada conditions with respect to c for all i, j such that $\lim_{c \to 0} u_c(\cdot, c) = +\infty$, and $\lim_{c \to +\infty} u_c(\cdot, c) = 0$.

To summarize, if say, Mrs i received an *iid* income described by $C =$

[2] This simplifying assumption can be motivated by considering $(1 - \beta)u$ as the indirect utility function to evaluate nontradable risk in the presence of otherwise competitive financial markets. If it is statistically independent from tradable risk then expected utility guarantees that the indirect utility function inherits monotonicity and risk aversion from the original utility function on total wealth (see Gollier, 2001).

(c_s, c_t), then her expected discounted life-time utility would amount to

$$Eu(i, C) = pu(i, c_s) + (1 - p)u(i, c_t).$$

We now define stable matches as in Chiappori and Reny (2006). That is, any woman i may either remain in autarky and consume W_i or she may engage in a social union with a man j to form a couple, denoted by $\langle i, j \rangle$. The analogous possibilities apply to any man j. Being matched in a couple is assumed to be a necessary condition for i and j to exchange units of the consumption good, possibly state-contingent.

Each individual may be in at most one couple at a time. If this criterion is satisfied for all i, j, then the collection of couples will be called a *matching.* In particular, we are interested in the properties of matchings which satisfy the criterion of *stability.*

Definition 6 *A match is* stable *if, at each point in time τ and each state of the world s_τ, there exist no couple $\langle i, j \rangle$ such that one or more of the following conditions hold.*

1. *i or j strictly prefer to revert to autarky.*

2. *there exists a man j_0, either single or matched, such that both i and j_0 strictly prefer $\langle i, j_0 \rangle$ over their current status.*

3. *there exists a woman i_0, either single or matched, such that both j and i_0 strictly prefer $\langle i_0, j \rangle$ over their current status.*

In addition we are going to assume that couples marry, understanding that they cannot remarry after divorce. This assumption is made for technical reasons as it permits us to pin down the outside option with respect to being in a couple. One way to motivate the assumption is to account for the social and pecuniary limits on the frequency of divorce.

Finally, we need to make our notions of assortative matchings precise.

Definition 7 *A matching is* negative assortative *if for every matched couple* $\langle i', j' \rangle$*, there exists no* $i'' > i'$*,* $j'' > j'$ *who are matched in* $\langle i'', j'' \rangle$*.*

Definition 8 *A matching is* positive assortative *if for every matched couple* $\langle i'', j' \rangle$*, there exists no* $i' < i''$*,* $j'' > j'$ *who are matched in* $\langle i', j'' \rangle$*.*

Suppose that all agents were matched, then a matching is negative assortative if and only if $\{\langle 1, N \rangle, \langle 2, N-1 \rangle, \ldots, \langle N, 1 \rangle\}$, and positive assortative if and only if $\{\langle 1, 1 \rangle, \langle 2, 2 \rangle, \ldots, \langle N, N \rangle\}$.

3.3 Full commitment

Even with fully enforceable sharing rules the equilibrium analysis will be complicated by non-transferable utilities. That is, the single technology to redistribute joint utility is to exchange the consumption good, which relates to utility in a nonlinear way thanks to risk aversion. Next to distributive considerations, any consumption arrangement must therefore also allocate household risk efficiently.

To see how this affects the optimal rules, suppose for a moment that the economy were composed of one woman i and one man j alone, i.e. $N = 1$. Suppose further that they lived through a single risky period as in Chiappori and Reny (2006). Finally, say that Mr j demands a minimum utility of $(1-\beta)v$ to join the couple.

Now let the efficient sharing rule be identified by the consumption of Mrs i, denoted by $C^*(i, j, v) = (c^*_s(i, j, v), c^*_t(i, j, v))$ with

$$C^*(i, j, v) \equiv \arg\max_C (1-\beta) Eu(i, C) \quad s.t. \quad Eu(j, Z - C) \geq v.$$

Thanks to our regularity assumptions on the utility functions we are able to use the necessary and sufficient condition from Borch (1962), namely that marginal rates of substitution between states

$$M(i, C) \equiv \frac{u_c(i, c_s)}{u_c(i, c_t)}$$

and agents be equated $M(i, C^*(i, j, v)) = M(j, Z - C^*(i, j, v))$. If $C^*(i, j, v)$ is preferred to W_i, then i and j may form a couple $\langle i, j \rangle$.

Consider now a similar situation but with infinitely-lived agents facing an *iid* risk. The nature of the income process together with the preference for smooth consumption coming from risk aversion allows us to restrict our attention to stationary sharing rules such that they do not depend on the history $s^{\tau-1}$, but only on the current state of the world.

This is turn implies immediately that every optimal rule must replicate a static efficient risk sharing arrangement. Any $C^*(i, j, v)$ which is optimal to grant an instantaneous utility requirement $(1-\beta)v$ must therefore also solve the dynamic problem in which Mr j requires a discounted lifetime utility of v with

$$C^*(i, j, v) = \arg\max_C Eu(i, C) \quad s.t. \quad Eu(j, Z - C) = v. \tag{3.0}$$

Consider now the case where $N > 1$. We are going to apply a condition from Legros and Newman (2007b) to infer properties of matching equilibria. We say that two matches are *payoff-equivalent* if for all i, j the level of expected discounted life-time utility is the same in both matchings. Legros and Newman (2007b) provide a sufficient condition for payoff equivalence with negative assortative matching (NAM). That is, if there exists a stable matching which is not negative assortative, then there always a exists a NAM which yields the same level of expected discounted utility for each agent.

Stated for any level of commitment, a rule $C(i, j, v)$ will be required to fulfill that for all $i'' > i'$, $j'' > j'$, and all feasible utility levels $v' =$

$Eu(j', Z - C(i, j, v'))$ and $v'' = Eu(j'', Z - C(i, j, v''))$ we have

$$C(i'', j'', v'') \succeq_{i''} C(i'', j', v') \quad \Longrightarrow \quad C(i', j'', v'') \succeq_{i'} C(i', j', v'), \qquad \text{(GDD)}$$

where $C \succeq_i (\succ_i) D \Longleftrightarrow Eu(i, C) \geq (>) Eu(i, D)$.

Condition GDD stands for a *generalized decreasing difference* condition, whose purpose can be illustrated in the following way. If the more risk-averse woman i'' had to compete with i' over partners, then if i'' preferred to match with j'' she could never outbid i'.

If, in addition, we seek to guarantee that indeed any stable match will be negative assortative, a stronger *generalized strictly decreasing difference* (GSDD) condition is needed:

$$C(i'', j'', v'') \succeq_{i''} C(i'', j', v') \quad \Longrightarrow \quad C(i', j'', v'') \succ_{i'} C(i', j', v'). \qquad \text{(GSDD)}$$

It is worth reiterating the subtleties of conditions GDD and GSDD. These are not the same as saying that a less risk-averse woman provides insurance more efficiently. Rather, they require her to have her *comparative advantage* in providing insurance to more risk-averse men.

Similarly, a *generalized increasing difference* condition (GID) guarantees payoff equivalence with positive assortative matching (PAM). It requires that for all $i'' > i', j'' > j'$, and all feasible utility levels v'' and v'

$$C(i', j'', v'') \succeq_{i'} C(i', j', v') \quad \Longrightarrow \quad C(i'', j'', v'') \succeq_{i''} C(i'', j', v'). \qquad \text{(GID)}$$

Similarly, the counterpart to (GSDD) becomes a *generalized strictly increasing difference condition* (GSID), which guarantees PAM:

$$C(i', j'', v'') \succeq_{i'} C(i', j', v') \quad \Longrightarrow \quad C(i'', j'', v'') \succ_{i''} C(i'', j', v'). \qquad \text{(GSID)}$$

Before generalizing the monotonicity of the matching correspondence we will state a useful property from risk theory. In the remainder of the paper we will make extensive use of the domains $\mathbf{C}^- \equiv \{(c_s, c_t) : c_s \leq c_t\}$ and $\mathbf{C}^+ \equiv \{(c_s, c_t) : c_s \geq c_t\}$, where $\mathbf{C}^-$ contains all binary lotteries which pay off at least as much in t as they do in s, and the reverse for lotteries in $\mathbf{C}^+$. Naturally, any full insurance allocation lies on the security line, described by $\mathbf{C}^+ \cap \mathbf{C}^-$.

Lemma 9 (Single Crossing) *Preferences $Eu(i, C)$ have a single crossing property in (i, c_s) on either side of the security line such that*

1. *for each $X, C \in \mathbf{C}^-$ with $c_s > x_s$ (for each $X, C \in \mathbf{C}^+$ with $c_s < x_s$)*

$$X \succeq_{i'} C \Longrightarrow X \succ_{i''} C;$$

2. *for each $X, C \in \mathbf{C}^-$ with $c_s < x_s$ (for each $X, C \in \mathbf{C}^+$ with $c_s > x_s$)*

$$X \succeq_{i''} C \Longrightarrow X \succ_{i'} C.$$

A direct consequence of comparative risk aversion and $M(i'', C) \geq (\leq) M(i', C)$ for all $C \in \mathbf{C}^-$ (for all $C \in \mathbf{C}^+$). See e.g. Pratt (1964).

The above Lemma provides single crossing properties of indifference curves as implied by the expected utility hypothesis and strict concavity of functions u in consumption. It is illustrated in Figure 3.1. When interpreting X as an initial position and C-X as a lottery, property 2 means that the less risk-averse agent i' finds more lotteries acceptable which are positively correlated with X. Property 1 says that she will only be more tolerant to negatively correlated additional risk if it is large enough, i.e. the final position C risky enough. This will prove to be a key property to adapt and apply assortative matching results from Legros and Newman (2007b).

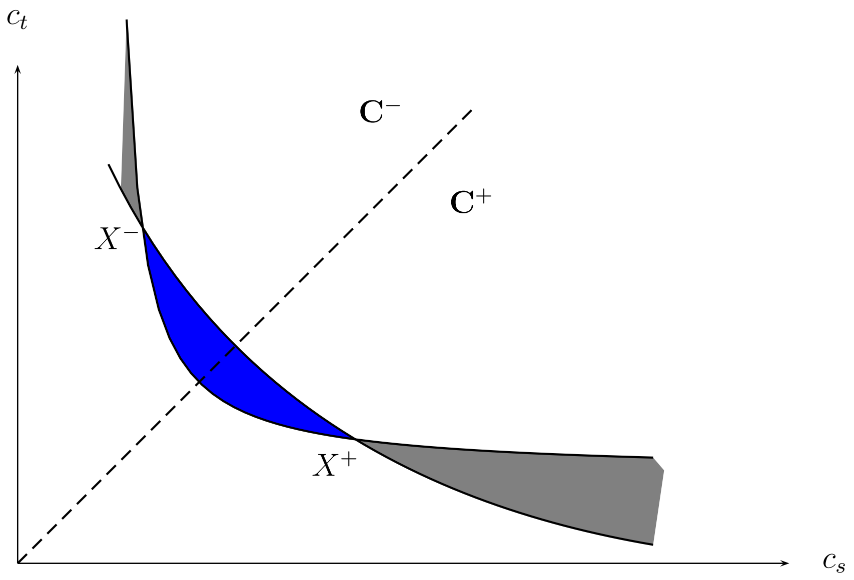

Figure 3.1: *Single crossing on each side of the security line.* A plot of indifference curves of i'' and i'. Only i'' prefers the blue area to the lotteries marked as X. Only i' prefers gray to the X.

The generalization to the case of infinitely lived agents follows directly from the optimality of stationary rules. We add to existing literature in also allowing for the absence of aggregate risk.

Proposition 11 *Suppose full commitment. Absent aggregate risk any stable matching must be payoff-equivalent to NAM. Under aggregate risk any stable matching must be NAM.*

Assume $C^*(i'', j'', v'') \succeq_{i''} C^*(i'', j', v')$. We first prove (GSDD) under aggregate risk $z_s > z_t$. Efficiency requires that both $C^*(i, j, v)$ and $Z - C^*(i, j, v)$ lie in $\mathbf{C}^+$ for any (i, j, v). Call $X(j, v) \in \mathbf{C}^+$ the crossing point which satisfies $X(j,v) \sim_i C^*(i,j,v)$ for both $i = i', i''$. By Lemma 9, GSDD obtains if

$$X(j'',v'') \succeq_{i''} X(j',v') \Longrightarrow x_s(j'',v'') > x_s(j',v'),$$

since the two conditions together imply $X(j'',v'') \succ_{i'} X(j',v')$ and hence $C^*(i',j'',v'') \succ_{i'} C^*(i',j',v')$. Efficient rules must satisfy $M(i, C^*(i,j,v)) = M(j, Z - C^*(i,j,v))$. Also, by comparative risk aversion $c_s^*(i,j,v)$ must be decreasing in i since $i'' > i' \Longrightarrow M(i',C) > M(i'',C)$ on $\mathbf{C}^+$ and therefore $c_s^*(i',j,v) > x_s(j,v) > c_s^*(i'',j,v)$. Lastly, call $Y \in \mathbf{C}^+$ the crossing point which satisfies $Eu(j',Y) = v'$ and $Eu(j'',Y) = v''$. If Mrs i'' preferes j'', then

$$z_s - c_s^*(i'',j'',v'') < y_s,$$

otherwise, by Lemma 9, we would have $Z - C^*(i'',j'',v'') \succ_{j'} Y$ such that the utility constraint of j' would be slack at $C^*(i'',j'',v'')$. The two conditions combined imply $x_s(j'',v'') > z_s - y_s$. Suppose first $z_s - y_s \geq c_s^*(i',j',v')$. This implies GSDD as it leads to $x_s(j'',v'') > x_s(j',v')$. Suppose instead $z_s - y_s < c_s^*(i',j',v')$. This implies $Z - C^*(i',j',v') \succ_{j''} Y$ and, again, i' strictly prefers to match with j'', which completes the proof for GSDD. Finally, we prove (GDD) absent aggregate risk. Efficiency requires full insurance and hence $C^*(i',\cdot,\cdot) = C^*(i'',\cdot,\cdot)$ for all j, v, which leads to (GDD).

Condition GSDD follows from Lemma 9, the fact that all efficient sharing rules must be such that $c_s > c_t$ and the fact that partnering up with a more risk-averse man involves taking up more of the household risk, i.e a greater c_s. This is where the less risk-averse woman has her comparative advantage.[3]

Our result goes beyond existing literature as we allow for idiosyncratic

[3]The GSDD result under aggregate risk obtains directly from the static models in Legros and Newman (2007b) and Chiappori and Reny (2006). We have proven it alternatively using single crossing conditions of the primitives, which will be useful for the remainder of the paper.

risk, where $z_s = z_t$. In this case GSDD has to be weakened to GDD. Since in this case no woman has a comparative advantage in providing full insurance to a particular man, it is always possible to find stable arrangements which are not necessarily NAM, but payoff-equivalent to it, i.e. the same full insurance arrangements with rearranged partners. This result also extends to static environments, as in Legros and Newman (2007b) and Chiappori and Reny (2006) when the assumption of aggregate risk is relaxed.

3.4 Limited Commitment

Suppose now that there is limited commitment on marriage markets in the sense that any risk sharing arrangements must be self-enforcing for all contingencies s_τ. Naturally, the utility possibility frontier within couples deteriorates by constraining rules to not only to be individually rational ex ante. Subsequently, the non-starred $C(i, j, v)$ are understood to solve the following problem

$$C(i, j, v) \equiv \arg\max_C Eu(i, C) \tag{3.-4}$$

subject to

$$\begin{aligned}
Eu(j, Z - C) &= v, \\
u(i, c_s) - u(i, w_{is}) &\geq {}^{\beta}/_{1-\beta} E[u(i, C) - u(i, W_i)], && (Is) \\
u(i, c_t) - u(i, w_{it}) &\geq {}^{\beta}/_{1-\beta} E[u(i, C) - u(i, W_i)], && (It) \\
u(j, z_s - c_s) - u(j, w_{js}) &\geq {}^{\beta}/_{1-\beta} E[u(j, Z - C) - u(j, W_j)], && (Js) \\
u(j, z_t - c_t) - u(j, w_{jt}) &\geq {}^{\beta}/_{1-\beta} E[u(j, Z - C) - u(j, W_j)]. && (Jt)
\end{aligned}$$

Next to the utility constraint the four *enforcement constraints* guarantee that autarky is worse than making the transfer prescribed by the rule C.

The optimality of stationary contracts allows us to reduce the number of enforcement constraints for every individual and history s^τ to just one constraint per individual per state of the world. Below we will say that C is *attainable* in (i, j, v) if it satisfies all of the five above constraints.

In particular we only consider trigger-strategies, as is standard in the literature following Kocherlakota (1996) and Ligon et al. (2002b). That is, if one of the members of a couple breaks the match in period τ, her former partner is not going to make any transfer for the remaining future, which has been shown to be the most severe subgame-perfect punishment for defection.

A useful way to reinterpret the enforcement constraints is in terms of *subjectively* distorted probabilities. Let $\widehat{p}^s = \beta p + (1 - \beta)$ and $\widehat{p}^t = \beta p$. Accordingly, let hatted random variables be distributed according to the distorted state probabilities with e.g. $\widehat{W}_i^s \sim (\widehat{p}^s, w_{is}; 1 - \widehat{p}^s, w_{it})$ or $\widehat{W}_i^t \sim (\widehat{p}^t, w_{is}; 1 - \widehat{p}^t, w_{it})$. The "conflict of interest" between ex-ante insurance and making transfers ex post can be rewritten in terms of subjective evaluations of future consumption. For instance, constraint (Is) can be rewritten as $\widehat{C}_i^s \succeq_i \widehat{W}_i^s$, and constraint (It) as $\widehat{C}_i^t \succeq_i \widehat{W}_i^t$ respectively.

Recall that the full commitment Proposition 11 relies on comparative statics properties of an interior solution. By contrast, absent commitment, the comparative advantages of agents may also be determined by how enforcement constraints vary with risk aversion. As a first step to solve the problem, we are going to eliminate redundant enforcement constraints.

Lemma 10 *For each couple at most two enforcement constraints will be binding at the same time:*

1. *if $M(i, W_i) > M(j, W_j)$, then (Is) and (Jt) will be slack;*

2. *if $M(i, W_i) < M(j, W_j)$, then (It) and (Js) will be slack.*

Promising strictly positive transfers in any state cannot be preferred to autarky. At least one enforcement constraint must therefore be slack. It cannot be optimal for both i and j to make net transfers in the same state. Thus (Is) and (Js) (resp. (It) and (Jt)) cannot be binding at the same time. If $M(i, W_i) > M(j, W_j)$ then autarky Pareto-dominates an arrangement with a positive transfer to j in state s. By the same argument, autarky Pareto-dominates a positive transfer to j in state t if $M(i, W_i) < M(j, W_j)$.

Lemma 10 follows from the observation that marginal rates of substitution govern the flow of transfers in equilibrium, i.e. the individual who will make a transfer in state s. This result reflects the basic insight that agents have a greater incentive to break up the match in states where they are supposed to give up some of their income to the benefit of their spouse.

3.4.1 No aggregate risk

Consider now an economy where, if they pooled their income, each household receives a sure income $z_s = z_t = z$. Without loss of generality, suppose that women receive more in state t. As a result, since $M(i, W_i) > M(j, W_j)$, Lemma 10 allows us to disregard the redundant enforcement constraints (Is) and (Jt).

The following counterexample shows that the payoff equivalence with NAM under idiosyncratic risk ceases to be an equilibrium property.

> **Example.** Suppose two women $i = 1, 2$ and two men $j = 1, 2$ with constant relative risk aversion $u(1, x) = -\frac{2}{\sqrt{x}}$ and $u(2, x) = -\frac{1}{x}$. Indeed $u(2, \cdot) = \phi(u(1, \cdot))$ with $\phi(x) = -0.25x^2$ increasing and concave on the nonpositive domain. Further suppose $\beta = 2/3$, $p = 1/2$, $W_i = (2, 4)$, and $W_j = (4, 2)$. Consider now the couple $\langle 2, 2 \rangle$. One can show that enforcement constraints (Is) and (Jt) will be satisfied with equality if there is a full insurance arrange-

ment with $(3, 3)$. Further, according to the parameters, Mrs 1 cannot promise expected utility of $v \geq -1/3$ to Mr 2 without violating her enforcement constraint (It). The same holds for Mr 1, who cannot offer utility $-2/\sqrt{3}$ to Mrs 2 without violating his constraint (Js). Hence $\langle 2, 2 \rangle$ must be part of any stable matching. Finally, there exist enforceable rules for $\langle 1, 1 \rangle$ which are preferred to autarky for both $i, j = 1$, for instance $C_1 = (2.2, 3.8)$.

This example shows that the magnitude of transfers which are enforceable for the more risk-averse Mrs 2 in state t are not compatible with (It) for Mrs 1. This makes Mrs 2 an attractive partner. In particular for Mr 2, whom she prefers weakly over Mr 1 for just the same qualities: he will be able to make a credible promise to transfer a greater share of his income in state s thanks to being relatively less constrained by (Js). It turns out that when the two risk-averse agents form a couple, they can even reach the Pareto-optimal full insurance allocation $(3, 3)$ while the couple of less risk-averse agents only achieves partial insurance.

The counterexample makes evident that it is necessary to go beyond standard comparative statics on the *willingness* to take on risk. In addition we also need to take into account the *ability* to make credible promises.

In particular, by $M(i, W_i) > M(j, W_j)$, a necessary condition for sharing rules C to be preferred to autarky for both i and j is that $c_s \geq w_{is}$. By Lemma 9 we can directly state the following.

Corollary 5 *Absent aggregate risk, suppose a $C \in \mathbf{C}^-$ with $c_s \geq w_{is}$. If promising C is credible for i' then it must also be credible for $i'' > i'$.*

Moreover, prior to deriving matching properties, we need to acknowledge that the utility constraints in (3.4) may be slack. Indeed, if (Js) is binding at C, then granting more than utility v is a Pareto-improvement if $M(i, C) >$

$\frac{\widehat{p}^s}{p}\frac{1-p}{1-\widehat{p}^s}M(j, Z-C)$. That is, a movement along the "twisted indifference curve" (Js) in the direction of c_s is mutually beneficial.

Call $\underline{v}_i$ the greatest maximizer of $Eu(C(i,j,v))$ with $Eu(C(i,j,v)) = Eu(C(i,j,\underline{v}_i))$ for all $v < \underline{v}_i$ with the possibility that, trivially, $\underline{v}_i = u(j,0)$. Further, we denote by $\underline{v}'_i$ (resp. $\underline{v}''_i$) the minimum utility that Mrs i grants to i' (resp. i''). The following states comparative statics properties of the minimum utility.

Lemma 11 *Suppose no aggregate risk and $W_i \in \mathbf{C}^-$. The following properties hold:*

1. *Consider a man j. More risk-averse women grant higher minimum utility $\underline{v}_{i''} \geq \underline{v}_{i'}$;*

2. *If $v'' \leq \underline{v}''_i$ then matching with j'' is preferred over matching with any $j' < j''$: $C(i, j'', \underline{v}''_i) \succeq_i C(i, j', \underline{v}'_i)$.*

Property 1. follows from $M(i'', C) \geq M(i', C)$ on $\mathbf{C}^-$ and the fact that among all C which satisfy (Js) with equality and with $Z - C \in \mathbf{C}^+$, $Eu(j, Z - C)$ increases with c_s thanks to $0 < \beta < 1$. To prove property 2 use Corollary 5 to show that $C(i, j', v')$ must also be attainable in (i, j'', v''), thus if $C(i, j'', v'')$ yields maximal utility for i in $\langle i, j'' \rangle$, then she must weakly prefer matching with j'' under $v < \underline{v}''_i$.

All candidate rules C must lie in $\mathbf{C}^-$ (resp. $Z - C$ lies in $\mathbf{C}^+$) since both the Pareto frontier of full insurance and the initial endowment of W_i lie in $\mathbf{C}^-$. For every proposal coming from Mrs i', Mrs i'' will be able to provide at least as much insurance without violating her enforcement constraint. Similarly, for any arrangement with j', there will be an enforceable arrangement with j'' involving equal or larger transfers in each state of the world. That is, the greater the amount of risk aversion, the more closely a couple may approach the Pareto frontier of full insurance.

Finally, recall that under full commitment less risk-averse women were always willing to promise more insurance to men. This property is reversed on $\mathbf{C}^-$, which allows us to state the following result.

Proposition 12 *Under limited commitment and no aggregate risk any stable match will be payoff-equivalent to a PAM. In general, there will be no payoff equivalence with NAM.*

The second claim follows from the example above. It remains to prove (GID), that is $C(i',j'',v'') \succeq_{i'} C(i',j',v') \Longrightarrow C(i'',j'',v'') \succeq_{i''} C(i'',j',v')$. The property follows immediately from Lemma 11 for all $v'' \leq \underline{v}''_{i''}$. Suppose instead $v'' > \underline{v}''_{i''}$. Consider first a slack enforcement constraint (It) at $C(i',j'',v'')$. This occurs when the rule prescribes either full insurance or partial insurance constrainder by (Js). By $v'' > \underline{v}''_{i''} > \underline{v}''_{i'}$ requirement v'' will be satisfied with equality. By Corollary 5 constraint (It) for i'' must be slack, which implies $C(i'',j'',v'') = C(i',j'',v'') \equiv X$. Since $v'' > \underline{v}''_{i''} > \underline{v}''_{i'}$, any $i = i', i''$ strictly prefers X over C, if C satisfies (Js) for any $j = j', j''$ and transfers are larger or equal than under X, i.e. $c_s > x_s$. Thus Lemma 9 delivers the desired result if we could show that all $C \succ_{i''} X$ with $c_s < x_s$ attainable in (i'',j',v') must also be attainable in (i',j',v'). Indeed $C \succ_{i''} X \Longrightarrow \widehat{C}^t \succ_{i''} \widehat{X}^t \succeq_{i''} \widehat{W}^t_i$ by $0 < \beta < 1$ and by X attainable in (i',j'',v''). This implies $\widehat{C}^t \succ_{i'} \widehat{W}^t_i$ by Lemma 9 as desired. Consider the remaining case where (It) is binding in (i',j'',v''). If Mrs i' preferred j'' over j' then $c_s(i',j'',v'') \geq z_s - y_s$ with Y defined by $Eu(j',Y) = v'$ and $Eu(j'',Y) = v''$. By Corollary 5 constraint (It) is slack for i'' at $C(i',j'',v'')$. Also, for all $C \in \mathbf{C}^-$ we have $M(i'',C) \geq M(j, Z - C)$. Together these imply $c_s(i'',j'',v'') \geq c_s(i',j'',v'') \geq z_s - y_s$ and $c_s(i'',j'',v'') \geq x_s(j'',v'') \geq c_s(i',j'',v'')$ where crossing point $X(j,v)$ is defined as $X(j,v) \sim_{i'} C(i',j,v)$ and $X(j,v) \sim_{i''} C(i'',j,v)$. Suppose first $z_s - y_s \geq c_s(i'',j',v')$. This implies GID as it leads to $x_s(j'',v'') \geq x_s(j',v')$. Otherwise $z_s - y_s < c_s(i'',j',v')$,

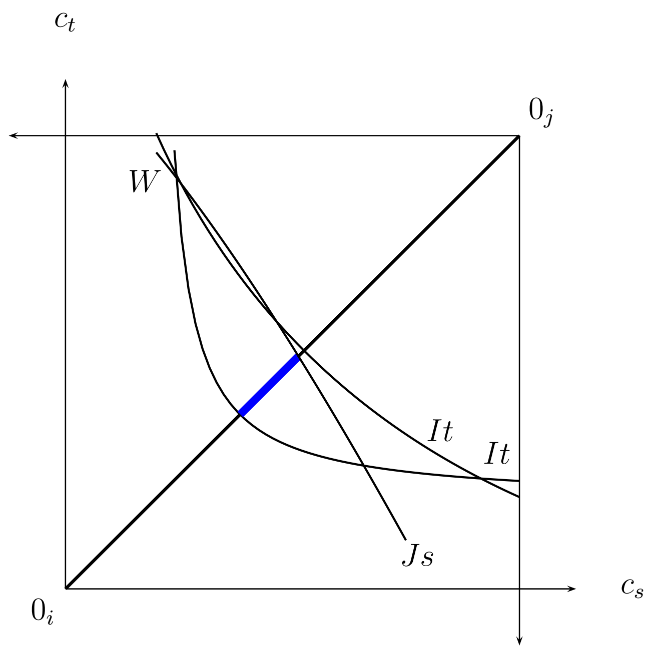

Figure 3.2: Two women compete for a man. The areas enclosed by It and Js mark the sets of enforceable sharing rules. For this specification only the more risk-averse woman $i'' > i'$ will be able to offer full insurance.

which implies $Eu(j'', Z - C(i'', j', v') > v''$ and hence GID.

The counterexample proved that NAM was no longer a general property as a less risk-averse spouse may face aggravated commitment problems. Proposition 12 goes further to show that more risk-averse woman will always be able to outbid a less risk-averse woman if the candidate rules lie on $\mathbf{C}^-$. Along the same reasoning, more risk-averse men will be more attractive partners since their enforcement constraints permit them to make greater transfers in state s. When combined these properties imply that, absent aggregate risk, couples will be formed by the most similar agents.

Figure 3.2 illustrates how the enforcement constraints favor the less risk-averse agent. Importantly, the slopes are different from the slopes of indifference curves. In particular, (Js) "underweights" consumption in state t while the curves (It) "overweight" state t.

The disciplining device of being sent into autarky for the remaining future is therefore less of a threat to less risk-averse agents. One might conjecture that this reasoning should hold more generally. The following section, however, shows that this is not true in general.

3.4.2 Uncorrelated risks

Suppose now that W_i and W_j are not correlated and that there is aggregate risk $z_s > z_t$. In the two states set-up this means that either W_i or W_j must be constant. Without loss of generality, suppose that W_i is risk-less such that $w_{is} = w_{it} \equiv w_i$ and therefore, since only men are exposed to aggregate risk, we have $M(i, W_i) < M(j, W_j)$ for all i, j.

Using Lemma 10 again, the direction of the insurance transfers will be identical to the previous section. That is, a necessary condition for sharing rules to be preferred to autarky for both i and j is that women will receive a transfer in s.

In contrast to the previous section both W_i and all unconstrained sharing rules C^* lie on $\mathbf{C}^+$. Accordingly, applying Lemma 9 yields that enforcement constraints will be more difficult to satisfy for more risk-averse women.

Corollary 6 *Absent income correlation, suppose a $C \in \mathbf{C}^+$ with $c_s \geq w_i$. If promising C is credible for i'' then it must also be credible for $i' < i''$.*

Making a positive transfer in t implies for women to give up a sure position. Applying the properties of comparative risk aversion to enforcement constraints, a less risk-averse woman will always be able to make larger transfers in t.

Again, as in the previous section, the utility constraint in (3.4) may be slack if β is very small such that $M(i,C) > \frac{\widehat{p}^s}{p}\frac{1-p}{1-\widehat{p}^s}M(j,Z-C)$ for some rules C which satisfy (Js) with equality. This is going to reverse the comparative statics properties of the minimum utility with respect to the idiosyncratic case.

Lemma 12 *Suppose no income correlation and a sure income for women. The following properties hold:*

1. *Consider a man j. Less risk-averse women grant higher minimum utility $\underline{v}_{i'} \geq \underline{v}_{i''}$;*
2. *If $v'' \leq \underline{v}''_i$ then matching with j'' is preferred over matching with any $j' < j''$: $C(i, j'', \underline{v}''_i) \succeq_i C(i, j', \underline{v}'_i)$.*

Property 1. follows from $M(i',C) \geq M(i'',C)$ on $\mathbf{C}^+$ and the fact that among all C which satisfy (Js) with equality and where $Z - C \in \mathbf{C}^+$, $Eu(j, Z - C)$ increases with c_s thanks to $0 < \beta < 1$. To prove property 2 proceed as for Lemma 11.

If there is aggregate risk, then all candidate rules C must be such that both C and $Z - C$ lie in $\mathbf{C}^+$ since the unconstrained efficient rules C^* and the initial endowment of W_i lie in $\mathbf{C}^+$ as well. Therefore, the intuitive comparative statics properties from the full commitment case obtain. Indeed, for every proposal coming from Mrs i'', Mrs i' will be willing and able to outbid her. Hence the following.

Proposition 13 *If incomes are uncorrelated, condition GSDD holds and any matching will be NAM.*

We want to show $C(i'', j'', v'') \succeq_{i''} C(i', j', v') \Longrightarrow C(i', j'', v'') \succ_{i'} C(i', j', v')$. If $v'' \leq \underline{v}''_{i'}$ then the desired property follows directly from Lemma 12. Suppose instead $v'' > \underline{v}''_{i''}$. Consider first partial insurance with (Js) binding and

(It) slack in (i', j'', v''). By $v'' > \underline{v}''_{i'} > \underline{v}''_{i''}$ requirement v'' will be satisfied with equality and (It) for i' will be slack thanks to Corollary 6, such that $C(i', j'', v'') = X = C(i', j'', v'')$. Proceeding in the proof as for Proposition 12 but with the reversed single-crossing properties on $\mathbf{C}^+$ yields GDD. Suppose now the remaining cases of either efficient risk-sharing or partial insurance with (It) binding for (i'', j'', v''). Since $M(i', C) > M(i'', C)$ for all $C \in \mathbf{C}^+$, and by Corollary 6 we must have $c_s(i'', j'', v'') > x_s(j'', v'') > c_s(i', j'', v'')$ where $X(j, v) \in \mathbf{C}^+$ is the crossing point $X(j, v) \sim_i C(i, j, v)$ for $i = i', i''$. If Mrs i'' preferred j'' over j' then the transfers must be such that $c_s(i', j'', v'') \geq z_s - y_s$ with Y defined by $Eu(j', Y) = v'$ and $Eu(j'', Y) = v''$. Proceeding as in the proof of Proposition 11 yields that either $x_s(j'', v'') > x_s(j', v')$ or $z_s - y_s < c(i', j', v')$ and thus GSDD.

To say that an individual i'' is more risk-averse than i' in the sense of Arrow and Pratt is equivalent to saying that if adding a given risk to a sure income is acceptable for i'' then it must be acceptable for i'. In the case of uncorrelated risks this implies two properties which both work in the direction of NAM. On the one hand, enforcement constraints of more risk-averse women are harder to satisfy since autarky means a sure income for the remaining future. On the other hand, for a potential spouse j, the less risk-averse woman is willing to make an offer which promises larger transfers in t, while the more risk-averse man is willing and able to give up a larger share of his consumption in s in return.

3.4.3 Discussion of the general case

The previous subsections covered two special cases where all agents who were willing to make larger transfers from an ex-ante perspective were also the ones whose enforcement constraints were weaker, which is not true in general.

Suppose for instance the case where there is aggregate risk but negative income correlation. Figure 3.1 reveals that the enforcement constraints for the less risk-averse agent may then be either weaker or more severe on $\mathbf{C}^+$, depending on how large a transfer will be made in state t.

That is, for β low enough, the less risk-averse woman may not be able to outbid i'' for a more risk-averse spouse. In particular, we may have a case where β so small that no enforceable risk sharing rule in $\langle i', j'' \rangle$ exists, while the couple $\langle i'', j'' \rangle$ may achieve partial insurance. On the other hand, as β approaches 1, the comparative advantage of the less risk-averse agent to take on more risk can be exploited without violating her enforcement constraint and NAM obtains.

Similarly, a low parameter β might make it impossible to match with agents if the gains from trade are small. Suppose therefore a potential couple $\langle i, j'' \rangle$ where the endowment W_i is close to the efficient risk-sharing allocation. For instance we might obtain a case where i would need to make a positive transfer in t, i.e. $M(i, W_i) > M(j'', W_j)$ but for β low enough we might find that

$$\frac{M(i, W_i)}{M(j'', W_j)} < \frac{\widehat{p}^t}{\widehat{p}^s} \frac{1 - \widehat{p}^s}{1 - \widehat{p}^t}.$$

This leaves Mrs i incapable of reaching a partial insurance arrangement with j''. At the same time, there may be a more risk-averse man $j''' > j''$ with large enough gains from trade $M(i, W_i) > M(j''', W_j)$. In particular, there may be a less risk-averse man j' such that $M(i, W_i) < M(j'', W_j)$, i.e. i would want to make a transfer in s, with gains from trade large enough

$$\frac{M(i, W_i)}{M(j', W_j)} < \frac{\widehat{p}^s}{\widehat{p}^t} \frac{1 - \widehat{p}^t}{1 - \widehat{p}^s}.$$

Hence there would be enforceable risk sharing arrangements with j''' and j', while no enforceable transfers could be made to j''.

Therefore, extending our matching predictions to more general environments requires to determine the potential spouses for each agent, depending on the gains from trade $\frac{M(i,W_i)}{M(j,W_j)}$ unless parameter β is sufficiently close to 1.

3.5 Conclusion

Recent literature provided predictions on the composition of risk attitudes within households. This paper has relaxed the assumption of legally binding insurance contracts. Our results predict that positive correlation between the risk attitudes of spouses may occur when men and women's income are negatively correlated.

Complementary work would investigate numerically whether our assortative matching predictions extend to more realistic stochastic environments to be then confronted both with survey data and evidence from the laboratory.

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